An Equality for Balanced Digraphs

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Abstract

Consider a directed multigraph D that is balanced (i.e., at each vertex, the indegree equals the outdegree). Let A be its set of arcs. Fix an integer k. Let s be a vertex of D. We show that the number of k-element subsets B of A that contain no cycles but contain a path from each vertex to s (we call them "s-convergences") is independent of s. This generalizes known facts about spanning arborescences, acyclic orientations and maximal acyclic subdigraphs (or, equivalently, minimum feedback arc sets). Moreover, this result can be generalized even further, replacing "contain no cycles" with "have a given set of cycles".

Mathematics Subject Classifications: 05C20, 05C45

1 The theorems

In this note, we shall discuss *balanced multidigraphs* – i.e., directed multigraphs (allowing loops and multiple arcs) in which each vertex satisfies "outdegree = indegree". We recall the relevant definitions in more detail:

A *multidigraph* (henceforth just *digraph*) means a triple (V, A, ψ), where V and A are two finite sets and $\psi : A \to V \times V$ is a map. The elements of V are called the *vertices* of this digraph, and the elements of A are called the *arcs* of this digraph. The *source* and *target* of an arc $a \in A$ are, respectively, the first and second entries of the pair $\psi(a)$. The *indegree* $\deg^- v$ of a vertex $v \in V$ means the number of arcs $a \in A$ whose target is v. The *outdegree* $\deg^+ v$ of a vertex $v \in V$ means the number of arcs $a \in A$ whose source is v. We say that a digraph (V, A, ψ) is *balanced* if and only if each vertex $v \in V$ satisfies $\deg^+ v = \deg^- v$. For further terminology on digraphs, we refer to [1].[1]

A *to-root* of a digraph D means a vertex s of D such that for each vertex v of D, the digraph D has a path (equivalently, a walk) from v to s.

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[1]The famous directed Euler–Hierholzer theorem (which will not be used in this note) says that a weakly connected digraph has an Eulerian circuit if and only if it is balanced. Thus, weakly connected balanced digraphs are also known as *Eulerian digraphs*.

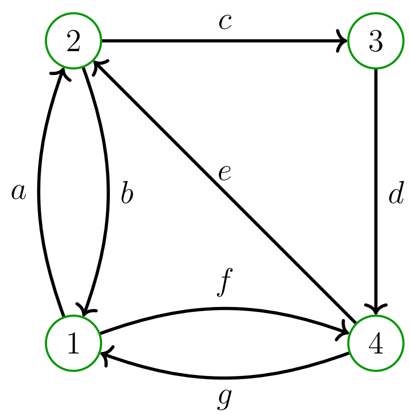
From now on, we **fix a balanced digraph** $D = (V, A, \psi)$. If B is any subset of A, then $D \langle B \rangle$ will denote the induced subdigraph $(V, B, \psi \mid_B)$. A subset B of A will be called *acyclic* if the subdigraph $D \langle B \rangle$ has no cycles. ("Cycle" always means "directed cycle", as in [1, Definition 4.5.1].)

Given a vertex s of D, we define an *s-convergence* to be an acyclic subset B of A such that s is a to-root of the subdigraph $D \langle B \rangle$. We can think of an s-convergence as a set B of arcs of D such that by following the B-arcs (i.e. the arcs in B) from any vertex $v \in V$, we will always arrive at s (no matter which B-arcs we take), and we will be stuck at s. (This intuition is formalized in Proposition 22.)

For any $k \in \mathbb{N}$ and $s \in V$, we let $\gamma_k(s)$ denote the number of s-convergences of size k (that is, with k arcs).[2] Our first main result is the following:

Theorem 1. *Let $k \in \mathbb{N}$. Then, the number $\gamma_k(s)$ does not depend on s. That is, $\gamma_k(s) = \gamma_k(t)$ for any $s, t \in V$.*

Example 2. Let D be the following balanced multidigraph:



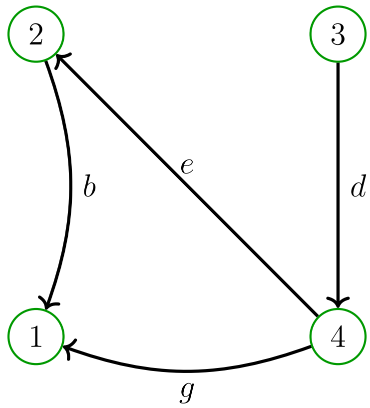
Then, the 1-convergences are the subsets

$$\{b, d, g\}, \qquad \{b, d, e\}, \qquad \{c, d, g\}, \qquad \{b, d, e, g\}, \qquad \{b, c, d, g\}.$$

Hence, $\gamma_3(1) = 3$, $\gamma_4(1) = 2$, and $\gamma_k(1) = 0$ for all $k \notin \{3, 4\}$. As a visual aid, below is

[2]The symbol $\mathbb{N}$ denotes the set $\{0, 1, 2, \ldots\}$.

the spanning subdigraph $D\langle B\rangle$ for $B = \{b, d, e, g\}$:



Theorem 1 says that for all $v \in \{1, 2, 3, 4\}$, we have $\gamma_3(v) = 3$, $\gamma_4(v) = 2$, and $\gamma_k(v) = 0$ for all $k \notin \{3, 4\}$. For example, the 2-convergences are the subsets

$$\{a, d, e\}, \qquad \{a, d, g\}, \qquad \{d, e, f\}, \qquad \{a, d, e, f\}, \qquad \{a, d, e, g\}.$$

Hence, $\gamma_3(2) = 3 = \gamma_3(1)$, $\gamma_4(2) = 2 = \gamma_4(1)$, and $\gamma_k(2) = 0 = \gamma_k(1)$ for all $k \notin \{3, 4\}$. The same holds for $v \in \{3, 4\}$.

As we will see in Section 2, Theorem 1 generalizes several results known from the literature. However, Theorem 1 can be generalized even further. Before we state the most general version, let us tease a corollary of it that is even easier to state than Theorem 1: It says that we can essentially drop the acyclicity requirement in Theorem 1!

Let us state this rigorously. Given a vertex s of D, we define an *s-preconvergence* to be a (not necessarily acyclic) subset B of A such that s is a to-root of the subdigraph $D\langle B\rangle$. (Thus, it is defined just like an s-convergence, but without requiring it to be acyclic.) For any $k \in \mathbb{N}$ and $s \in V$, we let $\delta_k(s)$ denote the number of s-preconvergences of size k (that is, with k arcs). Then we claim:

Theorem 3. *Let $k \in \mathbb{N}$. Then, the number $\delta_k(s)$ does not depend on s. That is, $\delta_k(s) = \delta_k(t)$ for any $s, t \in V$.*

Example 4. Let D be as in Example 2. Then, for each $s \in V$, the numbers $\delta_k(s)$ for $k = 0, 1, 2, 3, \ldots$ are $0, 0, 0, 3, 10, 12, 6, 1, 0, 0, 0, \ldots$. For instance, $\{c, d, f, g\}$ is a 1-preconvergence, one among the 10 counted in $\delta_4(1) = 10$.

Finally, it is time to let the cat out of the bag and state the most general version of our result. Rather than removing the acyclicity requirement from Theorem 1, we shall replace it by a precise set of cycles that we want our induced subdigraph $D\langle B\rangle$ to have.

