

RSK Linear Operators and the Vershik-Kerov-Logan-Shepp Curve

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Abstract

Stelzer and Yong (2024) studied the Robinson-Schensted-Knuth (RSK) correspondence as a linear operator on the coordinate ring of matrices. They showed that this operator is block diagonal and conjectured that, in a special block, most diagonal entries vanish. We establish this conjecture by identifying these zeros with certain Schensted insertion interactions and analyzing them probabilistically using the Vershik-Kerov-Logan-Shepp Limit Shape Theorem.

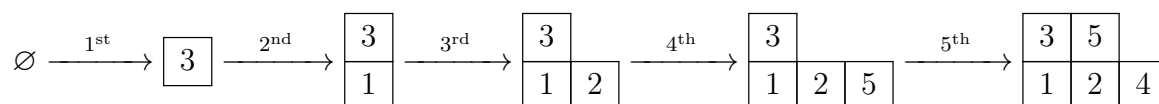
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1 Introduction

1.1 The Schensted insertion algorithm

Given a permutation in one-line notation, the *Schensted insertion* algorithm constructs a *standard Young tableau* by sequentially inserting each number while maintaining increasing order along rows and columns [8]. Starting with an empty tableau, each number is inserted into the first row. If it is larger than all entries in the row, it is appended to the end. Otherwise, it replaces the smallest greater number, which is “bumped” to the next row, where the process repeats. If a number is bumped from the last row, a new row is created.

Example 1. The permutation $31254 \in S_5$ is processed as follows.



We call a bump *vertical* if it moves a number directly upward within the same column; otherwise, we call it *lateral*. Since columns in standard Young tableaux (in French

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notation) are strictly increasing, a lateral bump always shifts a number strictly to the left. In Example 1, the second insertion bumps 3 from the first row and first column to the second row, but stays in the same column, so the bump is vertical. However, the fifth insertion bumps 5 from the third column to the second column, making it lateral.

Let V_n denote the set of all permutations in S_n whose Schensted insertion produces no lateral bumps. As we shall explain, the proof of the Stelzer-Yong conjecture (Theorem 3) follows from Theorem 2 below. Informally, Theorem 2 states that asymptotically almost all permutations have a lateral bump in their Schensted insertion, i.e., the probability under the uniform measure on S_n that a permutation has no lateral bumps tends to zero as $n \rightarrow \infty$. We deduce this probabilistic statement from the celebrated Vershik-Kerov-Logan-Shepp limit shape theorem concerning random partitions under the Plancherel measure.

Theorem 2. $\lim_{n \rightarrow \infty} \frac{|V_n|}{n!} = 0$.

Theorem 2 is closely related to the results of Romik and Śniady [7], who analyzed the bumping route created by the last number in a permutation. Their work shows that the asymptotic shape of this route depends on the value of the last number, leading to a family of limiting curves. Theorem 2 seems derivable from their results (although we could not do so). Instead, we give a self-contained proof based only on the Limit Shape Theorem and elementary arguments.

1.2 The Stelzer-Yong conjecture

The RSK correspondence is a well-studied combinatorial bijection between matrices with non-negative integer entries and pairs of semistandard Young tableaux (SSYT) of the same shape. It is a more general version of the Schensted insertion algorithm; restricting the RSK correspondence to permutation matrices and taking the first SSYT of the pair yields exactly the Schensted insertion. For a combinatorial perspective on RSK, see Stanley [9]; for its connections to representation theory, see Fulton [2]. Stelzer and Yong studied the RSK correspondence as a linear operator on the coordinate ring of matrix space, proving results on its diagonalizability, eigenvalues, trace, and determinant [10]. In the same paper, they also conjecture the vanishment of most diagonal entries of a special block in the RSK linear operator (see [10], Conjecture 8.7). As mentioned above, this conjecture is affirmed by Theorem 2 (see Corollary 9). To state their conjecture, we first describe the relevant notation from their work.

Let $p, q \in \mathbb{N} := \{0, 1, 2, \dots\}$, and let $\mathbf{Mat}_{p,q}(\mathbb{N})$ denote the set of all $p \times q$ matrices with non-negative integer entries. Identify the coordinate ring of the space $\mathbf{Mat}_{p,q}(\mathbb{N})$, given by

$$R_{p,q} := \mathbb{C}[z_{i,j}]_{1 \leq i \leq p, 1 \leq j \leq q}.$$

The space $R_{p,q}$ has a *monomial basis* given by

$$z^\alpha := \prod_{1 \leq i \leq p, 1 \leq j \leq q} z_{i,j}^{\alpha_{i,j}},$$

where $\alpha = (\alpha_{ij}) \in \mathbf{Mat}_{p,q}(N)$ is the exponent matrix. Another basis of $R_{p,q}$ is the *bitableaux basis*, introduced by Doubilet-Rota-Stein and denoted by $[P \mid Q]$, where (P, Q) ranges over pairs of semistandard Young tableaux of the same shape [1]. This basis also plays a central role in invariant theory, as it is closely related to the first and second fundamental theorems for general linear groups over arbitrary commutative rings. Let (P_α, Q_α) denote the image of $\alpha \in \mathbf{Mat}_{p,q}(\mathbb{N})$ under the RSK correspondence. Stelzer and Yong studied a linear operator between these two bases of the space $R_{p,q}$:

$$\mathrm{RSK}: z^\alpha \mapsto [P_\alpha \mid Q_\alpha].$$

Let $(\sigma, \pi) \in \mathbb{N}^p \times \mathbb{N}^q$ satisfy $|\sigma| = |\pi|$, where $|\sigma| := \sigma_1 + \cdots + \sigma_p$ and $|\pi| := \pi_1 + \cdots + \pi_q$. Define $\mathbf{Mat}_{\sigma,\pi}(\mathbb{N})$ as the set of all matrices $\alpha \in \mathbf{Mat}_{p,q}(\mathbb{N})$ whose row and column sums are given by (σ, π) , i.e.,

$$\begin{aligned} \sum_j \alpha_{i,j} &= \sigma_i, & 1 \leq i \leq p; \\ \sum_i \alpha_{i,j} &= \pi_j, & 1 \leq j \leq q. \end{aligned}$$

The *weight* of a Young tableau T is a vector $(c_1, c_2, \dots)$, where c_i denotes the number of times i appears in T . Under the RSK correspondence, if $\alpha \in \mathbf{Mat}_{\sigma,\pi}$, then (P_α, Q_α) has weights (σ, π) . We recall the notion of the *weight space* (see [10], Section 1.2) associated with (σ, π) as

$$R_{\sigma,\pi} = \mathrm{Span}\{z^\alpha \mid \alpha \in \mathbf{Mat}_{\sigma,\pi}(\mathbb{N})\}.$$

The bitableaux $[P \mid Q]$, where (P, Q) has weights (σ, π) , form a linear basis of $R_{\sigma,\pi}$ (see [10], Lemma 2.5). Consequently, the restriction of the linear operator RSK to $R_{\sigma,\pi}$, denoted by

$$\mathrm{RSK}_{\sigma,\pi}: R_{\sigma,\pi} \rightarrow R_{\sigma,\pi},$$

is well-defined. The linear operator $\mathrm{RSK}_{\sigma,\pi}$ on the subspace $R_{\sigma,\pi}$ can be expressed as a matrix with respect to the monomial basis $\{z^\alpha \mid \alpha \in \mathbf{Mat}_{\sigma,\pi}\}$. The entries of this matrix are given by the function

$$\mathrm{RSK}_{\sigma,\pi}(\beta, \alpha) = [z^\beta] [P_\alpha \mid Q_\alpha],$$

for all $\alpha, \beta \in \mathbf{Mat}_{\sigma,\pi}$, where $[z^\beta] [P_\alpha, Q_\alpha]$ denotes the coefficient of z^β in $[P_\alpha, Q_\alpha]$.

Let $1^n := (1, 1, \dots, 1)$; then the block $\mathrm{RSK}_{1^n, 1^n}$ plays a significant role in the analysis of [10] (see Section 4.1). This block arises from restricting the RSK linear operator to the subspace corresponding to permutation matrices, and can be viewed as a natural analogue of the classical restriction of the RSK correspondence to permutations, which recovers the Robinson-Schensted correspondence. In particular, Theorem 8.1 of [10] shows that the trace of this operator provides leading-term information about the trace of $\mathrm{RSK}_{p,q,n}$. Let

$$C_n = \{\alpha \in \mathbf{Mat}_{1^n, 1^n} \mid \mathrm{RSK}_{1^n, 1^n}(\alpha, \alpha) = 0\}.$$

We prove the following theorem, originally conjectured in [10] (Conjecture 8.7).

Theorem 3. $\lim_{n \rightarrow \infty} \frac{|C_n|}{n!} = 1$.

1.3 Organization of the Paper

A key ingredient in our proof of Theorem 2 is the Limit Shape Theorem, which describes the asymptotic shape of a random Young diagram under the Plancherel measure. Independently established by Vershik-Kerov [11] and Logan-Shepp [5], it has deep connections to combinatorics, probability, and representation theory. For historical context and further developments, see Kerov [4]. We revisit this theorem in Section 2.3, using Romik's reformulation [6].

This paper is organized as follows. In Section 2.1, we review the RSK correspondence. Section 2.2 recalls the concept of bitableaux and explains why Theorem 3 is equivalent to Theorem 2 (see Corollary 9). In Section 2.3, we state the Limit Shape Theorem. Finally, in Section 3, we provide a self-contained proof of Theorem 2 based solely on the Limit Shape Theorem and elementary arguments.

2 Preliminaries

2.1 RSK

A *partition* of a positive integer n is a sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ of positive integers such that $\lambda_1 \geq \dots \geq \lambda_k$ and $\lambda_1 + \dots + \lambda_k = n$. The set of partitions of n is denoted by $\text{Par}(n)$. A *Young diagram* of shape λ (in French notation) is a collection of boxes arranged in left-justified rows, with row lengths given by $\lambda_1, \dots, \lambda_k$. A *semistandard Young tableau* (SSYT) of shape λ is a filling of the boxes of the Young diagram with positive integer entries $a_{i,j}$ satisfying the following two conditions.

