Higher dimensional floorplans and Baxter *d*-permutations

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Abstract

A 2−dimensional mosaic floorplan is a partition of a rectangle by other rectangles with no empty rooms. These partitions (considered up to some deformations) are known to be in bijection with Baxter permutations. A d-floorplan is the generalisation of mosaic floorplans in higher dimensions, and a d-permutation is a $(d-1)$-tuple of permutations. Recently, in N. Bonichon and P.-J. Morel, *J. Integer Sequences* 25 (2022), Baxter d-permutations generalising the usual Baxter permutations were introduced.

In this paper, we consider mosaic floorplans in arbitrary dimensions, and we construct a generating tree for d-floorplans, which generalises the known generating tree structure for 2-floorplans. The corresponding labels and rewriting rules appear to be significantly more involved in higher dimensions. Moreover we give a bijection between the 2^{d-1}-floorplans and d-permutations characterized by forbidden vincular patterns. Surprisingly, this set of d-permutations is strictly contained within the set of Baxter d-permutations.

Mathematics Subject Classifications: 06A07, 05A05, 05A19

Contents

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1 Introduction

A *floorplan* of size n is a partition of a rectangle using n interior-disjoint rectangles. These combinatorial objects have been studied in various fields of computer science, architecture or discrete geometry.

The boundaries of the rectangles of a floorplan define a set of horizontal and vertical segments called block-segments. A floorplan is *generic* if the union of horizontal (resp. vertical) block-segments that share the same y-coordinate (resp. x-coordinate) is a single line. A *border* is a maximal (horizontal or vertical) segment in the inner boundaries. A *mosaic floorplan* is a generic floorplan with no border crossing. This condition is called the *tatami condition*.

In this paper we investigate a natural generalization of floorplans to higher dimensions. A *d-dimensional floorplan* (or a *box partition*) is a partition of a d-dimensional hyperrectangle with n interior-disjoint d-dimensional hyperrectangles (called blocks). The boundaries of the hyperrectangles of a d-dimensional floorplan define a set of $(d-1)$-hyperrectangles called block-facets. A *d-dimensional floorplan* is *generic* if the set of boundaries that share the same i-th coordinate is a single $(d-1)$-hyperrectangle. A *border* is a maximal $(d-1)$-hyperrectangle of the interior of the bounding d-hyperrectangle. A *d-floorplan* is a generic d-dimensional floorplan that has no border crossing (this being the *tatami condition* in higher dimensions). These objects have already been considered in the literature. In the case $d = 3$, they are called generic boxed Plattenbau in [FKU20].

In order to consider d-floorplans as combinatorial objects, we consider them up to the so-called *weak equivalence*, that preserves incidence and sideness between borders and blocks. In the two dimensional case, other equivalence relations such as the strong equivalence (which refines the weak equivalence by further preserving block adjacencies)

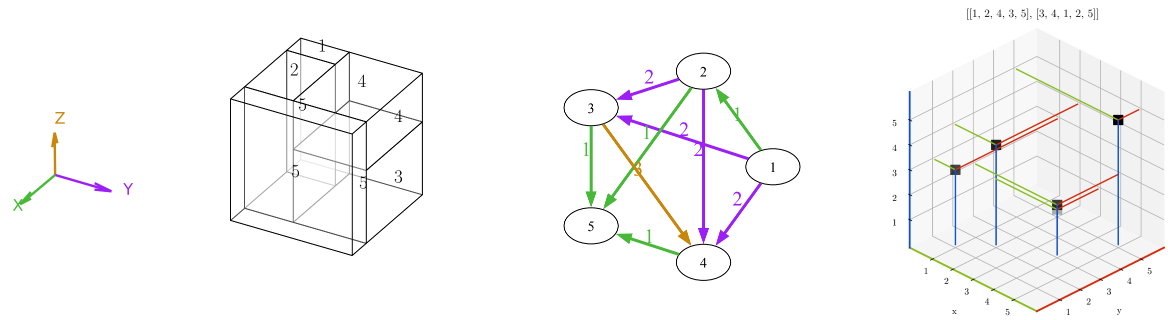

Figure 1: On the left an example of a 3−floorplan. In the middle the relative order of each blocks with respect to each direction (x, y, z). On the right the corresponding 3-permutation (considering the 3-floorplan as a 4-floorplan).

[Rea12] and the S-equivalence (defined on the segments of a floorplan) [ABBM$^+$13] have been considered.

Mosaic floorplans are known to be closely related to pattern-avoiding permutations. For instance, weakly equivalent mosaic floorplans are connected to Baxter permutations and separable permutations [ABP06], strongly equivalent floorplans are connected to 2-clumped permutations and S-equivalent floorplans to anti-Baxter permutations [ABBM$^+$13]. Additionally, mosaic floorplans are also known to be related to other combinatorial objects such as twin binary trees or lattice paths [ABB$^+$23, YCS03, ACFF24].

The study of the generating functions and the enumeration of families of pattern-avoiding mosaic floorplans is also an active field of research, see for example [AB24].

In [AM10], a generalization to arbitrary dimensions of a restricted family of mosaic floorplans, called *guillotine partitions*, was considered. The authors find a bijection between 2^{d-1}−dimensional guillotine partitions and separable d-permutations, which are a higher dimensional generalisation of separable permutations.

The main contribution of this paper is twofold. First, we exhibit a generation tree for d-floorplans. Given a floorplan, we can remove its *top* box and unequivocally fill the resulting empty space in order to obtain a smaller floorplan (using the tatami condition). This gives a natural definition for a generating tree. In the $d = 2$ case, the children of a given floorplan is determined by the number of boxes that touch the top and left boundaries. Moreover, determining these parameters for the children is straightforward [BBMF10]. However, for $d \geqslant 3$, we need to manage more involved parameters. This allows us to enumerate efficiently the set of all d-floorplans for $n \leqslant 10$.

The second main contribution is a generalization of the bijection between mosaic floorplans and Baxter permutations. A *d-permutation* (or *multipermutation*) is a tuple of $d-1$ permutations of size n. The presented generalization of the mapping from d-permutation to 2^{d-1}-floorplan is straightforward, but the characterization of the corresponding multipermutation is more involved. These multipermutations are defined by the avoidance of the vincular patterns of Baxter permutations and the dimension 3 patterns of the separable d-permutations. Surprisingly, this set is strictly included in the set of Baxter

d-permutations defined in [BM22]. A summary of the objects and the corresponding pattern avoiding permutations is given in Table 1. Moreover, this bijection generalizes the bijection between guillotine 2^{d-1}-floorplans and separable d-permutations of [AM10].

Objects	Permutations	Pattern avoidance
Slicing floorplans	Separable	Sym(2413)
Mosaic floorplans	Baxter	$\mathrm{Sym}(2413\|_{2,2})$
2^{d-1}-guillotine floorplans	d-Separable	Sym(2413), Sym((312, 213))
2^{d-1}-Floorplans	sub d-Baxter	$\mathrm{Sym}(2413\|_{2,2})$, Sym((312, 213))
	$d-$Baxter	$\mathrm{Sym}(2413\|_{2,2})$, $\mathrm{Sym}((312,213)\|_{1,2,.})$, $\mathrm{Sym}((3412,1432)\|_{2,2,.})$, $\mathrm{Sym}((2143,1423)\|_{2,2,.})$

Table 1: Table of the different class of floorplans and their corresponding permutation classes

2 Preliminaries and definitions

In this section, we give various definitions leading to the one of d-floorplans. We then introduce a notion of equivalence and some operations on these objects. The definitions of this section are natural generalisations of the 2−dimensional case (see [ABP06, HHC+00, ACFF24]).

2.1 Generalisation of rectangles in dimension d

Let us consider a d-dimensional Euclidean space with coordinates $(x_1, \ldots, x_d)$. A *hyperrectangle* is defined by the Cartesian product of d intervals of the form $[x_{i,min}, x_{i,max}]$, where i denotes the ith coordinate. An interval $[a, b]$ is *punctual* if $a = b$. The *dimension* of a hyperrectangle is the number of non-punctual intervals. The *interior* of a hyperrectangle R is the Cartesian product of the d intervals of R where each non-punctual interval $[x_{i,min}, x_{i,max}]$ is replaced by the open interval $]x_{i,min}, x_{i,max}[$. The *boundary* of a hyperrectangle is the difference between the hyperrectangle and its interior.

A *box* (resp. *facet*, *edge*) is a hyperrectangle of dimension d (resp. $d-1$, $d-2$). We say that a facet is of axis i if its i-th coordinate is punctual.

We say that a point $q = (x_1, \ldots, x_d)$ is a *corner of a hyperrectangle* $R = \Pi_{i=1..d}[x_{i,min}, x_{i,max}]$ if $x_i \in \{x_{i,min}, x_{i,max}\}$ for all i. We denote by $q_{\min}(R)$ (resp. $q_{\max}(R)$) the corner of the hyperrectangle with the minimal (resp. maximal) coordinates, we call these corners the *minimal* and the *maximal* corner of R. Note that R is fully determined by $q_{\min}(R)$ and $q_{\max}(R)$. We denote by $\bar{q}$ the opposite corner of q in R. By extension a *corner* is a point that is a corner of at least one box.

The boundary of a box possesses $2d$ facets, $2d \times (d-1)$ edges and 2^d corners. More precisely, it possesses two facets and four edges of each axis (one for each combination of maximal and minimal fixed coordinates). The positions of the facets, the edges and the corners are given by the boundaries of the intervals that define the boxes. Similarly, the boundary of a facet is composed of $2 \times (d-1)$ edges and 2^{d-1} corners.

Given a facet f of axis j of a box b, we say that f is the lower (and resp. upper) facet of axis j if it contains $q_{min}(b)$ (and resp. $q_{max}(b)$).

Definition 1. Let f be a facet of axis i and f' be a facet of axis j.

- We say that f and f' *crosses* each other if the intersection of their interiors is non empty.
- We also say that f *properly touches* f' if the interior of f intersect f' and they do not cross each other.
- We say that f touch f' if $f = f'$ or f properly touches f'.

Example 2. An example in dimension 3 of two facets crossing and touching is given in Figure 2.

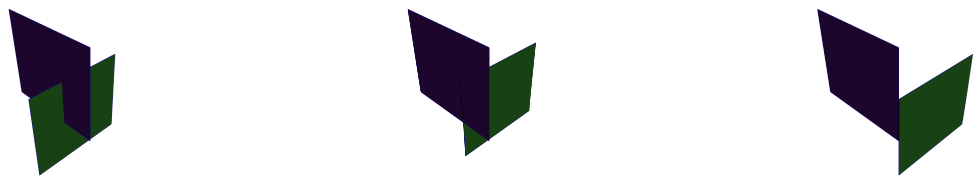

Figure 2: On the left two facets crossing. In the middle, facet B touches facet A. On the right, two facets touching each other.

Let us now give some explicit examples. When considering the $2-$, $3-$ and resp. 4-dimensional case, we denote the coordinates by (x, y), (x, y, z) and resp. (x, y, z, t). In these cases, a 3-box corresponds to a rectangular parallelepiped, a facet of dimension 3 of axis x to a rectangle drawn in the yz-plane and a 3-edge of axis yz to a segment drawn in the x direction. A 2-box corresponds to a rectangle, a facet of dimensions two and axis x to a segment drawn in the y direction and a 2-edge to a vertex.

2.2 Mosaic floorplans in dimension d

A d-dimensional floorplan $\mathcal{P}$ is a box partitioned into boxes. The interior boxes are called *blocks* and the facets induced by the boundaries of these blocks are called *block-facets*[1]. A *border* of axis j is the union of all block-facets of axis j that share the same position and that do not lie on the bounding box of $\mathcal{P}$. Given an inner block-facet f of $\mathcal{P}$, we denote by $b(f)$ the border of f.

[1]Notice that in d-dimensional floorplan, all block-facets are facets but some facets may also be a union of block-facets.

Definition 3. A d-dimensional floorplan is *generic* if all its borders are facets.

In dimension 2, a floorplan is generic if every border is connected. In dimension $d > 2$ a border may be connected but not be a facet as illustrated in Figure 3. In this particular example, the borders are composed of two block-facets whose unions are of the form  (instead of a rectangle), reminiscent of an ”L” in terms of shape.

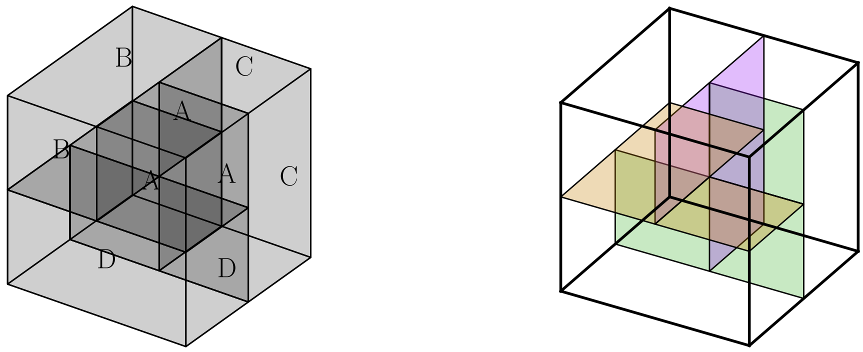

Figure 3: A non-generic 3-floorplan. On the left the boxes. On the right the borders with ”L” shapes.

Definition 4. A *mosaic d-dimensional floorplan* (or a *d-floorplan*) is a generic d-dimensional floorplan without border crossings. This last condition is referred to as the *tatami* condition.

In a d-floorplan, borders of different types are only allowed to touch each other. This lead to *block junctions* made of borders and with a $\perp$ shape (as in the right of Figure 2) that generalise the one made of segments in the two dimensional case.

Example 5. In dimension 2, Definitions 3 and 4 become the standard definition of a mosaic floorplan:

- A 2−dimensional floorplan is a partition of a rectangle into interior-disjoint rectangles, called *blocks*.
- In a 2−dimensional floorplan, borders are segments. A 2−dimensional floorplan is thus *generic* if there are no segments that share the same fixed coordinate.
- A generic 2−dimensional floorplan fulfills the tatami condition it no two segments cross each other. This condition imposes equivalently that no point can belong to the boundary of four rectangles. Examples of 2−floorplans are given in Figure 5.