To state this properly, we introduce some more notations. For any subset B of A, we let $\operatorname{Cycs} B$ denote the set of all cycles of the subdigraph $D\langle B\rangle$. (Thus, B is acyclic if and only if $\operatorname{Cycs} B = \varnothing$. Note that $\operatorname{Cycs} A$ is the set of all cycles of $D\langle A\rangle = D$.) For any $k \in \mathbb{N}$ and $s \in V$ and any subset Z of $\operatorname{Cycs} A$, we let $\gamma_{k,Z}(s)$ denote the number of s-preconvergences B of size k (that is, with k arcs) that satisfy $\operatorname{Cycs} B = Z$. Then our general result is the following:

Theorem 5. *Let $k \in \mathbb{N}$. Let Z be a subset of* $\operatorname{Cycs} A$. *Then, the number $\gamma_{k,Z}(s)$ does not depend on s. That is, $\gamma_{k,Z}(s) = \gamma_{k,Z}(t)$ for any $s, t \in V$.*

Example 6. Let D be as in Example 2, and let $Z = \{\mathbf{c}\}$, where $\mathbf{c}$ is the cycle $(1, a, 2, b, 1)$ (that is, the cycle with vertices 1 and 2 and arcs a and b). Then, the 1-preconvergences B satisfying $\operatorname{Cycs} B = Z$ are

$$\{a, b, d, e\}, \qquad \{a, b, d, g\}, \qquad \{a, b, d, e, g\}.$$

The 2-preconvergences B satisfying $\operatorname{Cycs} B = Z$ are exactly the same. The 3-preconvergences B satisfying $\operatorname{Cycs} B = Z$ are

$$\{a, b, c, e\}, \qquad \{a, b, c, g\}, \qquad \{a, b, c, e, g\}.$$

Thus, we find, for each $v \in \{1, 2, 3\}$, that $\gamma_{4,Z}(v) = 2$ and $\gamma_{5,Z}(v) = 1$ and $\gamma_{k,Z}(v) = 0$ for all other k. The reader can easily check that this also holds for $v = 4$.

Clearly, an s-convergence is the same as an acyclic s-preconvergence, i.e., an s-preconvergence B that satisfies $\operatorname{Cycs} B = \varnothing$ (since the condition $\operatorname{Cycs} B = \varnothing$ says precisely that $D\langle B\rangle$ has no cycles, i.e., that B is acyclic). Thus, for any $k \in \mathbb{N}$ and $s \in V$, we have $\gamma_k(s) = \gamma_{k,\varnothing}(s)$. Hence, Theorem 1 is the particular case of Theorem 5 for $Z = \varnothing$.

2 Particular cases

Theorem 1 was inspired by a talk of Karla Leipold (NORCOM 2025), which made the first author aware of [9, Lemma 4.1] and [11, §3.1]. The talk discussed no enumerative questions, yet was haunted by a perceptible aura of identities. The present note follows this aura to its source.

Some particular cases of Theorem 1 are known:

1. When $k = |V| - 1$, the s-convergences B of size k (or, more precisely, the respective subdigraphs $D\langle B\rangle$ of D) are precisely the spanning arborescences of D rooted to s (see [1, Definition 5.10.1 **(b)**] for the definition of this). Indeed, the condition $|B| = |V| - 1$, combined with the to-rootness of s, forces $D\langle B\rangle$ to be an arborescence rooted to s (by [1, Theorem 5.10.5]), and conversely, if $D\langle B\rangle$ is a spanning arborescence rooted to s, then [1, Theorem 5.10.5] shows that B is acyclic and $|B| = |V| - 1$. Thus, in the case $k = |V| - 1$, Theorem 1 is saying that the number of spanning arborescences of a balanced digraph D rooted to a vertex s does not depend on s. This is a classical result of van Aardenne-Ehrenfest and de Bruijn [2, Theorem 6] (also proved in [1, Corollary 5.12.1]).

   Likewise, if $k < |V| - 1$, then Theorem 1 is just saying that $0 = 0$, since a spanning subdigraph $D\langle B\rangle$ with fewer than $|V| - 1$ arcs cannot have a to-root.

2. If $D = G^{\text{bidir}}$ for some undirected multigraph $G = (V, E, \varphi)$ (this means that D is obtained from G by replacing each edge e with two arcs $e^{\rightarrow}$ and $e^{\leftarrow}$, going in

opposite directions), and if $k = |E| = |A|/2$, then the s-convergences B of size k are just the acyclic orientations of G with unique sink s (because the acyclicity condition forbids B from containing both $e^{\rightarrow}$ and $e^{\leftarrow}$ for any given edge $e \in E$, but the size condition $|B| = k = |E|$ forces B to contain at least one of these two arcs; see Proposition 22 for the uniqueness of the sink). Thus, in this case, Theorem 1 is saying that the number of acyclic orientations of a given multigraph G with unique sink s does not depend on s. This is part of a result by Greene and Zaslavsky [7, Theorem 7.3], proved using hyperplane arrangements, and has recently been reproved combinatorially by Foissy [5, Proposition 4.6]. It is also a consequence of [10, Proposition 5.3].

3. Up to reversing the directions of the arcs, [11, Proposition 3.7] is Theorem 1 for a specific value of k – namely, for the maximum possible that makes $\gamma_k(s)$ nonzero.[3]

   We note that when D is weakly connected, then this maximum k is also the maximum size of an acyclic subset of A (not just of an s-convergence). This is a consequence of [11, Theorem 3.4]. It thus follows that finding this maximum k is equivalent to the *maximum acyclic subdigraph problem* for Eulerian (= weakly connected balanced) digraphs, which is known to be NP-hard by [11, Theorem 3.10] (see also [4, §3.7.1 and Lemma 4.4.3]). In the terminology of algorithmic combinatorics, this problem is often stated in terms of the complement of the acyclic subset: the *feedback arc set*. In these terms, our $\gamma_k(s)$ counts the feedback arc sets of size $|A| - k$; this counting problem is #P-complete [12, Theorem 5].

As we already noticed, Theorem 1 is the particular case of Theorem 5 for $Z = \varnothing$. We can easily derive Theorem 3 from Theorem 5 as well:

Proof of Theorem 3 using Theorem 5. Let $s, t \in V$. We must prove that $\delta_k(s) = \delta_k(t)$.

If B is any s-preconvergence, then $\operatorname{Cycs} B \subseteq \operatorname{Cycs} A$ (since $D\langle B\rangle$ is a subdigraph of $D = D\langle A\rangle$, so that every cycle of $D\langle B\rangle$ is a cycle of $D\langle A\rangle$). Thus, by the sum rule,

$$\begin{aligned}
&(\text{number of all } s\text{-preconvergences of size } k) \\
&= \sum_{Z \subseteq \operatorname{Cycs} A} \underbrace{(\text{number of all } s\text{-preconvergences } B \text{ of size } k \text{ that satisfy } \operatorname{Cycs} B = Z)}_{\substack{= \gamma_{k,Z}(s) \\ (\text{by the definition of } \gamma_{k,Z}(s))}} \\
&= \sum_{Z \subseteq \operatorname{Cycs} A} \gamma_{k,Z}(s).
\end{aligned}$$

But $\delta_k(s)$ was defined as the number of all s-preconvergences of size k. Thus,

$$\delta_k(s) = (\text{number of all } s\text{-preconvergences of size } k) = \sum_{Z \subseteq \operatorname{Cycs} A} \gamma_{k,Z}(s).$$

[3] This maximum k has also been considered in [8, Theorem 1.11]. Its independence of s was observed in [8, paragraph below Theorem 1.9] as well.