(i) $(a_{i,j})$ is weakly increasing along rows, i.e.,

$$a_{i,1} \leq a_{i,2} \leq \dots \leq a_{i,\lambda_i}, \quad \text{for all } 1 \leq i \leq k.$$

(ii) $(a_{i,j})$ is strictly increasing along columns, i.e.,

$$a_{1,j} < a_{2,j} < \dots, \quad \text{for all } 1 \leq j \leq \lambda_1.$$

A *standard Young tableau* of shape $\lambda \in \text{Par}(n)$ is a SSYT with the additional restriction that the entries are precisely $\{1, 2, \dots, n\}$, with each appearing exactly once.

Example 4. Let $n = 7$. The following is a SSYT of the shape $\lambda = (4, 2, 1)$.

$$\begin{array}{|c|c|c|c|} \hline a_{31} & & & \\ \hline a_{21} & a_{22} & & \\ \hline a_{11} & a_{12} & a_{13} & a_{14} \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 4 & & & \\ \hline 2 & 2 & & \\ \hline 1 & 1 & 1 & 3 \\ \hline \end{array}$$

For a matrix $\alpha \in \text{Mat}_{p,q}(\mathbb{N})$, we construct a *biword* of α , denoted by $\text{biword}(\alpha)$, by reading the entries $\alpha_{i,j}$ down the columns from left to right and recording them as $(i i \dots i \mid j j \dots j)$, where $\alpha_{i,j}$ is the number of times that i appears on the left and j appears on the right.

Example 5. Consider $\alpha = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$. Then, we have

$$\text{biword}(\alpha) = (1322311 \mid 1122233).$$

Given $\text{biword}(\alpha) = (a_1, a_2, \dots, a_n \mid b_1, b_2, \dots, b_n)$, we can construct a pair of semistandard Young tableaux, known as the *insertion* and *recording* tableaux, using an algorithm called *row insertion*. Row insertion is a more general form of Schensted insertion, and it proceeds as follows: Start with an empty insertion tableau and an empty recording tableau. At each step i , insert a_i into the insertion tableau, beginning with the bottom row. If the row is empty, create a new box containing a_i . If the row is not empty and a_i is larger than all entries in the row, create a new box at the end containing a_i . Otherwise, replace the leftmost number $x > a_i$ in the row and *bump* x into the row above in the same manner, repeating this process as necessary for all rows. Whenever a new box is created in the insertion tableau, create a corresponding box in the recording tableau in the same position, containing b_i . After n steps, the resulting pair of tableaux is denoted (P_α, Q_α) .

Example 6. Consider the $\text{biword}(\alpha) = (1322311 \mid 1122233)$. The algorithm for constructing (P_α, Q_α) is as follows.

$$\begin{aligned} &(\emptyset, \emptyset), \left(\begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 3 \\ \hline 1 \end{array}, \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & 1 \end{array} \right), \\ &\left(\begin{array}{|c|c|c|} \hline 3 & & \\ \hline 1 & 2 & 2 \end{array}, \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 1 & 2 \end{array} \right), \left(\begin{array}{|c|c|c|c|} \hline 3 & & & \\ \hline 1 & 2 & 2 & 3 \end{array}, \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline 1 & 1 & 2 & 2 \end{array} \right), \\ &\left(\begin{array}{|c|c|c|c|} \hline 3 & & & \\ \hline 2 & & & \\ \hline 1 & 1 & 2 & 3 \end{array}, \begin{array}{|c|c|c|c|} \hline 3 & & & \\ \hline 2 & & & \\ \hline 1 & 1 & 2 & 2 \end{array} \right), \left(\begin{array}{|c|c|c|c|} \hline 3 & & & \\ \hline 2 & 2 & & \\ \hline 1 & 1 & 1 & 3 \end{array}, \begin{array}{|c|c|c|c|} \hline 3 & & & \\ \hline 2 & 3 & & \\ \hline 1 & 1 & 2 & 2 \end{array} \right) = (P_\alpha, Q_\alpha). \end{aligned}$$

2.2 Bitableaux and zero coefficients

For a pair (P, Q) of SSYT of the same shape λ , with entries $\{a_{i,j}\}$ and $\{b_{i,j}\}$ respectively, we construct minors Δ_j for each $1 \leq j \leq \lambda_1$ as follows. Consider an infinite matrix with entries $\{z_{i,j}\}_{i,j \geq 1}$. For each j , we extract a square submatrix whose rows are indexed by the entries of column j of P and whose columns are indexed by the entries of column j of Q . The determinant of this submatrix defines the minor Δ_j . The *bitableau* of (P, Q) is then defined as the product of all minors Δ_j , denoted by $[P \mid Q]$.

