Sesqui-regular graphs with smallest eigenvalue at least -3

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Abstract

Koolen, Yang and Yang showed that if a graph with smallest eigenvalue at least -3 has large minimum valency, then it is 2-integrable. In this paper, we show that sesqui-regular graphs with smallest eigenvalue at least -3 and parameters (n, k, c), where $c \geqslant 9$ and valency k is sufficiently large, are 1-integrable, and as a consequence we classify them.

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1 Introduction

In this paper, all graphs considered are finite, undirected and simple, and by the eigenvalues of a graph, we mean the eigenvalues of its adjacency matrix. For undefined notions, see [1] and [2].

1.1 Main result

In this subsection we present our main result.

Suppose that Γ is a graph with vertex set $V(\Gamma)$. For any two vertices x and y, we write $x \sim y$ if x and y are adjacent, and $x \not\sim y$ otherwise. Let x be a vertex of Γ. We

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denote by $\Gamma_i(x)$ the set of vertices at distance i from x. For a vertex y, we denote the distance between x and y by $d(x,y)$ and hence $d(x,y)=i$ if and only if $y\in\Gamma_i(x)$. If $i=1$, the cardinality $|\Gamma_1(x)|$ is the *valency* of x. Whenever the valency of every vertex of Γ is the same, say k, Γ is also said to be *regular* of valency k or *k-regular*.

A regular graph Γ with n vertices and valency k is *co-edge-regular* with parameters (n,k,c), if any two distinct non-adjacent vertices have exactly c common neighbors. As a generalization of co-edge-regular graphs, there are so-called sesqui-regular graphs. A regular graph Γ with n vertices and valency k is *sesqui-regular* with parameters (n,k,c), if any two vertices at distance 2 have exactly c common neighbors. Note that the class of sesqui-regular graphs is a much wider class of graphs than co-edge-regular graphs.

Before introducing our main result on sesqui-regular graphs, we recall strongly regular graphs.

A regular graph Γ with n vertices and valency k is *strongly regular* with parameters (n,k,a,c), if any two adjacent vertices have exactly a common neighbors and any two distinct non-adjacent vertices have exactly c common neighbors. Note that a strongly regular graph is disconnected if it is the disjoint union of at least 2 cliques with $a+1$ vertices.

Here is a family of important strongly regular graphs constructed from Steiner systems.

A *Steiner system* $S(2,t,v)$ is a pair $(\mathcal{P},\mathcal{B})$, where $\mathcal{P}$ is a v-element set called a *point set* and $\mathcal{B}$ is a family of t-element subsets of $\mathcal{P}$ called a *block set* such that each 2-element subset of $\mathcal{P}$ is contained in exactly one block. Take a Steiner system $S(2,t,v)$. Construct a graph as follows: its vertex set is the block set of this Steiner system, and two blocks are adjacent whenever they intersect in one point. This gives a geometric strongly regular graph with parameters $(\frac{v(v-1)}{t(t-1)},\frac{t(v-t)}{t-1},(t-1)^2+\frac{v-1}{t-1}-2,t^2)$ (see [4, p. 5]). We call this graph the *block graph* of the Steiner system. In particular, a Steiner system $S(2,3,v)$ is called a *Steiner triple system* and denoted by $STS(v)$, whose block graph is a strongly regular graph with parameters $(\frac{v(v-1)}{6},\frac{3(v-3)}{2},\frac{v+3}{2},9)$. A necessary and sufficient condition for the existence of an $STS(v)$ is that $v\equiv 1,3 \pmod 6$, see [13].

Now we present our main result of this paper.

Theorem 1. *There exists a positive integer κ_1 such that, for any connected sesqui-regular graph Γ with parameters (n,k,c), where $c\geqslant 9$, and smallest eigenvalue $\lambda_{\min}(\Gamma)\in[-3,-2)$, if $k\geqslant\kappa_1$, then one of the following holds:*

(i) *$c=9$ and Γ is the block graph of a Steiner triple system;*

(ii) *$c=k-1$ and Γ is the complement of the disjoint union of cycles of length at least 4;*

(iii) *$c=k$ and Γ is the complete multipartite graph $K_{\frac{n}{3}\times 3}$, that is, the complement of the disjoint union of cycles of length 3.*

1.2 Background and motivation

In this subsection we will give some background and motivation.

1.2.1 Strongly regular graphs

First we discuss strongly regular graphs with fixed smallest eigenvalue. A theorem of Sims is as follows.

Theorem 2 ([2, Theorem 8.6.4])**.** *Let $\lambda \geqslant 2$ be an integer. The strongly regular graphs with integral smallest eigenvalue $-\lambda$ are*

(i) *the complete multipartite graphs $K_{v\times\lambda}$,*

(ii) *Latin square graphs $\mathrm{LS}_\lambda(v)$,*

(iii) *block graphs of Steiner systems $S(2, \lambda, \lambda v + \lambda - v)$,*

(iv) *finitely many further graphs.*

For the definition of item (ii), see [2, p. 204]. As a consequence, the following holds.

Theorem 3. *Let $\lambda \geqslant 2$ be an integer. There exists a positive integer $\kappa_3(\lambda)$ such that a strongly regular graph with parameters (n, k, a, c) and integral smallest eigenvalue $-\lambda$ has either $c \in \{k, \lambda^2, \lambda(\lambda-1)\}$ or $k \leqslant \kappa_3(\lambda)$.*

Neumaier [22, Theorem 4.7], see also [2, Theorem 8.6.3], gave an explicit bound on the valency of graphs in item (iv) of Theorem 2. This bound has recently been improved by Koolen, Lv, Markowsky and Park [14]. Although the main result Theorem 1 looks similar to Theorem 2, its proof is much more involved than the proof of Theorem 2. To obtain Theorem 2, Neumaier first showed that for a strongly regular graph with parameters (n, k, a, c) and smallest eigenvalue $-\lambda$ (where λ is an integer), either $c = k$ or $c \leqslant \lambda^3(2\lambda-3)$ holds. To obtain this result, he used the Krein parameters of strongly regular graphs. Next, he showed that if $c < k$ and

$$a > \frac{1}{2}\lambda(\lambda-1)(c+1) + c - \lambda - 1, \tag{1}$$

then the strongly regular graph is in item (ii) or (iii) of Theorem 2. Note that if a strongly regular graph satisfies (1), then it has no induced $K_{1,\lambda+1}$'s. This is a very important tool in the proof of Neumaier. But to have this kind of tool, the amply-regularity condition plays a crucial role.

Now let us assume only sesqui-regularity and look at the recent developments in this direction.

1.2.2 Sesqui-regular graphs

In [18], Koolen, Gebremichel, Yang and Yang showed the following result for sesqui-regular graphs with large minimum valency and fixed smallest eigenvalue.

Theorem 4 ([18, Theorem 1.5])**.** *Let $\lambda \geqslant 2$ be an integer. There exists a positive integer $\kappa_2(\lambda)$ such that, for any connected sesqui-regular graph Γ with parameters (n, k, c) and smallest eigenvalue $\lambda_{\min}(\Gamma) \geqslant -\lambda$, if $k \geqslant \kappa_2(\lambda)$, then either $c \leqslant \lambda^2(\lambda-1)$ or $n - k - 1 \leqslant \frac{(\lambda-1)^2}{4} + 1$ holds.*

Note that the bound $n-k-1 \leqslant \frac{(\lambda-1)^2}{4}+1$ cannot be improved, as the complements of triangle-free Ramanujan graphs with valency k approximately attain equality, and there are infinitely many bipartite Ramanujan graphs with valency k, for any integer $k \geqslant 3$, as is shown in [21].

Also, Neumaier [22, Theorem 3.1] showed that for an integer $\lambda \geqslant 2$, a strongly regular graph with smallest eigenvalue $-\lambda$ has either $c = k$ or $c \leqslant \lambda^3(2\lambda-3)$. Note that Gillespie [8] conjectured that the equality occurs only for $\lambda = 2$ and $\lambda = 3$. We wonder whether the same is true for sesqui-regular graphs; namely, whether it is true that a sesqui-regular graph with parameters (n, k, c) and smallest eigenvalue at least $-\lambda$, where $\lambda \geqslant 2$ is an integer, has either $k - c \leqslant \lambda$ or $c \leqslant \lambda^3(2\lambda - 3)$. Even for the class of distance-regular graphs, we do not know it.

Our main theorem is an improvement of Theorem 4. The reason we can improve the result in the case $\lambda = 3$ is that we have more detailed information on regular graphs with smallest eigenvalue at least -3 and large valency. This will be explained in more detail in the next subsection. Note that we do not have an explicit bound for κ_1 in our Theorem 1. Using the method of Bose-Laskar-Hoffman theory in [16], it may be possible to obtain a reasonable bound.

To show how strong the notion of sesqui-regularity is, we introduce the notion of the level of a sesqui-regular graph. For any two adjacent vertices x and y, let $a(x, y)$ denote the number of common neighbors of x and y. The *level* $l(\Gamma)$ of a sesqui-regular graph Γ is defined as

$$l(\Gamma) := |\{a(x, y) \mid x \text{ and } y \text{ are adjacent vertices in } \Gamma\}|\,.$$

Observe that a strongly regular graph has level 1, and the s-clique-extension ($s \geqslant 2$) of a strongly regular graph, which will be defined later in this subsection, has level 2 and exactly 4 distinct eigenvalues.

In [7], Ge and Koolen constructed infinite families of connected co-edge-regular graphs with certain fixed smallest eigenvalue, exactly 4 distinct eigenvalues and levels 3 and 4. Furthermore, in [24], van Dam, Ge and Koolen showed that there exists an increasing sequence of integers $3 < a_1 < a_2 < a_3 < \cdots$ and for each $i = 1, 2, 3, \ldots$, there exist infinite families of co-edge-regular graphs with fixed smallest eigenvalue $-a_i$ such that the level of the graphs increases strictly with i.

These examples indicate that, under the assumption of sesqui-regularity alone, the level of a graph can be arbitrarily large, even within the class of co-edge-regular graphs having exactly 4 distinct eigenvalues. Consequently, one cannot expect a structural theory analogous to Neumaier's theorem to be obtained. Therefore, the study of sesqui-regular graphs requires methods substantially different from those Neumaier used.

1.2.3 Sesqui-regular graphs with smallest eigenvalue at least -3

To investigate sesqui-regular graphs with smallest eigenvalue at least -3, we will look at the positive semidefinite matrix $A+3I$, where A is the adjacency matrix of a sesqui-regular graph Γ with parameters (n, k, c) and with smallest eigenvalue in the interval $[-3, -2)$. In [19], Koolen, Yang and Yang have shown that if the minimum valency of Γ is large, then there exists a matrix N such that $A + 3I = N^T N$ holds, where every entry of N is in $\mathbb{Z} \cup \frac{1}{\sqrt{2}}\mathbb{Z}$. If we cannot find N such that all its entries are integral, then we show that $c \leqslant 8$. This is the reason for our assumption $c \geqslant 9$. Now we are in the case that all entries of N are integral. In this case, we are able to show $c = 9$ or $c \in \{k, k-1\}$ and determine the graphs. For the known sesqui-regular graphs with $c \leqslant 8$ and smallest eigenvalue in the interval $[-3, -2)$, see the end of this subsection.

It is not difficult to show that a non-complete connected sesqui-regular graph with parameters (n, k, c), where $n \geqslant 29$, and smallest eigenvalue at least -2 is one of the following:

(i) a cocktail party graph and $c = k$, or

(ii) the line graph of a complete graph and $c = 4$, or

(iii) the line graph of a complete bipartite graph and $c = 2$, or

(iv) the line graph of a regular graph with girth at least 5 and $c = 1$, or

(v) the line graph of a bipartite semiregular graph with no cycles of length 4 and $c = 1$.

The reason is that if a connected regular graph has smallest eigenvalue at least -2, then it is a line graph, a cocktail party graph, or has at most 28 vertices (see [3, Theorem 4.4, Theorem 4.11] and [11]).

The situation for -3 is very different. There exists a unique strongly regular graph with $c = 81$ and smallest eigenvalue -3, namely the strongly regular graph with parameters $(275, 162, 105, 81)$ which is the complement of the McLaughlin graph (see [2] and [10]). The largest open case by order of a strongly regular graph with smallest eigenvalue -3 has parameters $(1344, 221, 88, 26)$ [23]. It is conjectured by Koolen and Gebremichel [17] that a primitive strongly regular graph with smallest eigenvalue -3 has at most 276 vertices or $c \in \{6, 9\}$.

Here is a method to construct a new sesqui-regular graph from a given sesqui-regular graph.

For a positive integer s and a graph Γ, the s-clique-extension of Γ is the graph $\widetilde{\Gamma}$ obtained from Γ by replacing each vertex $x \in V(\Gamma)$ by an s-clique $\widetilde{X}$, such that $\tilde{x} \sim \tilde{y}$ (for $\tilde{x} \in V(\widetilde{X})$, $\tilde{y} \in V(\widetilde{Y})$, $\widetilde{X} \neq \widetilde{Y}$) in $\widetilde{\Gamma}$ if and only if $x \sim y$ in Γ. Note that the s-clique-extension of a connected sesqui-regular graph with parameters (n, k, c), where $n \geqslant 2$, and smallest eigenvalue $-\lambda$ is a connected sesqui-regular graph with parameters $(sn, sk + s - 1, sc)$ and smallest eigenvalue $-s\lambda + s - 1$.

In the remainder of this subsection we give examples of infinite families of sesqui-regular graphs with $c \leqslant 8$ and smallest eigenvalue at least -3.

Examples:

(i) Take the line graph Γ of a k-regular graph with girth at least 5. Then Γ is a sesqui-regular graph with $c = 1$ and smallest eigenvalue at least -2. Now the 2-clique-extension of Γ is a sesqui-regular graph with $c = 2$. This shows that there exists an infinite family of connected sesqui-regular graphs with $c = 2$, smallest eigenvalue at least -3, and unbounded diameter.

(ii) The Cartesian product of 3 complete graphs K_r, K_s, K_t with $2 \leqslant r \leqslant s \leqslant t$ is sesqui-regular with $c = 2$, diameter 3, and smallest eigenvalue -3.

(iii) The 2-clique-extension of the Cartesian product of 2 complete graphs K_r, K_s with $2 \leqslant r \leqslant s$ is sesqui-regular with $c = 4$, diameter 2, and smallest eigenvalue -3.

(iv) The Johnson graphs $J(v, 3)$ ($v \geqslant 6$) are sesqui-regular with $c = 4$, diameter 3, and smallest eigenvalue -3.

(v) The Latin square graphs $\mathrm{LS}_3(v)$ ($v \geqslant 3$) are sesqui-regular with $c = 6$ and smallest eigenvalue -3.

(vi) The 2-clique-extension of the triangular graphs $T(v)$ ($v \geqslant 4$), where $T(v)$ is the line graph of the complete graph K_v, are sesqui-regular with $c = 8$ and smallest eigenvalue -3.

(vii) Let $M_1, M_2, \ldots, M_{2v-1}$ be a Hoffman coloring of the triangular graph $T(2v)$ ($v \geqslant 3$), that is, each M_i is a co-clique of order v and $M_1 \cup M_2 \cup \cdots \cup M_{2v-1}$ forms a vertex partition of $T(2v)$ (for details on Hoffman colorings, see [7]). The graph obtained from the triangular graph $T(2v)$ by adding edges such that each M_i becomes a clique is a sesqui-regular graph with $c = 8$ and smallest eigenvalue -3.