Similarly,

$$\delta_k(t) = \sum_{Z \subseteq \operatorname{Cycs} A} \gamma_{k,Z}(t).$$

But the sums on the right hand sides of these two equalities are equal, since their respective addends agree (by Theorem 5). Hence, the left hand sides must also be equal. In other words, $\delta_k(s) = \delta_k(t)$. This proves Theorem 3. □

Theorem 3, too, can be specialized to a result about orientations of undirected graphs: If G is an undirected multigraph and s is a vertex of G, then the number of orientations of G that have s as a to-root is independent of s. This follows by applying Theorem 3 to $D = G^{\text{bidir}}$ and $k = |E|$ (the number of edges of G). Note that a similar (but not quite the same) fact is proved in [6, Theorem 3.4].

3 The proof

We will prove Theorem 5 through a sequence of lemmas, which are self-contained and might be of independent interest.

If P, Q and S are three sets, then the notation "$S = P \sqcup Q$" shall mean that $S = P \cup Q$ and $P \cap Q = \varnothing$. We note that if $S = P \sqcup Q$, then P and Q are subsets of S and $|S| = |P| + |Q|$.

For any subsets P and Q of V, we let $A(P,Q)$ denote the set of arcs in A whose source belongs to P and whose target belongs to Q. The following fact is a simple but crucial property of balanced digraphs:

Proposition 7. *Let P and Q be two subsets of V such that $V = P \sqcup Q$. Then,*

$$|A(P,Q)| = |A(Q,P)|.$$

Proof. By definition of an outdegree, $\sum\limits_{p \in P} \deg^+ p$ is the number of all arcs with source in P. But since $V = P \sqcup Q$, every arc with source in P has target either in P or in Q. Hence,

$$\sum_{p \in P} \deg^+ p = |A(P,P)| + |A(P,Q)|. \tag{1}$$

Similarly, every arc with target in P has source either in P or Q. Hence, by definition of an indegree, we have

$$\sum_{p \in P} \deg^- p = |A(P,P)| + |A(Q,P)|. \tag{2}$$

However, each $p \in P$ satisfies $\deg^+ p = \deg^- p$ (since D is balanced). Thus, the left hand sides of (1) and (2) are equal, so that their right hand sides are equal as well. That is,

$$|A(P,P)| + |A(P,Q)| = |A(P,P)| + |A(Q,P)|.$$

Subtracting $|A(P,P)|$ from this equality, we obtain $|A(P,Q)| = |A(Q,P)|$. Thus, Proposition 7 is proved. $\square$

Now, given a subset B of A and two vertices $v, w \in V$, we say that "*v can B-reach w*" if the digraph $D\langle B\rangle$ has a path (or, equivalently, a walk) from v to w.

Fix two vertices $s, t \in V$ and an integer $k \in \mathbb{N}$. We want to show that $\gamma_{k,Z}(s) = \gamma_{k,Z}(t)$ for any subset Z of $\operatorname{Cycs} A$.

For any subset B of A, we define the subsets

$$S(B) := \{v \in V \mid v \text{ can } B\text{-reach } s\} \qquad \text{and}$$
$$T(B) := \{v \in V \mid v \text{ can } B\text{-reach } t\} \qquad \text{of } V.$$

We call them the *attraction basins* of s and t with respect to B. Note that $s \in S(B)$ and $t \in T(B)$ always hold, since each vertex can B-reach itself (by a path of length 0).

We shall now prove several properties of attraction basins, which will be used in the proof of Lemma 20 later on. (The adventurous reader can treat them as exercises, while the impatient reader might prefer to skip them and return to them as the need arises.)

Lemma 8. *Let B be a subset of A. Let $v, w \in V$ be two vertices such that v can B-reach w. Assume that $w \in S(B)$. Then, $v \in S(B)$.*

Proof. We have assumed that v can B-reach w. In other words, the digraph $D\langle B\rangle$ has a path $\mathbf{p}$ from v to w. Consider this path $\mathbf{p}$.

Moreover, w can B-reach s (since $w \in S(B)$). In other words, the digraph $D\langle B\rangle$ has a path $\mathbf{q}$ from w to s. Consider this path $\mathbf{q}$.

Concatenating the path $\mathbf{p}$ (from v to w) with the path $\mathbf{q}$ (from w to s), we obtain a walk from v to s. Thus, the digraph $D\langle B\rangle$ has a walk from v to s (namely, the one we just obtained) and thus also a path from v to s. In other words, v can B-reach s. That is, $v \in S(B)$. This proves Lemma 8. $\square$

Lemma 9. *Let P and Q be two subsets of V such that $V = P \sqcup Q$. Let F be a subset of A such that $S(F) = P$. Then, $F \cap A(Q,P) = \varnothing$.*

Proof. Assume the contrary. Thus, there exists some arc $a \in F \cap A(Q,P)$. Consider this a. From $a \in F \cap A(Q,P)$, we obtain $a \in F$ and $a \in A(Q,P)$.

Since $a \in A(Q,P)$, the arc a has a source $q \in Q$ and a target $p \in P$. Consider these q and p. Clearly, the digraph $D\langle F\rangle$ has a walk from q to p (namely, the walk that consists of the single arc $a \in F$). In other words, q can F-reach p. Moreover, $p \in P = S(F)$. Hence, Lemma 8 (applied to $B = F$, $v = q$, and $w = p$) shows that $q \in S(F)$. Thus, $q \in S(F) = P$. But this contradicts $q \in Q$ (since $V = P \sqcup Q$ shows that P and Q are disjoint). Thus, Lemma 9 is proven. $\square$

Lemma 10. *Let P and Q be two subsets of V such that $V = P \sqcup Q$. Let F be a subset of A such that $T(F) = Q$. Then, $F \cap A(P,Q) = \varnothing$.*

Proof. This is analogous to Lemma 9. (Just switch the roles of s and t as well as the roles of P and Q.) $\square$

Lemma 11. *Let P and Q be two subsets of V such that $V = P \sqcup Q$. Let F be a subset of A. Let C be a subset of $A(P, Q)$. Then, $S(F) = P$ if and only if $S(F \cup C) = P$.*

Proof. First, we observe that if B_1 and B_2 are two subsets of A satisfying $B_1 \subseteq B_2$, then any path of $D\langle B_1\rangle$ is a path of $D\langle B_2\rangle$, and thus we have $S(B_1) \subseteq S(B_2)$. Hence, $S(F) \subseteq S(F \cup C)$ (since $F \subseteq F \cup C$).

We must prove the equivalence $(S(F) = P) \iff (S(F \cup C) = P)$. We shall verify the $\Longrightarrow$ and $\Longleftarrow$ directions separately:

$\Longrightarrow$: Assume that $S(F) = P$. We must show that $S(F \cup C) = P$.

By assumption, we have $P = S(F) \subseteq S(F \cup C)$. It remains to prove the converse inclusion.

Let $v \in S(F \cup C)$. Thus, v can $F \cup C$-reach s, so the digraph $D\langle F \cup C\rangle$ has a path $\mathbf{p}$ from v to s. Consider this path $\mathbf{p}$.

We claim that all arcs of $\mathbf{p}$ belong to F. Indeed, let us assume the contrary. Then, at least one of the arcs of $\mathbf{p}$ does not belong to F. Let c be the **last** arc of $\mathbf{p}$ that does not belong to F, and let q be the target of this arc c.

The arc c belongs to $F \cup C$ (since it is part of the path $\mathbf{p}$, which is a path of $D\langle F \cup C\rangle$). Since it does not belong to F, it must thus belong to C. Thus, $c \in C \subseteq A(P, Q)$, so that the target q of c belongs to Q.