Example 7. Considering the pair of SSYT in Example 6, we have

$$[P_\alpha \mid Q_\alpha] = \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \cdot \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix} \cdot |z_{12}| \cdot |z_{32}|.$$

For each $\alpha \in \text{Mat}_{1^n, 1^n}$, the matrix α can be viewed as a permutation matrix represented by $w_\alpha = w_1 \dots w_n \in S_n$, where the entries satisfy $\alpha_{w_i, i} = 1$ for all $i \in [n] := \{1, 2, \dots, n\}$ and all other entries are zero. In this case, $\text{biword}(\alpha) = (w_1, \dots, w_n \mid 1, \dots, n)$, so (P_α, Q_α) is a pair of standard Young tableaux with weights $(1^n, 1^n)$. The algorithm that constructs P_α is precisely the Schensted insertion of the permutation w_α .

Lemma 8. *For $\alpha \in \text{Mat}_{1^n, 1^n}$, we have $\text{RSK}_{1^n, 1^n}(\alpha, \alpha) = 0$ if and only if w_α has a lateral bump in its Schensted insertion.*

Proof. We observe that

$$z^\alpha = z_{w_1, 1} z_{w_2, 2} \dots z_{w_n, n}$$

is a monomial of degree n , and

$$[P_\alpha \mid Q_\alpha] = \Delta_1 \Delta_2 \dots$$

is a homogeneous polynomial of degree n . Suppose that w_α undergoes a lateral bump during its Schensted insertion. We consider the last insertion w_i that causes a lateral bump. Then w_i is inserted at the bottom of some k^{th} column of P_α , and the new box containing i is created in the l^{th} column of Q_α , where $l < k$. By the choice of w_i , any subsequent insertion $w_{i+1}, \dots, w_n$ undergoes only vertical bumps, so w_i always remains in the k^{th} column of P_α . Since w_i is in the k^{th} column of P_α while i is in the l^{th} column of Q_α , the variable $z_{w_i, i}$ does not appear in any minor, implying that the coefficient of z^α in $[P_\alpha \mid Q_\alpha]$ is zero.

Conversely, if w_α has no lateral bumps, then by a similar argument as above, each $z_{w_i, i}$ for $i \in [n]$ appears in a unique minor Δ_k . Since $[P_\alpha \mid Q_\alpha]$ is a homogeneous polynomial of degree n , constructing the degree- n monomial z^α requires selecting exactly one subterm with coefficient ± 1 from each Δ_k . Here, a subterm refers to a product of some $z_{w_i, i}$ for $i \in [n]$. Therefore, the coefficient of z^α is ± 1 . $\square$

Corollary 9. $\lim_{n \rightarrow \infty} \frac{|V_n|}{n!} = 0$ if and only if $\lim_{n \rightarrow \infty} \frac{|C_n|}{n!} = 1$.

Proof. Lemma 8 establishes that Theorem 3 is equivalent to Theorem 2. $\square$

2.3 Limit Shape Theorem

The restriction of the RSK correspondence to pairs of standard Young tableaux is known as the Schensted correspondence, which provides a bijection between permutations and pairs of standard Young tableaux of the same shape. For $\lambda \in \text{Par}(n)$, let f_λ denote the number of standard Young tableaux of shape λ . Define the function

$$\text{sh}: S_n \rightarrow \text{Par}(n), \quad w \mapsto \lambda_w$$

where λ_w is the shape of the tableau obtained by applying the Schensted insertion algorithm to w . Then, by the Schensted correspondence, we obtain

$$|\text{sh}^{-1}(\lambda)| = |\{w \in S_n \mid \text{sh}(w) = \lambda\}| = f_\lambda^2.$$

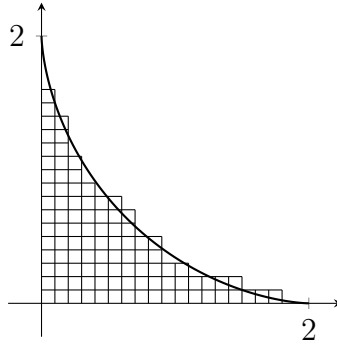


Figure 1: The limit shape Γ and a $\frac{1}{10}$ -rescaled partition $\lambda \in \text{Par}(100)$.

Now, consider a *uniform probability measure* $\mathbb{P}_n$ on S_n , where

$$\mathbb{P}_n(w) := \frac{1}{n!}, \quad \text{for all } w \in S_n.$$

The *Plancherel measure* $\mathbb{P}_n^*$ on $\text{Par}(n)$ is defined as the pushforward measure of the uniform measure on S_n via sh (see [3]), i.e.,

$$\mathbb{P}_n^*(\lambda) := \mathbb{P}_n(\text{sh}(w) = \lambda) = \frac{f_\lambda^2}{n!}, \quad \text{for all } \lambda \in \text{Par}(n).$$

The *limit shape* is a deterministic curve that describes the asymptotic shape of a random Young diagram under the Plancherel measure after scaling by $1/\sqrt{n}$. We recall Romik's reformulation of the Vershik-Kerov-Logan-Shepp limit shape theorem in the standard Cartesian coordinate system from [6].