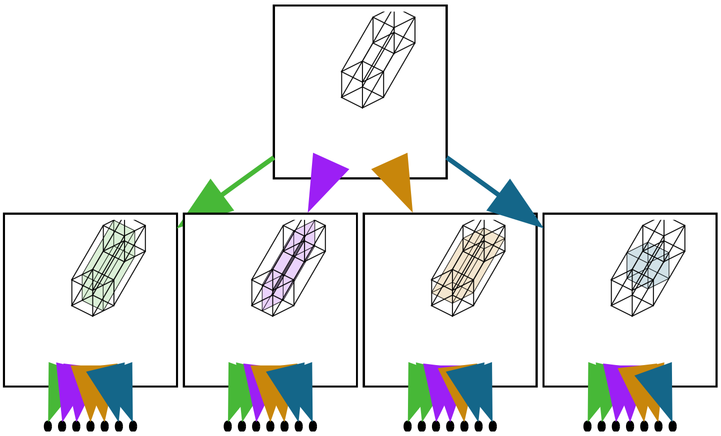

Figure 4: The first 4-floorplans.

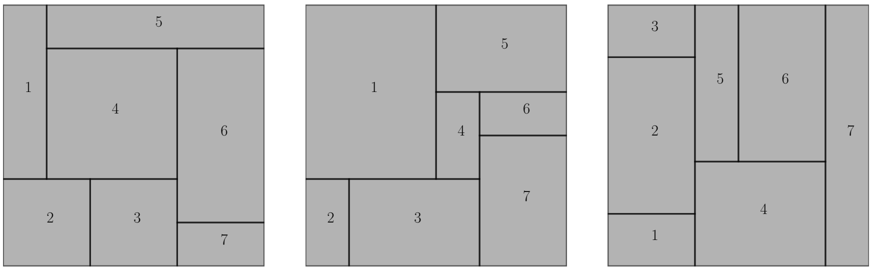

Figure 5: Examples of 2−floorplans

In dimension 3, the rectangles of the 2−dimensional case are replaced by rectangular parallelepipeds. The blocks are now separated by rectangles and the tatami condition imposes that two such rectangles of different axis cannot cross each other. Some examples of 3−floorplans are given in Figure 6. Moreover, Figure 3 shows an example of a 3−floorplan that is not generic and Figure 5 shows examples of configurations that do not respect that tatami condition in two and three dimensions.

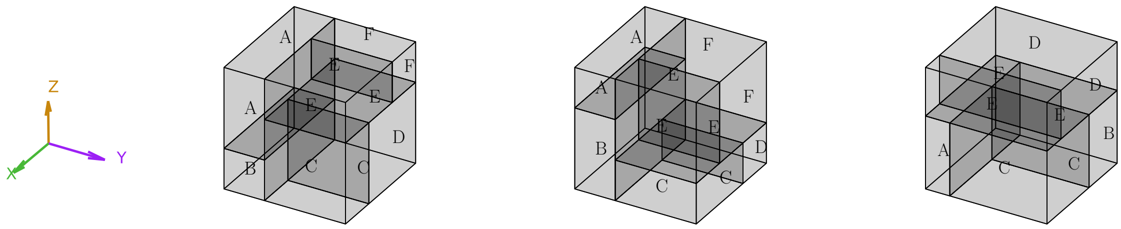

Figure 6: Examples of 3−floorplans

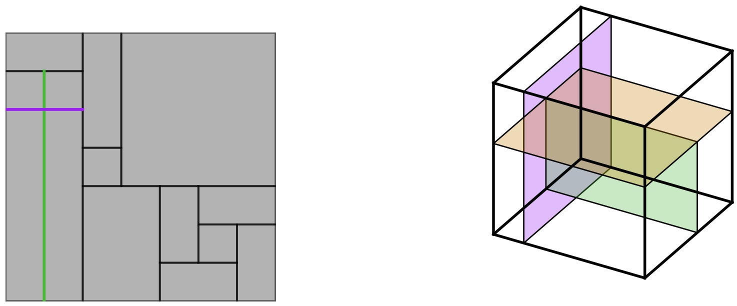

Figure 7: Configurations that do not respect the tatami rule.

2.3 Equivalence relation for d-floorplans

As in [ABP06, HHC+00, ACFF24], we consider in this paper d-floorplans up to an *equivalence relation*. This relation is a straightforward generalisation of the so-called *weak equivalence*. Let us first define the direction relations between the blocks of a d-floorplan; there are d possible such relations, one per axis.

Definition 6. Let $\mathcal{P}$ be a d-floorplan and let B_1 and B_2 be two blocks in $\mathcal{P}$. The neighborhood relation "*B_1 is a left neighbor of B_2 in the j direction*" describes the following situation:

- $x_{j,max}(B_1) = x_{j,min}(B_2) = x_j$ and there is a border of axis j in $\mathcal{P}$ that contains the block-facet of axis j and position x_j of the two blocks.

We define the direction relation B_1 *precedes* B_2 *in the j direction* (denoted $B_1 \overset{j}{\twoheadleftarrow} B_2$ with $j = 1, \ldots, d$) as the transitive and reflexive closure of this neighborhood relation.

We also define similarly the opposite relation, "B_2 *follows* B_1" in the j direction, denoted by $B_2 \overset{j}{\twoheadrightarrow} B_1$.

Example 7. In dimension 2, there are two types of neighborhood relations, these are shown in Figure 8. Figure 9 shows the 3 types of neighborhood relations of the blocks of

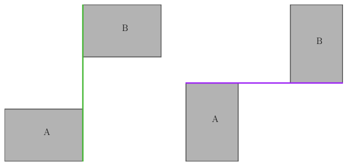

Figure 8: Two neighbor rectangles with respect to the x and y axis.

3−floorplans.

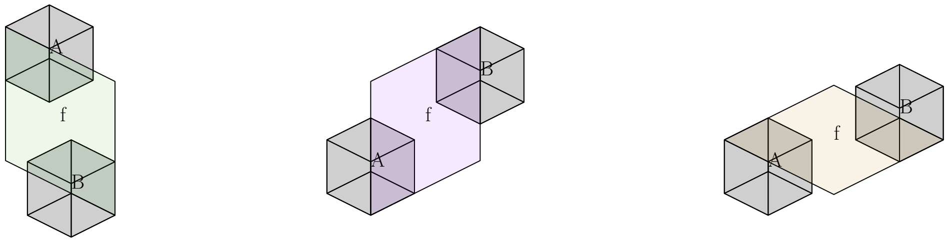

Figure 9: The relations A and B are $x-$neighbors, resp. $y-$neighbors, and resp. $z-$neighbors for the blocks of a 3−floorplan.

Remark 8. The relations $\overset{j}{\twoheadleftarrow}$ and $\overset{j}{\twoheadrightarrow}$ are partial orders for all j.

Definition 9. Two d-floorplans are *weakly equivalent* if there exists a labeling of their blocks by $[n]$ such that, for every direction, the partial orders induced by the direction relations on $[n]$ are identical for the two floorplans.

From now on, when we refer to a $d-$floorplan, we refer to its whole equivalence class. This equivalence relation can be seen from two aspects. On the one hand, one can look at the adjacency graph of the blocks of a d-floorplan as in Figure 12. Two equivalent d-floorplans are then identified if their adjacency graphs are isomorph.

On the other hand, one can consider local modifications of the blocks. Two equivalent d-floorplans can be identified if one can be obtained from the other by a sequence of block junctions's shifts (that are made of borders). This alternative definition of weakly equivalent $d-$floorplans comes from the fact that the block-borders adjacencies of a $d-$floorplan are preserved during a junction shift. The direction relation of blocks are thus invariant under this operation.

Loosely speaking, considering the equivalence given in Definition 9 is equivalent to say that in a d-floorplan, we are only interested in the relative positions of the blocks with respect to each other and to segments.

Notice also that, knowing all direction relations $\overset{j}{\leftarrow}$ of the blocks of an equivalence class of d-floorplans is sufficient to be able to reconstruct one of its representatives. Hence, equivalence classes of d-floorplans can be represented by sets of d partial orders on the same ground set. However, not every set of d partial orders represents an equivalence class of d-floorplans.

Example 10. In Figure 5, the first two 2−floorplans are equivalent and the third one is not. Figure 10 shows the adjacency graphs of the block direction relations for these two equivalent 2−floorplans. Additionally, Figure 11 shows a sequence of junction shiftings that brings the first 2−floorplan to the second. At each step the 2−floorplan obtained is weakly equivalent to the others of the figure.

Similarly, the first two 3−floorplans in Figure 6 are equivalent. Figure 12 shows the adjacency graphs of the block direction relations of these equivalent 3−floorplans.

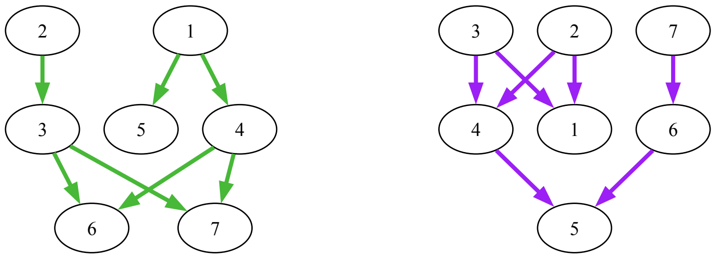

Figure 10: Horizontal and vertical order relations between blocks of the 2-floorplans on the left of Figure 5.

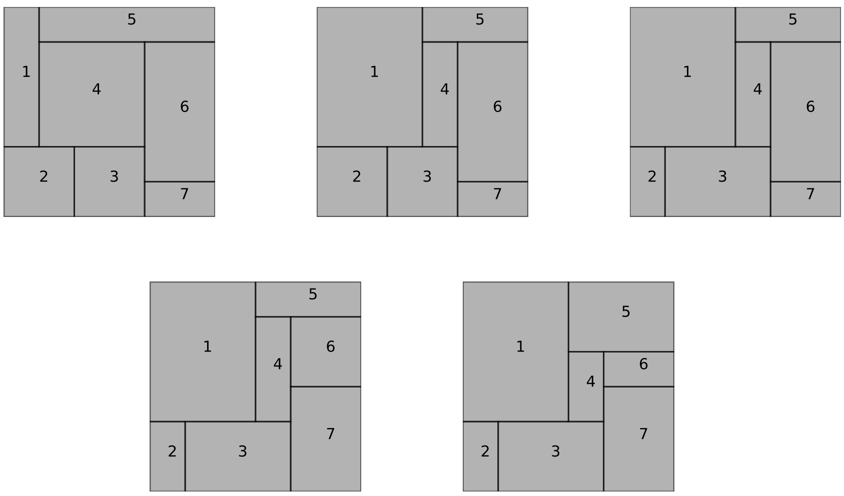

Figure 11: Shifting of the segments junctions of the first floorplan of figure 5

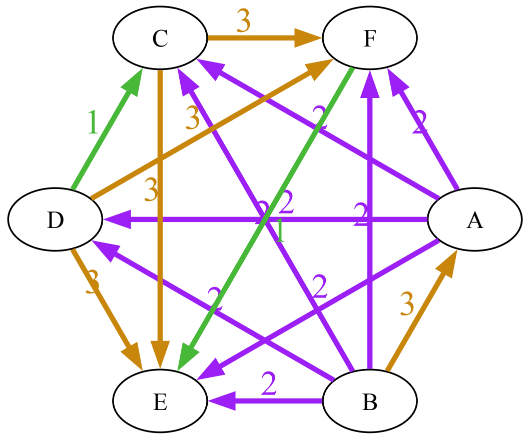

Figure 12: Adjacency graphs of the 3−floorplans on the left of Figure 6.

It is also interesting to mention that, to any equivalence class of d-floorplans, there exists a representative for which all the blocks are intersected by the line defined by the two corners q_{min} and q_{max} of the bounding d-rectangle, such d-floorplans are said to be diagonal. This will be further explained in Section 5 where the algorithm used to construct a 2^{d-1}-floorplan from a d-permutation will produce such 2^{d-1}-floorplans.

2.4 Block deletion in d-floorplans

In the context of 2-floorplan [ABP06, HHC+00] a block deletion operation was introduced. It consists of removing the block incident to a specific corner of the bounding box and then filling the resulting empty space by shifting one block-facet of the deleted block. In this section, we generalize this operation in higher dimensions.

Let q be a corner of a d-floorplan $\mathcal{P}$, let also B be the block incident to q and let $\bar{q}$ be its opposite corner in B. A *shifting facet* is a block-facet of B incident to $\bar{q}$ such that $b(f) = f$.

A *block deletion* operation with respect to q consists of removing the block B by shifting the facet f until it reaches q.

In the rest of this sub-section, we show that for each corner q of a $d-$floorplan there is a unique block B incident to q and, more importantly, B has a unique shifting facet (Lemma 2.2).

Let $\mathcal{F}_{B,q}$ be the set of block-facets of B that are incident to a corner q. We denote by $b(\mathcal{F}_{B,q})$ the set of borders of the facets of $\mathcal{F}_{B,q}$.

Lemma 2.1. Let B be a block of a d-floorplan $\mathcal{P}$ and q corner of B that is not on the boundary of $\mathcal{P}$. The relation touch (see Definition 1) is a total order on $b(\mathcal{F})_{B,q}$.

Proof. Let f and f' be two block-facets of $\mathcal{F}_{B,q}$.

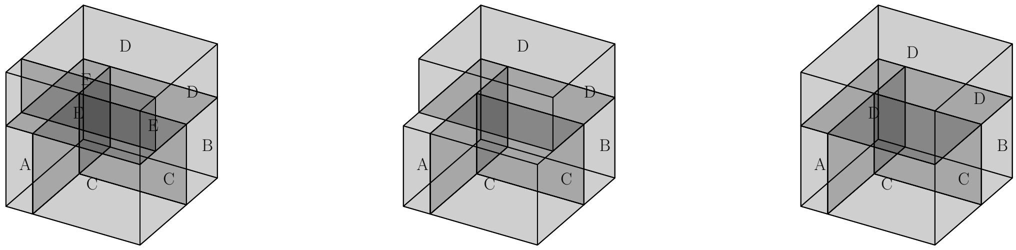

Figure 13: An example of a block deletion operation of a 3−floorplan.

- **Reflexivity.** By definition $b(f)$ touches itself.

- **Antisymmetry.** If $b(f)$ and $b(f')$ touch each other, then either $b(f)$ and $b(f')$ cross each other or $b(f) = b(f')$. Since $\mathcal{P}$ is a d-floorplan, there are no crossings, hence the latter case occurs.