This shows that there are many sesqui-regular graphs with $c \leqslant 8$ and smallest eigenvalue at least -3. Whether it is possible to classify them even for $c = 8$ and large k remains unclear.

1.3 Integrability of sesqui-regular graphs

In this subsection we will introduce the notion of the s-integrability of graphs and give the results on the integrability of sesqui-regular graphs with smallest eigenvalue at least -3.

Let s be a positive integer and Γ a graph with smallest eigenvalue $\lambda_{\min}(\Gamma)$. We say that the graph Γ is *s-integrable*, if for each vertex x of Γ, there exists an integral vector $\mathbf{x}$ such that the following equations hold:

$$(\mathbf{x}, \mathbf{y}) = \begin{cases} s\lceil -\lambda_{\min}(\Gamma)\rceil & \text{if } x = y, \\ s & \text{if } x \sim y, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where $(\cdot, \cdot)$ denotes the standard inner product. Note that if a graph is s- and t-integrable, then it is also $(s + t)$-integrable.

The notion of the s-integrability for graphs is derived from the notion of the s-integrability for integral lattices, where the latter was introduced by Conway and Sloane [5] in 1989. Note that a graph Γ with the adjacency matrix $A(\Gamma)$ and smallest eigenvalue $\lambda_{\min}(\Gamma)$ is s-integrable if and only if there exists an integral matrix N such that the equation $A(\Gamma) + \lceil -\lambda_{\min}(\Gamma)\rceil I = \frac{1}{s}N^T N$ holds, and if and only if the integral lattice with the

Gram matrix $A(\Gamma)+\lceil -\lambda_{\min}(\Gamma)\rceil I$ is s-integrable. Note that if a graph is s-integrable with small s, then this gives strong information about the graph. For example, a connected non-complete graph with smallest eigenvalue at least -2 is 1-integrable if and only if it is a generalized line graph (see [6, p. 7]).

Using this notion, a consequence of the result of Cameron, Goethals, Seidel and Shult [3] in 1976 can now be stated as follows:

Theorem 5. *Any connected graph Γ with smallest eigenvalue $\lambda_{\min}(\Gamma) \geqslant -2$ is s-integrable for any integer $s \geqslant 2$. Moreover, if Γ has more than 36 vertices, then Γ is 1-integrable.*

Later in 2018, Koolen et al. [19] studied graphs with smallest eigenvalue at least -3 and proved that:

Theorem 6. *There exists a positive integer κ_4 such that any graph Γ with smallest eigenvalue $\lambda_{\min}(\Gamma) \geqslant -3$ and minimum valency at least κ_4 is s-integrable, for any integer $s \geqslant 2$.*

The proof of Theorem 6 gives more detailed information on the graph Γ. By using the detailed information in Theorem 6, we will, in this paper, investigate the integrability of sesqui-regular graphs with smallest eigenvalue at least -3 to obtain Theorem 8, and finally prove Theorem 1.

In order to show Theorem 1, it is sufficient to show the following results.

Theorem 7. *Let Γ be a connected sesqui-regular graph with parameters (n,k,c) and smallest eigenvalue $\lambda_{\min}(\Gamma) \in [-3,-2)$.*

(i) *If $c = k$, then $n \geqslant 6$ and Γ is the complete multipartite graph $K_{\frac{n}{3}\times 3}$, that is, the complement of the disjoint union of cycles of length 3.*

(ii) *If $c = k-1$, then $n \geqslant 7$ and Γ is the complement of the disjoint union of cycles of length at least 4, or is the 3-cube.*

In particular, Γ is 1-integrable when $c = k$ or $k-1$.

Theorem 8. *There exists a positive integer κ_5 such that for any connected sesqui-regular graph Γ with parameters (n,k,c), where $c \geqslant 9$, and smallest eigenvalue $\lambda_{\min}(\Gamma) \in [-3,-2)$, if $k \geqslant \kappa_5$, then Γ is 1-integrable.*

Theorem 9. *Let Γ be a connected sesqui-regular graph with parameters (n,k,c), where $k-2 \geqslant c \geqslant 9$, and smallest eigenvalue $\lambda_{\min}(\Gamma) \in [-3,-2)$. If Γ is 1-integrable, then $c = 9$ and Γ is the block graph of a Steiner triple system.*

We will proceed as follows: In Section 2, we give the proof of Theorem 7. As we need Hoffman graphs to show Theorem 8, we first prove Theorem 9 in Section 3. In the last section, we will introduce the terminology related to Hoffman graphs and show Theorem 8.

2 Proof of Theorem 7

Let Γ be a connected sesqui-regular graph with parameters (n, k, c) and smallest eigenvalue $\lambda_{\min}(\Gamma)$. Assume

$$-3 \leqslant \lambda_{\min}(\Gamma) < -2.$$

We immediately have $k \geqslant 3$ and $c \geqslant 1$. In this section, we deal with the special case where $k-1 \leqslant c \leqslant k$ and prove Theorem 7. First, let us look at the case where $c = k$.

Lemma 10. *Suppose that Γ is a connected sesqui-regular graph with parameters (n, k, k) and smallest eigenvalue $\lambda_{\min}(\Gamma) \in [-3, -2)$. Then $n \geqslant 6$ and Γ is the complete multipartite graph $K_{\frac{n}{3}\times 3}$.*

Proof. As the parameters of Γ are (n, k, k), any two vertices at distance 2 in Γ have the same neighbors. This implies that the diameter of Γ is 2, and for any two vertices x and y of Γ, $\Gamma_1(x) = \Gamma_1(y)$ holds if $x \not\sim y$. Now we define a relation $\mathcal{R}$ on the vertex set of Γ such that for any two vertices x and y, $x\mathcal{R}y$ if and only if $\Gamma_1(x) = \Gamma_1(y)$. It is not hard to find that $\mathcal{R}$ is an equivalence relation, and for any two distinct vertices x and y, $x\mathcal{R}y$ if and only if $x \not\sim y$. This means that the vertices in the same equivalence class are pairwise non-adjacent, and the vertices in different equivalence classes are always adjacent. Considering that Γ is regular, we conclude that all of these equivalence classes have the same cardinality, and hence Γ is the complete multipartite graph $K_{\frac{n}{p}\times p}$ for some integer p. We have $\frac{n}{p} \geqslant 2$ immediately, as Γ is connected. Note that the smallest eigenvalue of the graph $K_{\frac{n}{p}\times p}$ equals $-p$. We have $p = 3$, as $\lambda_{\min}(\Gamma) \in [-3, -2)$. Therefore, the lemma holds. $\square$

Before discussing the case where $c = k-1$, we need the following eigenvalue interlacing results.

Lemma 11. (i) ([9, Theorem 9.1.1]) *If Γ is a graph and Γ' is an induced subgraph of Γ, then the smallest eigenvalue of Γ' is at least the smallest eigenvalue of Γ.*

(ii) ([9, Lemma 9.6.1]) *Suppose that Γ is a graph and π is a partition of its vertex set $V(\Gamma)$ with cells $V_1, \ldots, V_r$. Define the* quotient matrix *of Γ relative to π to be the $r \times r$ matrix B such that the (i, j)-entry of B is the average number of neighbors in V_j of vertices in V_i. Then the smallest eigenvalue of B is at least the smallest eigenvalue of Γ.*

Now we are going to prove the following lemma.

Lemma 12. *Suppose that Γ is a connected sesqui-regular graph with parameters $(n, k, k-1)$, smallest eigenvalue $\lambda_{\min}(\Gamma) \in [-3, -2)$ and diameter $D(\Gamma)$.*

(i) *If $D(\Gamma) = 2$, then $n \geqslant 7$ and Γ is the complement of the disjoint union of cycles of length at least 4.*

(ii) *If $D(\Gamma) \geqslant 3$, then $D(\Gamma) = 3$ and Γ is the 3-cube.*

Proof. (i) Assume $D(\Gamma) = 2$. Since Γ is sesqui-regular with parameters $(n, k, k-1)$, we have, for any two distinct vertices w_1 and w_2,

$$|\Gamma_1(w_1) \cap \Gamma_1(w_2)| = k - 1 \text{ if } w_1 \not\sim w_2. \tag{3}$$

Pick a vertex x of Γ. It is not hard to see $\Gamma_2(x) \neq \emptyset$, as otherwise $k = n-1$ and Γ is the complete graph K_n which has smallest eigenvalue -1. This leads to a contradiction, as $\lambda_{\min}(\Gamma) \in [-3, -2)$. By (3), we have, for any $y \in \Gamma_2(x)$,

$$|\Gamma_1(y) \cap \Gamma_1(x)| = k - 1 \text{ and } |\Gamma_1(y) \cap \Gamma_2(x)| = 1. \tag{4}$$

This implies that the induced subgraph of Γ on $\Gamma_2(x)$ is the disjoint union of K_2's. We claim $|\Gamma_2(x)| = 2$. Otherwise, we can find $\{y_1, y_1', y_2, y_2'\} \subseteq \Gamma_2(x)$ with $y_1 \sim y_1'$ and $y_2 \sim y_2'$. Let Γ' be the induced subgraph of Γ on the set $\{x\} \cup \Gamma_1(x) \cup \{y_1, y_1', y_2, y_2'\}$. Then, it is straightforward to check that the quotient matrix Q of Γ' relative to the partition $\{\{x\}, \Gamma_1(x), \{y_1, y_1', y_2, y_2'\}\}$, by (4), is as follows

$$\begin{pmatrix} 0 & k & 0 \\ 1 & \alpha - 1 - \frac{4(k-1)}{k} & \frac{4(k-1)}{k} \\ 0 & k-1 & 1 \end{pmatrix}, \text{ where } \alpha \leqslant k.$$

Denote by $\lambda_{\min}(Q)$ the smallest eigenvalue of Q. By Lemma 11, $\lambda_{\min}(Q) \geqslant \lambda_{\min}(\Gamma') \geqslant \lambda_{\min}(\Gamma) \geqslant -3$ holds. This shows $k = 3$. But as $y_1 \not\sim y_2$, we have $|\Gamma_1(y_1) \cap \Gamma_1(y_2)| = k - 1 = 2$ by (3). Note that $\Gamma_1(y_1) = \{y_1'\} \cup (\Gamma_1(y_1) \cap \Gamma_1(x))$ and $|\Gamma_1(y_1) \cap \Gamma_1(x)| = 2$ by (4). It follows $\Gamma_1(y_1) \cap \Gamma_1(y_2) = \Gamma_1(y_1) \cap \Gamma_1(x)$, as $y_1' \not\sim y_2$. Similarly, we have $\Gamma_1(y_1) \cap \Gamma_1(y_2') = \Gamma_1(y_1) \cap \Gamma_1(x)$. Let $z \in \Gamma_1(y_1) \cap \Gamma_1(x)$. Then $\{x, y_1, y_2, y_2'\} \subseteq \Gamma_1(z)$, which contradicts $|\Gamma_1(z)| = k = 3$. Hence $|\Gamma_2(x)| = 2$ and the induced subgraph of Γ on $\Gamma_2(x)$ is a K_2. This reveals that the complement $\bar{\Gamma}$ of Γ is 2-regular and the two neighbors of each vertex in $\bar{\Gamma}$ are non-adjacent. In other words, $\bar{\Gamma}$ is a disjoint union of cycles of length at least 4. Note that $\lambda_{\min}(\Gamma) \in [-3, -2)$ implies that the second largest eigenvalue of $\bar{\Gamma}$ is greater than 1. Hence, either Γ is the complement of a cycle of length at least 7 or the complement of the disjoint union of at least 2 cycles of length at least 4. This shows that $n \geqslant 7$.

(ii) Assume $D(\Gamma) \geqslant 3$. Let $x_0 \sim x_1 \sim x_2 \sim x_3$ be a path where $x_i \in \Gamma_i(x_0)$ for $i = 1, 2, 3$ and let Γ'' be the subgraph of Γ induced on the set $\{x_0\} \cup \Gamma_1(x_0) \cup (\Gamma_2(x_0) \cap \Gamma_1(x_3)) \cup \{x_3\}$. It follows immediately that

(a) for any vertex $w_1 \in \Gamma_2(x_0) \cap \Gamma_1(x_3)$, it has exactly $k-1$ neighbors in the set $\Gamma_1(x_0)$, as $d(x_0, w_1) = 2$.

We claim that $d(w_2, x_3) = 2$ for any vertex $w_2 \in \Gamma_1(x_0)$, which will be shown at the end of the proof. Thus,

(b) for any vertex $w_2 \in \Gamma_1(x_0)$, it also has exactly $k-1$ neighbors in the set $\Gamma_2(x_0) \cap \Gamma_1(x_3)$ as $d(w_2, x_3) = 2$.

By double-counting the edges between the two sets $\Gamma_1(x_0)$ and $\Gamma_2(x_0) \cap \Gamma_1(x_3)$, we have $|\Gamma_2(x_0) \cap \Gamma_1(x_3)| = |\Gamma_1(x_0)| = k$ and thus $\Gamma_2(x_0) \cap \Gamma_1(x_3) = \Gamma_1(x_3)$. At this point, it is not hard to see that every vertex in Γ'' has valency k and we conclude $\Gamma'' = \Gamma$, as Γ is connected. From (a) and (b), we find that the quotient matrix of Γ relative to the partition $\{\{x_0\}, \Gamma_1(x_0), \Gamma_2(x_0) \cap \Gamma_1(x_3), \{x_3\}\}$ is the 4×4 matrix

$$\begin{pmatrix} 0 & k & 0 & 0 \\ 1 & 0 & k-1 & 0 \\ 0 & k-1 & 0 & 1 \\ 0 & 0 & k & 0 \end{pmatrix}$$

with smallest eigenvalue $-k$. By Lemma 11 (ii), we have $-k \geqslant \lambda_{\min}(\Gamma) \geqslant -3$ and hence $k = 3$. Using (a) and (b) again, we conclude that Γ is the 3-cube.

Now we prove our claim. As $d(x_1, x_3) = 2$, we have $2 \leqslant k - 1 = |\Gamma_1(x_1) \cap \Gamma_1(x_3)| \leqslant |\Gamma_2(x_0) \cap \Gamma_1(x_3)|$. There exist two distinct vertices w_3 and w_4 in $\Gamma_2(x_0) \cap \Gamma_1(x_3)$. From (a) we obtain, for $i = 3, 4$, that

$$|\Gamma_1(w_i) \cap \Gamma_1(x_0)| = k - 1, \tag{5}$$

and further

$$\Gamma_1(w_i) = (\Gamma_1(w_i) \cap \Gamma_1(x_0)) \cup \{x_3\}.$$

This tells us $w_3 \not\sim w_4$. Note that $d(w_3, w_4) = 2$. Thus, $k - 1 = |\Gamma_1(w_3) \cap \Gamma_1(w_4)| = |(\Gamma_1(w_3) \cap \Gamma_1(w_4) \cap \Gamma_1(x_0)) \cup \{x_3\}|$, that is,

$$|\Gamma_1(w_3) \cap \Gamma_1(w_4) \cap \Gamma_1(x_0)| = k - 2. \tag{6}$$

By the inclusion-exclusion principle and (5) and (6), we have

$$\begin{aligned}
&|(\Gamma_1(w_3) \cup \Gamma_1(w_4)) \cap \Gamma_1(x_0)| \\
=&|(\Gamma_1(w_3) \cap \Gamma_1(x_0)) \cup (\Gamma_1(w_4) \cap \Gamma_1(x_0))| \\
=&|\Gamma_1(w_3) \cap \Gamma_1(x_0)| + |\Gamma_1(w_4) \cap \Gamma_1(x_0)| - |\Gamma_1(w_3) \cap \Gamma_1(w_4) \cap \Gamma_1(x_0)| \\
=&2(k-1) - (k-2) \\
=&k = |\Gamma_1(x_0)|.
\end{aligned}$$

Hence, $\Gamma_1(x_0) \subseteq \Gamma_1(w_3) \cup \Gamma_1(w_4)$ and the claim holds. The proof is completed. □

Proof of Theorem 7. To prove this theorem, it is sufficient to show that the complement of the disjoint union of cycles and the 3-cube are 1-integrable by Lemma 10 and Lemma 12.