However, c is the **last** arc of $\mathbf{p}$ that does not belong to F. Thus, all the arcs of $\mathbf{p}$ that come after c must belong to F. Therefore, the arcs of $\mathbf{p}$ that come after c form a path of the digraph $D\langle F\rangle$. This path starts at q (the target of c) and ends at s. Hence, we have shown that $D\langle F\rangle$ has a path from q to s. In other words, q can F-reach s, meaning that $q \in S(F) = P$. Thus, $q \notin Q$ (since P is disjoint from Q). This contradicts $q \in Q$.

This contradiction shows that our assumption was false. Hence, all arcs of $\mathbf{p}$ belong to F. Therefore, $\mathbf{p}$ is a path of $D\langle F\rangle$. Hence, v can F-reach s (by the path $\mathbf{p}$). That is, $v \in S(F) = P$.

Since we have proved this for each $v \in S(F \cup C)$, we thus conclude that $S(F \cup C) \subseteq P$. Combining this with $P \subseteq S(F \cup C)$, we obtain $S(F \cup C) = P$. This proves the "$\Longrightarrow$" direction of Lemma 11.

$\Longleftarrow$: Assume that $S(F \cup C) = P$. We must show that $S(F) = P$.

We have $S(F) \subseteq S(F \cup C) = P$ (by assumption). It remains to prove the converse inclusion.

Let $p \in P$. Then, $p \in P = S(F \cup C)$ (by assumption). Hence, p can $F \cup C$-reach s, so the digraph $D\langle F \cup C\rangle$ has a path $\mathbf{p}$ from p to s. Any vertex v of this path $\mathbf{p}$ must itself belong to $S(F \cup C)$ (since it can $F \cup C$-reach s by walking along $\mathbf{p}$ from v to s), so it cannot belong to Q (since $S(F \cup C) = P$ is disjoint from Q). Therefore, no arc of this path $\mathbf{p}$ can belong to $A(P, Q)$ (since this would require its target to belong to Q). Thus, all arcs of this path $\mathbf{p}$ belong to $(F \cup C) \setminus \underbrace{A(P, Q)}_{\supseteq C} \subseteq (F \cup C) \setminus C \subseteq F$. Hence, $\mathbf{p}$ is a path of $D\langle F\rangle$. Consequently, the vertex p can F-reach s (via the path $\mathbf{p}$). In other words, $p \in S(F)$. Since we have proved this for each $p \in P$, we conclude that $P \subseteq S(F)$.

Therefore, $S(F) = P$ (since $S(F) \subseteq P$). This proves the "$\Longleftarrow$" direction of Lemma 11. $\square$

Lemma 12. *Let P and Q be two subsets of V such that $V = P \sqcup Q$. Let F be a subset of A such that $S(F) = P$. Let C be a subset of $A(P,Q)$. Then,* $\operatorname{Cycs}(F \cup C) = \operatorname{Cycs} F$.

Proof. First, we observe that if B_1 and B_2 are two subsets of A satisfying $B_1 \subseteq B_2$, then any cycle of $D\langle B_1\rangle$ is a cycle of $D\langle B_2\rangle$, and thus we have $\operatorname{Cycs} B_1 \subseteq \operatorname{Cycs} B_2$ (since $\operatorname{Cycs} B$ denotes the set of all cycles of $D\langle B\rangle$). Applying this to $B_1 = F$ and $B_2 = F \cup C$, we obtain $\operatorname{Cycs} F \subseteq \operatorname{Cycs}(F \cup C)$ (since $F \subseteq F \cup C$).

We shall now prove that $\operatorname{Cycs}(F \cup C) \subseteq \operatorname{Cycs} F$. Indeed, let $\mathbf{c} \in \operatorname{Cycs}(F \cup C)$ be arbitrary. Thus, $\mathbf{c}$ is a cycle of $D\langle F \cup C\rangle$. We want to show that $\mathbf{c}$ is a cycle of $D\langle F\rangle$ as well.

Assume the contrary. Thus, $\mathbf{c}$ is not a cycle of $D\langle F\rangle$. In other words, not all arcs of $\mathbf{c}$ belong to F. But $\mathbf{c}$ is a cycle of $D\langle F \cup C\rangle$; thus, each arc of $\mathbf{c}$ belongs to $F \cup C$. In other words, each arc of $\mathbf{c}$ belongs to F or to C. Hence, some arc of $\mathbf{c}$ belongs to C (because not all arcs of $\mathbf{c}$ belong to F). This arc must therefore belong to $A(P,Q)$ as well (since $C \subseteq A(P,Q)$); in other words, this arc has source in P and target in Q. Thus, the cycle $\mathbf{c}$ contains both a vertex in P and a vertex in Q (namely, the source and the target of this arc). Consequently, it must cross from Q to P at some point (since $V = P \sqcup Q$). In other words, it contains an arc $a \in A(Q,P)$. Consider this arc a.

Lemma 9 yields $F \cap A(Q,P) = \varnothing$. Hence, $a \notin F$ (since $a \in A(Q,P)$). Combining this with $a \in F \cup C$ (since a is an arc of $\mathbf{c}$, which is a cycle of $D\langle F \cup C\rangle$), we obtain $a \in (F \cup C) \setminus F \subseteq C \subseteq A(P,Q)$. Thus, the source of a belongs to P. On the other hand, the source of a belongs to Q (since $a \in A(Q,P)$). The preceding two sentences contradict each other, since $P \cap Q = \varnothing$.

This contradiction shows that our assumption was wrong. Hence, we have shown that $\mathbf{c}$ is a cycle of $D\langle F\rangle$. In other words, $\mathbf{c} \in \operatorname{Cycs} F$.

Forget that we fixed $\mathbf{c}$. We thus have shown that $\mathbf{c} \in \operatorname{Cycs} F$ for each $\mathbf{c} \in \operatorname{Cycs}(F \cup C)$, so that $\operatorname{Cycs}(F \cup C) \subseteq \operatorname{Cycs} F$. Combining this with $\operatorname{Cycs} F \subseteq \operatorname{Cycs}(F \cup C)$, we obtain $\operatorname{Cycs}(F \cup C) = \operatorname{Cycs} F$. This proves Lemma 12. $\square$

Lemma 13. *Let P and Q be two subsets of V such that $V = P \sqcup Q$ and $t \in Q$. Let F be a subset of A that is disjoint from $A(P,Q)$. Then, $T(F) \subseteq Q$.*

Proof. Let $x \in T(F)$ be any vertex. We must show that $x \in Q$.

Assume the contrary. Then, $x \in P$ (since $V = P \sqcup Q$). But x can F-reach t (since $x \in T(F)$). So the digraph $D\langle F\rangle$ has a path $\mathbf{x}$ from x to t. This path $\mathbf{x}$ is a path of $D\langle F\rangle$; thus, all its arcs belong to F.