- **Totality.** Assume that neither $b(f)$ touches $b(f')$ nor $b(f')$ touches $b(f)$, and let us find a contradiction (see Figure 2 on the right). Let us consider the edge $e = f \cap f'$. Since $b(f)$ doesn't touch $b(f')$, e doesn't intersect the interior of $b(f')$. By symmetry, it doesn't intersect the interior of $b(f)$. Hence e is part of the boundary of $b(f)$ and $b(f')$. Let ϵ be a positive number arbitrarily small. Let p_b be a point on the segment $q, \bar{q}$ at distance ϵ from q. Let p_f and p'_f be two points such that p_b is on the segment $[p_f, p'_f]$ and the segments $]p_b, p_f[$ and $]p_b, p'_f[$ intersect respectively block-facets f and f'. Let p'_b be the symmetric of p_b with respect to the edge e. Since the the border $b(f')$ doesn't go beyond e, p_f and p'_b belongs to the same block B'. Similarly, p'_f and p'_b belongs also to B'. Since p_b is between p_f and p'_f and not in B', this means that B' is not convex, which is a contradiction.

- **Transitivity.** If $d = 2$, $\mathcal{F}_{B,q}$ contains only 2 block-facets, so the property trivially hold. Let us consider the case $d > 2$. Let f'' be another block-facet of $\mathcal{F}_{B,q}$ such that $b(f)$ touches $b(f')$ and $b(f')$ touches $b(f'')$. Since the touch relation is total, either $b(f)$ touches $b(f'')$ or $b(f'')$ touches $b(f)$. Let us assume the latter case, and let us find a contradiction. To do so we proceed as for the totality property. Let $e = f \cap f'$. Let ϵ be a positive number arbitrarily small. Let p_b be a point on the segment $q, \bar{q}$ at distance ϵ from q. Let $p"_f$ and p_e be two points such that p_b is on the segment $[p"_f, p_e]$ and the segments $]p_b, p"_f[$ and $]p_b, p_e[$ intersect respectively block-facet f” and and the edge e. Let p_1 be the symmetric of $p"_f$ with respect to the hyperplane containing f'. Let p_2 be the symmetric of p_1 with respect to the hyperplane containing f. Since the border of $b(f')$ doesn't cross f”, $p"_f$ and p_1 belong to the same block B'. Similarly, p_1 and p_2 belong also to B' since $b(f)$ doesn't cross f' and p_2 and p_e belong to the same block since $b(f$”) doesn't cross f. We deduce that $p"_f$ and p_e belong to B' and that B' is not convex which is a contradiction. An example of a non transitive configuration is shown in Figure 14.

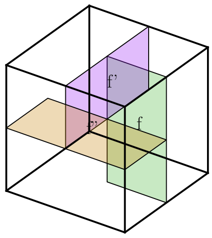

Figure 14: Non-transitive configuration.

□

Lemma 2.2. Let B be a block of a d-floorplan $\mathcal{P}$ that is incident to a corner q of $\mathcal{P}$. B has a unique shifting facet.

Proof. By Lemma 2.1, there is a unique block-facet f of $\mathcal{F}_{B,\bar{q}}$ such that $b(f)$ touches every border of $b(\mathcal{F}_{B,\bar{q}})$. Moreover, since f and $b(f')$ have a common point $\overline{q}$, the block-facet f touches $b(f')$ for every $f' \in \mathcal{F}_{B,\bar{q}}$. Hence f touches each border of $b(\mathcal{F}_{B,\bar{q}})$. This implies that $b(f) = f$.

Moreover, for each other block-facet f', f touches $b(f')$, which means that the intersection of f with the interior of $b(f')$ is non-empty. As f and f' are the block-facets of the same block, the intersection of f with the interior of f' is empty. Hence $b(f') \neq f'$. In other words, f is the unique shifting facet of B. □

In the study of d-floorplans, the block deletion plays a crucial role as it gives a natural definition of the parent of a d-floorplan and thus of the generating tree of d-floorplans. The encoding of this generating tree is explained in Section 3.

3 A generating tree of *d*-floorplans

Let us define a generating tree based on the block deletion operation. Given a d-floorplan $\mathcal{P}$ with more than one block, we define the parent of $\mathcal{P}$ denoted $p(\mathcal{P})$ as the floorplan obtained after a block deletion with respect to the maximal corner $q_{\max}$ of $\mathcal{P}$. This defines a generating tree on d-floorplans whose root is the d-floorplan with only one block. The *children* of a d-floorplan $\mathcal{P}$ are the floorplans $\mathcal{P}'$ such that $p(\mathcal{P}') = \mathcal{P}$.

Example 11. In Figure 15 and 16 are shown the first levels of the generating trees of 2-floorplans and 3-floorplans induced by the block deletion.

In the rest of the section we will first show how to characterize the children of a d-floorplan. For this purpose, we will introduce the notion of *pushable facet* and *pushable*

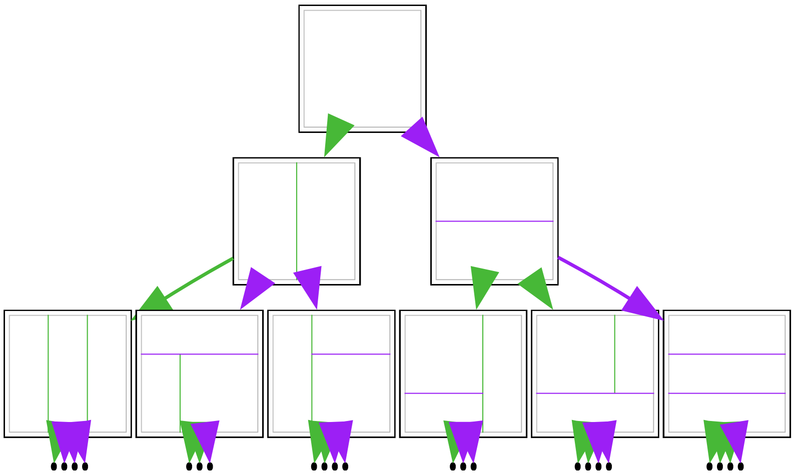

Figure 15: The first levels of the 2-floorplan's generating tree

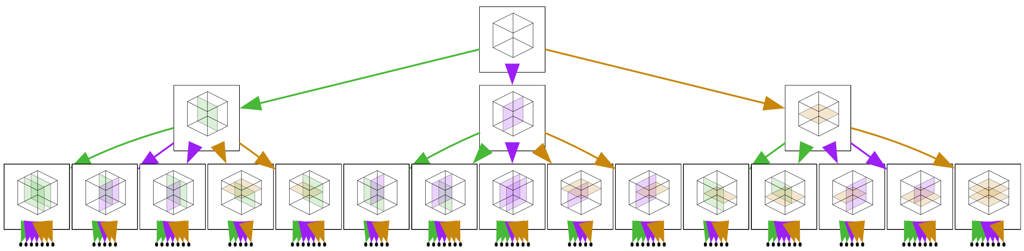

Figure 16: The first levels of the 3-floorplan's generating tree

corner. Then we will show how to compute the pushable corners of the children of a d-floorplan, knowing only the pushable corners of the parent. Based on this, we will define a generating tree isomorphic to the one on d-floorplans, which allows us to enumerate efficiently the first d-floorplans.

3.1 Pushable corners and block insertion

In order to extract the structure of the generating tree, one first needs to introduce the inverse of a block deletion, an operation which we call a *block insertion*. We define here this block insertion with respect to the maximal corner q_{max}, the definition for the other corners follow. We call *maximal block*, the block that contains q_{max}.

Before a block deletion with respect to q_{max}, the shifting facet f of the maximal block is its lower block-facet of a given axis i. Moreover, it was also the union of upper block-facets of axis i of blocks of $\mathcal{P}$. After the block deletion, f becomes a facet that is included in the upper facet of axis i of the bounding box of $\mathcal{P}$, such that it contains q_{max} and it is the union of upper block-facets of axis i of blocks of $\mathcal{P}$.

A *pushable facet* f of axis i (with respect to q_{max}) is a facet of axis i included in the bounding box of $\mathcal{P}$, that contains q_{max} and that is the union of upper block-facets of axis i of blocks of $\mathcal{P}$. A *pushable corner* of axis i is a corner q that is the minimal corner of a pushable facet of axis i. Remark that a pushable corner must be the minimal corner of an upper block-facet of a block, but not all minimal corners of upper block-facets of blocks are pushable corners. Given a pushable corner q, we denote by f_q its corresponding pushable facet.

In Figures 17 and 18, the pushable corners are materialized with arrows. Given a pushable corner q of axis i and its associated pushable facet f_q, we define the *block insertion* associated with q as the d-floorplan obtained by flattening the blocks below the f_q in the direction i and inserting a block in the newly created space. This operation defines a mapping between the pushable corners of a d-floorplan and its children. Let q_{new} be the opposite corner of $q_{\max}$ of the newly created block. $q_{new} = (x_1, \dots, x_i - \alpha, x_{i+1}, \dots, x_d)$ for a value of α small enough so that $x_i - \alpha$ is greater than the i-th coordinate of borders of axis i of $\mathcal{P}$. We denote q^j_{new} the projection of q_{new} on the maximal facet of axis j of the bounding box of $\mathcal{P}$.

In 2-floorplans, there is exactly one pushable corner of axis x per rightmost block and one pushable corner of axis y per topmost block. In higher dimensions, the set of pushable corners is more complex to describe.

3.2 Pushable corners of the children of a d-floorplan

We say that a pushable corner $q' = (x'_1, \dots, x'_d)$ of axis i *is shadowed by* a pushable corner $q = (x_1, \dots, x_d)$ if there exists $j \neq i$ such that $x'_j > x_j$. One can remark that the pushable facets of a given axis are nested. This induces a total order on the pushable corners. The pushable corners of axis i that are shadowed by a pushable corner q of the same axis i are exactly the pushable corners in the interior of the pushable facet f_q. In 2-floorplans,

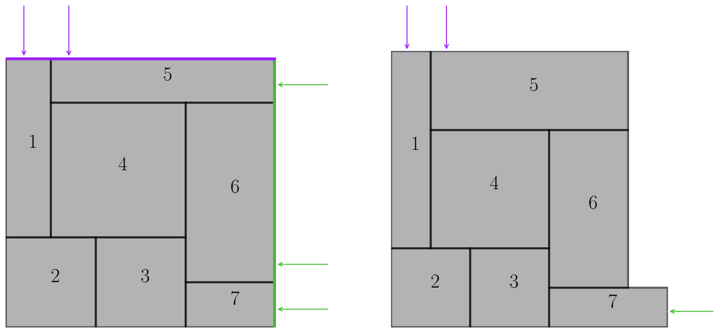

Figure 17: Block insertion on a 2-floorplan. Here we push on the pushable corner of block 6.

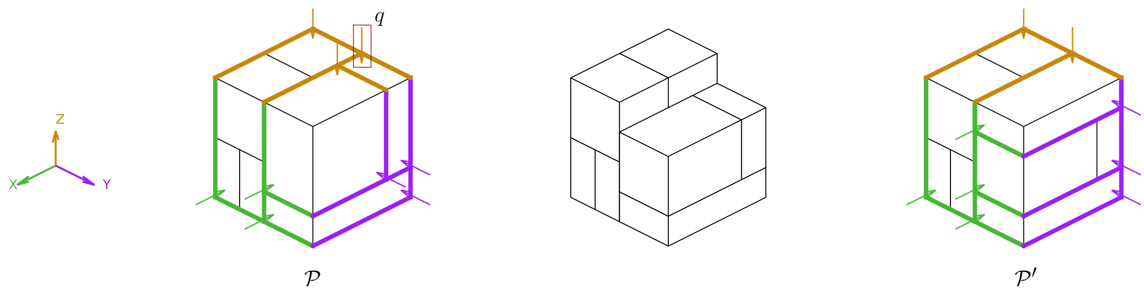

Figure 18: An example of a block insertion on a 3-floorplan.

there are no other shadowed pushable corners. As one can see in Figure 18, this is not the case in higher dimensions.

Lemma 3.1. Let $\mathcal{P}$ be a d-floorplan and q a pushable corner of axis i. The pushable corners of the child $\mathcal{P}'$ of $\mathcal{P}$ with respect to q are the pushable corners of $\mathcal{P}$ that are not shadowed by q and the corners created by the block insertion in $\mathcal{P}$ using q (these corners are given by $\{q^j_{new}, 1 \leqslant j \leqslant d\}$).

Proof. To prove this lemma, we will prove the following four properties that directly imply the result:

1. The shadowed pushable corners are not pushable in $\mathcal{P}'$.
2. The non-shadowed pushable corners are pushable in $\mathcal{P}'$.
3. $\{q^j_{new}, 1 \leqslant j \leqslant d\}$ are pushable corners in $\mathcal{P}'$.
4. A corner that is not pushable in $\mathcal{P}$ is not pushable in $\mathcal{P}'$.

Let us prove these four properties.

1. *The shadowed pushable corners are not pushable in $\mathcal{P}'$.* Let q' be a shadowed corner. First, assume that q' is of axis i. In $\mathcal{P}'$, the corresponding block no longer intersects the upper facet of axis i of the bounding box of the floorplan, thus q' is a point of the interior of $\mathcal{P}'$. Hence it is not a pushable corner of $\mathcal{P}'$. Now assume that q' is of axis $j \neq i$. In $\mathcal{P}'$, let $f_{q',q_{max}}$ be the facet defined by the two corners q_{max} and q'. Because q shadowed q', q^j_{new} is not in $f_{q',q_{max}}$. Hence $f_{q^j_{new}}$ is only partially included in $f_{q',q_{max}}$. So $f_{q',q_{max}}$ is not the union of some block facets, which implies that q' is not pushable in $\mathcal{P}'$.

2. *The non-shadowed pushable corners are pushable in $\mathcal{P}'$.* Let q' be a non-shadowed pushable corner. Suppose first that $f_{q'}$ is, in $\mathcal{P}$, a pushable facet of axis i. As it is non-shadowed, the facet f_q is included in the facet $f_{q'}$. In $\mathcal{P}'$, f_q becomes a single block facet, hence $f_{q'}$ is a union of block-facets (and is a pushable facet) in both $\mathcal{P}$ and $\mathcal{P}'$. Let now q' be of axis j, since q' is non shadowed, the facet $f_{q^j_{new}}$, introduced in $\mathcal{P}'$, is included in $f_{q'}$. In $\mathcal{P}$, the facet $f_{q'}$ is a union of block-facets. Hence $f_{q'}$ is also a union of block-facets in $\mathcal{P}'$.

3. *$\{q^j_{new}, 1 \leqslant j \leqslant d\}$ are pushable corners in $\mathcal{P}'$.* Let $1 \leqslant j \leqslant d$. The facet $f_{q^j_{new}}$ is a single facet in $\mathcal{P}'$. Hence it is a pushable facet.