Definition 10. The limit shape Γ in the $\mathbb{R}^2$ plane is the region (see Figure 1) bounded by the coordinate axes and the following parametric curve:

$$\begin{aligned} \gamma &= (\gamma_x, \gamma_y) : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \mathbb{R}^2, \\ \gamma_x(\theta) &= \left(\frac{2\theta}{\pi} + 1\right) \sin \theta + \frac{2}{\pi} \cos \theta, \\ \gamma_y(\theta) &= \left(\frac{2\theta}{\pi} - 1\right) \sin \theta + \frac{2}{\pi} \cos \theta. \end{aligned}$$

Definition 11. Given a partition $\lambda = (\lambda_1, \dots, \lambda_k)$, we define its *planar set* as

$$S_\lambda = \bigcup_{1 \leq j \leq k, 1 \leq i \leq \lambda_j} ([i-1, i] \times [j-1, j]) \subseteq \mathbb{R}^2.$$

Example 12. Consider $w = 25143 \in S_5$. Then, $\text{sh}(w) = (2, 2, 1)$. Its corresponding planar set is shown in Figure 2.

Theorem 13 (Limit Shape Theorem, Theorem 1.26 of [6]). *For a fixed $\epsilon \in (0, 1)$, under the Plancherel measure $\mathbb{P}_n^*$ on $\text{Par}(n)$, we have*

$$\mathbb{P}_n^* \left((1 - \epsilon) \Gamma \subseteq \frac{1}{\sqrt{n}} S_\lambda \subseteq (1 + \epsilon) \Gamma \right) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

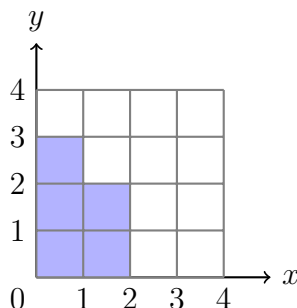


Figure 2: Planar set of $\lambda = (2, 2, 1)$.

3 Proof of Theorem 2

We briefly explain the role of the limit shape theorem in the proof. The limit shape theorem is used only to establish Lemma 23, which asserts that under the Plancherel measure, the first row of a typical Young diagram has length approximately $2\sqrt{n}$, and in particular at least $\sqrt{2n}$ with high probability. Assuming that V_n has positive asymptotic density in S_n , this first-row property must also hold for a positive proportion of permutations in V_n . This leads to an improved estimate for $|V_{n+1}|$ in terms of $|V_n|$, which ultimately yields a contradiction.

Definition 14. An n -word $a_1 a_2 \dots a_n$, where each $a_i \in \mathbb{R}$, is called *injective* if

$$a_i \neq a_j \quad \text{for all } 1 \leq i < j \leq n.$$

Definition 15. The *flattening operation* Flat on the set of injective n -words is given by

$$\text{Flat}: a_1 a_2 \dots a_n \mapsto b_1 b_2 \dots b_n,$$

where $b_i := |\{j \in [n] \mid a_j \leq a_i\}|$. Evidently, $b_1 b_2 \dots b_n$ is a permutation on $[n]$.

Definition 16. Two injective n -words $a_1 a_2 \dots a_n$ and $c_1 c_2 \dots c_n$ are said to have the same *relative order* if, for all $i, j \in [n]$, we have $a_i < a_j$ if and only if $c_i < c_j$. In this case, we also write $a_1 a_2 \dots a_n \simeq c_1 c_2 \dots c_n$.

Lemma 17. *For any injective n -word $a_1 a_2 \dots a_n$, we have*

$$a_1 a_2 \dots a_n \simeq \text{Flat}(a_1 a_2 \dots a_n).$$

Moreover, for two injective n -words $a_1 a_2 \dots a_n$ and $c_1 c_2 \dots c_n$, we have

$$a_1 a_2 \dots a_n \simeq c_1 c_2 \dots c_n \quad \text{if and only if} \quad \text{Flat}(a_1 a_2 \dots a_n) = \text{Flat}(c_1 c_2 \dots c_n).$$

Proof. Let $\text{Flat}(a_1 a_2 \dots a_n) = b_1 b_2 \dots b_n$. By definition, if $a_i < a_j$, then

$$\{k \mid a_k \leq a_i\} \sqcup \{a_j\} \subseteq \{k \mid a_k \leq a_j\},$$

so $b_i < b_j$. Hence,

$$a_1 a_2 \dots a_n \simeq b_1 b_2 \dots b_n = \text{Flat}(a_1 a_2 \dots a_n).$$

The second claim follows immediately. $\square$

Now, for positive integer n , we define a map $\varphi_n: S_{n+1} \rightarrow S_n$ by

$$\varphi_n(w_1 w_2 \dots w_n w_{n+1}) := \text{Flat}(w_1 w_2 \dots w_n).$$

Example 18. Let $w = 14352 \in S_5$. Then,

$$\varphi_5(14352) = \text{Flat}(1435) = 1324 \in S_4.$$

In the following discussion, for each $k \in \mathbb{Z}$, denote $k^* := k + \frac{1}{2}$.