The path $\mathbf{x}$ starts at a vertex in P (namely, at $x \in P$) and ends at a vertex in Q (namely, at $t \in Q$). So this path $\mathbf{x}$ must cross from P to Q at some point (since $V = P \sqcup Q$). In other words, it must contain an arc that belongs to $A(P,Q)$. But this is impossible, since all arcs of $\mathbf{x}$ belong to the set F, which is disjoint from $A(P,Q)$. This contradiction shows that $x \in Q$. Since we have proved this for each $x \in T(F)$, we thus conclude that $T(F) \subseteq Q$. This proves Lemma 13. $\square$

Lemma 14. *Let P and Q be two subsets of V such that $V = P \sqcup Q$ and such that Q is nonempty. Let B be a subset of A such that $S(B) = P$ and $S(B) \cup T(B) = V$. Let $C = B \cap A(P, Q)$. Then, $T(B \setminus C) = Q$.*

Proof. From $S(B) \cup T(B) = V$ and $S(B) = P$, we obtain $\underbrace{(S(B) \cup T(B))}_{=V} \setminus \underbrace{S(B)}_{=P} = V \setminus P = Q$ (since $V = P \sqcup Q$), so that

$$Q = (S(B) \cup T(B)) \setminus S(B) = T(B) \setminus S(B).$$

Next, we claim that $t \in Q$. Indeed, there exists some vertex $v \in Q$ (since Q is nonempty). Consider this v. Then, $v \in Q = T(B) \setminus S(B)$, so that $v \in T(B)$ and $v \notin S(B)$. From $v \in T(B)$, we see that v can B-reach t. If we had $t \in S(B)$, then Lemma 8 (applied to $w = t$) would thus show that $v \in S(B)$, which would contradict $v \notin S(B)$. Hence, we cannot have $t \in S(B)$. Hence, $t \notin S(B) = P$, so that $t \in Q$ (since $V = P \sqcup Q$).

From $C = B \cap A(P, Q)$, we obtain

$$B \setminus C = B \setminus (B \cap A(P, Q)) = B \setminus A(P, Q). \tag{3}$$

Hence, the set $B \setminus C$ is disjoint from $A(P, Q)$ (since $B \setminus A(P, Q)$ clearly is). Thus, Lemma 13 (applied to $F = B \setminus C$) yields that $T(B \setminus C) \subseteq Q$ (since $t \in Q$).

Now, we shall show that $Q \subseteq T(B \setminus C)$. Indeed, let $q \in Q$ be any vertex. Then, $q \in Q = T(B) \setminus S(B)$. That is, $q \in T(B)$ and $q \notin S(B)$. In particular, q can B-reach t (since $q \in T(B)$), so the digraph $D\langle B \rangle$ has a path $\mathbf{q}$ from q to t.

Let w be any vertex of this path $\mathbf{q}$. Then, q can B-reach w as well (by following the path $\mathbf{q}$ from q until w). If we had $w \in S(B)$, then this would entail $q \in S(B)$ (by Lemma 8, applied to $v = q$), which would contradict $q \notin S(B)$. Thus, we cannot have $w \in S(B)$. In other words, we have $w \notin S(B) = P$.

Forget that we fixed w. Thus, we have showed that $w \notin P$ for any vertex w of the path $\mathbf{q}$. In other words, no vertex of the path $\mathbf{q}$ belongs to P. Therefore, no arc of the path $\mathbf{q}$ belongs to $A(P, Q)$ (since the source of such an arc would be a vertex of $\mathbf{q}$ and belong to P). Thus, all arcs of $\mathbf{q}$ belong to $B \setminus A(P, Q) = B \setminus C$ (by (3)). Hence, $\mathbf{q}$ is a path of $D\langle B \setminus C \rangle$. This shows that q can $B \setminus C$-reach t (via the path $\mathbf{q}$); that is, $q \in T(B \setminus C)$. Since we have proved this for each $q \in Q$, we thus conclude that $Q \subseteq T(B \setminus C)$.

Combining this with $T(B \setminus C) \subseteq Q$ yields $T(B \setminus C) = Q$. This proves Lemma 14. $\square$

Now, we take aim at proving Theorem 5. Let us fix a subset Z of $\operatorname{Cycs} A$.

For any $i \in \mathbb{Z}$, we let $\mathcal{P}_i(A)$ denote the set of all i-element subsets of A. Define the subset

$$U_{k,Z} := \{B \in \mathcal{P}_k(A) \mid S(B) \cup T(B) = V \text{ and } \operatorname{Cycs} B = Z\} \qquad \text{of } \mathcal{P}_k(A).$$

We observe the following:

Lemma 15. *We have*

$$|\{B \in U_{k,Z} \ \mid \ S(B) = V\}| = \gamma_{k,Z}(s).$$

Proof. Note that a subset B of A is an s-preconvergence
if and only if s is a to-root of $D\langle B\rangle$,
i.e., if and only if each vertex $v \in V$ has a path to s in $D\langle B\rangle$,
i.e., if and only if each vertex $v \in V$ can B-reach s,
i.e., if and only if $S(B) = V$
(since $S(B)$ is defined as the set of all vertices $v \in V$ that can B-reach s). Thus, an s-preconvergence is the same thing as a subset B of A that satisfies $S(B) = V$.

Hence, the number $\gamma_{k,Z}(s)$ of all s-preconvergences B of size k that satisfy $\operatorname{Cycs} B = Z$ can also be described as the number of all subsets B of A of size k that satisfy $S(B) = V$ and $\operatorname{Cycs} B = Z$. In other words,

$$\gamma_{k,Z}(s) = |\{B \in \mathcal{P}_k(A) \ \mid \ S(B) = V \text{ and } \operatorname{Cycs} B = Z\}|. \tag{4}$$

But any $B \in \mathcal{P}_k(A)$ satisfying $S(B) = V$ and $\operatorname{Cycs} B = Z$ must also satisfy $B \in U_{k,Z}$ (since $\underbrace{S(B)}_{=V} \cup T(B) = V \cup T(B) = V$). Conversely, the definition of $U_{k,Z}$ shows that any $B \in U_{k,Z}$ belongs to $\mathcal{P}_k(A)$ and satisfies $\operatorname{Cycs} B = Z$. These two facts show that

$$\{B \in \mathcal{P}_k(A) \ \mid \ S(B) = V \text{ and } \operatorname{Cycs} B = Z\} = \{B \in U_{k,Z} \ \mid \ S(B) = V\}.$$

Hence, we can rewrite (4) as

$$\gamma_{k,Z}(s) = |\{B \in U_{k,Z} \ \mid \ S(B) = V\}|.$$

This proves Lemma 15. $\square$

Now we claim the following:

Lemma 16. *We have*

$$\gamma_{k,Z}(s) = |U_{k,Z}| - \sum_{\substack{P,Q \subseteq V \text{ are nonempty;} \\ V = P \sqcup Q}} |\{B \in U_{k,Z} \ \mid \ S(B) = P\}|.$$

Proof. First, we note that the set V is nonempty (since $s \in V$).

For each $B \in U_{k,Z}$, there is a unique pair (P, Q) of subsets $P, Q \subseteq V$ satisfying $V = P \sqcup Q$ such that $S(B) = P$ (indeed, the condition $S(B) = P$ uniquely determines P, while the condition $V = P \sqcup Q$ forces Q to be $V \setminus P$). Moreover, in this unique pair (P, Q), the set P is nonempty (since $s \in S(B) = P$). Hence, for each $B \in U_{k,Z}$, there is a unique pair (P, Q) of subsets $P, Q \subseteq V$ satisfying "P is nonempty" and $V = P \sqcup Q$

such that $S(B) = P$. Therefore, we can count the elements $B \in U_{k,Z}$ by the sum rule, according to the value of this pair:

$$\begin{aligned}
|U_{k,Z}| &= \sum_{\substack{P,Q\subseteq V;\\ P \text{ is nonempty;}\\ V=P\sqcup Q}} |\{B \in U_{k,Z} \ \mid \ S(B) = P\}| \\
&= \sum_{\substack{P,Q\subseteq V \text{ are nonempty;}\\ V=P\sqcup Q}} |\{B \in U_{k,Z} \ \mid \ S(B) = P\}| + \underbrace{|\{B \in U_{k,Z} \ \mid \ S(B) = V\}|}_{\substack{=\gamma_{k,Z}(s)\\ \text{(by Lemma 15)}}} \\
&= \sum_{\substack{P,Q\subseteq V \text{ are nonempty;}\\ V=P\sqcup Q}} |\{B \in U_{k,Z} \ \mid \ S(B) = P\}| + \gamma_{k,Z}(s),
\end{aligned}$$

where the second equality comes from splitting off the addend for $(P, Q) = (V, \varnothing)$ from the sum (this addend indeed exists because V is nonempty).