4. *A corner that is not pushable in $\mathcal{P}$ is not pushable in $\mathcal{P}'$.* Assume that there is a pushable corner q' in $\mathcal{P}'$ that is not pushable in $\mathcal{P}$. The facet $f_{q'}$ is a union of block-facets in $\mathcal{P}$. All the block-facets in $f_{q'}$ are also block-facets of $\mathcal{P}$ except the newly created facet f_{new} of axis j. There are then two cases: 1. $f_{q'}$ is of axis i 2. $f_{q'}$ is of axis $j \neq i$. In the the first case, by construction, f_{new} is a union of block-facets of $\mathcal{P}$. Hence $f_{q'}$ is also a pushable facet of $\mathcal{P}$, which is a contradiction. In the second case, f_{new} is not a union of block-facets of $\mathcal{P}$. In $\mathcal{P}$, the facet defined by the two corners q' and q_{max}, is a union of block-facets in $\mathcal{P}$ which is obtained from $f_{q'}$ (which is defined by the same corners) by removing f_{new} from it and extending some of its block facets to fill the resulting space (as it is done in a block deletion). Hence, the facet defined by q' and q_{max} is a pushable facet in $\mathcal{P}$, implying that q' is also a pushable corner in $\mathcal{P}$, which is a contradiction. □

3.3 Encoding the generating tree

In the generating tree of d-floorplans, the number of children of a floorplan is determined by its set of pushable corners. One can remark that the relevant information about pushable corners is not the values of the coordinates of the corners, but simply the order of the values of the coordinates of the corners. We can thus encode the set of pushable corners by replacing their coordinates by their ranks among the pushable corners with respect to each axis.

Let us consider a set of pushable corners $\mathcal{Q}$. Let q be a pushable corner of axis i. We encode q by a vertical vector by replacing for all $j \neq i$ its j-th entry by the rank (starting at 0) of its j-th coordinate among corners of $\mathcal{Q}$ and we replace its i-th coordinate by ".", meaning that it has the largest rank on this coordinate. The *label* of a d-floorplan $\mathcal{P}$ with $\mathcal{Q}$ as a list of pushable corners is the concatenation of the vector of the pushable corners of $\mathcal{P}$. For instance, the label associated with the 3-floorplan on the right in Figure 18 is $\begin{pmatrix} . & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & . & . & . & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 1 & 2 & . & . \end{pmatrix}$.

A vector is of axis i if its i-th entry is ".".

A vector v' of axis i' is *shadowed* by another vector v of axis i, if $v'_j > v_j$ for some $j \notin \{i, i'\}$.

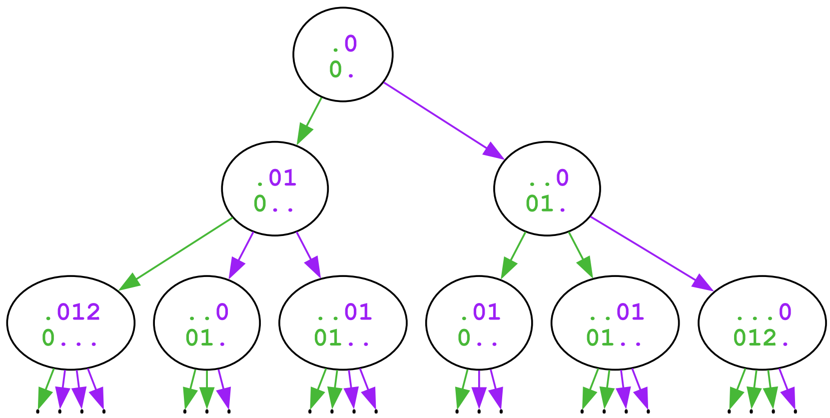

Figure 19: The first three levels of the generating tree T_d^l.

Given $\mathcal{L}$ a list of vectors, and i an axis, we define the *new core vector associated with* $\mathcal{L}$ *on the axis* i :

$$v_{\text{new},\mathcal{L}} := \begin{matrix} \max_1 \\ \vdots \\ \max_{i-1} \\ \max_i + 1 \\ \max_{i+1} \\ \vdots \\ \max_d \end{matrix}$$

where max_j is the maximum of the j-th entry of the vectors in $\mathcal{L}$. Let $v^j_{\text{new},\mathcal{L}}$ be the vector obtained by replacing the j-th entry by ".".

Now we are ready to define a generating tree T_d^l on vectors that is isomorphic to the generating tree of d-floorplans T_d (note that the "l" in T_d^l stands for labels).

The root of T_d^l has the label composed of a list of d-vectors such that the entries of the i-th vector are 0 except on the i-th entry which is ".".

Each node with label $\mathcal{L}$ has $|\mathcal{L}|$ children, one per vector. The label of the child associated with the vector v of axis i is $\mathcal{L}' \cup \{v^j_{\text{new},\mathcal{L}'}, 1 \leqslant j \leqslant d\}$, where $\mathcal{L}'$ are the vectors of $\mathcal{L}$ that are not shadowed by v and where $v_{\text{new},\mathcal{L}'}$ is the new core vector associated with $\mathcal{L}'$ on the axis i.

Theorem 12. *The generating tree of $d-$floorplans T_d is isomorphic to the tree T_d^l.*

Using the labeling and the rewriting rules encoding the structure of T_d^l, one can find the first numbers of the enumeration sequence of d-floorplans for different values of d, see Table 2

Note that the sequences found for $d \geqslant 3$ do not appear in OEIS.

$n\backslash d$	2	3	4	5
1	1	1	1	1
2	2	3	4	5
3	6	15	28	45
4	22	93	244	505
5	92	651	2392	6365
6	422	4917	25204	86105
7	2074	39111	278788	1221565
8	10754	322941	3193204	17932745
9	58202	2742753	37547284	270120905
10	326240	23812341	450627808	4151428385
11	1882960	210414489	5497697848	64839587065
12	11140560	1886358789	67979951368	1026189413865
13	67329992	17116221531	850063243936	
14	414499438	156900657561		
15	2593341586	1450922198319		

Table 2: Values of $|F_n^d|$ for the first values of n.

Example 13. Figure 27 shows the first levels of the generating trees of $2-$floorplans and $3-$floorplans obtained using the labeling introduced in this section.

4 Bijection between higher dimensional floorplans and *d*-permutations

In [ABP06], a bijection between 2-floorplans and Baxter permutations was proposed. This bijection relies on peeling orders of the blocks of the floorplans with respect to two corners of the floorplans (see Figure 10).

In this section, we propose a generalisation of this bijection between 2^{d-1}-floorplans and the set of d-permutations characterised by some forbidden patterns. In this generalisation, we consider d peeling orders. As it will become clear in the sequel, the choice of the peeling corners is crucial.

Without any loss of generality, we consider partitions of the unitary d-dimensional cube ($x_{i,min}(\mathcal{P}) = 0$ and $x_{i,max}(\mathcal{P}) = 1$, for $i = 1, \ldots, d$).

4.1 Definition of the bijection

In higher dimensions, permutations are generalised by *multidimensional permutations*, also called d-permutations [AM10, BM22]. A d-permutation of $[n]$ is a sequence of $d-1$ permutations $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_{d-1})$. Given a d-permutation $\boldsymbol{\pi}$, π_0 is the identity permutation on $[n]$.

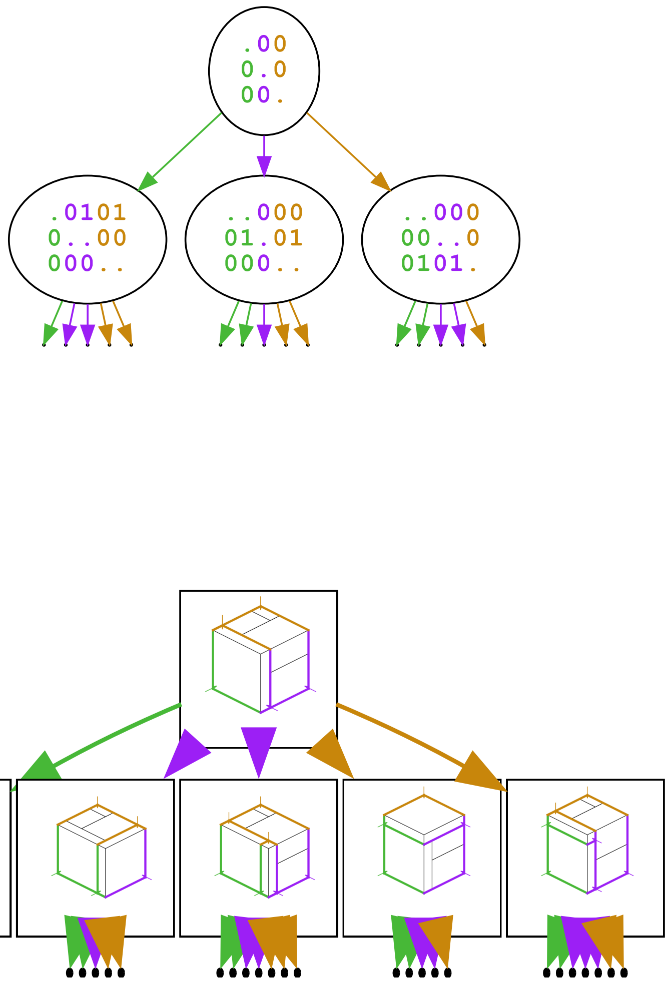

Figure 20: Above: The first two levels of the generating tree of the labels of 3-floorplans. Below: Part of the generating tree in $3d$. More complete versions of the trees can be found in Figure 27.

The *diagram* of a d-permutation $\boldsymbol{\pi}$ is the set of points in $P_{\boldsymbol{\pi}} := \{(\pi_0(i), \pi_1(i), \ldots, \pi_{d-1}(i)), i \in [n]\}$.

A permutation can be considered as a total order of $[n]$, similarly a d-permutation can be considered as a $(d-1)$-tuple of total orders of $[n]$. Thus, any set of d peeling orders obtained from a 2^{d-1}-floorplan can be used to define a d-permutation and yields a mapping ϕ from 2^{d-1}-floorplans to d-permutations. Similarly, one can define a mapping ψ from d-permutations to lists of 2^{d-1} partial orders (on the same ground set) as in Definitions 19

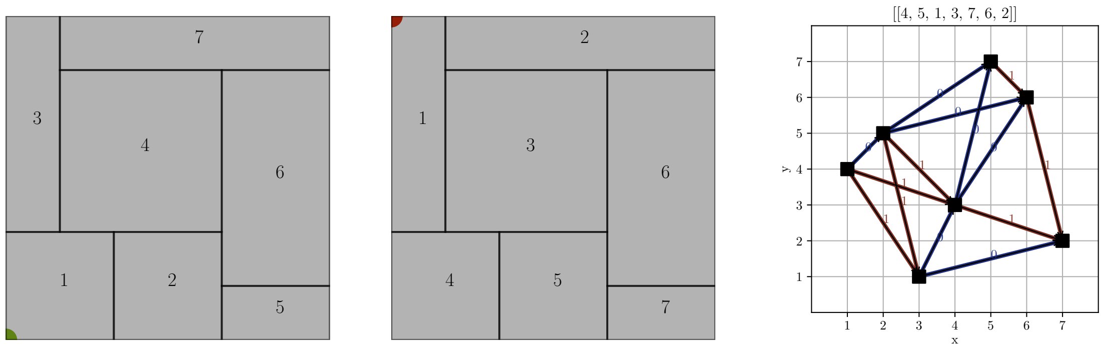

Figure 21: A mosaic floorplan, the labels of the blocks are given by the peeling order of the floorplan with respect to the top-left corner (on the left) and the bottom left corner (in the middle). On the right, the corresponding Baxter permutation 4513762.

and 20. It is shown in section 5 that these two mappings are bijective.

The block deletion operation also provides a notion of total orders of the blocks of a d-floorplan.

Definition 14. Given a corner q of a d-floorplan $\mathcal{P}$ one can perform recursive block deletions using this corner until there remains a single block. We call this operation the *peeling* of $\mathcal{P}$ with respect to q.

The *peeling order* $\nwarrow^q$ of the blocks of $\mathcal{P}$ is the total order defined by the order of deletion of the blocks during the peeling of $\mathcal{P}$ with respect to the corner q. For two blocks A and B, one has $A \nwarrow^q B$ if A is deleted before B during the peeling.

Example 15. In Figure 21 are given the labels in the peeling orders of the blocks of the first two mosaic floorplans given in Figure 5. On the left, the label of the blocks is given by the peeling order with respect to the bottom-left corner. On the right, the label is given by the one with respect to the top-left corner.

Figure 22 shows the last 3-floorplan of Figure 6. The label of the blocks is given by their labels in the peeling orders with respect to the corner $q'_0 = (0, 0, 0)$ (on the left), the corner $q'_1 = (0, 1, 0)$ (in the middle), and the corner $q'_2 = (0, 0, 1)$ (on the right).

Knowing the order of deletion of the blocks with respect to a corner q gives also the order of deletion with respect to the opposite corner of q. Hence, in a d-floorplan, there are 2^{d-1} independent peeling orders, one per pair of corners.

Definition 16. Let $\mathcal{P}$ be a 2^{d-1}-floorplan, let also $\mathbf{c} = c_0 \dots c_{d-1}$ be a set of corners of $\mathcal{P}$ and let σ_{c_i} be the permutation defined by the list of the blocks of $\mathcal{P}$, ordered according to $\nwarrow^{c_i}$. The d-permutation of $\mathcal{P}$ with respect to $\mathbf{c}$ is defined as $\boldsymbol{\pi} := (\sigma_{c_1}\sigma_{c_0}^{-1}, \dots, \sigma_{c_{d-1}}\sigma_{c_0}^{-1})$. This defines a mapping $\chi_{\mathbf{c}}$ from 2^{d-1}-floorplans to d-permutations given by $\chi_{\mathbf{c}}(\mathcal{P}) := \boldsymbol{\pi}$.

Even though any set of d peeling orders of a 2^{d-1}-floorplan can be used to define a d-permutation, the mapping $\chi_{\mathbf{c}}$ is not bijective for any such set. For $\chi_{\mathbf{c}}$ to be bijective, the set of corners $\mathbf{c}$ must fulfill the condition that half of the coordinates of each corner

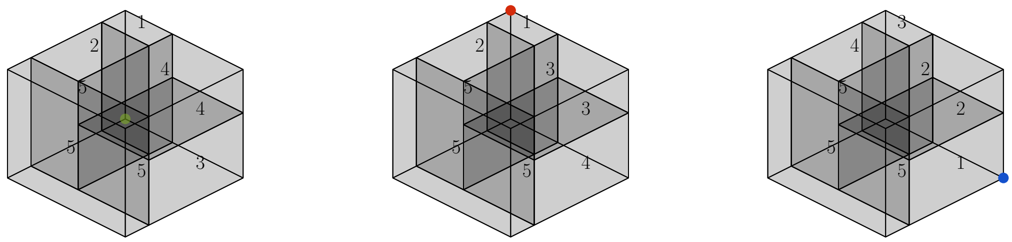

Figure 22: Peeling orders of the blocks of a 3-floorplan with respect to the corners $q = (1,1,1)$ (on the left), $q = (0,0,1)$ (in the middle), and $q = (0,1,0)$ (on the right).

are different from the coordinates of the other ones. This condition allows us to prove the different lemmas and theorems of Section 5. Note that, even with such a condition on the corner set, there are still several possible choices for $\mathbf{c}$. In this paper, we consider an example of such a corner set that is given in Definition 17.