Proposition 19. If $\varphi_n(w_1 w_2 \dots w_n w_{n+1}) = v_1 v_2 \dots v_n$, then

$$w_1 w_2 \dots w_n w_{n+1} \simeq v_1 v_2 \dots v_n k^*,$$

where $k = w_{n+1} - 1$.

Proof. Lemma 17 implies that $w_1 w_2 \dots w_n \simeq v_1 v_2 \dots v_n$. As a result,

$$v_i = \begin{cases} w_i, & \text{if } w_i < w_{n+1}, \\ w_i - 1, & \text{if } w_i > w_{n+1}, \end{cases}$$

for any $i \in [n]$. Therefore, if $w_i < w_{n+1}$, then $v_i < k^*$. Conversely, if $w_i > w_{n+1}$, we have $v_i > k^*$. Combining these with the fact that

$$w_1 w_2 \dots w_n \simeq v_1 v_2 \dots v_n,$$

we obtain the conclusion. $\square$

Lemma 20. For any positive integer n and any $v \in S_n$, we have

$$|\varphi_n^{-1}(v)| = n + 1.$$

Proof. By Proposition 19, each permutation $w \in \varphi_n^{-1}(v)$ is of the form $\text{Flat}(v k^*)$ for $k = w_{n+1} - 1 \in \{0, 1, \dots, n\}$. Conversely, since both Flat and φ_n preserve the relative order of the first n entries, we have

$$\varphi_n(\text{Flat}(v k^*)) = \text{Flat}(v) = v \quad \text{for all } k \in \{0, 1, \dots, n\}.$$

Thus, $\text{Flat}(v k^*) \in \varphi_n^{-1}(v)$ for all $k \in \{0, 1, \dots, n\}$. Since the last entry of $\text{Flat}(v k^*)$ equals to $k + 1$, those are $n + 1$ distinct permutations in S_{n+1} . Hence, $|\varphi_n^{-1}(v)| = n + 1$. $\square$

Example 21. Consider $v = 3124 \in S_4$. To compute $\varphi_4^{-1}(v)$, we append each of the values $0^*, 1^*, 2^*, 3^*, 4^*$ to the end of v , and then apply **Flat** to each resulting 5-word. This yields the following five elements in $\varphi_4^{-1}(v) \subseteq S_5$:

$$\begin{aligned}\text{Flat}(31240^*) &= 42351, \\ \text{Flat}(31241^*) &= 41352, \\ \text{Flat}(31242^*) &= 41253, \\ \text{Flat}(31243^*) &= 31254, \\ \text{Flat}(31244^*) &= 31245.\end{aligned}$$

Recall from Section 1.1 that V_n is the set of all permutations in S_n whose Schensted insertion contains no lateral bumps.

Lemma 22. *For any positive integer n , if $w \in V_{n+1}$, then $\varphi_n(w) \in V_n$.*

Proof. Let $w = w_1w_2 \dots w_{n+1} \in V_{n+1}$, and suppose $\varphi_n(w) = v_1v_2 \dots v_n \in S_n$. Let $k = w_{n+1} - 1$. By Proposition 19, we have

$$v_1v_2 \dots v_n k^* \simeq w_1w_2 \dots w_{n+1}.$$

Since lateral bumps in Schensted insertion depend only on the relative order of entries, the absence of lateral bumps in $w_1w_2 \dots w_{n+1}$ implies their absence in $v_1v_2 \dots v_n$, i.e.,

$$\varphi_n(w) = v_1v_2 \dots v_n \in V_n. \quad \square$$

Since $\varphi_n(V_{n+1}) \subseteq V_n$ by Lemma 22, we can define the map ψ_n as the restriction of φ_n to the domain V_{n+1} , that is, $\psi_n: V_{n+1} \rightarrow V_n$. Considering all the maps ψ_n for $n \in \mathbb{N}$, we obtain a *rooted tree*, where the root is the identity permutation in S_1 . (We adopt graph-theoretic terminology for convenience.) The rooted tree is illustrated in Figure 3. For each $w \in V_n$, we view the elements in $\psi_n^{-1}(w) \subseteq V_{n+1}$ as the *children* of w . By Lemma 20, we have

$$|\psi_n^{-1}(w)| \leq n + 1, \quad (1)$$

and consequently,

$$|V_{n+1}| \leq (n + 1) |V_n|. \quad (2)$$

For $\lambda \in \text{Par}(n)$, suppose that $\lambda = (\lambda_1, \dots, \lambda_k)$. We define $L(\lambda) := \lambda_1$ as the length of the first row of the shape λ . For example, for $\lambda = (2, 2, 1)$, we have $L(\lambda) = 2$. Recall the Plancherel measure from Section 2.3, and we state the following lemma.