Solving this for $\gamma_{k,Z}(s)$, we obtain the claim of the lemma. $\square$

Similarly, we find:

Lemma 17. *We have*

$$\gamma_{k,Z}(t) = |U_{k,Z}| - \sum_{\substack{P,Q\subseteq V \text{ are nonempty;}\\ V=P\sqcup Q}} |\{B \in U_{k,Z} \ \mid \ T(B) = Q\}| .$$

Proof. Analogous to the proof of Lemma 16. (Just switch the roles of s and t and also the roles of P and Q.) $\square$

Our goal is to prove $\gamma_{k,Z}(s) = \gamma_{k,Z}(t)$. In light of Lemma 16 and Lemma 17, it will suffice to show the following:

Proposition 18. *Let P and Q be two nonempty subsets of V such that $V = P \sqcup Q$. Then,*

$$|\{B \in U_{k,Z} \ \mid \ S(B) = P\}| = |\{B \in U_{k,Z} \ \mid \ T(B) = Q\}| .$$

We will prove this by finding analogous expressions for both sides. First, we need some notation:

Definition 19. If P and Q are two subsets of V satisfying $V = P \sqcup Q$, and if $i \in \mathbb{Z}$ is arbitrary, then we set

$$X_{i,Z}^{P,Q} := \{B \in \mathcal{P}_i(A) \ \mid \ S(B) = P \text{ and } T(B) = Q \text{ and } \operatorname{Cycs} B = Z\} .$$

Lemma 20. *Let P and Q be two nonempty subsets of V such that $V = P \sqcup Q$. Then,*

$$|\{B \in U_{k,Z} \ \mid \ S(B) = P\}| = \sum_{m=0}^{|A(P,Q)|} \binom{|A(P,Q)|}{m} \cdot \left| X_{k-m,Z}^{P,Q} \right| .$$

Proof of Lemma 20. By the sum rule,

$$\begin{aligned} &\left|\left\{B \in U_{k,Z} \ \mid \ S\left(B\right) = P\right\}\right| \\ &= \sum_{C \subseteq A(P,Q)} \left|\left\{B \in U_{k,Z} \ \mid \ S\left(B\right) = P \text{ and } B \cap A\left(P,Q\right) = C\right\}\right|, \end{aligned} \tag{5}$$

since the intersection $B \cap A\left(P,Q\right)$ is always a subset of $A\left(P,Q\right)$.

Fix a subset $C \subseteq A(P,Q)$, and let

$$Y_{k,Z} := \left\{B \in U_{k,Z} \ \mid \ S\left(B\right) = P \text{ and } B \cap A\left(P,Q\right) = C\right\}.$$

We want to show that $\left|Y_{k,Z}\right| = \left|X^{P,Q}_{k-|C|,Z}\right|$. Define

$$\text{the map } \Phi : Y_{k,Z} \to X^{P,Q}_{k-|C|,Z} \quad \text{by } \Phi(B) = B \setminus C$$

and

$$\text{the map } \Psi : X^{P,Q}_{k-|C|,Z} \to Y_{k,Z} \quad \text{by } \Psi(F) = F \cup C.$$

Let us first show that these maps are well-defined:

Claim 1. The map Φ is well-defined. That is, $B \setminus C \in X^{P,Q}_{k-|C|,Z}$ for all $B \in Y_{k,Z}$.

Proof of Claim 1. Let $B \in Y_{k,Z}$. Then $B \in U_{k,Z}$ as well as $S\left(B\right) = P$ and $B \cap A\left(P,Q\right) = C$ (by the definition of $Y_{k,Z}$). From $B \in U_{k,Z}$, we see that B is a k-element subset of A satisfying $S\left(B\right) \cup T\left(B\right) = V$ and $\operatorname{Cycs} B = Z$ (by the definition of $U_{k,Z}$). In particular, $|B| = k$.

We must show that $B \setminus C \in X^{P,Q}_{k-|C|,Z}$. In other words, we must show that $B \setminus C$ is a set in $\mathcal{P}_{k-|C|}\left(A\right)$ and satisfies $S\left(B \setminus C\right) = P$, $T\left(B \setminus C\right) = Q$, and $\operatorname{Cycs}\left(B \setminus C\right) = Z$.

From $C = B \cap A\left(P,Q\right) \subseteq B$, we obtain

$$|B \setminus C| = |B| - |C| = k - |C| \qquad \left(\text{since } |B| = k\right).$$

Thus, $B \setminus C \in \mathcal{P}_{k-|C|}\left(A\right)$. Also, from $C \subseteq B$, we obtain $\left(B \setminus C\right) \cup C = B$.

Let us now prove that $S\left(B \setminus C\right) = P$. Indeed, Lemma 11 (applied to $F = B \setminus C$) yields that $S\left(B \setminus C\right) = P$ if and only if $S\left(\left(B \setminus C\right) \cup C\right) = P$. Since $S\left(\left(B \setminus C\right) \cup C\right) = P$ does hold (because $S\Big(\underbrace{\left(B \setminus C\right) \cup C}_{=B}\Big) = S\left(B\right) = P$), we thus conclude that $S\left(B \setminus C\right) = P$.

Hence, Lemma 12 (applied to $F = B \setminus C$) shows that $\operatorname{Cycs}\left(\left(B \setminus C\right) \cup C\right) = \operatorname{Cycs}\left(B \setminus C\right)$ (since $C \subseteq A\left(P,Q\right)$). Thus, $\operatorname{Cycs}\left(B \setminus C\right) = \operatorname{Cycs}\underbrace{\left(\left(B \setminus C\right) \cup C\right)}_{=B} = \operatorname{Cycs} B = Z$.

Finally, Lemma 14 shows that $T\left(B \setminus C\right) = Q$ (since $C = B \cap A\left(P,Q\right)$).

Altogether, we now have shown that $B \setminus C$ is a set in $\mathcal{P}_{k-|C|}\left(A\right)$ and satisfies $S\left(B \setminus C\right) = P$, $T\left(B \setminus C\right) = Q$, and $\operatorname{Cycs}\left(B \setminus C\right) = Z$. In other words, $B \setminus C \in X^{P,Q}_{k-|C|,Z}$ (by the definition of $X^{P,Q}_{k-|C|,Z}$). Claim 1 is thus proved. $\square$

Claim 2. The map Ψ is well-defined. That is, $F \cup C \in Y_{k,Z}$ for all $F \in X^{P,Q}_{k-|C|,Z}$.

Proof of Claim 2. Let $F \in X^{P,Q}_{k-|C|,Z}$. Then F is a $(k-|C|)$-element subset of A satisfying $S(F) = P$, $T(F) = Q$, and $\operatorname{Cycs} F = Z$ (by the definition of $X^{P,Q}_{k-|C|,Z}$).

We must show that $F \cup C \in Y_{k,Z}$. In other words, we must show that $F \cup C \in U_{k,Z}$, $S(F \cup C) = P$ and $(F \cup C) \cap A(P,Q) = C$.