Assume that Γ is the complement of the disjoint union of m cycles $C_1, C_2, \ldots, C_m$, where the length of C_i is ℓ_i and $V(C_i) = \{x_{i,1}, \ldots, x_{i,\ell_i}\}$, and that two vertices $x_{i,p}$ and $x_{i,q}$ are adjacent in the cycle C_i if and only if $p - q \equiv \pm 1 \pmod{\ell_i}$. For each vertex $x_{i,p}$ of Γ, define the integral vector $\mathbf{x}_{i,p}$ as follows:

$$\mathbf{x}_{i,p} = \begin{cases} \mathbf{e} + \mathbf{e}_{i,p} - \mathbf{e}_{i,p+1} & \text{if } p < \ell_i, \\ \mathbf{e} + \mathbf{e}_{i,\ell_i} - \mathbf{e}_{i,1} & \text{if } p = \ell_i, \end{cases}$$

where $\{\mathbf{e}, \mathbf{e}_{i,p_i} \mid i = 1, \ldots, m, p_i = 1, \ldots, \ell_i\}$ is a set of orthonormal integral vectors. This shows that Γ is 1-integrable.

As for the case where Γ is the 3-cube, it is straightforward to check that Γ is 1-integrable. This completes the proof. $\square$

3 Proof of Theorem 9

In this section, we will mainly prove Theorem 9. So we will always assume, in this section, that Γ is a connected 1-integrable sesqui-regular graph with parameters (n, k, c), where $c \leqslant k-2$, and smallest eigenvalue $\lambda_{\min}(\Gamma) \in [-3, -2)$, and that $\{\mathbf{x} \mid x \in V(\Gamma)\}$ is a family of integral vectors which satisfy (2) with $s = 1$.

Assume $\{\mathbf{x} \mid x \in V(\Gamma)\} \subset \mathbb{Z}^m$ for some integer m. For $i = 1, \ldots, m$, let $\mathbf{e}_i$ be the vector whose i-th entry is 1 and the others are zero. Since the norm of the integral vector $\mathbf{x}$ is 3, $\mathbf{x}$ is a $(0, \pm 1)$-vector, that is, there exist $i_1, i_2, i_3 \in \{1, 2, \ldots, m\}$ and $a_{i_1}, a_{i_2}, a_{i_3} \in \{1, -1\}$ such that $\mathbf{x}$ can be uniquely expressed as

$$\mathbf{x} = a_{i_1}\mathbf{e}_{i_1} + a_{i_2}\mathbf{e}_{i_2} + a_{i_3}\mathbf{e}_{i_3}.$$

We call the set $\{i_1, i_2, i_3\}$ the *support* of $\mathbf{x}$, and denote it by $\operatorname{supp}(\mathbf{x})$. Also, we define a mapping $\sigma_x : \{1, 2, \ldots, m\} \to \{0, \pm 1\}$ such that

$$\sigma_x(i) = \begin{cases} a_i & \text{if } i \in \operatorname{supp}(\mathbf{x}), \\ 0 & \text{otherwise.} \end{cases}$$

Let x and y be two distinct vertices of Γ. It follows that

$$(\mathbf{x}, \mathbf{y}) = \sum\nolimits_{\ell=1}^{m} \sigma_x(\ell)\sigma_y(\ell) = \sum\nolimits_{\ell \in \operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(\mathbf{y})} \sigma_x(\ell)\sigma_y(\ell) = \begin{cases} 1 & \text{if } x \sim y, \\ 0 & \text{if } x \not\sim y. \end{cases} \tag{7}$$

Note that if $\operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(\mathbf{y}) = \{i\}$, we will write $\mathbf{e}_{\operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(\mathbf{y})}$ instead of $\mathbf{e}_i$ for convenience.

There is an easy observation as follows:

Lemma 13. *For any two distinct vertices x and y of Γ, $|\operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(\mathbf{y})| = 1$ or 3 holds if $x \sim y$, and $|\operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(\mathbf{y})| = 0$ or 2 holds if $x \not\sim y$.*

Proof. This follows directly from (7). $\square$

Let

$$S = \{x \in V(\Gamma) \mid \text{ there exists } x' \neq x \text{ with } \operatorname{supp}(\mathbf{x}') = \operatorname{supp}(\mathbf{x})\}.$$

Lemma 14. *If $S \neq \emptyset$, then for a given vertex $x \in S$, there exists a unique vertex $x' \neq x$ such that $\operatorname{supp}(\mathbf{x}') = \operatorname{supp}(\mathbf{x})$ holds. In particular, x' is adjacent to x.*

Proof. Assume $\mathbf{x} = a_{i_1}\mathbf{e}_{i_1} + a_{i_2}\mathbf{e}_{i_2} + a_{i_3}\mathbf{e}_{i_3}$, where $a_{i_1}, a_{i_2}, a_{i_3} \in \{1, -1\}$. If a vertex y distinct from x satisfies $\mathrm{supp}(\mathbf{y}) = \mathrm{supp}(\mathbf{x}) = \{i_1, i_2, i_3\}$, then $y \sim x$ by Lemma 13 and hence $(\mathbf{y}, \mathbf{x}) = 1$. This implies that $\mathbf{y}$ equals one of the following vectors:

$$-a_{i_1}\mathbf{e}_{i_1} + a_{i_2}\mathbf{e}_{i_2} + a_{i_3}\mathbf{e}_{i_3},\ a_{i_1}\mathbf{e}_{i_1} - a_{i_2}\mathbf{e}_{i_2} + a_{i_3}\mathbf{e}_{i_3},\ a_{i_1}\mathbf{e}_{i_1} + a_{i_2}\mathbf{e}_{i_2} - a_{i_3}\mathbf{e}_{i_3}. \tag{8}$$

Since any two distinct vectors in (8) have inner product -1, we obtain the uniqueness. □

For each vertex $x \in S$, we call the unique vertex x' with $\mathrm{supp}(\mathbf{x}') = \mathrm{supp}(\mathbf{x})$ the *mate* of x.

Lemma 15. *Given a vertex x,*

(i) *if a vertex y which is not the mate of x is adjacent to x, then $|\mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y})| = 1$ and $(\mathbf{x}, \mathbf{e}_{\mathrm{supp}(\mathbf{x})\cap\mathrm{supp}(\mathbf{y})}) = (\mathbf{y}, \mathbf{e}_{\mathrm{supp}(\mathbf{x})\cap\mathrm{supp}(\mathbf{y})})$.*

(ii) *if $\{i_1, i_2\} \subset \mathrm{supp}(\mathbf{x})$, then $|\{y \in V(\Gamma) \mid (\mathbf{y}, \sigma_x(i_1)\mathbf{e}_{i_1} + \sigma_x(i_2)\mathbf{e}_{i_2}) = 2\}| \leqslant 2$, with equality if and only if $x \in S$.*

Proof. (i) This is straightforward by Lemma 13 and Lemma 14, since x and y are not mates.

(ii) Assume $\mathrm{supp}(\mathbf{x}) = \{i_1, i_2, i_3\}$ and $\mathbf{x} = a_{i_1}\mathbf{e}_{i_1} + a_{i_2}\mathbf{e}_{i_2} + a_{i_3}\mathbf{e}_{i_3}$. If y is a vertex distinct from x with $(\mathbf{y}, a_{i_1}\mathbf{e}_{i_1} + a_{i_2}\mathbf{e}_{i_2}) = 2$, then $\mathbf{y} = a_{i_1}\mathbf{e}_{i_1} + a_{i_2}\mathbf{e}_{i_2} - a_{i_3}\mathbf{e}_{i_3}$ as $(\mathbf{x}, \mathbf{y}) \in \{0, 1\}$. In other words, y is the mate of x and $x \in S$. This shows (ii). □

For the set S, we claim the following.

Claim 16. *$S = \emptyset$ holds if $c \geqslant 9$.*

In order to make the proof of Theorem 9 clearer and more logical, we will assume $S = \emptyset$ until Theorem 9 is proved. After that, we will prove Claim 16, as it is quite involved.

Lemma 17. *Assume $S = \emptyset$ and $c \geqslant 8$. Given a vertex x, if there is a vertex y satisfying $|\mathrm{supp}(\mathbf{y}) \cap \mathrm{supp}(\mathbf{x})| = 2$, then for each vertex $z \in \Gamma_2(x)$, $|\mathrm{supp}(\mathbf{z}) \cap \mathrm{supp}(\mathbf{x})| = 2$ holds.*

Proof. By Lemma 13, we have $x \not\sim y$ and then $(\mathbf{x}, \mathbf{y}) = 0$ follows. Without loss of generality, we may assume

$$\mathbf{x} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3,\ \mathbf{y} = \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_4.$$

Suppose $z \in \Gamma_2(x)$ and $|\mathrm{supp}(\mathbf{z}) \cap \mathrm{supp}(\mathbf{x})| = 0$. We have $(\mathbf{z}, \mathbf{e}_4) \in \{0, 1\}$, as $(\mathbf{z}, \mathbf{y}) \in \{0, 1\}$. Now we are going to give a contradiction by counting the common neighbors of x and z. A fact is that for each vertex $w \in \Gamma_1(x) \cap \Gamma_1(z)$,

$$|\mathrm{supp}(\mathbf{w}) \cap \mathrm{supp}(\mathbf{x})| = 1 \text{ and } |\mathrm{supp}(\mathbf{w}) \cap \mathrm{supp}(\mathbf{z})| = 1 \tag{9}$$

hold by Lemma 15 (i), as $S = \emptyset$. Depending on the value of $(\mathbf{z}, \mathbf{e}_4)$, we have the following two cases to deal with.

(i) If $(\mathbf{z}, \mathbf{e}_4) = 1$, then we may let

$$\mathbf{z} = \mathbf{e}_4 + \mathbf{e}_5 + \mathbf{e}_6.$$

For each vertex $w \in \Gamma_1(x) \cap \Gamma_1(z)$, we obtain, by (9), that

$$(\mathbf{w}, \mathbf{e}_i + \mathbf{e}_j) = 2 \text{ for some } \{i, j\} \in \{\{1,5\}, \{1,6\}, \{2,4\}, \{3,4\}, \{3,5\}, \{3,6\}\},$$

considering $(\mathbf{w}, \mathbf{y}) \in \{0, 1\}$. Thus,

$$|\Gamma_1(x) \cap \Gamma_1(z)| \leqslant |\{\{1,5\}, \{1,6\}, \{2,4\}, \{3,4\}, \{3,5\}, \{3,6\}\}| = 6$$

by Lemma 15 (ii).

(ii) If $(\mathbf{z}, \mathbf{e}_4) = 0$, then we may let

$$\mathbf{z} = \mathbf{e}_5 + \mathbf{e}_6 + \mathbf{e}_7.$$

Similarly, we have, for each vertex $w \in \Gamma_1(x) \cap \Gamma_1(z)$, that

$$(\mathbf{w}, \mathbf{e}_i + \mathbf{e}_j) = 2 \text{ for some } \{i, j\} \in \{\{1,5\}, \{1,6\}, \{1,7\}, \{2,4\}, \{3,5\}, \{3,6\}, \{3,7\}\},$$

and hence

$$|\Gamma_1(x) \cap \Gamma_1(z)| \leqslant |\{\{1,5\}, \{1,6\}, \{1,7\}, \{2,4\}, \{3,5\}, \{3,6\}, \{3,7\}\}| = 7.$$

This contradicts that $c \geqslant 8$. Therefore $|\operatorname{supp}(\mathbf{z}) \cap \operatorname{supp}(\mathbf{x})| = 2$ by Lemma 13. Our lemma is proved. $\square$

Lemma 18. *Assume $S = \emptyset$ and $c \geqslant 4$. For each vertex x, there exists a vertex $y \in \Gamma_2(x)$ with $|\operatorname{supp}(\mathbf{y}) \cap \operatorname{supp}(\mathbf{x})| = 0$.*

Proof. Assume that the lemma does not hold. We may let $\mathbf{x} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$. From the assumption, we find that $y \in \Gamma_2(x)$ implies $(\mathbf{y}, \mathbf{e}_{i_1} - \mathbf{e}_{i_2}) = 2$ for some $\{i_1, i_2\} \in \{\{1,2\}, \{1,3\}, \{2,3\}\}$. Note that there exists no other vertex y' such that $(\mathbf{y}', -\mathbf{e}_{i_1} + \mathbf{e}_{i_2}) = 2$, since $(\mathbf{y}', \mathbf{y})$ should equal 1 or 0. This means

$$|\Gamma_2(x)| \leqslant |\{\{1,2\}, \{1,3\}, \{2,3\}\}| = 3 \tag{10}$$

by Lemma 15 (ii), as $S = \emptyset$.

We next claim that

$$\Gamma_i(x) = \emptyset \qquad \text{for every } i \geqslant 3.$$

Otherwise, there is a path $x \sim x_1 \sim x_2 \sim x_3$ with $x_i \in \Gamma_i(x)$ for $i = 1, 2, 3$. Since $d(x_1, x_3) = 2$, sesqui-regularity gives $\left|\Gamma_1(x_1) \cap \Gamma_1(x_3)\right| = c$. Moreover, every vertex in $\Gamma_1(x_1) \cap \Gamma_1(x_3)$ belongs to $\Gamma_2(x)$. Hence

$$c = \left|\Gamma_1(x_1) \cap \Gamma_1(x_3)\right| \leqslant |\Gamma_2(x)| \leqslant 3,$$

This contradicts $c \geqslant 4$. Therefore, every vertex of Γ is at distance at most 2 from x.