Definition 17. Let $\mathcal{P}$ be a 2^{d-1}-floorplan with $n > 1$ blocks. We define the set of *canonical corners* $\boldsymbol{q}$ as the set of d corners $q_0 \dots q_{d-1}$ for which the coordinates of q_i are given by an alternation of packets of 2^{d-1-i} zeros and ones such that

$$q_i(\mathcal{P}) = \Big(\underbrace{0, \dots, 0}_{2^{d-1-i}}, \underbrace{1, \dots, 1}_{2^{d-1-i}}, \underbrace{0, \dots, 0}_{2^{d-1-i}}, \ \dots \ \Big) \,. \tag{1}$$

We further call the set of peeling orders with respect to these corners the *canonical peeling orders*.

For instance, for a 2-floorplan, the canonical corners are: $q_0 = (0,0)$ and $q_1 = (0,1)$. For a 4-floorplan $q_0 = (0,0,0,0)$, $q_1 = (0,0,1,1)$, and $q_2 = (0,1,0,1)$. Note that a 3-floorplan can also be seen as a 4-floorplan, where each block has width 1 on the last coordinate. In that case, the canonical peeling orders can also be computed with the following corners $q'_0 = (0,0,0)$, $q'_1 = (0,0,1)$, and $q'_2 = (0,1,0)$ (see Figure 22).

We can now define the mapping ϕ as a restriction of the mapping $\chi_{\mathbf{c}}$ to the set of canonical corners $\mathbf{q}$:

Definition 18. We call ϕ, the mapping $\chi_{\mathbf{q}}$ where $\mathbf{q}$ is the canonical set of corners. Starting from a 2^{d-1}-floorplan $\mathcal{P}$, the mapping ϕ gives a d-permutation $\phi(\mathcal{P})$ called the *canonical d-permutation* of $\mathcal{P}$.

Let us also define a mapping ψ that extracts partial orders from the points of a d-permutation. A *direction* $\mathbf{dir}$ is an element of $\{+1,-1\}^d$. A direction is *positive* if its first element is $+1$. For a given d, there are 2^{d-1} positive directions. Let also $\mathbf{dir}$ be a positive direction, the *opposite* of $\mathbf{dir}$, denoted $(-\mathbf{dir})$, is the direction such that $(-\mathbf{dir}) = (-1) \times \mathbf{dir}$.

Definition 19. Let $\boldsymbol{\pi}$ be a d-permutation with n points and let p_1 resp. p_2 be two points in $\boldsymbol{\pi}$. The **direction** between the two points (p_1, p_2), denoted $\mathbf{dir}(p_1, p_2)$, is defined as the direction $\mathbf{dir}$ such that $(\text{sign}(x_0(p_2) - x_0(p_1)), \ldots, \text{sign}(x_{d-1}(p_2) - x_{d-1}(p_1))) = \mathbf{dir}$. We say that p_1 *precedes* p_2 with respect to the positive direction $\mathbf{dir}$ if $p_1 = p_2$ or $\mathbf{dir}(p_1, p_2) = (\mathbf{dir})$. Similarly, we say that p_1 follows p_2 in the positive direction $\mathbf{dir}$ if $p_1 = p_2$ or $\mathbf{dir}(p_1, p_2) = (-\mathbf{dir})$. These two relations define two partial orders $<_{\mathbf{dir}}$ and $>_{\mathbf{dir}}$ of the points of $\boldsymbol{\pi}$.

Definition 20. Let $\boldsymbol{\pi}$ be a d-permutation. Let $F = \{\mathbf{dir}^1, \ldots, \mathbf{dir}^{2^{d-1}}\}$ be the set of positive directions in dimension d. Let also $<_{\mathbf{dir}^i}$ and $>_{\mathbf{dir}^i}$ be the partial orders of the points of $\boldsymbol{\pi}$ with respect to the direction $\mathbf{dir}^i$. The mapping ψ gives the 2^{d-1} partial orders $(<_{\mathbf{dir}^1} | \ldots | <_{\mathbf{dir}^{2^{d-1}}})$ of a d-permutation $\boldsymbol{\pi}$. $\left(\psi(\boldsymbol{\pi}) := (<_{\mathbf{dir}^1} | \ldots | <_{\mathbf{dir}^{2^{d-1}}})\right)$

As explained in subsection 2.4, a 2^{d-1}-floorplan can be described by a set of 2^{d-1} partial orders on the same ground set. These partial orders correspond to the direction relations of the blocks of this 2^{d-1}-floorplan. Additionally, the partial orders of a d-permutation have the same properties as the direction relations of the blocks of a 2^{d-1}-floorplan. One can thus try to build a 2^{d-1}-floorplan from the partial orders of the points of a d-permutation (the algorithm to do so is provided in section 5). This is done by associating to each point of a permutation, a block in the 2^{d-1}-floorplans to be built. However, as it is proven in section 5, only sets of partial orders coming from a subset of the d-permutations can be used to build a 2^{d-1}-floorplans (this subset is defined in the next section).

4.2 Forbidden pattern and main theorem

We first recall from [BM22] some useful definitions on d-permutations and forbidden patterns. We then give the main theorem of this paper.

Definition 21. Let $\mathbf{i} = i_1 \ldots i_{d'}$ be a sequence of indices in $\{0, \ldots, d\}$, let also $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_{d-1})$ be a d-permutation of size n. The *projection* on $\mathbf{i}$ of $\boldsymbol{\sigma}$ is the d'−permutation given by $\text{proj}_{\mathbf{i}}(\boldsymbol{\sigma}) := (\sigma_{i_2}\sigma_{i_1}^{-1}, \ldots, \sigma_{i_{d'}}\sigma_{i_1}^{-1})$. A projection is *direct* if $i_1 < i_2 < \cdots < i_{d'}$.

Definition 22. Let the d-permutation $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_{d-1}) \in S_n^{d-1}$ and the d'−permutation $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_{d'-1}) \in S_k^{d'-1}$ with $k \leqslant n$ and $d' \leqslant d$. Then $\boldsymbol{\sigma}$ *contains the pattern* $\boldsymbol{\pi}$ if there exists a direct projection $\boldsymbol{\sigma'} = \text{proj}_i(\boldsymbol{\sigma})$ of dimension d' and a set of indices $c_1 < \ldots < c_k$ s.t. $\sigma'_i(c_1) \ldots \sigma'_i(c_k)$ is order-isomorphic to π_i for all i. A permutation avoids a pattern if it doesn't contain it. Let s be a symmetry operation of the $[n]^d$ grid (seen as a d-cube), $s(P_{\boldsymbol{\pi}})$ is a diagram of a d-permutation that we denote $s(\boldsymbol{\pi})$. We also denote by $\text{Sym}(\boldsymbol{\pi})$ the family of permutations obtained by applying the symmetries of the d-cube on $\boldsymbol{\pi}$.

Let also p_i and p_j be the i^{th} and j^{th} points with respect to the axis 0 in a d-permutation σ. We say that p_i and p_j are *k-adjacents* if one has $\sigma_k(p_i) = \sigma_k(p_j) \pm 1$. We also say that p_i and p_j are 0-adjacent if $i = j \pm 1$. One can now define a *generalised vincular pattern* as follows:

Definition 23. A *generalised vincular pattern* $\boldsymbol{\pi}|_{X_0,\ldots,X_{d-1}}$ is a d-permutation $\boldsymbol{\pi}$ of size k along with a list of *adjacencies* given by subsets of $[k-1]$. Given a d-permutation σ, we say that the set of points $p_1 \ldots p_k$ is an occurrence of $\boldsymbol{\pi}|_{X_0,\ldots,X_{d-1}}$ if it is an occurrence of $\boldsymbol{\pi}$ and if it satisfies that for any j in any X_k the j^{th} and the $j+1^{th}$ point (of the occurrence) with respect to the axis k are k-adjacent.

Let us denote by S_n^{d-1} the set of d-permutations with n points and by $S_n^{d-1}(\pi_1, \ldots, \pi_m)$ the set of such d-permutations avoiding a list of m patterns $\pi_1 \ldots \pi_m$ (that can be vincular).

Example 24. Consider the patterns $k_1 = (132, 213)$, $k_2 = 231$, $k_3 = (132, 213)|_{1,.,.}$ and $k_4 = (132, 213)|_{2,1,.}$. The 3-permutation $\boldsymbol{\pi} = (1432, 3124)$ contains an occurrence of k_1, given by the $1^{st}, 3^{rd}$ and 4^{th} points of $\boldsymbol{\pi}$, it also contains an occurrence of k_4 but not one of k_3 because of the adjacency constraint on these patterns. As $\mathrm{proj}_{1,2}(\boldsymbol{\pi}) = 3421$, $\boldsymbol{\pi}$ also contains an occurrence of k_2.

One can consider classes of pattern-avoiding d-permutations, examples of such a class are the separable d-permutations that avoid $\mathrm{Sym}((312, 213))$ and $\mathrm{Sym}(2413)$ [AM10] and the Baxter d-permutations that avoid $\mathrm{Sym}((312, 213)|_{1,2,.})$ $\mathrm{Sym}((3412, 1432)|_{2,2,.})$, $\mathrm{Sym}(2413|_{2,2})$, and $\mathrm{Sym}((2143, 1423)|_{2,2,.})$ [BM22].

In this paper, we deal with $d-$permutations avoiding $\mathrm{Sym}((312, 213)), \mathrm{Sym}(2413|_{2,2})$. We call this permutation class F^{d-1}, and we denote its set of elements with n points as F_n^{d-1}.

Let us now state the main result of this section.

Theorem 4.1. The mappings ϕ and ψ (Def. 20 and 18) define a bijection between F_n^{d-1} and the set of 2^{d-1}-floorplans with n blocks. One has $\psi = \phi^{-1}$.

As noted in [AM10, Corollary 3.3] we can naturally extend the previous theorem to floorplans of arbitrary dimensions (not necessarily a power of 2).

Corollary 4.2. Let $q \leqslant 2^{d-1}$. There is a bijection between the set of d-permutations avoiding $2^{d-1}-q$ (among 2^{d-1}) d-permutations of size 2 and $\mathrm{Sym}((312, 213)), \mathrm{Sym}(2413|_{2,2})$ with the set of $2^{d-1} - q$-dimensional floorplans.

A *guillotine partition* is a d-floorplan recursively defined as follows: a d-floorplan with one block is a guillotine partition and two guillotine floorplans merged along a common extremal facet of their bounding boxes is a guillotine partition. Guillotine partitions of dimension 2 are called *slicing floorplans*.

A d-permutation is *separable* if it is of size 1 or if its diagram can be partitioned into 2 parts P_1 and P_2 such that there exists a direction **dir** such that $\forall p_1 \in P_1, \forall p_2 \in P_2$, $\mathbf{dir}(p_1, p_2) = \mathbf{dir}$. In [AM10] it is shown that separable d-permutations are in bijection with 2^{d-1}-dimensional guillotine partitions. In the same paper the authors also show that separable d-permutations are the d-permutations avoiding $\mathrm{Sym}((312, 213))$ and $\mathrm{Sym}(2413)$. From the recursive definitions of guillotine partitions and separable d-permutations, we get the following corollary:

Corollary 4.3. The bijection defined by Theorem 4.1 is a bijection between 2^{d-1}-dimensional guillotine partitions and separable d-permutations.

Example 25. In dimension 2, the bijection given above reduces to the one known between mosaic floorplans and Baxter permutations (see [ABP06]). One can summarize the bijection between a 2-floorplan and a permutation in the following table:

Permutation	2-floorplans
points	blocks
positive direction between two points	direction relation between two blocks
x-coordinate of the points	peeling order w.r.t bottom left corner
y-coordinate of the points	peeling order w.r.t top left corner

Figure 21 shows a 2-floorplan and its Baxter permutation obtained through the bijection given above. This defines the permutation $\pi = 4513762$.

In the 3-dimensional case, Figure 23 shows an illustration of the extended bijection. To obtain the relation between the 3-floorplan and the 3-permutation, one considers a 4-floorplan for which there is an empty direction relation and that can thus be projected into a 3-floorplan. In the permutation space this results in considering 3-permutations that avoid a pattern of size 2.

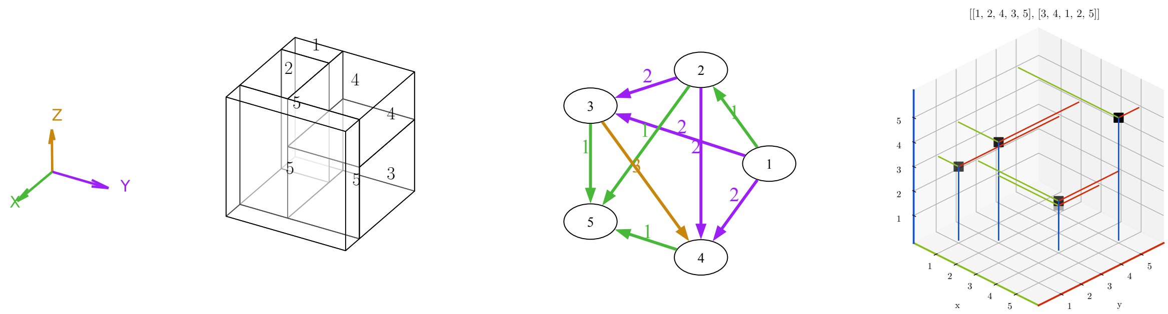

Figure 23: On the left, a 3-floorplan of size 5. In the middle, the 3 corresponding orders $(\overset{1}{\twoheadleftarrow}, \overset{2}{\twoheadleftarrow}, \overset{3}{\twoheadleftarrow})$. On the right, the 3-permutation $\boldsymbol{\pi} = (12435, 34125)$ corresponding to the 3-floorplan (seen as a 4-floorplan in which there are no covering relations in $\overset{4}{\twoheadleftarrow}$).

5 Proof of the bijection

We prove in this section the bijection of Theorem 4.1. The proof strategy is the following:

- We prove different properties of the direction relations and of the peeling orders of 2^{d-1}-floorplans. These properties are the main tools used at the different steps of the proof of the bijection.