From $S(F) = P$, we immediately obtain $S(F \cup C) = P$ by Lemma 11.

Moreover, Lemma 10 yields $F \cap A(P,Q) = \varnothing$. Also, Lemma 9 shows that $F \cap A(Q,P) = \varnothing$. Furthermore, Lemma 12 yields $\operatorname{Cycs}(F \cup C) = \operatorname{Cycs} F = Z$.

The set $A(P,Q)$ is disjoint from $A(Q,P)$, since the source of an arc cannot belong to P and Q at the same time (because of $V = P \sqcup Q$). Hence, the set C (being a subset of $A(P,Q)$) must be disjoint from $A(Q,P)$ as well. In other words, $C \cap A(Q,P) = \varnothing$.

Also, note that C is a subset of $A(P,Q)$, and thus we have

$$F \cap \underbrace{C}_{\subseteq A(P,Q)} \subseteq F \cap A(P,Q) = \varnothing.$$

Hence,

$$F \cap C = \varnothing, \tag{6}$$

so that

$$|F \cup C| = |F| + |C| = k \qquad \left(\text{since } |F| = k - |C|\right).$$

Therefore, $F \cup C \in \mathcal{P}_k(A)$.

Furthermore,

$$(F \cup C) \cap A(P,Q) = \underbrace{(F \cap A(P,Q))}_{=\varnothing} \cup \underbrace{(C \cap A(P,Q))}_{\substack{=C \\ (\text{since } C \subseteq A(P,Q))}} = \varnothing \cup C = C.$$

If v is a vertex in $S(F)$, then v can F-reach s and thus can $F \cup C$-reach s as well (since $F \subseteq F \cup C$); thus it belongs to $S(F \cup C)$. This shows that $S(F) \subseteq S(F \cup C)$. Similarly, $T(F) \subseteq T(F \cup C)$. Now,

$$V = P \sqcup Q = \underbrace{P}_{=S(F) \subseteq S(F \cup C)} \cup \underbrace{Q}_{=T(F) \subseteq T(F \cup C)} \subseteq S(F \cup C) \cup T(F \cup C).$$

Of course, this entails $S(F \cup C) \cup T(F \cup C) = V$.

This shows that $F \cup C \in U_{k,Z}$ (since $F \cup C \in \mathcal{P}_k(A)$ and $\operatorname{Cycs}(F \cup C) = Z$).

So, we have shown that $F \cup C \in U_{k,Z}$, $S(F \cup C) = P$, and $(F \cup C) \cap A(P,Q) = C$. All together, this yields $F \cup C \in Y_{k,Z}$. Claim 2 is thus proved. □

Claim 3. The maps Φ and Ψ are mutually inverse.

Proof of Claim 3. For each $B \in Y_{k,Z}$, we have $C = B \cap A(P,Q) \subseteq B$. Thus, for each $B \in Y_{k,Z}$, we have

$$\Psi(\Phi(B)) = \Psi(B \setminus C) = (B \setminus C) \cup C = B \qquad \text{(since } C \subseteq B\text{)}.$$

In other words, $\Psi \circ \Phi = \mathrm{id}$.

On the other hand, for each $F \in X^{P,Q}_{k-|C|}$, we have $F \cap C = \varnothing$ (see the equality (6) in the proof of Claim 2) and thus

$$\Phi(\Psi(F)) = \Phi(F \cup C) = (F \cup C) \setminus C = F \qquad \text{(since } F \cap C = \varnothing\text{)}.$$

This shows that $\Phi \circ \Psi = \mathrm{id}$.

From $\Phi \circ \Psi = \mathrm{id}$ and $\Psi \circ \Phi = \mathrm{id}$, we conclude that the maps Φ and Ψ are mutually inverse. This proves Claim 3. $\square$

Claim 3 shows that the map Ψ is a bijection from $X^{P,Q}_{k-|C|,Z}$ to $Y_{k,Z}$. Thus,

$$\left|X^{P,Q}_{k-|C|,Z}\right| = |Y_{k,Z}| = |\{B \in U_{k,Z} \mid S(B) = P \text{ and } B \cap A(P,Q) = C\}|$$

(by the definition of $Y_{k,Z}$).

We have proved this equality for each subset C of $A(P,Q)$. Summing it over all such subsets, we obtain

$$\begin{aligned}\sum_{C \subseteq A(P,Q)} \left|X^{P,Q}_{k-|C|,Z}\right| &= \sum_{C \subseteq A(P,Q)} |\{B \in U_{k,Z} \mid S(B) = P \text{ and } B \cap A(P,Q) = C\}| \\ &= |\{B \in U_{k,Z} \mid S(B) = P\}| \qquad \text{(by (5))}.\end{aligned}$$

Thus,

$$\begin{aligned}|\{B \in U_{k,Z} \mid S(B) = P\}| &= \sum_{C \subseteq A(P,Q)} \left|X^{P,Q}_{k-|C|,Z}\right| \\ &= \sum_{m=0}^{|A(P,Q)|} \sum_{\substack{C \subseteq A(P,Q);\\ |C|=m}} \underbrace{\left|X^{P,Q}_{k-|C|,Z}\right|}_{\substack{=\left|X^{P,Q}_{k-m,Z}\right| \\ \text{(since } |C|=m)}} \\ &= \sum_{m=0}^{|A(P,Q)|} \sum_{\substack{C \subseteq A(P,Q);\\ |C|=m}} \left|X^{P,Q}_{k-m,Z}\right| \\ &= \sum_{m=0}^{|A(P,Q)|} \binom{|A(P,Q)|}{m} \cdot \left|X^{P,Q}_{k-m,Z}\right|,\end{aligned}$$

where the last equality holds since, for each fixed m, the inner sum consists of $\binom{|A(P,Q)|}{m}$ many equal addends. This proves Lemma 20. $\square$

Lemma 21. *Let P and Q be two nonempty subsets of V such that $V = P \sqcup Q$. Then,*

$$\left|\left\{B \in U_{k,Z} \ \mid \ T(B) = Q\right\}\right| = \sum_{m=0}^{|A(Q,P)|} \binom{|A(Q,P)|}{m} \cdot \left|X^{P,Q}_{k-m,Z}\right|.$$

Proof. Analogous to the proof of Lemma 20. (Just switch the roles of s and t and also the roles of P and Q.) ☐

Proof of Proposition 18. Lemma 20 yields

$$\begin{aligned}
\left|\left\{B \in U_{k,Z} \ \mid \ S(B) = P\right\}\right| &= \sum_{m=0}^{|A(P,Q)|} \binom{|A(P,Q)|}{m} \cdot \left|X^{P,Q}_{k-m,Z}\right| \\
&= \sum_{m=0}^{|A(Q,P)|} \binom{|A(Q,P)|}{m} \cdot \left|X^{P,Q}_{k-m,Z}\right| \\
&\qquad \left(\begin{array}{c}\text{since Proposition 7} \\ \text{yields } |A(P,Q)| = |A(Q,P)|\end{array}\right) \\
&= \left|\left\{B \in U_{k,Z} \ \mid \ T(B) = Q\right\}\right| \qquad \text{(by Lemma 21)}.
\end{aligned}$$

This proves Proposition 18. ☐

Now, Lemma 16 yields

$$\begin{aligned}
\gamma_{k,Z}(s) &= |U_{k,Z}| - \sum_{\substack{P,Q \subseteq V \text{ are nonempty;} \\ V = P \sqcup Q}} \underbrace{\left|\left\{B \in U_{k,Z} \ \mid \ S(B) = P\right\}\right|}_{\substack{= \left|\left\{B \in U_{k,Z} \ \mid \ T(B) = Q\right\}\right| \\ \text{(by Proposition 18)}}} \\
&= |U_{k,Z}| - \sum_{\substack{P,Q \subseteq V \text{ are nonempty;} \\ V = P \sqcup Q}} \left|\left\{B \in U_{k,Z} \ \mid \ T(B) = Q\right\}\right| = \gamma_{k,Z}(t)
\end{aligned}$$

(by Lemma 17). This completes the proof of Theorem 5.