Now let $y \in \Gamma_2(x)$. Then every neighbor of y belongs to $\Gamma_1(x) \cup \Gamma_2(x)$, and hence

$$\left|\Gamma_1(y) \cap \Gamma_2(x)\right| = k - \left|\Gamma_1(y) \cap \Gamma_1(x)\right| = k - c \geqslant 2.$$

Together with (10), this implies

$$2 \leqslant \left|\Gamma_1(y) \cap \Gamma_2(x)\right| \leqslant |\Gamma_2(x)| - 1 \leqslant 2.$$

Consequently, $|\Gamma_2(x)| = 3$ and the subgraph induced on $\Gamma_2(x)$ is a K_3. Now we may assume $\Gamma_2(x) = \{y, y_1, y_2\}$ and $\mathbf{y} = \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_4$. For $t = 1, 2$, since $y_t \sim y$, we have

$$|\operatorname{supp}(\mathbf{y_t}) \cap \operatorname{supp}(\mathbf{y})| = 1$$

by Lemma 15 (i). Also, from the assumption, we have

$$|\operatorname{supp}(\mathbf{y_t}) \cap \operatorname{supp}(\mathbf{x})| = 2.$$

More precisely, $(\mathbf{y_{t_1}}, \mathbf{e}_1 - \mathbf{e}_3) = 2$, $(\mathbf{y_{t_1}}, \mathbf{e}_2) = 0$, $(\mathbf{y_{t_2}}, -\mathbf{e}_2 + \mathbf{e}_3) = 2$, $(\mathbf{y_{t_2}}, \mathbf{e}_1) = 0$ and $(\mathbf{y_{t_1}}, \mathbf{e}_4) = (\mathbf{y_{t_2}}, \mathbf{e}_4) = 0$, where $\{t_1, t_2\} = \{1, 2\}$. But in this case, $(\mathbf{y_{t_1}}, \mathbf{y_{t_2}}) \leqslant 0$, which contradicts $(\mathbf{y_{t_1}}, \mathbf{y_{t_2}}) = 1$ as $y_{t_1} \sim y_{t_2}$. Therefore, the lemma holds. $\square$

Lemma 19. *Assume $S = \emptyset$ and $c \geqslant 9$. For any two distinct vertices x and y, the following hold:*

(i) $|\operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(\mathbf{y})| = 1$ *holds if $x \sim y$, and* $|\operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(\mathbf{y})| = 0$ *holds if $x \not\sim y$.*

(ii) *If $d(x, y) = 2$, then for any $i \in \operatorname{supp}(\mathbf{x})$ and $j \in \operatorname{supp}(\mathbf{y})$, there exists $w \in \Gamma_1(x) \cap \Gamma_1(y)$ such that $(\mathbf{w}, \sigma_x(i)\mathbf{e}_i + \sigma_y(j)\mathbf{e}_j) = 2$ holds.*

(iii) $c = 9$.

(iv) *The diameter of Γ is* 2.

(v) *If $\operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(\mathbf{y}) = \{\ell\}$, then for any $i \in \operatorname{supp}(\mathbf{x}) - \{\ell\}$ and $j \in \operatorname{supp}(\mathbf{y}) - \{\ell\}$, there exists $w \in \Gamma_1(x) \cap \Gamma_1(y)$ such that $(\mathbf{w}, \sigma_x(i)\mathbf{e}_i + \sigma_y(j)\mathbf{e}_j) = 2$ holds.*

Proof. We may let $\mathbf{x} = a_{i_1}\mathbf{e}_{i_1} + a_{i_2}\mathbf{e}_{i_2} + a_{i_3}\mathbf{e}_{i_3}$ and $\mathbf{y} = b_{j_1}\mathbf{e}_{j_1} + b_{j_2}\mathbf{e}_{j_2} + b_{j_3}\mathbf{e}_{j_3}$, where $a_{i_1}, a_{i_2}, a_{i_3}, b_{j_1}, b_{j_2}, b_{j_3} \in \{1, -1\}$.

(i) If $x \sim y$, it follows from Lemma 15 (i). If $x \not\sim y$, it follows from Lemma 13, Lemma 17 and Lemma 18 as $c \geqslant 9$.

(ii)–(iii) We have, by (i), that $i_1, i_2, i_3, j_1, j_2, j_3$ are pairwise distinct, and for each vertex $z \in \Gamma_1(x) \cap \Gamma_1(y)$, $|\operatorname{supp}(\mathbf{z}) \cap \operatorname{supp}(\mathbf{x})| = |\operatorname{supp}(\mathbf{z}) \cap \operatorname{supp}(\mathbf{y})| = 1$. Thus, there exist $i' \in \{i_1, i_2, i_3\}$ and $j' \in \{j_1, j_2, j_3\}$ such that $(\mathbf{z}, a_{i'}\mathbf{e}_{i'} + b_{j'}\mathbf{e}_{j'}) = 2$. By Lemma 15 (ii), we have

$$\begin{aligned}
9 \leqslant c &= |\Gamma_1(x) \cap \Gamma_1(y)| \\
&= |\{z \mid (\mathbf{z}, a_{i'}\mathbf{e}_{i'} + b_{j'}\mathbf{e}_{j'}) = 2, i' \in \{i_1, i_2, i_3\}, j' \in \{j_1, j_2, j_3\}\}| \\
&\leqslant |\{\{i_1, j_1\}, \{i_1, j_2\}, \{i_1, j_3\}, \{i_2, j_1\}, \{i_2, j_2\}, \{i_2, j_3\}, \{i_3, j_1\}, \{i_3, j_2\}, \{i_3, j_3\}\}| \\
&\leqslant 9.
\end{aligned}$$

Hence (ii) and (iii) hold.

(iv) Let x' and y' be two vertices of distance 2. To prove (iv), it is sufficient to prove that $d(z, x') \leqslant 2$ for any vertex $z \in \Gamma_1(y')$. Without loss of generality, we may assume $z \not\sim x'$. As $z \sim y'$, we have $|\operatorname{supp}(\mathbf{z}) \cap \operatorname{supp}(\mathbf{y}')| = 1$ by (i). From (ii), we find that there must be a vertex $w' \in \Gamma_1(x') \cap \Gamma_1(y')$, such that $\operatorname{supp}(\mathbf{w}') \cap \operatorname{supp}(\mathbf{y}') = \operatorname{supp}(\mathbf{z}) \cap \operatorname{supp}(\mathbf{y}')$. Thus, $|\operatorname{supp}(\mathbf{w}') \cap \operatorname{supp}(\mathbf{z})| = 1$ and $w' \sim z$, which implies $d(z, x') = 2$. Thus, (iv) holds.

(v) We may assume that $i_3 = j_3 = \ell$. Without loss of generality, we may consider the case where $i = i_1$ and $j = j_1$. Let z be a vertex in $\Gamma_2(y)$. Then $|\operatorname{supp}(\mathbf{z}) \cap \operatorname{supp}(\mathbf{y})| = 0$ follows. By (ii), we can find at least two vertices w_1 and w_2 in $\Gamma_1(y) \cap \Gamma_1(z)$ such that $(\mathbf{w_1}, b_{j_1}\mathbf{e}_{j_1}) = (\mathbf{w_2}, b_{j_1}\mathbf{e}_{j_1}) = 1$ and $(\mathbf{w_1}, \mathbf{e}_{j_2}) = (\mathbf{w_2}, \mathbf{e}_{j_2}) = (\mathbf{w_1}, \mathbf{e}_\ell) = (\mathbf{w_2}, \mathbf{e}_\ell) = 0$. If either $(\mathbf{w_1}, a_{i_1}\mathbf{e}_{i_1})$ or $(\mathbf{w_2}, a_{i_1}\mathbf{e}_{i_1})$ equals 1, then the existence of w is shown. Otherwise $(\mathbf{w_1}, \mathbf{e}_{i_1}) = (\mathbf{w_2}, \mathbf{e}_{i_1}) = 0$. Note that now we have either $(\mathbf{w_1}, \mathbf{e}_{i_2}) = 0$ or $(\mathbf{w_2}, \mathbf{e}_{i_2}) = 0$, as $|\operatorname{supp}(\mathbf{w_1}) \cap \operatorname{supp}(\mathbf{w_2})| \leqslant 1$ by (i). This means that at least one of w_1 and w_2 is not adjacent to x. Let us assume $w_1 \not\sim x$. Since $d(x, w_1) = 2$ by (iv), using (ii) again, we obtain the existence of w in the set $\Gamma_1(x) \cap \Gamma_1(w_1)$. □

Now we give the proof of Theorem 9.

Proof of Theorem 9. Since $S = \emptyset$ by Claim 16, we may let $\mathcal{P} = \bigcup_{x \in V(\Gamma)} \operatorname{supp}(\mathbf{x})$ and $\mathcal{B} = \{\operatorname{supp}(\mathbf{x}) \mid x \in V(\Gamma)\}$ a family of 3-element subsets of $\mathcal{P}$. Assume $i, j \in \mathcal{P}$ are distinct with $i \in \operatorname{supp}(\mathbf{x})$ and $j \in \operatorname{supp}(\mathbf{y})$. If $x = y$, then $\{i, j\}$ is contained in the block $\operatorname{supp}(\mathbf{x})$. If $x \neq y$, Lemma 19 (iv), (ii) and (v) guarantee that there exists $w \in \Gamma_1(x) \cap \Gamma_1(y)$ such that $\{i, j\}$ is contained in the block $\operatorname{supp}(\mathbf{w})$. The uniqueness of the block containing $\{i, j\}$ follows from Lemma 19 (i). Hence we conclude that $(\mathcal{P}, \mathcal{B})$ is a Steiner triple system. By Lemma 19 (i), the block graph of this Steiner triple system is exactly the graph Γ. Therefore, the theorem holds. □

For the rest of this section, we will prove Claim 16.

Lemma 20. *Assume $S \neq \emptyset$ and $c \geqslant 7$. For any $x \in S$ and $y \in \Gamma_2(x)$, $|\operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(\mathbf{y})| = 0$ holds.*

Proof. Suppose $|\operatorname{supp}(\mathbf{y}) \cap \operatorname{supp}(\mathbf{x})| = 2$. As $y \not\sim x$, $(\mathbf{x}, \mathbf{y}) = 0$ follows. Without loss of generality, we may assume

$$\mathbf{x} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3,\ \mathbf{y} = \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_4.$$

Let x' be the mate of x. Considering $(\mathbf{x}', \mathbf{y}) = 1$ or 0, we obtain

$$\mathbf{x}' = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3,$$

and thus $x' \not\sim y$. By Lemma 15 (i), for any vertex $w \in \Gamma_1(x) - \{x'\}$,

$$|\operatorname{supp}(\mathbf{w}) \cap \operatorname{supp}(\mathbf{x})| = 1 \text{ and } (\mathbf{w}, \mathbf{e}_{\operatorname{supp}(\mathbf{w}) \cap \operatorname{supp}(\mathbf{x})}) = 1$$

hold. This means that for any vertex $w \in \Gamma_1(x) - \{x'\}$, $\operatorname{supp}(\mathbf{w}) \cap \operatorname{supp}(\mathbf{x}) \neq \{3\}$ as $(\mathbf{w}, \mathbf{x}') \in \{0, 1\}$, and furthermore, if $w \sim y$, then

$$(\mathbf{w}, \mathbf{e}_1) = 1 \text{ and } (\mathbf{w}, \mathbf{e}_i) = 0 \text{ for } i = 2, 3, 4; \tag{11}$$

and if $w \not\sim y$, then

$$(\mathbf{w}, \mathbf{e}_1 - \mathbf{e}_4) = 2 \text{ or } (\mathbf{w}, \mathbf{e}_2 + \mathbf{e}_4) = 2. \tag{12}$$

Assume that there exists a vertex $w_1 \in \Gamma_1(x) - \{x'\} - \Gamma_1(y)$ such that $(\mathbf{w_1}, \mathbf{e}_1 - \mathbf{e}_4) = 2$. Let $\mathrm{supp}(\mathbf{w_1}) = \{1, 4, \ell_1\}$ for some $5 \leqslant \ell_1 \leqslant m$. We find that in this case, for any vertex $u \in \Gamma_1(y) - \Gamma_1(x)$,

$$\mathrm{supp}(\mathbf{u}) \cap \mathrm{supp}(\mathbf{x}) = \emptyset \text{ and } (\mathbf{u}, \mathbf{e}_4 + \sigma_{w_1}(\ell_1)\mathbf{e}_{\ell_1}) = 2.$$

Note that $2 \leqslant k - c = |\Gamma_1(y) - \Gamma_1(x)| = |\{u \in V(\Gamma) \mid \mathrm{supp}(\mathbf{u}) \cap \mathrm{supp}(\mathbf{x}) = \emptyset \text{ and } (\mathbf{u}, \mathbf{e}_4 + \sigma_{w_1}(\ell_1)\mathbf{e}_{\ell_1}) = 2\}| \leqslant 2$ by Lemma 15 (ii). Hence, $k - c = 2$ and there exist two vertices $u_1, u_2 \in \Gamma_1(y)$ such that

$$\mathbf{u_1} = \mathbf{e}_4 + \sigma_{w_1}(\ell_1)\mathbf{e}_{\ell_1} + \mathbf{e}_{\ell_2},\ \mathbf{u_2} = \mathbf{e}_4 + \sigma_{w_1}(\ell_1)\mathbf{e}_{\ell_1} - \mathbf{e}_{\ell_2}$$

for some $\ell_2 \notin \{1, 2, 3, 4, \ell_1\}$. From (11), we easily obtain that for every $v \in \Gamma_1(x) \cap \Gamma_1(y)$, $\mathbf{v}$ satisfies $(\mathbf{v}, \mathbf{e}_1) = 1$, $(\mathbf{v}, \mathbf{e}_i) = 0$ for $i = 2, 3, 4$. Since each of $(\mathbf{v}, \mathbf{u_1})$, $(\mathbf{v}, \mathbf{u_2})$ and $(\mathbf{v}, \mathbf{w_1})$ equals 1 or 0, we have $(\mathbf{v}, \mathbf{e}_{\ell_1}) = (\mathbf{v}, \mathbf{e}_{\ell_2}) = 0$, and thus $v \not\sim u_1$. Notice that $d(v, u_1) = 2$, as $v \sim y$ and $u_1 \sim y$, but $c = |\Gamma_1(v) \cap \Gamma_1(u_1)| \leqslant |\Gamma_1(v) - \{x, x', w_1\}| = k - 3$. This gives a contradiction. Therefore, for any $w \in \Gamma_1(x) - \{x'\}$ and $w \not\sim y$, $(\mathbf{w}, \mathbf{e}_2 + \mathbf{e}_4) = 2$ holds by (12), and

$$k - c = |\Gamma_1(x) - \Gamma_1(y)| \leqslant |\{x'\} \cup \{w \in V(\Gamma) \mid (\mathbf{w}, \mathbf{e}_2 + \mathbf{e}_4) = 2\}| \leqslant 1 + 2 = 3.$$

Let $w_2 \in \Gamma_1(x) - \Gamma_1(y)$ be a vertex such that $(\mathbf{w_2}, \mathbf{e}_2 + \mathbf{e}_4) = 2$. Let $\mathrm{supp}(\mathbf{w_2}) = \{2, 4, \ell_3\}$ for some $5 \leqslant \ell_3 \leqslant m$. If $d(y, w_2) = 2$, then we can easily obtain the following:

$$\begin{aligned} k = |\Gamma_1(y)| &\geqslant |(\Gamma_1(y) \cap \Gamma_1(x)) \cup (\Gamma_1(y) \cap \Gamma_1(w_2))| \\ &= |\Gamma_1(y) \cap \Gamma_1(x)| + |\Gamma_1(y) \cap \Gamma_1(w_2)| - |\Gamma_1(y) \cap \Gamma_1(x) \cap \Gamma_1(w_2)| \\ &\geqslant c + c - |\{w \in V(\Gamma) \mid (\mathbf{w}, \mathbf{e}_1 + \sigma_{w_2}(\ell_3)\mathbf{e}_{\ell_3}) = 2\}| \\ &\geqslant 2c - 2 \geqslant 2(k-3) - 2. \end{aligned}$$

This means $k \leqslant 8$ and thus $c \leqslant 6$ as $k - c \geqslant 2$, which contradicts the given condition $c \geqslant 7$. So now we may assume $d(y, w_2) \geqslant 3$ and look at the vertices in $\Gamma_1(y) - \Gamma_1(x)$. Note that for any vertex $z \in \Gamma_1(y) - \Gamma_1(x)$,

$$\mathrm{supp}(\mathbf{z}) \cap \mathrm{supp}(\mathbf{x}) = \emptyset \text{ and } (\mathbf{z}, \mathbf{e}_4 - \sigma_{w_2}(\ell_3)\mathbf{e}_{\ell_3}) = 2,$$

as $z \not\sim w_2$. Using an argument similar to the one above, we can also find vertices z_1 and z_2 in $\Gamma_1(y) - \Gamma_1(w_2)$ such that

$$\mathbf{z_1} = \mathbf{e}_4 - \sigma_{w_2}(\ell_3)\mathbf{e}_{\ell_3} + \mathbf{e}_{\ell_4},\ \mathbf{z_2} = \mathbf{e}_4 - \sigma_{w_2}(\ell_3)\mathbf{e}_{\ell_3} - \mathbf{e}_{\ell_4}$$

for some $\ell_4 \notin \{1, 2, 3, 4, \ell_3\}$.