Lemma 23. *Under the Plancherel measure $\mathbb{P}_n^*$ on $\text{Par}(n)$, we have*

$$\mathbb{P}_n^* \left(L(\lambda) \geq \sqrt{2n} \right) \longrightarrow 1 \quad \text{as } n \longrightarrow \infty.$$

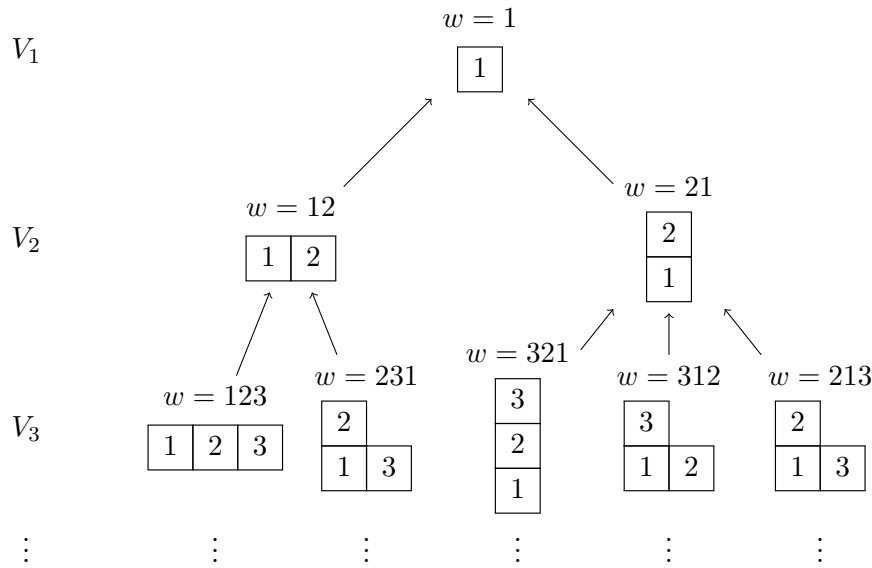


Figure 3: The rooted tree of permutations with no lateral bumps.

Proof. Substituting $\epsilon = 1 - \frac{\sqrt{2}}{2}$ in Theorem 13, which implies

$$\mathbb{P}_n^* \left(\frac{\sqrt{2}}{2} \Gamma \subseteq \frac{1}{\sqrt{n}} S_\lambda \right) \longrightarrow 1 \quad \text{as } n \longrightarrow \infty.$$

If $\frac{\sqrt{2}}{2} \Gamma \subseteq \frac{1}{\sqrt{n}} S_\lambda$, then $L(\lambda) \geq \sqrt{2n}$. Therefore, we have

$$\mathbb{P}_n^* \left(L(\lambda) \geq \sqrt{2n} \right) \geq \mathbb{P}_n^* \left(\frac{\sqrt{2}}{2} \Gamma \subseteq \frac{1}{\sqrt{n}} S_\lambda \right).$$

By the squeeze theorem, it follows that

$$\mathbb{P}_n^* \left(L(\lambda) \geq \sqrt{2n} \right) \longrightarrow 1 \quad \text{as } n \longrightarrow \infty. \quad \square$$

Lemma 24. *For any $\lambda \in \text{Par}(n)$, if $L(\lambda) \geq \sqrt{2n}$, then the shape λ has two columns of the same height.*

Proof. Suppose all $L(\lambda)$ columns of λ have distinct heights. Then the total number of boxes in λ is at least the sum of the $L(\lambda)$ smallest distinct positive integers:

$$|\lambda| \geq 1 + 2 + \cdots + L(\lambda) = \frac{1}{2} L(\lambda) (L(\lambda) + 1).$$

But since $|\lambda| = n$, this implies

$$n \geq \frac{1}{2} L(\lambda) (L(\lambda) + 1) > \frac{1}{2} L(\lambda)^2.$$

Hence, $L(\lambda) < \sqrt{2n}$. This contradicts the assumption that $L(\lambda) \geq \sqrt{2n}$, so λ must have at least two columns of the same height. $\square$

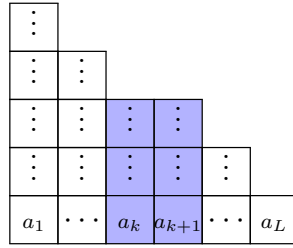


Figure 4: Example λ with 2 columns, k and $k + 1$, of the same height.

Lemma 25. *For any $v \in V_n$, if the shape $\lambda_v := \text{sh}(v)$ has two columns with the same height, then $|\psi_n^{-1}(v)| \leq n$, i.e., v has at most n children.*

Proof. Suppose that k^{th} and $(k + 1)^{\text{th}}$ columns of $\lambda_v := \text{sh}(v)$ have the same height (see Figure 4). Let $a_1, a_2, \dots, a_L$ be the entries in the first row of λ_v , and define

$$w := \text{Flat}(v a_k^*) \in S_{n+1}.$$

In the final insertion of $v a_k^*$, i.e., inserting a_k^* into the tableau λ_v , the number a_k^* replaces a_{k+1} , bumping a_{k+1} to the second row. If there are consecutive vertical bumps in the $(k + 1)^{\text{th}}$ column, then the bumping process will get stuck at the top of the column. Therefore, the final insertion causes a lateral bump. Since lateral bumps depend only on the relative order of entries, the Schensted insertion of w also undergoes a lateral bump. Thus, combining with Lemma 20, we obtain

$$|\psi_n^{-1}(v)| \leq |\varphi_n^{-1}(v) \setminus \{w\}| = n,$$

meaning that v has at most n children. □

Lemma 26.