As we already said, Theorem 1 and Theorem 3 follow from Theorem 5.

4 Further remarks

1. One might wonder whether our proof of Theorem 5 is, or can be made, bijective. In the given form, it is not, as it uses subtraction twice: once in proving Proposition 7 and once again in the "subtractive flip" that is involved in Lemma 16 (and Lemma 17). Both of these instances of subtraction can be made bijective using the Garsia–Milne involution principle [13, §4.6].

In the case of Lemma 16, only the simplest case of the involution principle is needed: Let $\Gamma_{k,Z}(s)$ be the set of all s-preconvergences B of size k that satisfy $\operatorname{Cycs} B = Z$, and let $\Gamma_{k,Z}(t)$ be the set of all t-preconvergences B of size k that satisfy $\operatorname{Cycs} B = Z$. Our above proof of Proposition 18 gives a bijection

$$\{B \in U_{k,Z} \mid S(B) = P\} \to \{B \in U_{k,Z} \mid T(B) = Q\}$$

for every pair of nonempty subsets $P, Q \subseteq V$ satisfying $V = P \sqcup Q$ (essentially, this bijection sends each $B \in U_{k,Z}$ satisfying $S(B) = P$ to the set

$$(B \setminus A(P,Q)) \cup \rho_{P,Q}(B \cap A(P,Q)),$$

where $\rho_{P,Q}$ is a fixed size-preserving bijection from $A(P,Q)$ to $A(Q,P)$). Combining these bijections for all such pairs (P,Q), we obtain a bijection $\phi : U_{k,Z} \setminus \Gamma_{k,Z}(s) \to U_{k,Z} \setminus \Gamma_{k,Z}(t)$. Now we need to construct a bijection $\psi : \Gamma_{k,Z}(s) \to \Gamma_{k,Z}(t)$ from it. The involution principle tells us that this ψ acts on a given element $B \in \Gamma_{k,Z}(s)$ by repeatedly applying ϕ^{-1} to it until it no longer belongs to $U_{k,Z} \setminus \Gamma_{k,Z}(t)$ (which means that it belongs to $\Gamma_{k,Z}(t)$). It is not hard to see that this does not require more than $|V|$ many iterations, since $|T(B)|$ increases with each application of ϕ^{-1}.

In the case of Proposition 7, however, we can also give a direct bijective proof: By the directed Euler–Hierholzer theorem, each weak component of the balanced digraph D has an Eulerian circuit. Pick such a circuit for each weak component of D. Note that on each of these circuits, arcs from $A(P,Q)$ and arcs from $A(Q,P)$ alternate (if we remove the arcs from $A(P,P)$ and $A(Q,Q)$). Thus, we can define a bijection $A(P,Q) \to A(Q,P)$ that sends each arc $a \in A(P,Q)$ to the next arc $b \in A(Q,P)$ following it on the chosen Eulerian circuit. At a second thought, this does not even require Eulerian circuits; it suffices to pick any decomposition of A into (arc-disjoint) circuits.

2. We can also characterize s-convergences in terms of sinks. Recall that a *sink* of a digraph means a vertex with no outgoing arcs (i.e., a vertex with outdegree 0). Now our alternative characterization of s-convergences is as follows:

Proposition 22. *Let B be an acyclic subset of A. Let $s \in V$. Then, B is an s-convergence if and only if s is the only sink of the subdigraph $D\langle B\rangle$.*

Proof. $\Longrightarrow$: Assume that B is an s-convergence. Thus, s is a to-root of $D\langle B\rangle$.

We shall first show that s is a sink of $D\langle B\rangle$. Indeed, assume the contrary; thus, there exists an arc a of $D\langle B\rangle$ with source s. Let v be the target of this arc a. Since s is a to-root of $D\langle B\rangle$, there exists a path from v to s in $D\langle B\rangle$. This path, together with the arc a, creates a cycle[4] in $D\langle B\rangle$, which contradicts the acyclicity of B. So, we see that s is a sink of $D\langle B\rangle$.

Furthermore, no vertex $v \neq s$ of $D\langle B\rangle$ can also be a sink of $D\langle B\rangle$, for $D\langle B\rangle$ must have a path from v to s (because s is a to-root of $D\langle B\rangle$) and this path must begin with an arc with source v. Hence, s is the only sink of $D\langle B\rangle$.

$\Longleftarrow$: Assume that s is the only sink of $D\langle B\rangle$. Let $v \in V$ be any vertex.

[4] If $v = s$, then this cycle consists of just a single loop.

The digraph $D\langle B\rangle$ has no cycles (since B is acyclic). Thus, any walk of $D\langle B\rangle$ is a path (since a walk that is not a path would contain a cycle). Consequently, the walks of $D\langle B\rangle$ are precisely the paths of $D\langle B\rangle$. Hence, the digraph $D\langle B\rangle$ has only finitely many walks that start at v (since it clearly has only finitely many **paths** that start at v). Moreover, there is at least one such walk (the length-0 walk (v)).

Thus, there is a longest such walk. Let $\mathbf{p}$ be such a walk. Then, $\mathbf{p}$ must end at a sink of $D\langle B\rangle$ (since otherwise, we could extend $\mathbf{p}$ by an additional arc at the end, obtaining an even longer walk that starts at v). Since the only sink of $D\langle B\rangle$ is s, this means that $\mathbf{p}$ must end at s. Thus, $\mathbf{p}$ is a walk from v to s, hence a path from v to s (since any walk of $D\langle B\rangle$ is a path). Hence, $D\langle B\rangle$ has a path from v to s (namely, $\mathbf{p}$). Since v was arbitrary, this shows that s is a to-root of $D\langle B\rangle$. Therefore, B is an s-convergence (since B is acyclic). □

3. In Lemma 16, Lemma 17, Lemma 20, Lemma 21 and Proposition 22, there is no need for the digraph D to be balanced.

4. The relationships between the above theorems and some results in the literature prompt some natural questions:

(a) Let $G = (V, E, \varphi)$ be an undirected multigraph, and let $D = G^{\text{bidir}}$. Then, in [10, Proposition 5.3], Novik, Postnikov and Sturmfels show that the number $\delta_{|E|}(s)$ for any $s \in V$ is the Möbius invariant of the intersection lattice of the graphic arrangement of G. Are there some similar interpretations for $\delta_k(s)$ for other k, and perhaps for the general case when $D \neq G^{\text{bidir}}$?

(b) Again, let $G = (V, E, \varphi)$ be an undirected multigraph. In [6, Theorem 3.4], Gioan shows that the number of indegree functions of orientations of G that have a given vertex s as a to-root (or "unique quasi-sink" in his lingo) is independent of s, and equals the evaluation $t(G; 1, 1)$ of the Tutte polynomial of G. This is not quite our $\delta_{|E|}(s)$, since we are counting the orientations themselves rather than their indegree functions; but it suggests that the Tutte polynomial is not far away. The Tutte polynomial can be generalized to digraphs [3]; does the relationship generalize?

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