We will verify that $d(z_1, w_2) = 2$. Take a vertex $v \in \Gamma_1(x) \cap \Gamma_1(y)$. By (11), we have $(\mathbf{v}, \mathbf{e}_i) = 0$ for $i = 2, 3, 4$. Since $d(y, w_2) \geqslant 3$ and $v \sim y$, we have $v \not\sim w_2$. It follows that

$0 = (\mathbf{v}, \mathbf{w_2}) = \sigma_{w_2}(\ell_3)(\mathbf{v}, \mathbf{e}_{\ell_3})$, and thus $(\mathbf{v}, \mathbf{e}_{\ell_3}) = 0$. As both $(\mathbf{v}, \mathbf{z_1})$ and $(\mathbf{v}, \mathbf{z_2})$ belong to $\{0, 1\}$, $(\mathbf{v}, \mathbf{e}_{\ell_4}) = 0$ holds and hence $v \not\sim z_1$.

Now x is a common neighbor of v and w_2, while y is a common neighbor of v and z_1. Thus $d(v, w_2) = d(v, z_1) = 2$. By sesqui-regularity,

$$|\Gamma_1(v) \cap \Gamma_1(w_2)| = |\Gamma_1(v) \cap \Gamma_1(z_1)| = c.$$

Also,

$$\begin{aligned} &|\Gamma_1(v) \cap \Gamma_1(w_2) \cap \Gamma_1(z_1)| \\ \geqslant &|\Gamma_1(v) \cap \Gamma_1(w_2)| + |\Gamma_1(v) \cap \Gamma_1(z_1)| - |\Gamma_1(v)| \\ = &2c - k = c - (k - c) \geqslant c - 3 \geqslant 4. \end{aligned}$$

Hence $\Gamma_1(z_1) \cap \Gamma_1(w_2) \neq \emptyset$ and $d(z_1, w_2) = 2$. We can now apply a similar argument to z_1, z_2, w_2, y as we did for x, x', y, w_1 before and obtain a contradiction. Therefore, our lemma holds. $\square$

Lemma 21. *Assume $S \neq \emptyset$ and $c \geqslant 9$. For any $x \in S$ and $y \in \Gamma_2(x)$, $y \notin S$ holds.*

Proof. Suppose $y \in S$. By Lemma 20, we have $|\operatorname{supp}(\mathbf{y}) \cap \operatorname{supp}(\mathbf{x})| = 0$. Thus we may assume $\mathbf{x} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ and $\mathbf{y} = \mathbf{e}_4 + \mathbf{e}_5 + \mathbf{e}_6$. Let x' and y' be the mates of x and y respectively. Without loss of generality, we may also assume $\mathbf{x'} = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$ and $\mathbf{y'} = \mathbf{e}_4 + \mathbf{e}_5 - \mathbf{e}_6$. Now it is easy to see

$$\begin{aligned} 9 \leqslant c &= |\Gamma_1(x) \cap \Gamma_1(y)| \\ &\leqslant |\{w \in V(\Gamma) \mid (\mathbf{w}, \mathbf{e}_i + \mathbf{e}_j) = 2, i \in \{1, 2\}, j \in \{4, 5\}\}| \\ &\leqslant 2|\{\{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}\}| = 8 \end{aligned}$$

by Lemma 15 (ii). This gives a contradiction. Hence $y \notin S$. $\square$

Lemma 22. *Assume $S \neq \emptyset$ and $c \geqslant 9$. Let $x \in S$ and $y \in V(\Gamma)$ be two vertices such that $d(x, y) = 2$, and let x' be the mate of x. If $\operatorname{supp}(\mathbf{x}) = \{i_1, i_2, i_3\}$ and $\sigma_x(i_1) = \sigma_{x'}(i_1)$, $\sigma_x(i_2) = \sigma_{x'}(i_2)$, then the following hold:*

(i) *$y \in V(\Gamma) - S$ and $\operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(\mathbf{y}) = \emptyset$.*

(ii) *$|\Gamma_1(x) \cap \Gamma_1(y) \cap S| = 6$.*

(iii) *$\bigcap_{w \in \Gamma_1(x) \cap \Gamma_1(y) \cap S} \operatorname{supp}(\mathbf{w}) \subset \{i_1, i_2\}$ and $|\bigcap_{w \in \Gamma_1(x) \cap \Gamma_1(y) \cap S} \operatorname{supp}(\mathbf{w})| = 1$.*

(iv) *For $i \in \bigcap_{w \in \Gamma_1(x) \cap \Gamma_1(y) \cap S} \operatorname{supp}(\mathbf{w})$ and $j \in \operatorname{supp}(\mathbf{y})$, there exist $z_1 \in \Gamma_1(x) \cap \Gamma_1(y) \cap S$ and its mate z_1' such that $(\mathbf{z_1}, \sigma_x(i)\mathbf{e}_i + \sigma_y(j)\mathbf{e}_j) = (\mathbf{z_1'}, \sigma_x(i)\mathbf{e}_i + \sigma_y(j)\mathbf{e}_j) = 2$ hold.*

(v) *For $i \in \{i_1, i_2\} - \bigcap_{w \in \Gamma_1(x) \cap \Gamma_1(y) \cap S} \operatorname{supp}(\mathbf{w})$ and $j \in \operatorname{supp}(\mathbf{y})$, there exists $z_2 \in \Gamma_1(x) \cap \Gamma_1(y) - S$ such that $(\mathbf{z_2}, \sigma_x(i)\mathbf{e}_i + \sigma_y(j)\mathbf{e}_j) = 2$ holds.*

(vi) *$c = 9$.*

Proof. (i) This follows from Lemma 21 and Lemma 20 immediately.

(ii)–(vi) We may assume $\mathbf{x} = \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}$, $\mathbf{x}' = \mathbf{e}_{i_1} + \mathbf{e}_{i_2} - \mathbf{e}_{i_3}$, $\mathbf{y} = \mathbf{e}_{j_1} + \mathbf{e}_{j_2} + \mathbf{e}_{j_3}$, and pay our attention to the set $\Gamma_1(x) \cap \Gamma_1(y)$. For any $w \in \Gamma_1(x) \cap \Gamma_1(y)$,

$$|\operatorname{supp}(\mathbf{w}) \cap \operatorname{supp}(\mathbf{x})| = 1, \ |\operatorname{supp}(\mathbf{w}) \cap \operatorname{supp}(\mathbf{y})| = 1 \tag{13}$$

hold by Lemma 15 (i) and (i), and hence there exist $\ell_1 \in \{i_1, i_2\}$ and $\ell_2 \in \{j_1, j_2, j_3\}$ such that $(\mathbf{w}, \mathbf{e}_{\ell_1} + \mathbf{e}_{\ell_2}) = 2$. Let

$$\begin{aligned}
T_{1,i_1} &= \{\{i_1, \ell\} \mid (\mathbf{w}, \mathbf{e}_{i_1} + \mathbf{e}_\ell) = 2, w \in \Gamma_1(x) \cap \Gamma_1(y) \cap S, \ell \in \{j_1, j_2, j_3\}\},\\
T_{1,i_2} &= \{\{i_2, \ell\} \mid (\mathbf{w}, \mathbf{e}_{i_2} + \mathbf{e}_\ell) = 2, w \in \Gamma_1(x) \cap \Gamma_1(y) \cap S, \ell \in \{j_1, j_2, j_3\}\}, \text{ and}\\
T_2 &= \{\{\ell_1, \ell_2\} \mid (\mathbf{w}, \mathbf{e}_{\ell_1} + \mathbf{e}_{\ell_2}) = 2, w \in \Gamma_1(x) \cap \Gamma_1(y) - S, \ell_1 \in \{i_1, i_2\}, \ell_2 \in \{j_1, j_2, j_3\}\}.
\end{aligned}$$

For a pair $\{i, \ell\}$, where $i \in \{i_1, i_2\}$ and $\ell \in \{j_1, j_2, j_3\}$, define

$$\mathcal{C}(\{i, \ell\}) := \{w \in \Gamma_1(x) \cap \Gamma_1(y) \mid (\mathbf{w}, \mathbf{e}_i + \mathbf{e}_\ell) = 2\}.$$

By (13), every vertex $w \in \Gamma_1(x) \cap \Gamma_1(y)$ determines a unique pair $\{i, \ell\}$ of this form.

Suppose first that $\{i, \ell\} \in T_{1,i_1} \cup T_{1,i_2}$. By the definition of T_{1,i_1} and T_{1,i_2}, there exists $w \in \Gamma_1(x) \cap \Gamma_1(y) \cap S$ such that $(\mathbf{w}, \mathbf{e}_i + \mathbf{e}_\ell) = 2$. Let w' be the mate of w. By Lemma 15 (ii), the only vertices $u \in V(\Gamma)$ satisfying $(\mathbf{u}, \mathbf{e}_i + \mathbf{e}_\ell) = 2$ are w and w'. Moreover, $\mathcal{C}(\{i, \ell\}) = \{w, w'\}$ and $|\mathcal{C}(\{i, \ell\})| = 2$.

Suppose next that $\{i, \ell\} \in T_2$. Then there exists $w \in \Gamma_1(x) \cap \Gamma_1(y) - S$ such that $(\mathbf{w}, \mathbf{e}_i + \mathbf{e}_\ell) = 2$. As $w \notin S$, Lemma 15 (ii) implies $\mathcal{C}(\{i, \ell\}) = \{w\}$ and $|\mathcal{C}(\{i, \ell\})| = 1$.

It follows in particular that T_{1,i_1}, T_{1,i_2}, and T_2 are pairwise disjoint. We obtain

$$\begin{aligned}
|T_{1,i_1}| + |T_{1,i_2}| + |T_2| &\leqslant |\{\{\ell_1, \ell_2\} \mid \ell_1 \in \{i_1, i_2\}, \ell_2 \in \{j_1, j_2, j_3\}\}| = 6,\\
2|T_{1,i_1}| + 2|T_{1,i_2}| + |T_2| &= |\Gamma_1(x) \cap \Gamma_1(y)| = c \geqslant 9
\end{aligned} \tag{14}$$

Consequently,

$$|T_{1,i_1}| + |T_{1,i_2}| \geqslant 3. \tag{15}$$

If both T_{1,i_1} and T_{1,i_2} are not empty, then (15) implies that there exist $\ell_3, \ell_4 \in \{j_1, j_2, j_3\}$ such that $\ell_3 \neq \ell_4$ and $\{i_1, \ell_3\} \in T_{1,i_1}$, $\{i_2, \ell_4\} \in T_{1,i_2}$. Let w_{i_1,ℓ_3} and w_{i_2,ℓ_4} be the vertices in $\Gamma_1(x) \cap \Gamma_1(y) \cap S$ such that $(\mathbf{w_{i_1,\ell_3}}, \mathbf{e}_{i_1} + \mathbf{e}_{\ell_3}) = 2$ and $(\mathbf{w_{i_2,\ell_4}}, \mathbf{e}_{i_2} + \mathbf{e}_{\ell_4}) = 2$. After looking at the integral vectors corresponding to the mates of w_{i_1,ℓ_3} and w_{i_2,ℓ_4}, we can easily obtain $|\operatorname{supp}(\mathbf{w_{i_1,\ell_3}}) \cap \operatorname{supp}(\mathbf{w_{i_2,\ell_4}})| = 0$ and thus $w_{i_1,\ell_3} \not\sim w_{i_2,\ell_4}$. Since both w_{i_1,ℓ_3} and w_{i_2,ℓ_4} are adjacent to x, $d(w_{i_1,\ell_3}, w_{i_2,\ell_4}) = 2$ follows. This contradicts Lemma 21, as both w_{i_1,ℓ_3} and w_{i_2,ℓ_4} are in S and have distance 2. So now we may assume $T_{1,i_2} = \emptyset$. Then (15) forces $|T_{1,i_1}| = 3$, and (iii) and (iv) hold. Since $|\Gamma_1(x) \cap \Gamma_1(y) \cap S| = 2|T_{1,i_1}|$, (ii) also follows. By (14) we obtain $|T_2| = 3$ and $T_2 = \{\{i_2, j_1\}, \{i_2, j_2\}, \{i_2, j_3\}\}$ immediately. Now it is obvious to see both (v) and (vi) hold. $\square$

Lemma 23. *Assume $S \neq \emptyset$ and $c \geqslant 9$. Then the subgraph of Γ induced on S is a clique.*

Proof. Suppose not. Then let $d = \min\{d(x,y) \mid x,y \in S \text{ and } d(x,y) \geqslant 2\}$. Assume $x_0, x_d \in S$ and $d(x_0,x_d) = d$. There is a path $x_0 \sim x_1 \sim \cdots \sim x_d$ in Γ with $x_i \in \Gamma_i(x_0)$ for $i = 1,\ldots,d$. If $d \geqslant 3$, then $x_{d-1} \notin S$ as $d(x_0,x_d) = \min\{d(x,y) \mid x,y \in S \text{ and } d(x,y) \geqslant 2\}$. By Lemma 22 (ii), there exists $z \in \Gamma_1(x_{d-2}) \cap \Gamma_1(x_d)$ such that $z \in S$. Note that $d(x_0,z) = d-1 \geqslant 2$. It follows that $d-1 = d(x_0,z) \geqslant \min\{d(x,y) \mid x,y \in S \text{ and } d(x,y) \geqslant 2\} = d$, which gives a contradiction. Thus $d = 2$. But Lemma 21 tells that this is impossible. The proof is completed. ☐

Lemma 24. *Assume $S \neq \emptyset$ and $c \geqslant 9$. Then $|\bigcap_{x\in S} \operatorname{supp}(\mathbf{x})| = 1$ holds. In particular, there exists $a \in \{1,-1\}$, such that for every $y \in S$, $(\mathbf{y}, \mathbf{e}_{\bigcap_{x\in S}\operatorname{supp}(\mathbf{x})}) = a$ holds.*

Proof. It is sufficient to prove $|\bigcap_{x\in S} \operatorname{supp}(\mathbf{x})| = 1$. From Lemma 22 (ii), we have $|S| \geqslant 8$. Let x_1, x_2, x_1', x_2' be four vertices in S, where x_i' is the mate of x_i for $i = 1,2$. Note that by Lemma 23, the subgraph of Γ induced on S is a clique. Hence, we may assume

$$\mathbf{x_1} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3,\ \mathbf{x_1'} = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3,\ \mathbf{x_2} = \mathbf{e}_1 + \mathbf{e}_4 + \mathbf{e}_5,\ \mathbf{x_2'} = \mathbf{e}_1 + \mathbf{e}_4 - \mathbf{e}_5$$

by Lemma 15 (i).