$$\lim_{n \rightarrow \infty} \left(\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \right) = 0.$$

Proof. Let

$$a_n = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} = \frac{(2n)!}{2^{2n}(n!)^2}.$$

We recall Stirling's approximation

$$\lim_{n \rightarrow \infty} \frac{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n}{n!} = 1.$$

By applying the approximation, we obtain

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\sqrt{4\pi n} \left(\frac{2n}{e}\right)^{2n}}{2^{2n}(2\pi n) \left(\frac{n}{e}\right)^{2n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{\pi n}} = 0. \quad \square$$

Proof of Theorem 2. Define $p_n := \frac{|V_n|}{n!}$. For all $n \in \mathbb{N}$, using (2), we have

$$p_{n+1} = \frac{|V_{n+1}|}{(n+1)!} \leq \frac{(n+1)|V_n|}{(n+1)!} = p_n.$$

Thus, the sequence $\{p_n\}_{n=1}^{\infty}$ is monotone decreasing and bounded below by 0. Therefore, it converges to some limit $\epsilon \geq 0$. Suppose $\epsilon > 0$. By Lemma 23, for any $\delta > 0$, there exists $N \in \mathbb{N}$ such that

$$\mathbb{P}_n^* \left(L(\lambda) \geq \sqrt{2n} \right) > 1 - \delta, \quad \text{for all } n \geq N.$$

By the relation of the measures $\mathbb{P}_n$ on S_n and $\mathbb{P}_n^*$ on $\text{Par}(n)$ in Section 2.3, we obtain

$$\mathbb{P}_n \left(L(\lambda_w) \geq \sqrt{2n} \right) = \mathbb{P}_n^* \left(L(\lambda) \geq \sqrt{2n} \right) > 1 - \delta, \quad \text{for all } n \geq N.$$

We denote H_n as the set of permutations $w \in S_n$ such that λ_w has columns of different heights. By Lemma 24, if $L(\lambda_w) \geq \sqrt{2n}$, then λ_w has two columns of the same height, i.e., $w \notin H_n$. Therefore, we obtain

$$\mathbb{P}_n(w \notin H_n) \geq \mathbb{P}_n \left(L(\lambda_w) \geq \sqrt{2n} \right) > 1 - \delta, \quad \text{for all } n \geq N.$$

This implies that

$$\mathbb{P}_n(w \in H_n \cap V_n) \leq \mathbb{P}_n(w \in H_n) < \delta, \quad \text{for all } n \geq N.$$

We denote U_n as the set of permutations in V_n that have $n+1$ children. By Lemma 25, if $w \in U_n$, then $w \in H_n$. Thus, we obtain

$$|U_n| \leq |V_n \cap H_n| = n! \times \mathbb{P}_n(w \in H_n \cap V_n) < n! \times \delta.$$

Choosing $\delta = \frac{1}{2}\epsilon$, we have

$$|U_n| < \delta n! = \frac{1}{2}\epsilon n! \leq \frac{1}{2}|V_n|,$$

because $\frac{|V_n|}{n!} = p_n \geq \epsilon$. By (1) and the definition of U_n , any $w \in V_n \setminus U_n$ has at most n children. Thus,

$$|V_{n+1}| \leq n \cdot |V_n \setminus U_n| + (n+1) \cdot |U_n| \leq \left(n + \frac{1}{2} \right) |V_n|.$$

Therefore, we obtain

$$p_{n+1} \leq \frac{n + \frac{1}{2}}{n+1} p_n = \left(1 - \frac{1}{2(n+1)} \right) p_n.$$

Since this inequality holds for all $n \geq N$, we can apply it repeatedly as follows:

$$\begin{aligned}
 p_n &\leq \left(1 - \frac{1}{2n}\right) p_{n-1} \\
 &\leq \left(1 - \frac{1}{2n}\right) \left(1 - \frac{1}{2(n-1)}\right) p_{n-2} \\
 &\quad \vdots \\
 &\leq \left(1 - \frac{1}{2n}\right) \left(1 - \frac{1}{2(n-1)}\right) \cdots \left(1 - \frac{1}{2(N+1)}\right) p_N \\
 &= \prod_{i=N+1}^n \left(1 - \frac{1}{2i}\right) p_N.
 \end{aligned}$$

From Lemma 26, we obtain

$$p_n \leq \prod_{i=N+1}^n \left(1 - \frac{1}{2i}\right) p_N = \frac{a_n}{a_N} p_N \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Therefore, by the squeeze theorem, we conclude that

$$\lim_{n \rightarrow \infty} p_n = 0.$$

This leads to a contradiction, as we initially assumed $\lim_{n \rightarrow \infty} p_n = \epsilon > 0$. Hence, we must have $\epsilon = 0$. $\square$

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