- We prove that ϕ maps 2^{d-1}-floorplans to d-permutations in F_n^{d-1} and that this mapping is injective.
- We prove that the partial orders extracted by the mapping ψ from a $d-$permutation in F_n^{d-1} can be used to construct a 2^{d-1}-floorplan. We provide an algorithm performing the construction.
- We prove that, applying this algorithm to a $d-$permutation in F_n^{d-1} and applying the mapping ϕ on the resulting 2^{d-1}-floorplans gives back the original d-permutation.

5.1 Properties of the peeling orders

Remark 26. Let $\mathcal{P}$ be a d-floorplan with $n > 1$ blocks. Performing a block deletion in $\mathcal{P}$ doesn't change the $\overset{j}{\twoheadleftarrow}$ relations of the other blocks.

Lemma 5.1. Let $\mathcal{P}$ be a d-floorplan and let A and B be two blocks in $\mathcal{P}$. Then, there exists a unique j such that $A \overset{j}{\twoheadleftarrow} B$ or $A \overset{j}{\twoheadrightarrow} B$.

Proof. By Remark 26, one can delete blocks of $\mathcal{P}$ (using different corners) without changing the direction relation of A and B. For each corner of $\mathcal{P}$, delete blocks with respect to this corner until either A or B contains it and call the resulting floorplan $\mathcal{P}'$. A block cannot contain both a corner of a d-floorplan and its opposite without being the only block in this d-floorplan. Thus, in $\mathcal{P}'$ the block A will contain half of the corners of $\mathcal{P}'$ and B the other half. This implies that the blocks A and B have a single block-facet of the same type that is not contained in the boundary of $\mathcal{P}'$ and that there may be a smaller d-dimensional floorplan between these two faces in $\mathcal{P}'$. This situation is shown in Figure 24.

We thus have $A \overset{j}{\twoheadleftarrow} B$ or $B \overset{j}{\twoheadleftarrow} A$ for a single j (given by the axis of the border not contained in the boundaries of $\mathcal{P}'$) in $\mathcal{P}'$. According to Remark 26 we have the same relation between the two blocks in $\mathcal{P}$. ☐

Remark 27. If a block A follows immediately a block B in any peeling order, then there exists a j such that A and B are j-neighbors.

Given two relations $<_1$ and $<_2$ on a set X, we define the *union* $<_1 \cup <_2$ of these two relations: $A <_1 \cup <_2 B$ if $A <_1 B$ or $A <_2 B$. Similarly, we define the *intersection* $<_1 \cap <_2$ of these two relations: $A <_1 \cap <_2 B$ if $A <_1 B$ and $A <_2 B$. Given a corner q, a peeling order $\nwarrow^q$ is *compatible* with the direction relation $\overset{j}{\twoheadleftarrow}$ if the j-th coordinate of q equals 0 and is compatible with the direction relation $\overset{j}{\twoheadrightarrow}$ if the j-th coordinate of q equals 1. A peeling order $\nwarrow^q$ is *canonical* if q or $\overline{q}$ is in the canonical set of corners.

Remark 28. If $A \overset{j}{\twoheadleftarrow} B$ and $\nwarrow^q$ is compatible with $\overset{j}{\twoheadleftarrow}$ then $A \nwarrow^q B$.

Lemma 5.2. Given a corner q, the peeling order $\nwarrow^q$ is the union of the d direction relations that are compatible with it $\nwarrow^q$.

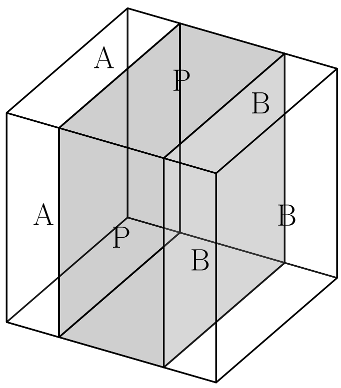

Figure 24: Illustration of Lemma 5.1.

Proof. Without loss of generality assume that $q = q_0$ and let us show that $\nwarrow^{q} = \bigcup_{j=1}^{d} \overset{j}{\twoheadleftarrow}$. By remark 27, $A \nwarrow^{q} B$ implies $A(\bigcup_{j=1}^{d} \overset{j}{\twoheadleftarrow})B$. By remark 28 we get the other implication. □

Lemma 5.3. Given a 2^{d-1}-floorplan, the direction relation $\overset{j}{\twoheadleftarrow}$ (resp. $\overset{j}{\twoheadrightarrow}$) is the intersection of the d canonical peeling orders that are compatible with $\overset{j}{\twoheadleftarrow}$ (resp. $\overset{j}{\twoheadrightarrow}$).

Proof. Without loss of generality assume that $j = 1$ and let us show that $\overset{1}{\twoheadleftarrow} = \bigcap_{i=0}^{d-1} \nwarrow^{q_i}$. By remark 27 we have that $A \overset{1}{\twoheadleftarrow} B$ implies $A \bigcap_{i=0}^{d-1} \nwarrow^{q_i} B$. Now, let us assume that A and B are not comparable with respect to $\overset{1}{\twoheadleftarrow}$. By Lemma 5.1 they must be comparable with respect to $\overset{j}{\twoheadleftarrow}$ for some $j \neq 1$. By remark 27 $A \nwarrow^{q}$ for each canonical corner q that is compatible with $\overset{j}{\twoheadleftarrow}$. By the construction of the canonical corners, at least one of these compatible corners is of the form $\overline{q}_i$. Hence we don't have $A \nwarrow^{q_i} B$. Hence we don't have $A \bigcap_{i=0}^{d-1} \nwarrow^{q_i} B$, which shows the other implication. □

Consider a 4−floorplan, the peeling orders can be written as the union of the partial orders:

$$\nwarrow^{q_0} = \overset{x}{\twoheadleftarrow} \cup \overset{y}{\twoheadleftarrow} \cup \overset{z}{\twoheadleftarrow} \cup \overset{t}{\twoheadleftarrow},$$

$$\nwarrow^{q_1} = \overset{x}{\twoheadleftarrow} \cup \overset{y}{\twoheadleftarrow} \cup \overset{z}{\twoheadrightarrow} \cup \overset{t}{\twoheadrightarrow},$$

$$\nwarrow^{q_2} = \overset{x}{\twoheadleftarrow} \cup \overset{y}{\twoheadrightarrow} \cup \overset{z}{\twoheadleftarrow} \cup \overset{t}{\twoheadrightarrow}.$$

The partial orders can be written as the intersection of the peeling orders with respect to the canonical corners:

$$\overset{x}{\twoheadleftarrow} = \nwarrow^{q_0} \cap \nwarrow^{q_1} \cap \nwarrow^{q_2},$$
$$\overset{y}{\twoheadleftarrow} = \nwarrow^{q_0} \cap \nwarrow^{q_1} \cap \nwarrow^{\overline{q_2}},$$
$$\overset{z}{\twoheadleftarrow} = \nwarrow^{q_0} \cap \nwarrow^{\overline{q_1}} \cap \nwarrow^{q_2},$$
$$\overset{t}{\twoheadleftarrow} = \nwarrow^{q_0} \cap \nwarrow^{\overline{q_1}} \cap \nwarrow^{\overline{q_2}}.$$

Remark 29 is a direct consequence of Lemmas 5.2 and 5.3.

Remark 29. Let four blocks A, B, C, D (two blocks can be the same) of a $2^{d-1}-$floorplan. If for two canonical corners q_1 and q_2, one has

$$A \nwarrow^{q_1} B \text{ and } A \nwarrow^{q_2} B$$

$$C \nwarrow^{q_1} D \text{ and } D \nwarrow^{q_2} C,$$

then the axis j that defines the direction relation between A and B is different than the one defining the direction relation between C and D. The converse is also true.

5.2 From 2^{d-1}- floorplans to d-permutations

Lemma 5.4. For any $2^{d-1}-$floorplan $\mathcal{P}$, the d-permutation $\boldsymbol{\pi} = \phi(\mathcal{P})$ is in F_n^{d-1}.

Proof. Let us show by contradiction that any permutation containing one of the forbidden patterns of F_n^{d-1} cannot be obtained from a $2^{d-1}-$floorplan. For each family of patterns, it suffices to prove this for one of their representative, the proofs of the other patterns follow by symmetry.

Let $\mathcal{P}$ be a $2^{d-1}-$floorplan with n blocks and let us do the following suppositions:

- The d-permutation $\phi(\mathcal{P})$ contains $2413|_{2,2}$. This implies that there are four blocks A, B, C, D in $\mathcal{P}$ such that

  $$A \nwarrow^{q_l} B \nwarrow^{q_l} C \nwarrow^{q_l} D,$$

  $$C \nwarrow^{q_m} A \nwarrow^{q_m} D \nwarrow^{q_m} B,$$

  for some canonical corner q_l and q_m ($l < m$). From the adjacency conditions of the patterns, one also has that B and C are i neighbors and that A and D are j neighbors by Remark 27. Between the pairs $A - D$ and $B - C$, one has

  $$A \nwarrow^{q_l} D \quad \text{and} \quad A \nwarrow^{q_m} D\,,$$

  $$B \nwarrow^{q_l} C \quad \text{and} \quad C \nwarrow^{q_m} B\,.$$

  .

  By Remark 29, one also has that i and j are not the same axis. Consider now the $2^{d-1}-$floorplan $\mathcal{P}'$ where:

– All the blocks preceding A in the peeling order with respect to q_l have been removed by a sequence of block deletions using the corner q_l.

– All the blocks coming after D in this same peeling order have been removed by a sequence of block deletion using the opposite corner of q_l in $\mathcal{P}$.

– All the blocks preceding C in the peeling order with respect to q_m have been removed by a sequence of block deletions using the corner q_m.

– All the blocks coming after B in this same peeling order have been removed by a sequence of block deletion using the opposite corner of q_m in $\mathcal{P}$.

The block A thus contains the corner q_l of $\mathcal{P}'$ and D its opposite, similarly the block C contains the corner q_m and B its opposite in $\mathcal{P}'$ (see Figure 25). Recall also that C and B are i neighbors, because of the previous point there must be a border of type i that slices $\mathcal{P}'$ in two parts. A similar argument can be applied to A and D, thus in $\mathcal{P}'$ there must also be a border of type j that cuts $\mathcal{P}'$ in two which is not possible as this leads to two intersecting borders. Since peeling operation can not create intersecting borders, $\mathcal{P}$ contains also intersecting borders, which leads to a contradiction. This argument is a straightforward generalization of the one in [ABP06] to higher dimensions.

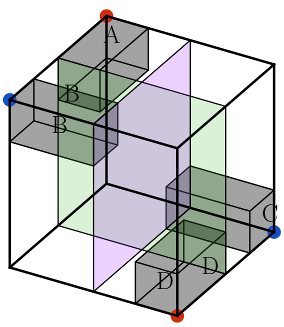

Figure 25: Floorplan configuration associated with $\mathrm{Sym}(2413|_{2,2})$.

- The d-permutation $\phi(\mathcal{P})$ contains $(312, 213)$. There are thus three blocks A, B, C such that

$$A \nwarrow^{q_k} B \nwarrow^{q_k} C,$$

$$B \nwarrow^{q_l} C \nwarrow^{q_l} A,$$

$$B \nwarrow^{q_m} A \nwarrow^{q_m} C.$$

Without loss of generality we assume that $k = 0$, $l = 1$ and $m = 2$. Let us start with an argument for a $4-$floorplan. First, remove all the blocks preceding A and following C in the peeling order with respect to q_0, using the block deletion with respect to q_0 and its opposite corner in the floorplan. Do the same operation for blocks preceding B and following A in the peeling order with respect to q_1 (using now q_1 and its opposite corner for the block deletions). Do it also for the blocks preceding B and following C in the peeling order with respect to q_2 (using q_2 and its opposite corner). Call the resulting floorplan $\mathcal{P}'$. Now suppose one deletes the block B and all the blocks preceding A in the peeling order with respect to q_2 and call the resulting floorplan $\mathcal{P}''$. In $\mathcal{P}''$, the block A contains thus the three corners q_0, $\overline{q}_1$ and q_2. This implies that in this floorplan A must contain any corner whose position is of the form $p = (*, *, 0, *)$ where the star symbol can either be a zero or a one. For any such corner one must have in $\mathcal{P}$ (by Remark 26) that $A \nwarrow^p C$. Similarly one can perform the same deletion procedure for the block A resp. C and all the blocks coming after C resp. B in the peeling orders with respect to q_1 resp. q_0 to find that for any corner of the form $p' = (1, *, *, *)$ resp. $p" = (*, *, *, 1)$ one must have $C \nwarrow^{p'} B$ resp. $B \nwarrow^{p"} A$ in $\mathcal{P}$. Thus, for the corner $p^* = (1, 0, 0, 1)$ (which fits the three types of corner described above) one must have $A \nwarrow^{p^*} C \nwarrow^{p^*} B \nwarrow^{p^*} A$ which leads to a contradiction and concludes the proof in that case.

Let us comment on this argument. In dimension 4, the coordinates of the corners that are contained by one of the blocks A, B or C is given by 3 free coordinates and one fixed coordinates. The fixed coordinate is found by looking at the position of the three canonical corners (q_1 q_2 q_3 and their opposites $\bar{q}_1$ $\bar{q}_2$ $\bar{q}_3$) touched by the corresponding block. This coordinate is given by the one coordinate matching for the three touched corners. Additionally, from Definition 17 it follows that in any dimension and for three canonical corners, there are $1/4^{th}$ of the coordinates that match and the rest that differs. For example, in dimension 4 and for the block A in the previous argument, the fixed coordinate was the z coordinate which is the only one with the same value in the position of the three corners q_1, $\bar{q}_2$ and q_3.

The previous argument can be generalised to higher dimensions but one cannot easily express the position of a corner where the contradictions occurs. However, it is possible to prove that there always exists such a corner. It suffices to show that the fixed coordinates of the position of the corners contained by a block that also contains the corners $\bar{q}_0$, q_1, q_2 resp. q_0, $\bar{q}_1$, q_2 resp. q_0, q_1, $\bar{q}_2$ are all different. The position of the corners where the contradictions occurs is then given by fixing the coordinates to their values fixed in each cases and letting free the other ones. For any two corners q_A and q_B, the pairs $q_1A - q_B$ and $\bar{q_A} - q_B$ are contained in one of three sets (different between the two pairs). Thus, by definition of the canonical set of corners (Definition 17) the fixed coordinates cannot overlap between the three sets of corners and there exist a corner p^* such that $A \nwarrow^{p^*} C \nwarrow^{p^*} B \nwarrow^{p^*} A$ which leads to a contradiction and concludes the proof.