Suppose that this lemma does not hold. Then we may assume that there exists a vertex $x_3 \in S$ such that $\operatorname{supp}(\mathbf{x_3}) \cap \operatorname{supp}(\mathbf{x_1}) \neq \operatorname{supp}(\mathbf{x_3}) \cap \operatorname{supp}(\mathbf{x_2})$. Let x_3' be the mate of x_3. It follows that $(\mathbf{x_3}, \mathbf{e}_2 + \mathbf{e}_4) = (\mathbf{x_3'}, \mathbf{e}_2 + \mathbf{e}_4) = 2$. We may assume

$$\mathbf{x_3} = \mathbf{e}_2 + \mathbf{e}_4 + \mathbf{e}_\ell,\ \mathbf{x_3'} = \mathbf{e}_2 + \mathbf{e}_4 - \mathbf{e}_\ell \text{ for some } 6 \leqslant \ell \leqslant m.$$

As $|S| \geqslant 8$, there must be a vertex x_4 in $S - \{x_1, x_1', x_2, x_2', x_3, x_3'\}$. But it is impossible to define $\mathbf{x_4}$ such that both $(\mathbf{x_4}, \mathbf{x_i}) = 1$ and $(\mathbf{x_4}, \mathbf{x_i'}) = 1$ hold for every $i \in \{1,2,3\}$. Therefore, our lemma is proved. ☐

From now on, if $S \neq \emptyset$, then we may always assume $S = \{x_1, x_1', x_2, x_2', \ldots, x_s, x_s'\}$, where $s \geqslant 4$ and for each $i = 1,2,\ldots,s$, the vertices x_i and x_i' are mates, and

$$\mathbf{x_i} = \mathbf{e}_i + \mathbf{e}_{s+i} + \mathbf{e}_m,\ \mathbf{x_i'} = \mathbf{e}_i - \mathbf{e}_{s+i} + \mathbf{e}_m$$

with $m = \max\{i \mid i \in \bigcup_{x\in V(\Gamma)} \operatorname{supp}(\mathbf{x})\}$.

Lemma 25. *Assume $S \neq \emptyset$ and $c \geqslant 9$. For any $x \in S$ and $y \in V(\Gamma) - S$, $d(x,y) \leqslant 2$ holds.*

Proof. Suppose $d = d(x,y) \geqslant 3$. Then we can find a path $x \sim y_1 \sim y_2 \sim \cdots \sim y_d = y$ with $y_i \in \Gamma_i(x)$ for $i = 1,\ldots,d$. Since $y_i \not\sim x$ for $i \geqslant 2$, we have $y_i \notin S$ by Lemma 23. We may assume $y_1 \in S$, otherwise we can find a vertex $z_1 \in \Gamma_1(x) \cap \Gamma_1(y_2) \cap S$ by Lemma 22 (ii) and replace y_1 by z_1. Now we look at the vertices y_1 and y_3. Using Lemma 22 (ii) again, we can find a vertex z_2 in $\Gamma_1(y_1) \cap \Gamma_1(y_3) \cap S$. Notice that z_2 is also adjacent to x, as $z_2 \in S$. This implies $d(y_3,x) \leqslant 2$, which contradicts $y_3 \in \Gamma_3(x)$. Therefore, our lemma holds. ☐

Lemma 26. *If $S \neq \emptyset$ and $c \geqslant 9$, then for any $y \in V(\Gamma) - S$, there exists $x \in S$ such that $d(x,y) = 2$ holds.*

Proof. Suppose not. Then for every $x \in S$, y is adjacent to x by Lemma 25. It follows, for $i = 1, \dots, s$, that $|\operatorname{supp}(\mathbf{y}) \cap \operatorname{supp}(\mathbf{x_i})| = 1$, $(\mathbf{y}, \mathbf{e}_{\operatorname{supp}(\mathbf{y}) \cap \operatorname{supp}(\mathbf{x_i})}) = 1$ and $|\operatorname{supp}(\mathbf{y}) \cap \operatorname{supp}(\mathbf{x'_i})| = 1$, $(\mathbf{y}, \mathbf{e}_{\operatorname{supp}(\mathbf{y}) \cap \operatorname{supp}(\mathbf{x'_i})}) = 1$. Note that $|S| \geqslant 8$ by Lemma 22 (ii). Thus $m \in \operatorname{supp}(\mathbf{y})$, $(\mathbf{y}, \mathbf{e}_m) = 1$ and $\operatorname{supp}(\mathbf{y}) \cap \{1, \dots, 2s\} = \emptyset$. We may assume $\mathbf{y} = \mathbf{e}_m + \mathbf{e}_{j_1} + \mathbf{e}_{j_2}$, where $j_1, j_2 \in \{2s+1, \dots, m-1\}$.

Let z be a vertex in $\Gamma_2(x_1)$ with $\operatorname{supp}(\mathbf{z}) = \{\ell_1, \ell_2, \ell_3\}$. Lemma 22 (iv) implies that $\ell_1, \ell_2, \ell_3 \in \{1, 2, \dots, s\}$ and $\mathbf{z} = \mathbf{e}_{\ell_1} + \mathbf{e}_{\ell_2} + \mathbf{e}_{\ell_3}$. Note that y and z are not adjacent, as $(\mathbf{y}, \mathbf{z}) = 0$, and both are adjacent to x_{ℓ_1}. So $d(y, z) = 2$ follows and $|\Gamma_1(y) \cap \Gamma_1(z)| = 9$ by Lemma 22 (vi). Now let us look at the common neighbors of y and z. For $w \in \Gamma_1(y) \cap \Gamma_1(z)$, either $(\mathbf{w}, \mathbf{e}_m + \mathbf{e}_{\ell_p}) = 2$ or $(\mathbf{w}, \mathbf{e}_{j_q} + \mathbf{e}_{\ell_p}) = 2$, where $q \in \{1, 2\}$ and $p \in \{1, 2, 3\}$. Hence

$$\begin{aligned}
9 = c =& |\Gamma_1(y) \cap \Gamma_1(z)| \\
=& |\{w \mid (\mathbf{w}, \mathbf{e}_m + \mathbf{e}_{\ell_p}) = 2,\ p \in \{1,2,3\}\}| \\
&+ |\{w \mid (\mathbf{w}, \mathbf{e}_{j_q} + \mathbf{e}_{\ell_p}) = 2,\ q \in \{1,2\},\ p \in \{1,2,3\}\}| \\
=& |\{x_{\ell_1}, x'_{\ell_1}, x_{\ell_2}, x'_{\ell_2}, x_{\ell_3}, x'_{\ell_3}\}| + |\{w \mid (\mathbf{w}, \mathbf{e}_{j_q} + \mathbf{e}_{\ell_p}) = 2,\ q \in \{1,2\},\ p \in \{1,2,3\}\}|.
\end{aligned}$$

This means that there must be a vertex $w_1 \in \Gamma_1(y) \cap \Gamma_1(z)$ such that $(\mathbf{w_1}, \mathbf{e}_{j_{q'}} + \mathbf{e}_{\ell_{p'}}) = 2$ for some $q' \in \{1, 2\}$ and $p' \in \{1, 2, 3\}$. For convenience, we may let $q' = 1$ and $p' = 1$.

As $\operatorname{supp}(\mathbf{w_1}) \cap \operatorname{supp}(\mathbf{y}) = \{j_1\}$, we have $(\mathbf{w_1}, \mathbf{e}_m) = 0$. Now it is not hard to see $w_1 \not\sim x_{\ell_2}$ or $w_1 \not\sim x_{\ell_3}$, as $(\mathbf{w_1}, \mathbf{e}_{j_1} + \mathbf{e}_{\ell_1}) = 2$ and $(\mathbf{w_1}, \mathbf{e}_m) = 0$ imply either $(\mathbf{w_1}, \mathbf{x_{\ell_2}}) = 0$ or $(\mathbf{w_1}, \mathbf{x_{\ell_3}}) = 0$. Let us assume $w_1 \not\sim x_{\ell_2}$. Observe that both w_1 and x_{ℓ_2} are adjacent to y. In other words, $w_1 \in \Gamma_2(x_{\ell_2})$. Lemma 22 (iv) forces $j_1 \in \{1, \dots, s\}$, and this gives a contradiction. We complete the proof. $\square$

Lemma 27. *Assume $c \geqslant 9$ and $S \neq \emptyset$. For every $y \in V(\Gamma) - S$,* $\operatorname{supp}(\mathbf{y})$ *is a 3-element subset of $\{1, 2, \dots, s\}$. In particular, y has exactly 6 neighbors in S.*

Proof. By Lemma 26, there exists $x \in S$ such that $d(x, y) = 2$ holds. Thus there exist $j_1, j_2, j_3 \in \{1, 2, \dots, s\}$ such that $\operatorname{supp}(\mathbf{y}) = \{j_1, j_2, j_3\}$ and $\mathbf{y} = \mathbf{e}_{j_1} + \mathbf{e}_{j_2} + \mathbf{e}_{j_3}$ by Lemma 22 (iv). Now it is not hard to see that $\Gamma_1(y) \cap S = \{x_{j_1}, x'_{j_1}, x_{j_2}, x'_{j_2}, x_{j_3}, x'_{j_3}\}$. Therefore, our lemma holds. $\square$

Proof of Claim 16. Suppose that $S \neq \emptyset$. Let $\mathcal{P}' = \{1, 2, \dots, s\}$ and let $\mathcal{B}' = \{\operatorname{supp}(\mathbf{y}) \mid y \in V(\Gamma) - S\}$. Then $\mathcal{B}'$ is a family of 3-element subsets of $\mathcal{P}'$ by Lemma 27. Now we claim that $(\mathcal{P}', \mathcal{B}')$ is a Steiner triple system $STS(s)$.

Let $\{i_1, i_2\}$ be a 2-element subset of $\mathcal{P}'$. We look at the sets $\Gamma_1(x_{i_1}) - S$ and $\Gamma_1(x_{i_2}) - S$. Since S is a clique and Γ is a connected regular graph, there exists a positive integer r, such that, for every $x \in S$,

$$|\Gamma_1(x) - S| = r \tag{16}$$

holds. Thus both $\Gamma_1(x_{i_1}) - S$ and $\Gamma_1(x_{i_2}) - S$ are not empty. Let y be a vertex in $\Gamma_1(x_{i_1}) - S$. Then $(\mathbf{y}, \mathbf{e}_{i_1}) = 1$ by Lemma 27. If $y \sim x_{i_2}$, then $(\mathbf{y}, \mathbf{e}_{i_2}) = 1$ and $\{i_1, i_2\}$ is contained in the block $\operatorname{supp}(\mathbf{y})$. If $y \not\sim x_{i_2}$, then $d(y, x_{i_2}) = 2$ by Lemma 25. From Lemma 22 (v), there exists $z \in \Gamma_1(x_{i_2}) \cap \Gamma_1(y) - S$ such that $i_1, i_2 \in \operatorname{supp}(\mathbf{z})$, that is, $\{i_1, i_2\}$ is contained in the block supp($\mathbf{z}$). For the uniqueness of blocks containing $\{i_1, i_2\}$, it follows from Lemma 15 (ii). Therefore $(\mathcal{P}', \mathcal{B}')$ is a Steiner triple system $STS(s)$ and its block graph is the subgraph of Γ induced on $V(\Gamma) - S$.

Let Γ_1 be the subgraph induced on S and Γ_2 the subgraph induced on $V(\Gamma) - S$. Then Γ_1 is a clique with $2s$ vertices, and Γ_2 is a strongly regular graph with parameters $(\frac{s(s-1)}{6}, \frac{3(s-3)}{2}, \frac{s+3}{2}, 9)$ (see [4, p. 5]). From (16), every vertex in Γ_1 has exactly r neighbors in Γ_2, and from Lemma 27, every vertex in Γ_2 has exactly 6 neighbors in Γ_1. Thus

$$r \cdot 2s = 6 \cdot \frac{s(s-1)}{6}. \tag{17}$$

Considering that Γ is regular, we have

$$(2s-1) + r = \frac{3(s-3)}{2} + 6. \tag{18}$$

Solving (17) and (18), we obtain $s = 3$, which contradicts the fact $s \geqslant 4$.

The proof is completed. □

4 Proof of Theorem 8

In this section, we will give the proof of Theorem 8. But before that, we need to introduce our main tool: Hoffman graphs.

4.1 Hoffman graphs

In this subsection, we will give the basic definitions and lemmas related to Hoffman graphs. For more details, we refer the reader to [15].

Definition 28. A Hoffman graph $\mathfrak{h}$ is a pair (H, ℓ), where $H = (V(H), E(H))$ is a graph and $\ell : V(H) \to \{f, s\}$ is a labeling map satisfying the following conditions:

(i) vertices with label f are pairwise non-adjacent,

(ii) every vertex with label f is adjacent to at least one vertex.

We call a vertex with label s a *slim vertex*, and a vertex with label f a *fat vertex*. We denote by $V_{\text{slim}}(\mathfrak{h})$ (resp. $V_{\text{fat}}(\mathfrak{h})$) the set of slim (resp. fat) vertices of $\mathfrak{h}$.

For a vertex x of $\mathfrak{h}$, we define $N_{\mathfrak{h}}^{\text{slim}}(x)$ (resp. $N_{\mathfrak{h}}^{\text{fat}}(x)$) to be the set of slim (resp. fat) neighbors of x in $\mathfrak{h}$. If every slim vertex of $\mathfrak{h}$ has a fat neighbor, then we call $\mathfrak{h}$ *fat*. In a similar fashion, we define $N_{\mathfrak{h}}^{\text{slim}}(x_1, x_2)$ (resp. $N_{\mathfrak{h}}^{\text{fat}}(x_1, x_2)$) to be the set of common slim (resp. fat) neighbors of x_1 and x_2 in $\mathfrak{h}$.

The *slim graph* of $\mathfrak{h} := (H, \ell)$ is the subgraph of H induced on $V_{\text{slim}}(\mathfrak{h})$. Note that any graph can be considered as a Hoffman graph with only slim vertices, and vice versa. We will not distinguish between Hoffman graphs with only slim vertices and graphs.

The *special graph* of $\mathfrak{h}$ is the signed graph

$$\mathcal{S}(\mathfrak{h}) := (V(\mathcal{S}(\mathfrak{h})), E^+(\mathcal{S}(\mathfrak{h})), E^-(\mathcal{S}(\mathfrak{h}))),$$

where $V(\mathcal{S}(\mathfrak{h})) = V_{\text{slim}}(\mathfrak{h})$ and

$$\begin{aligned} E^+(\mathcal{S}(\mathfrak{h})) &= \{\{x,y\} \mid x,y \in V_{\text{slim}}(\mathfrak{h}), \{x,y\} \in E(\mathfrak{h}), N_{\mathfrak{h}}^{\text{fat}}(x,y) = \emptyset\}, \\ E^-(\mathcal{S}(\mathfrak{h})) &= \{\{x,y\} \mid x,y \in V_{\text{slim}}(\mathfrak{h}), \{x,y\} \in E(\mathfrak{h}), |N_{\mathfrak{h}}^{\text{fat}}(x,y)| \geqslant 2\} \\ &\quad \cup \{\{x,y\} \mid x,y \in V_{\text{slim}}(\mathfrak{h}), x \neq y, \{x,y\} \notin E(\mathfrak{h}), N_{\mathfrak{h}}^{\text{fat}}(x,y) \neq \emptyset\}. \end{aligned}$$

A Hoffman graph $\mathfrak{h}_1 := (H_1, \ell_1)$ is called an *induced* (resp. *proper induced*) *Hoffman subgraph* of $\mathfrak{h} := (H, \ell)$, if H_1 is an induced (resp. proper induced) subgraph of H and $\ell_1(x) = \ell(x)$ holds for every vertex x of H_1.

Let W be a subset of $V_{\text{slim}}(\mathfrak{h})$. An induced Hoffman subgraph of $\mathfrak{h}$ generated by W, denoted by $\langle W \rangle_{\mathfrak{h}}$, is the Hoffman subgraph of $\mathfrak{h}$ induced on $W \cup \{f \in V_{\text{fat}}(\mathfrak{h}) \mid f \sim w \text{ for some } w \in W\}$.

For a fat vertex f of $\mathfrak{h}$, a *quasi-clique* (with respect to f) is a subgraph of the slim graph of $\mathfrak{h}$ induced on the slim vertices adjacent to f in $\mathfrak{h}$, and we denote it by $Q_{\mathfrak{h}}(f)$.

Definition 29. Let $\mathfrak{h}$ be a Hoffman graph with n slim vertices. The special matrix $Sp(\mathfrak{h})$ of $\mathfrak{h}$ is an $n \times n$ matrix indexed by the slim vertex set of $\mathfrak{h}$ such that for any two slim vertices x and y of $\mathfrak{h}$,

$$Sp(\mathfrak{h})_{x,y} = \begin{cases} -|N_{\mathfrak{h}}^{\text{fat}}(x)| & \text{if } x = y, \\ 1 - |N_{\mathfrak{h}}^{\text{fat}}(x,y)| & \text{if } x \sim y, \\ -|N_{\mathfrak{h}}^{\text{fat}}(x,y)| & \text{otherwise.} \end{cases}$$

The *eigenvalues* of $\mathfrak{h}$ are the eigenvalues of its special matrix $Sp(\mathfrak{h})$, and the smallest eigenvalue of $\mathfrak{h}$ is denoted by $\lambda_{\min}(\mathfrak{h})$.

For the smallest eigenvalues of Hoffman graphs and their induced Hoffman subgraphs, Woo and Neumaier showed the following inequality.

Lemma 30 ([25, Corollary 3.3]). *If $\mathfrak{h}_1$ is an induced Hoffman subgraph of a Hoffman graph $\mathfrak{h}$, then $\lambda_{\min}(\mathfrak{h}_1) \geqslant \lambda_{\min}(\mathfrak{h})$ holds.*

Definition 31. Let $\mathfrak{h}^1$ and $\mathfrak{h}^2$ be two Hoffman graphs. A Hoffman graph $\mathfrak{h}$ is the sum of $\mathfrak{h}^1$ and $\mathfrak{h}^2$, denoted by $\mathfrak{h} = \mathfrak{h}^1 \uplus \mathfrak{h}^2$, if $\mathfrak{h}$ satisfies the following condition:

There exists a partition $\{V_{\text{slim}}^1(\mathfrak{h}), V_{\text{slim}}^2(\mathfrak{h})\}$ of $V_{\text{slim}}(\mathfrak{h})$ such that the induced Hoffman subgraphs generated by $V_{\text{slim}}^i(\mathfrak{h})$ are $\mathfrak{h}^i$ for $i = 1, 2$ and

$$Sp(\mathfrak{h}) = \begin{pmatrix} Sp(\mathfrak{h}^1) & O \\ O & Sp(\mathfrak{h}^2) \end{pmatrix}$$

with respect to the partition $\{V_{\text{slim}}^1(\mathfrak{h}), V_{\text{slim}}^2(\mathfrak{h})\}$ of $V_{\text{slim}}(\mathfrak{h})$.

We can check that $\mathfrak{h}$ is a sum of two non-empty Hoffman graphs if and only if its special matrix $Sp(\mathfrak{h})$ is a block matrix with at least two non-empty diagonal blocks. If $\mathfrak{h} = \mathfrak{h}^1 \uplus \mathfrak{h}^2$ for some non-empty Hoffman subgraphs $\mathfrak{h}^1$ and $\mathfrak{h}^2$, then we call $\mathfrak{h}$ *decomposable* with $\{\mathfrak{h}^1, \mathfrak{h}^2\}$ as a *decomposition* and call $\mathfrak{h}^1, \mathfrak{h}^2$ *factors* of $\mathfrak{h}$. Otherwise, $\mathfrak{h}$ is called *indecomposable*. Note that a Hoffman graph is indecomposable if and only if its special graph is connected.

There is an equivalent way to define the sum of two Hoffman graphs as follows.

Lemma 32 ([20, Lemma 2.11]). *Let $\mathfrak{h}$ be a Hoffman graph and $\mathfrak{h}^1$ and $\mathfrak{h}^2$ be two induced Hoffman subgraphs of $\mathfrak{h}$. The Hoffman graph $\mathfrak{h}$ is the sum of $\mathfrak{h}^1$ and $\mathfrak{h}^2$ if and only if $\mathfrak{h}^1$, $\mathfrak{h}^2$, and $\mathfrak{h}$ satisfy the following conditions:*

(i) $V(\mathfrak{h}) = V(\mathfrak{h}^1) \cup V(\mathfrak{h}^2)$;

(ii) $\{V_{\text{slim}}(\mathfrak{h}^1), V_{\text{slim}}(\mathfrak{h}^2)\}$ *is a partition of* $V_{\text{slim}}(\mathfrak{h})$;

(iii) *if* $x \in V_{\text{slim}}(\mathfrak{h}^i)$, $f \in V_{\text{fat}}(\mathfrak{h})$ *and* $x \sim f$, *then* $f \in V_{\text{fat}}(\mathfrak{h}^i)$;

(iv) *if* $x \in V_{\text{slim}}(\mathfrak{h}^1)$ *and* $y \in V_{\text{slim}}(\mathfrak{h}^2)$, *then x and y have at most one common fat neighbor, and they have one if and only if they are adjacent.*

Definition 33. For a Hoffman graph $\mathfrak{h}$ and a positive integer m,

(i) if there is a mapping $\phi : V(\mathfrak{h}) \to \mathbb{Z}^m$ satisfying

$$(\phi(x), \phi(y)) = \begin{cases} t & \text{if } x = y \text{ and } x, y \in V_{\text{slim}}(\mathfrak{h}), \\ 1 & \text{if } x = y \text{ and } x, y \in V_{\text{fat}}(\mathfrak{h}), \\ 1 & \text{if } x \sim y, \\ 0 & \text{otherwise,} \end{cases}$$

then $\mathfrak{h}$ has an *integral representation* of norm t.

(ii) if there is a mapping $\psi : V_{\text{slim}}(\mathfrak{h}) \to \mathbb{Z}^m$ satisfying

$$(\psi(x), \psi(y)) = \begin{cases} t - |N_{\mathfrak{h}}^{\text{fat}}(x)| & \text{if } x = y, \\ 1 - |N_{\mathfrak{h}}^{\text{fat}}(x, y)| & \text{if } x \sim y, \\ -|N_{\mathfrak{h}}^{\text{fat}}(x, y)| & \text{otherwise,} \end{cases}$$

then $\mathfrak{h}$ has an *integral reduced representation* of norm t.

Lemma 34 (cf. [12, Theorem **2.8**]). *For a Hoffman graph $\mathfrak{h}$, the following conditions are equivalent:*

(i) $\mathfrak{h}$ *has an integral representation of norm t;*

(ii) $\mathfrak{h}$ *has an integral reduced representation of norm t.*

Theorem 35 (cf. [12, Theorem **3.7**]). *Given a fat indecomposable Hoffman graph with smallest eigenvalue at least -3, if one of its slim vertices has at least two fat neighbors, then this Hoffman graph has an integral reduced representation of norm* 3.

4.2 Sesqui-regular graphs and Hoffman graphs

In this subsection, we will use Hoffman graphs as our main tool to complete the proof of Theorem 8. Let Γ be a connected sesqui-regular graph with parameters (n, k, c), where $c \geqslant 9$, and smallest eigenvalue $\lambda_{\min}(\Gamma) \in [-3, -2)$. To prove that Γ is 1-integrable when k is sufficiently large, the only case needed to be considered is $c \leqslant k-2$ by Theorem 7. Note that $c \leqslant k-2$ implies directly that $n-k-1 \geqslant 1+(k-c) \geqslant 3$.

A crucial result is as follows:

Theorem 36 ([19, Lemma 5.1] and [18, Theorem 4.1])**.** *There exists a positive constant κ_6 such that, for any connected sesqui-regular graph Γ with parameters (n, k, c), where $n-k-1>2$, and with smallest eigenvalue at least -3, if $k \geqslant \kappa_6$, then there exists a fat Hoffman graph $\mathfrak{h}$ which satisfies the following:*

(i) *$\mathfrak{h}$ has Γ as its slim graph,*

(ii) *$\mathfrak{h}$ has smallest eigenvalue at least -3,*

(iii) *every quasi-clique in $\mathfrak{h}$ is a clique.*

Theorem 37. *Let Γ be a connected sesqui-regular graph with parameters (n, k, c), where $n-k-1>2$, and smallest eigenvalue $\lambda_{\min}(\Gamma) \in [-3, -2)$. Let $\mathfrak{h}$ be a fat Hoffman graph which satisfies the conditions* (i)–(iii) *in Theorem 36. If $k \geqslant 106$ and $k-75 \geqslant c \geqslant 9$, then $\mathfrak{h}$ has an integral reduced representation of norm* 3.

Proof. It is sufficient to prove that every indecomposable factor of $\mathfrak{h}$ has an integral reduced representation of norm 3. Assume that $\mathfrak{g}$ is an indecomposable factor which has no integral reduced representation of norm 3. Then $\mathfrak{g}$ is fat by Lemma 32 (iii), as $\mathfrak{h}$ is fat, and

$$|N_{\mathfrak{g}}^{\text{fat}}(x)| = 1 \text{ for } x \in V_{\text{slim}}(\mathfrak{g}) \tag{19}$$

by Theorem 35. Let $\mathcal{S}(\mathfrak{g})$ be the special graph of $\mathfrak{g}$. Since every quasi-clique in $\mathfrak{h}$ is a clique, for any two non-adjacent slim vertices x and y in $\mathfrak{h}$, $N_{\mathfrak{h}}^{\text{fat}}(x, y) = \emptyset$ holds. In particular, if both x and y are in $\mathfrak{g}$, then $N_{\mathfrak{g}}^{\text{fat}}(x, y) = \emptyset$ also holds. Taking into consideration (19), we have $E^{-}(\mathcal{S}(\mathfrak{g})) = \emptyset$. Hence, we can regard $\mathcal{S}(\mathfrak{g})$ as a graph. The graph $\mathcal{S}(\mathfrak{g})$ is connected, since $\mathfrak{g}$ is indecomposable.

Notice that $Sp(\mathfrak{g}) = A(\mathcal{S}(\mathfrak{g})) - I$, where $A(\mathcal{S}(\mathfrak{g}))$ is the adjacency matrix of $\mathcal{S}(\mathfrak{g})$. Let $\lambda_{\min}(\mathcal{S}(\mathfrak{g}))$ be the smallest eigenvalue of $\mathcal{S}(\mathfrak{g})$. From Lemma 30, we have $\lambda_{\min}(\mathcal{S}(\mathfrak{g})) - 1 = \lambda_{\min}(\mathfrak{g}) \geqslant \lambda_{\min}(\mathfrak{h}) \geqslant -3$, that is, $\lambda_{\min}(\mathcal{S}(\mathfrak{g})) \geqslant -2$. Also, $\mathcal{S}(\mathfrak{g})$ is not a complete graph, as $\mathfrak{g}$ has no integral reduced representation of norm 3. So

$$-2 \leqslant \lambda_{\min}(\mathcal{S}(\mathfrak{g})) < -1. \tag{20}$$

Using the fact that $\mathfrak{g}$ has no integral reduced representation of norm 3 again, we conclude that $\mathcal{S}(\mathfrak{g})$ is not 1-integrable. By Theorem 5, we obtain

$$|V_{\text{slim}}(\mathfrak{g})| = |V(\mathcal{S}(\mathfrak{g}))| \leqslant 36. \tag{21}$$

Claim 38. *Let x and y be two adjacent vertices in $\mathcal{S}(\mathfrak{g})$. There exist two distinct vertices y_1, y_2 $(\neq y)$ in $\mathcal{S}(\mathfrak{g})$ such that both of them are adjacent to x in $\mathcal{S}(\mathfrak{g})$ and satisfy $N_{\mathfrak{g}}^{\mathrm{fat}}(y_1) = N_{\mathfrak{g}}^{\mathrm{fat}}(y_2) = N_{\mathfrak{g}}^{\mathrm{fat}}(y)$ in $\mathfrak{g}$.*

Before showing Claim 38, we finish the proof of Theorem 37 by using this claim. Since the quasi-clique with respect to a fixed fat vertex in $\mathfrak{g}$ is a clique and every slim vertex in $\mathfrak{g}$ has exactly one fat neighbor in $\mathfrak{g}$ by (19), we observe that for any two vertices z_1, z_2 in $\mathcal{S}(\mathfrak{g})$,

$$z_1 \not\sim z_2 \text{ in } \mathcal{S}(\mathfrak{g}) \text{ if } N_{\mathfrak{g}}^{\mathrm{fat}}(z_1) = N_{\mathfrak{g}}^{\mathrm{fat}}(z_2) \text{ in } \mathfrak{g}. \tag{22}$$

Now under the assumption that Claim 38 holds, we easily figure out that $\mathcal{S}(\mathfrak{g})$ contains an induced subgraph with smallest eigenvalue less than -2 as follows. Since $\mathcal{S}(\mathfrak{g})$ is connected, there are two adjacent vertices x and y in $\mathcal{S}(\mathfrak{g})$. By Claim 38, there exist two more vertices y_1 and y_2 which are adjacent to x in $\mathcal{S}(\mathfrak{g})$ and have the same fat neighbor as y in $\mathfrak{g}$. By (22), the three vertices y, y_1, y_2 are pairwise non-adjacent in $\mathcal{S}(\mathfrak{g})$. Since the roles of x and y are symmetric, applying Claim 38 again, we can also find two more vertices x_1 and x_2 which are adjacent to y in $\mathcal{S}(\mathfrak{g})$ and satisfy $N_{\mathfrak{g}}^{\mathrm{fat}}(x_1) = N_{\mathfrak{g}}^{\mathrm{fat}}(x_2) = N_{\mathfrak{g}}^{\mathrm{fat}}(x)$ in $\mathfrak{g}$. Similarly, we have that x, x_1, x_2 are pairwise non-adjacent in $\mathcal{S}(\mathfrak{g})$ by (22).