□

Lemma 5.5. One has $\psi \circ \phi = \mathrm{Id}$.

Proof. This is a direct consequence of Lemmas 5.2 and 5.3. Two 2^{d-1}-floorplans do not have the same direction relations of their blocks. Thus they also do not have the same peeling orders, which implies that no two 2^{d-1}-floorplans produce the same $d-$permutation. □

5.3 From d-permutations to 2^{d-1}-floorplans

In the mapping ϕ from 2^{d-1}-floorplans to d-permutations, we associate each block of a 2^{d-1}-floorplan to a point of a $d-$permutation. Given a block A, each label of this block in a canonical peeling order gives a coordinate of the corresponding point p_A in the $d-$permutation. Here, this association is chosen such that the peeling order of the blocks with respect to the canonical corner q_i gives the x_i coordinates of the $d-$permutation points. This association between peeling orders and coordinates of points also corresponds to an *axis-direction association*.

Given two points p_A and p_B of a $d-$permutation $\boldsymbol{\pi} = (\pi_1, \dots, \pi_d)$. The relation $\pi_i(p_A) < \pi_i(p_B)$ can be written as the union of 2^{d-1} partial order relations of these points (defined by the different possible directions between two points; this is a direct consequence of the definition of the partial orders). This construction is similar to the one between the canonical peeling orders and the direction relations of the blocks of 2^{d-1}-floorplans. To any axis j, there is a corresponding positive direction **dir** such that, for any 2^{d-1}-floorplan $\mathcal{P}$, the partial order $\overset{j}{\twoheadleftarrow}$ on the blocks of $\mathcal{P}$ is the same as the partial order $<_{\mathbf{dir}}$ on the points of $\phi(\mathcal{P})$. We say that j is the *associated axis* of **dir** which we denote by $ax(\mathbf{dir})$.

Given a positive direction **dir**, we denote by $\mathbf{dir}(0)$ its first coordinate, $\mathbf{dir}(1)$ its second *etc.* (which are all equal to $+1$ or -1). Additionally given a canonical corner q_i, we call its *signed canonical corner* the corner $q_i^{\mathbf{dir}(i)}$ such that $q_i^{+1} = q_i$ and $q_i^{-1} = \overline{q_i}$. The associated axis $\mathrm{ax}(\mathbf{dir})$ is defined by the condition

$$\overset{ax(\mathbf{dir})}{\twoheadleftarrow} = \nwarrow_{q_0^{\mathbf{dir}(0)}} \cap \dots \cap \nwarrow_{q_{d-1}^{\mathbf{dir}(d-1)}} .$$

As explained above, this axis is defined such that if $A \overset{ax(\mathbf{dir})}{\twoheadleftarrow} B$ in a 2^{d-1}-floorplan $\mathcal{P}$, one has $p_A <_{\mathbf{dir}} p_B$ in $\phi(\mathcal{P})$.

Using these associated axes, We can define an algorithm that realises a 2^{d-1}-floorplan $\mathcal{P}(\psi(\boldsymbol{\pi}))$, from the set of 2^{d-1} partial orders $\psi(\boldsymbol{\pi})$ obtained from a d-permutation $\boldsymbol{\pi} \in F_n^{d-1}$ (Alg. 1). This algorithm is a direct generalisation of Algorithm BP2FP in [ABP06]. We denote here, in the d-permutation $\boldsymbol{\pi}$, the point whose first coordinate is equal to i as p_i.

The following remarks can be deduced by seeing that at each step i of Algorithm 1, the block i newly inserted contains the point $(i-1, \dots, i-1)$ in the resulting 2^{d-1}-floorplan.

Remark 30. For any sub-Baxter d-permutation, the output of Algorithm 1 is a diagonal 2^{d-1}-floorplan.

Lemma 5.6. Let π be a d-permutation in F_n^{d-1}. The object $\mathcal{P}(\pi)$ obtained by applying the algorithm d-perm2FP on π is a 2^{d-1}-floorplan with n blocks such that any block in the output 2^{d-1}-floorplan corresponds to a point p_A in the input $d-$permutation.

Algorithm 1 d-perm2FP

Input: A sub-Baxter d-permutation $\boldsymbol{\pi}$

Output: A $2^{d-1}-$floorplan with n blocks.

Setup a list *saillant*$[j]$ **for** each axis j;
Setup a list *blocks to push*;
Create a block of length n in each direction, name it 1 and add it to *saillant*$[j]$ for all axes j;
for $k = 2, \dots n$ **do**
    $f = \mathbf{dir}\,(p_{k-1}, p_k)$;
    for $i \in$ *saillant*$[\mathbf{ax}(f)]$ **do**
        if $\mathbf{dir}\,(p_i, p_k) = f$ **then**
            Add the block i to *blocks to push*;
        end if
    end for
    Create a new block called k from the corner $\bar{q}_0$ by pushing for $n-k$ units along the axis $\mathbf{ax}$(f) all the blocks in *blocks to push*;
    Remove all the blocks in *blocks to push* from *saillant*$[\mathbf{ax}(\text{f})]$;
    Remove all blocks from *blocks to push*;
    Add k to *saillant*[j] **for** each axis j;
end for

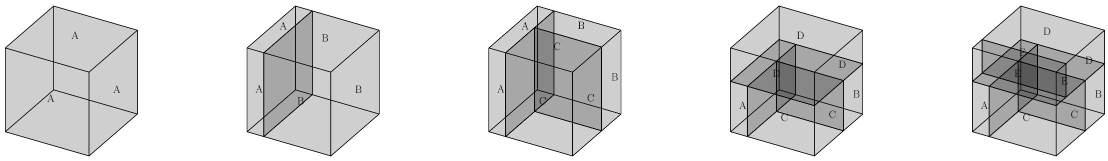

Figure 26: The execution of Algorithm 1 of the 3-permutation $\boldsymbol{\pi} = (12435, 34125)$.

Additionally, for two blocks A and B in $\mathcal{P}(\pi)$, corresponding to two points p_A and p_B such that $\mathbf{dir}(p_A, p_B)$ is a positive direction, one has $A \overset{ax(\mathbf{dir}(p_A,p_B))}{\twoheadleftarrow} B$.

Proof. We prove this by induction on the inserted blocks. Let $\mathcal{P}_k$ be the object obtained after the $(k)^{th}$ step of algorithm d-perm2FP. We suppose that:

- $\mathcal{P}_k$ is a 2^{d-1}-floorplan obtained by performing a block insertion in $\mathcal{P}_{k-1}$.
- *saillant*[l] corresponds to the list of blocks in $\mathcal{P}_k$ with a block-facet lying in the upper boundary of axis l that we call f_l.
- The axis j used to perform the block insertion in $\mathcal{P}_{k-1}$ is defined as $ax(\mathbf{dir}(p_{k-1}, p_k))$
- Let D_k be the block named i_{min}, for which i_{min} is the smallest i in *saillant*$[j]$ such that $\mathbf{dir}\,(p_i, p_k) = \mathbf{dir}\,(p_{k-1}, p_k)$. This block generates a pushable facet $f_{push,k}$ in

$\mathcal{P}_{k-1}$. This pushable facet is the one used to perform the block insertion of the block named k.

- For two blocks i and m in $\mathcal{P}_k$, one has $i \overset{ax(\mathbf{dir}(p_i,p_m))}{\twoheadleftarrow} m$.

At step $k = 2$, this is clearly the case. Let us now prove that this is true at the step $k + 1$. This amounts to prove that:

- Supposing that $\text{ax}(\mathbf{dir}\,(p_k, p_{k+1})) = j$, the block D, named i_{min} in the algorithm generates a pushable facet f_{push}(i_{min} is now the smallest i in *saillant*[j] such that $\mathbf{dir}\,(p_i, p_{k+1}) = \mathbf{dir}\,(p_k, p_{k+1})$).
- The list *blocks to push* corresponds to the list of blocks that have a block-facet lying in f_{push}. This implies that, pushing the blocks in *blocks to push* from the boundary of $\mathcal{P}_k$, and drawing $k + 1$ in the space created, is equivalent to performing a block insertion using f_{push}.
- At the end of the step $k + 1$, the list *saillant*[l] corresponds to the list of blocks in $\mathcal{P}_{k+1}$ with a facet lying in the upper boundary of axis l.
- For any block $w < k + 1$, one has $w \overset{ax(\mathbf{dir}(p_i,p_k))}{\twoheadleftarrow} k$.

Let us denote the upper boundary of axis j in $\mathcal{P}_k$ as f_j. Let also $\pi_v(p_L)$ be the coordinate v of the point p_L in π,

Proof of the first and second item: In order to prove the first point, we first prove that the list *blocks to push* corresponds to the end of the list *saillant*[j], whose first element is D. In other words, we prove that any block C in *saillant*[j] with $i_{min} < C$ is in *blocks to push*.

We prove this by contradiction, suppose this is not the case, there are then blocks in *saillant*[j] called $C_1C_2 \dots C_qC$ such that $i_{min} < C_1 < \dots C_q < C$ $(q \geqslant 0)$ such that $\mathbf{dir}(p_{C_1}, p_{k+1}) \neq \mathbf{dir}(p_k, p_{k+1}) \dots \mathbf{dir}(p_C, p_{k+1}) \neq \mathbf{dir}(p_k, p_{k+1})$. Let us consider the block $C + 1$, there are two possibilities, either $C + 1$ is in *saillant*[j] or not. If it is one has $\mathbf{dir}(p_D, p_{k+1}) = \mathbf{dir}(p_{C+1}, p_{k+1})$. If it is not, let B be a block in *saillant*[j] that is at the end of the chain of blocks that links $C + 1$ and f_j (i.e a block whose block-facet is contained in f_j such that $C + 1 \overset{j}{\twoheadleftarrow} B$).

Let p_{k+1}, p_B,p_{c+1}, p_C and p_D be the points of π associated to the block k+1, B, C+1, C and D. Since B and D are in *blocks to push* (B is in this list because $C < B$), one has $\mathbf{dir}(p_B, p_{k+1}) = \mathbf{dir}(p_D, p_{k+1})$. Additionally, since $B \overset{j}{\twoheadleftarrow} C + 1$ one has $\mathbf{dir}(p_{C+1}, p_B) = \mathbf{dir}(p_B, p_{k+1})$ which implies by transitivity $\mathbf{dir}(p_{C+1}, p_{k+1}) = \mathbf{dir}(p_B, p_{k+1})$. One thus has $\mathbf{dir}(p_D, p_{k+1}) = \mathbf{dir}(p_{C+1}, p_{k+1})$. Finally, since D was in *saillant*[j] at both steps $C + 1$ and $k+1$, one has $\mathbf{dir}(p_D, p_{k+1}) \neq \mathbf{dir}(p_D, p_{C+1})$. By Remark 29, the previous statements translate as:

$$\pi_0(p_D) < \pi_0(p_C) < \pi_0(p_{C+1}) < \pi_0(p_{k+1}),$$

$$\text{for at least one } l: \quad \begin{array}{c} \pi_l(p_{C+1}) < \pi_l(p_D) < \pi_l(p_{k+1}) \\ \text{or} \\ \pi_l(p_{k+1}) < \pi_l(p_D) < \pi_l(p_{C+1}) \end{array},$$

$$\text{for all } m \neq l \neq 0: \quad \begin{array}{c} \pi_m(p_D) < \pi_m(p_{C+1}) < \pi_m(p_{k+1}) \\ \text{or} \\ \pi_m(p_{C+1}) < \pi_m(p_D) < \pi_m(p_{k+1}) \\ \text{or} \\ \pi_m(p_{k+1}) < \pi_m(p_{C+1}) < \pi_m(p_D) \\ \text{or} \\ \pi_m(p_{k+1}) < \pi_m(p_D) < \pi_m(p_{C+1}) \end{array}.$$

Here, one additionally has $\pi_0(p_D) = i_{min}$, $\pi_0(p_C) = C$, $\pi_0(p_{C+1}) = C + 1$ and $\pi_0(p_{k+1}) = k + 1$. Since C is not in *blocks to push*, one has that $\mathbf{dir}(p_C, p_{k+1}) \neq \mathbf{dir}(p_B, p_{k+1}) = \mathbf{dir}(p_D, p_{k+1})$. By Remark 29, There must thus be at least one coordinate v such that:

$$\begin{array}{c} \pi_v(p_{C+1}), \pi_v(p_D) < \pi_v(p_{k+1}) < \pi_v(p_C) \\ \text{or} \\ \pi_v(p_C) < \pi_v(p_{k+1}) < \pi_v(p_{C+1}), \pi_v(p_D) \end{array}$$

If $v = l$, one has

$$\pi_0(p_D) < \pi_0(p_C) < \pi_0(p_{C+1}) < \pi_0(p_{k+1}),$$

$$\begin{array}{c} \pi_l(p_{C+1}) < \pi_l(p_D) < \pi_l(p_{k+1}) < \pi_l(p_C) \\ \text{or} \\ \pi_l(p_C) < \pi_l(p_{k+1}) < \pi_l(p_D) < \pi_l(p_{C+1}), \end{array}$$

which are occurrences of the forbidden patterns in $\mathrm{Sym}(2413|_2)$. As proven in [BM22], any occurrence of $2413|_2$ or $3142|_2$ is also an occurrence of $2413|_{2,2}$ or $3142|_{2,2}$, thus avoiding $\mathrm{Sym}(2413|_2)$ is equivalent to avoiding $\mathrm{Sym}(2413|_{2,2})$. In the 2-dimensional case, this is the only possibility and any block in *saillant*$[j]$ generates a pushable facet. In this case the proof is thus completed. In arbitrary dimensions, one has to consider more configurations.

If $v \neq l$ there are then six possibilities for π_l:

$$\begin{array}{ll} 1. & \pi_l(p_{C+1}) < \pi_l(p_D) < \pi_l(p_C) < \pi_l(p_{k+1}) \\ 2. & \pi_l(p_{C+1}) < \pi_l(p_C) < \pi_l(p_D) < \pi_l(p_{k+1}) \\ 3. & \pi_l(p_C) < \pi_l(p_{C+1}) < \pi_l(p_D) < \pi_l(p_{k+1}) \\ 4. & \pi_l(p_{k+1}) < \pi_l(p_C) < \pi_l(p_D) < \pi_l(p_{C+1}) \\ 5. & \pi_l(p_{k+1}) < \pi_l(p_D) < \pi_l(p_C) < \pi_l(p_{C+1}) \\ 6. & \pi_l(p_{k+1}) < \pi_l(p_D) < \pi_l(p_{C+1}) < \pi_l(p_C)\,. \end{array}$$

Additionally, there are two possibilities for π_v:

$$\begin{array}{ll} \text{a.} & \pi_v(p_D) < \pi_v(p_{C+1}) < \pi_v(p_{k+1}) < \pi_v(p_C), \\ \text{b.} & \pi_v(p_C) < \pi_v(p_{k+1}) < \pi_v(p_{C+1}) < \pi_v(p_D)\,. \end{array}$$

For the configurations: $1.a.$, $1.b.$, $2.a.$, $2.b.$, $4.a.$, $4.b.$, $5.a.$, $5.b.$; the configuration of the points p_C, p_{C+1} and p_{k+1} corresponds to occurrences of forbidden patterns in Sym (312, 213)). For the other configurations, the points p_C, p_D and p_{k+1} leads also to occurrence of the same patterns. This proves that any block C in *saillant*$[j]$ with $i_{min} < C$ is in *blocks to push*.

To complete the first point, it thus remain to prove that a corner of D generates the pushable facet f_{push} and that the list *blocks to push* corresponds to the blocks in $\mathcal{P}_k$ with a block-facet contained in f_{push}.

At step i_{min}, the minimal corner of D generates f_{push}. Let us prove that this is also the case at step $k+1$. The only possibility for this to not be the case is that, at some step E, the list *blocks to push* corresponds to a pushable facet whose generating corner shadows the minimal corner of D. Again, we prove that this is not possible by contradiction.

Since $D < E$, as for $C+1$ before, it is either in *saillant*$[j]$ at step $k+1$ or not. In both cases one has $\mathbf{dir}(p_D, p_{k+1}) = \mathbf{dir}(p_E, p_{k+1})$. As for $C+1$ and D before, one must have $\mathbf{dir}(p_D, p_{k+1}) \neq \mathbf{dir}(p_D, p_E)$. These statements leads to the conditions:

$$\pi_0(p_D) < \pi_0(p_E) < \pi_0(p_{k+1}),$$

$$\text{for at least one } l: \quad \begin{array}{c} \pi_l(p_E) < \pi_l(p_D) < \pi_l(p_{k+1}) \\ \text{or} \\ \pi_l(p_{k+1}) < \pi_l(p_D) < \pi_l(p_E) \end{array}$$

$$\text{for all } m \neq l \neq 0.: \quad \begin{array}{c} \pi_m(p_D) < \pi_m(p_E) < \pi_m(p_{k+1}) \\ \text{or} \\ \pi_m(p_E) < \pi_m(p_D) < \pi_m(p_{k+1}) \\ \text{or} \\ \pi_m(p_{k+1}) < \pi_m(p_E) < \pi_m(p_D) \\ \text{or} \\ \pi_m(p_{k+1}) < \pi_m(p_D) < \pi_m(p_E) \end{array}$$

Because at step E we insert a block using a generating corner that shadows the minimal corner of D, there must be a block $F < D$ in *saillant*[j], such that $\mathbf{dir}(p_F, p_D) \neq \mathbf{dir}(p_F, p_E)$. Because $F < D$, the block F is not in *blocks to push*, thus $\mathbf{dir}(p_F, p_{k+1}) \neq \mathbf{dir}(p_D, p_{k+1})$. This implies that for a coordinate l and a coordinate v

$$\pi_0(p_F) < \pi_0(p_D) < \pi_0(p_E) < \pi_0(p_{k+1}),$$

1. $\pi_l(p_D) < \pi_l(p_F) < \pi_l(p_E) < \pi_l(p_{k+1})$
2. $\pi_l(p_E) < \pi_l(p_F) < \pi_l(p_D) < \pi_l(p_{k+1})$
3. $\pi_l(p_{k+1}) < \pi_l(p_E) < \pi_l(p_F) < \pi_l(p_D)$
4. $\pi_l(p_{k+1}) < \pi_l(p_D) < \pi_l(p_F) < \pi_l(p_E)$

a. $\pi_v(p_D) < \pi_v(p_E) < \pi_v(p_{k+1}) < \pi_v(p_F)$
b. $\pi_v(p_E) < \pi_v(p_D) < \pi_v(p_{k+1}) < \pi_v(p_F)$
c. $\pi_v(p_F) < \pi_v(p_{k+1}) < \pi_v(p_E) < \pi_v(p_D)$
d. $\pi_v(p_F) < \pi_v(p_{k+1}) < \pi_k(p_D) < \pi_k(p_E)$

For the configurations $1.a.$, $1.b.$, $1.c.$, $1.d.$, $3.a.$, $3.b.$, $3.c.$, $3.d.$, the points p_D, p_F, p_{k+1} are occurrences of forbidden patterns in Sym (312, 213)). Similarly for the other configurations, the points p_E, p_F, p_{k+1} are also occurrences of patterns in Sym (312, 213)). It proves that at step $k+1$ the minimal corner of D generates f_{push}.

From the discussion above, the blocks in *blocks to push* are the end of the list *saillant*[j], starting at D. Additionally, at each step $i_{min} < l < k+1$, it has been proven that there was no block insertion performed by the algorithm, that uses a generating corner which shadows the minimal corner of D. Thus at each of these steps, the upper block-facet of axis j of the block inserted lies within f_{push}. All the block in *blocks to push* have thus their upper facet of axis j contained in f_{push}. Finally, for a block facet to be included in f_{push} it must be a facet of a block D' with $D < D'$ (a block facet can be contained in a pushable facet if and only if it is inserted after the block whose minimal corner generates the pushable facet). Thus there cannot be a block with a block-facet included f_{push} that is not in *blocks to push*.

This concludes the proof of the first and second item.

Proof of the third item: At the beginning of the step $k+1$, for any axis l, *saillant*[l] corresponds to the list of blocks in $\mathcal{P}_k$ with a block-facet lying in the upper boundary of axis l. At the end of the step $k+1$, the algorithm removes from *saillant*[j] all the blocks in *blocks to push*. These blocks correspond to the blocks pushed when $k+1$ is inserted in $\mathcal{P}_k$. It then adds in all *saillant*[l] the block $k+1$. Thus, these updates in *saillant*[l], for any axis l, correspond exactly to the updates of the list of the blocks with a block-facet lying in the upper boundary of axis l, after the block insertion of $k+1$ in $\mathcal{P}_k$.

Proof of the fourth item: Let us now consider a block A such that $A < k+1$. There are two possibilities:

- $\mathbf{dir}(p_A, p_{k+1})) = \mathbf{dir}(p_k, p_{k+1}))$ and A is in *saillant*$[j]$
- $\mathbf{dir}(p_A, p_{k+1})) = \mathbf{dir}(p_k, p_{k+1}))$ but A is not in *saillant*$[j]$
- $\mathbf{dir}(p_A, p_{k+1})) \neq \mathbf{dir}(p_k, p_{k+1}))$

In the first case, it is clear that $A \overset{ax(\mathbf{dir}(p_A,p_{k+1}))}{\twoheadleftarrow} k+1$.

In the second case, there is a block C such that $A < C < k+1$, $\mathbf{dir}(p_C, p_{k+1}) = \mathbf{dir}(p_k, p_{k+1})$ and $\mathbf{dir}(p_A, p_C) = \mathbf{dir}(p_k, p_{k+1})$. From the first case and by the induction hypothesis one has $A \overset{ax(\mathbf{dir}(p_k,p_{k+1}))}{\twoheadleftarrow} C \overset{ax(\mathbf{dir}(p_k,p_{k+1}))}{\twoheadleftarrow} k+1$. By transitivity, one thus has $A \overset{ax(\mathbf{dir}(p_k,p_{k+1}))}{\twoheadleftarrow} k+1$ which is equivalent to $A \overset{ax(\mathbf{dir}(p_A,p_{k+1}))}{\twoheadleftarrow} k+1$.

In the third case, let us reconsider the block D. For any block C such that $C < D < k+1$ and $\mathbf{dir}(p_C, p_{k+1}) \neq \mathbf{dir}(p_k, p_{k+1})$, one has by the induction hypothesis and by the definition of D (it is the block that generates the pushable facet f_{push}) that $A \overset{ax(\mathbf{dir}(p_A,p_D))}{\twoheadleftarrow} D$ and $A \overset{ax(\mathbf{dir}(p_A,p_D))}{\twoheadleftarrow} k+1$. One must thus show that $\mathbf{dir}(p_A, p_D) = \mathbf{dir}(p_A, p_{k+1})$ if $\mathbf{dir}(p_A, p_{k+1}) \neq \mathbf{dir}(p_D, p_{k+1})$. Suppose this is not the case, one must then have in the permutation $\boldsymbol{\pi}$ the following conditions for at least one l and one m:

$$\pi_0(p_A) < \pi_0(p_D) < \pi_0(p_{k+1}) ,$$
$$\pi_l(p_A) < \pi_i(p_{k+1}) < \pi_i(p_D) \text{ or } \pi_l(p_D) < \pi_l(p_{k+1}) < \pi_l(p_A) ,$$
$$\pi_m(p_{k+1}) < \pi_m(p_A) < \pi_m(p_D) \text{ or } \pi_m(p_D) < \pi_m(p_A) < \pi_m(p_{k+1}) .$$

These are occurrences of the patterns $(132, 213)$,$(132, 231)$,$(312, 213)$ and $(312, 231)$ which belong to the forbidden patterns in Sym$((312, 213))$.

This concludes the proof of the third item.

□

5.4 ψ is the inverse of ϕ

We now have all the element to prove Theorem 4.1. It remains to show that applying ψ on a $d-$permutation in F_n^{d-1} and then applying the mapping ϕ on the resulting 2^{d-1}-floorplans gives back the original d-permutation.

Proof of theorem 4.1 : Using Lemma 5.6, for any $\boldsymbol{\pi} \in F_n^{d-1}$, the output of algorithm d-perm2FP is a 2^{d-1}-floorplan that we call $\mathcal{P}(\psi(\boldsymbol{\pi}))$. Using Lemmas 5.4 and 5.5, the mapping ϕ is injective and maps 2^{d-1}-floorplans to $d-$permutations in F_n^{d-1}, The bijection is thus proven by showing that $\phi(\mathcal{P}(\psi(\boldsymbol{\pi})) = \boldsymbol{\pi}$.

Let $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_{d-1})$ be a d-permutation in F_n^{d-1}. By Lemma 5.6, one has that the direction relation of any two blocks A and B in $\mathcal{P}(\psi(\boldsymbol{\pi}))$ is given by the associated axis of the direction of their corresponding points in $\boldsymbol{\pi}$. Recall that in $\mathcal{P}(\psi(\boldsymbol{\pi}))$, the blocks are labeled by the first coordinate of their corresponding point in $\boldsymbol{\pi}$. By the definition of the associated axes and of Algorithm 1, one has that:

- The peeling order with respect to the canonical corner q_0 is given by $1, 2, \ldots, n$.

- The peeling orders of the blocks with respect to the canonical corner q_i (for $d-1 > i \geqslant 1$) in $\mathcal{P}(\psi(\boldsymbol{\pi}))$ is the total order defined by π_i.

Finally, using definition 18, one has: $\phi(\mathcal{P}(\psi(\boldsymbol{\pi})) = \boldsymbol{\pi}$ □

Concluding remarks

Several perspectives for future work can be exhibited. First, as the rewriting rule is more involved for $d \geqslant 2$, deriving an equation for the generating function of d-floorplans is a highly non trivial open problem, even for the 3−dimensional case. In [AP25, AB24], families of pattern avoiding floorplans were considered and their generating functions studied. Analogously, one can consider subfamilies of pattern avoiding d-floorplans, for which the generating tree structure is more simple and for which more results on the enumeration can be inferred.

Moreover, in this work we have considered d-floorplans up to the weak equivalence. One can naturally extend the definitions of the strong and S equivalence considered in [Rea12] and [ABBM+13] to this higher dimensional framework. In the two dimensional case, it was shown that these classes of floorplans are in bijection with 2-clumped permutations and anti-Baxter permutations (permutations avoiding Sym(2143$|_2$)). It would be interesting to generalize the results of [Rea12, ABBM+13] to d-floorplans.

Additionally, in [LR12], a *pivot operation* on diagonal mosaic floorplans (floorplans considered under the weak equivalence) was introduced. This operation replaces horizontal adjacencies of the block of a floorplan with vertical ones. It was then shown that the set of diagonal floorplans equipped with such operation has a natural lattice structure. It should be possible to generalize such operation to d- floorplan. Some open questions appear then to be:

- to investigate whether this operation leads to a lattice structure in higher dimensions or not.
- to, through the bijection with d-permutations found in this work, investigate the resulting operation on subbaxter d-permutations and the possible order relation induced on the permutations.

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A Generating tree of 3-floorplans

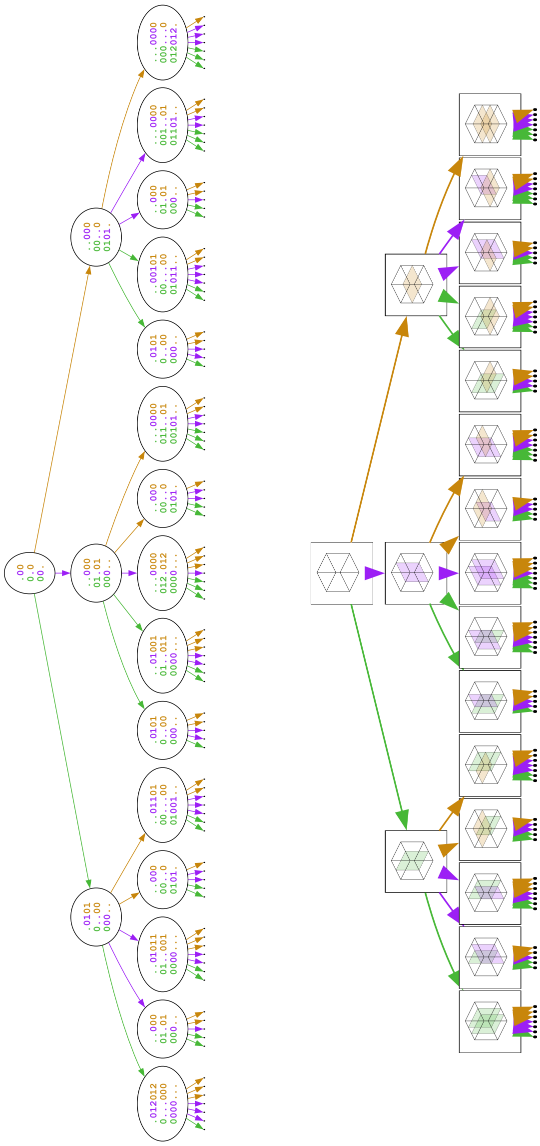

Figure 27: The first three levels of the generating tree of 3-floorplans and their labels.