(i) If either y_1 or y_2 is adjacent to at least one of x_1 and x_2, then $\mathcal{S}(\mathfrak{g})$ will contain one of the graphs in Figure 1

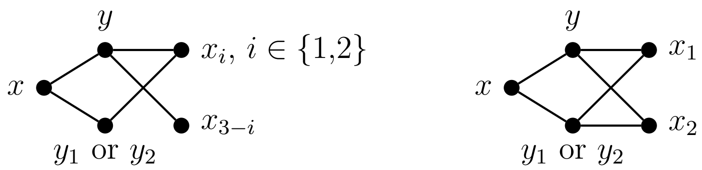

Figure 1: Two graphs with smallest eigenvalue less than -2

which have smallest eigenvalue less than -2 as an induced subgraph.

(ii) If both y_1 and y_2 are adjacent to neither x_1 nor x_2, then applying Claim 38 to the adjacent vertices x and y_1 again, we will find that there are two more vertices $x_{1,1}$ and $x_{1,2}$ which are adjacent to y_1 in $\mathcal{S}(\mathfrak{g})$ and satisfy $N_{\mathfrak{g}}^{\mathrm{fat}}(x_{1,1}) = N_{\mathfrak{g}}^{\mathrm{fat}}(x_{1,2}) = N_{\mathfrak{g}}^{\mathrm{fat}}(x)$ in $\mathfrak{g}$. Note that $x, x_1, x_2, x_{1,1}, x_{1,2}$ have the same fat neighbor in $\mathfrak{g}$. This means that these five vertices are pairwise non-adjacent in $\mathcal{S}(\mathfrak{g})$ by (22). Now if either y or y_2 is adjacent to one of $x_{1,1}$ and $x_{1,2}$, then $\mathcal{S}(\mathfrak{g})$ contains an induced subgraph isomorphic to one of the graphs in Figure 1, which has smallest eigenvalue less than -2. If both y and y_2 are adjacent to neither $x_{1,1}$ nor $x_{1,2}$, then $\mathcal{S}(\mathfrak{g})$ contains the graph in Figure 2 with smallest eigenvalue less than -2 as an induced subgraph.

This contradicts Lemma 11 (i), as $\mathcal{S}(\mathfrak{g})$ has smallest eigenvalue at least -2 by (20).

Hence every indecomposable factor of $\mathfrak{h}$ has an integral reduced representation of norm 3. This completes the proof. □

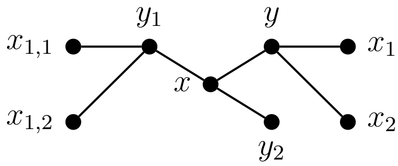

Figure 2: A graph with smallest eigenvalue less than -2

Proof of Claim 38. Assume $N_{\mathfrak{g}}^{\text{fat}}(x) = \{f_1\}$ and $N_{\mathfrak{g}}^{\text{fat}}(y) = \{f_2\}$. Let

$$F = \left(\bigcup_{w \in N_{\mathfrak{g}}^{\text{slim}}(x)} N_{\mathfrak{g}}^{\text{fat}}(w) \right) - \{f_2\}.$$

Considering $|V_{\text{slim}}(\mathfrak{g})| \leqslant 36$ by (21), we have

$$|F| \leqslant |\{w \mid w \in N_{\mathfrak{g}}^{\text{slim}}(x)\}| - 1 \leqslant |V_{\text{slim}}(\mathfrak{g}) - \{x\}| - 1 \leqslant 34 \tag{23}$$

by (19) immediately.

Let $z_1, z_2, z_3 \in V_{\text{slim}}(\mathfrak{h})$ be three vertices. If $|N_{\mathfrak{h}}^{\text{fat}}(z_1) \cap N_{\mathfrak{h}}^{\text{fat}}(z_2) \cap N_{\mathfrak{h}}^{\text{fat}}(z_3)| \geqslant 2$, then z_1, z_2 and z_3 are pairwise adjacent in $\mathfrak{h}$, as they are in the quasi-clique with respect to a fat vertex in $N_{\mathfrak{h}}^{\text{fat}}(z_1) \cap N_{\mathfrak{h}}^{\text{fat}}(z_2) \cap N_{\mathfrak{h}}^{\text{fat}}(z_3)$, which is actually a clique. Thus, the following Hoffman graph

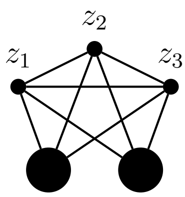

Figure 3: A Hoffman graph with smallest eigenvalue -4

with smallest eigenvalue -4 is an induced Hoffman subgraph of $\mathfrak{h}$, which contradicts $\lambda_{\min}(\mathfrak{h}) \geqslant -3$ by Lemma 30. This yields the following fact:

Fact 1. For any two distinct fat vertices f_3 and f_4 in $V_{\text{fat}}(\mathfrak{h})$, $|N_{\mathfrak{h}}^{\text{slim}}(f_3, f_4)| \leqslant 2$ holds.

From Fact 1 and (23), we obtain

$$\left| \bigcup_{f \in F \cup \{f_1\}} N_{\mathfrak{h}}^{\text{slim}}(f_2, f) \right| \leqslant 2(|F| + 1) \leqslant 70. \tag{24}$$

There is one more essential fact following from Lemma 32 (iv) and (19):

Fact 2. Let $w_1 \in V_{\text{slim}}(\mathfrak{g})$ and $w_2 \in V_{\text{slim}}(\mathfrak{h}) - V_{\text{slim}}(\mathfrak{g})$ be two slim vertices and let $N_{\mathfrak{g}}^{\text{fat}}(w_1) = \{f\}$. Then w_1 and w_2 are adjacent in $\mathfrak{h}$ if and only if $w_2 \in N_{\mathfrak{h}}^{\text{slim}}(f)$.

We look at the set $N_{\mathfrak{h}}^{\text{slim}}(y)$. By Fact 2, a vertex $w_2 \in V_{\text{slim}}(\mathfrak{h}) - V_{\text{slim}}(\mathfrak{g})$ is in the set $N_{\mathfrak{h}}^{\text{slim}}(y)$ if and only if $w_2 \in N_{\mathfrak{h}}^{\text{slim}}(f_2)$. Then $k = |N_{\mathfrak{h}}^{\text{slim}}(y)| = |N_{\mathfrak{g}}^{\text{slim}}(y)| +$

$|N_{\mathfrak{h}}^{\mathrm{slim}}(y) - N_{\mathfrak{g}}^{\mathrm{slim}}(y)| = |N_{\mathfrak{g}}^{\mathrm{slim}}(y)| + |\bigcup_{f\in F\cup\{f_1\}} N_{\mathfrak{h}}^{\mathrm{slim}}(f_2, f)| + |N_{\mathfrak{h}}^{\mathrm{slim}}(f_2) - N_{\mathfrak{g}}^{\mathrm{slim}}(f_2) - \bigcup_{f\in F\cup\{f_1\}} N_{\mathfrak{h}}^{\mathrm{slim}}(f_2, f)| \geqslant 106$. From (21) and (24), we easily obtain $|N_{\mathfrak{g}}^{\mathrm{slim}}(y)| \leqslant 35$ and $|\bigcup_{f\in F\cup\{f_1\}} N_{\mathfrak{h}}^{\mathrm{slim}}(f_2, f)| \leqslant 70$. Thus, $|N_{\mathfrak{h}}^{\mathrm{slim}}(f_2) - N_{\mathfrak{g}}^{\mathrm{slim}}(f_2) - \bigcup_{f\in F\cup\{f_1\}} N_{\mathfrak{h}}^{\mathrm{slim}}(f_2, f)| \geqslant 1$, and we are able to find a vertex z in $N_{\mathfrak{h}}^{\mathrm{slim}}(f_2) - N_{\mathfrak{g}}^{\mathrm{slim}}(f_2) - \bigcup_{f\in F\cup\{f_1\}} N_{\mathfrak{h}}^{\mathrm{slim}}(f_2, f)$. As $z \notin N_{\mathfrak{h}}^{\mathrm{slim}}(f_1)$, we have $z \notin N_{\mathfrak{h}}^{\mathrm{slim}}(x)$ by Fact 2. Notice that both x and z are adjacent to y. The distance between x and z in the slim graph Γ of $\mathfrak{h}$ is 2, and thus,

$$9 \leqslant c = |N_{\mathfrak{h}}^{\mathrm{slim}}(x, z)| = |(V_{\mathrm{slim}}(\mathfrak{h}) - V_{\mathrm{slim}}(\mathfrak{g})) \cap N_{\mathfrak{h}}^{\mathrm{slim}}(x, z)| + |V_{\mathrm{slim}}(\mathfrak{g}) \cap N_{\mathfrak{h}}^{\mathrm{slim}}(x, z)|. \quad (25)$$

By Fact 2, we have $(V_{\mathrm{slim}}(\mathfrak{h}) - V_{\mathrm{slim}}(\mathfrak{g})) \cap N_{\mathfrak{h}}^{\mathrm{slim}}(x, z) = (V_{\mathrm{slim}}(\mathfrak{h}) - V_{\mathrm{slim}}(\mathfrak{g})) \cap N_{\mathfrak{h}}^{\mathrm{slim}}(z, f_1)$. For any $w_1 \in V_{\mathrm{slim}}(\mathfrak{g})$, if $w_1 \in N_{\mathfrak{h}}^{\mathrm{slim}}(f_1)$, then $N_{\mathfrak{h}}^{\mathrm{fat}}(w_1) = \{f_1\}$ by (19). Furthermore, w_1 and z are not adjacent in $\mathfrak{h}$, as $z \notin N_{\mathfrak{h}}^{\mathrm{slim}}(f_1)$. In other words, $w_1 \notin N_{\mathfrak{h}}^{\mathrm{slim}}(z)$. This means $(V_{\mathrm{slim}}(\mathfrak{h}) - V_{\mathrm{slim}}(\mathfrak{g})) \cap N_{\mathfrak{h}}^{\mathrm{slim}}(z, f_1) = N_{\mathfrak{h}}^{\mathrm{slim}}(z, f_1)$.

Let $W = N_{\mathfrak{h}}^{\mathrm{slim}}(z, f_1)$. We will show $|W| \leqslant 6$. Without loss of generality, we may assume that $W \neq \emptyset$. Since the quasi-clique with respect to the fat vertex f_1 in $\mathfrak{h}$ is a clique, we have $N_{\mathfrak{g}}^{\mathrm{slim}}(f_1) - \{x\} \subset N_{\mathfrak{h}}^{\mathrm{slim}}(x)$. Also by Fact 2, we have $N_{\mathfrak{h}}^{\mathrm{slim}}(f_1) - N_{\mathfrak{g}}^{\mathrm{slim}}(f_1) = N_{\mathfrak{h}}^{\mathrm{slim}}(x) - N_{\mathfrak{g}}^{\mathrm{slim}}(x)$. It follows that

$$\begin{aligned}
|N_{\mathfrak{h}}^{\mathrm{slim}}(f_1)| &= |(N_{\mathfrak{h}}^{\mathrm{slim}}(f_1) - N_{\mathfrak{g}}^{\mathrm{slim}}(f_1)) \cup N_{\mathfrak{g}}^{\mathrm{slim}}(f_1)| \\
&= |(N_{\mathfrak{h}}^{\mathrm{slim}}(x) - N_{\mathfrak{g}}^{\mathrm{slim}}(x)) \cup N_{\mathfrak{g}}^{\mathrm{slim}}(f_1)| \\
&= |(N_{\mathfrak{h}}^{\mathrm{slim}}(x) \cup N_{\mathfrak{g}}^{\mathrm{slim}}(f_1)) - (N_{\mathfrak{g}}^{\mathrm{slim}}(x) - N_{\mathfrak{g}}^{\mathrm{slim}}(f_1))| \\
&\geqslant |N_{\mathfrak{h}}^{\mathrm{slim}}(x) \cup \{x\}| - |N_{\mathfrak{g}}^{\mathrm{slim}}(x)| \\
&\geqslant k + 1 - 35 = k - 34 \geqslant c + 41.
\end{aligned}$$

As $|W| \leqslant |N_{\mathfrak{h}}^{\mathrm{slim}}(x, z)| \leqslant c$, there exists $W' \subset N_{\mathfrak{h}}^{\mathrm{slim}}(f_1) - W$ such that $|W'| = 41$. Using the fact that the quasi-clique with respect to f_1 in $\mathfrak{h}$ is a clique again, we obtain that the quotient matrix of the subgraph of Γ induced on $\{z\} \cup W \cup W'$ relative to the partition $\{\{z\}, W, W'\}$ is the matrix

$$M := \begin{pmatrix} 0 & |W| & 0 \\ 1 & |W| - 1 & 41 \\ 0 & |W| & 40 \end{pmatrix}.$$

By Lemma 11 (ii) and Lemma 11 (i), the smallest eigenvalue of M is at least -3. Thus $\det(M + 3I) = -37|W| + 258 \geqslant 0$, that is, $|W| \leqslant 6$. By (25), we obtain

$$|V_{\mathrm{slim}}(\mathfrak{g}) \cap N_{\mathfrak{h}}^{\mathrm{slim}}(x, z)| \geqslant 3. \quad (26)$$

For any $w \in V_{\mathrm{slim}}(\mathfrak{g})$, if $w \in N_{\mathfrak{h}}^{\mathrm{slim}}(x, z)$, then $N_{\mathfrak{h}}^{\mathrm{fat}}(w) \subseteq N_{\mathfrak{h}}^{\mathrm{fat}}(z)$ by Fact 2, as w and z are adjacent in $\mathfrak{h}$. If $N_{\mathfrak{h}}^{\mathrm{fat}}(w) \neq \{f_2\}$, then $z \in \bigcup_{f\in F\cup\{f_1\}} N_{\mathfrak{h}}^{\mathrm{slim}}(f_2, f)$, which contradicts the choice of z. So $N_{\mathfrak{h}}^{\mathrm{fat}}(w) = \{f_2\}$. This leads to $V_{\mathrm{slim}}(\mathfrak{g}) \cap N_{\mathfrak{h}}^{\mathrm{slim}}(x, z) = V_{\mathrm{slim}}(\mathfrak{g}) \cap N_{\mathfrak{g}}^{\mathrm{slim}}(x, f_2)$. By (26), $|V_{\mathrm{slim}}(\mathfrak{g}) \cap N_{\mathfrak{g}}^{\mathrm{slim}}(x, f_2)| \geqslant 3$ follows.

Our claim follows immediately, if we choose two distinct vertices other than y in the set $V_{\mathrm{slim}}(\mathfrak{g}) \cap N_{\mathfrak{g}}^{\mathrm{slim}}(x, f_2)$. This completes the proof of Claim 38. □

Proof of Theorem 8. By Theorem 7, only the case where $c \leqslant k-2$ should be considered. As $k-c \geqslant 2$, $n-k-1>2$ follows. Now let $\kappa_5 = \max\{\kappa_2(3), \kappa_6, 106\}$, where $\kappa_2(3)$ and κ_6 are such that Theorem 4 and Theorem 36 hold respectively.

Since $k \geqslant \kappa_6$ and $n-k-1>2$, Theorem 36 yields a fat Hoffman graph $\mathfrak{h}$ satisfying conditions (i)–(iii) therein. Moreover, since $k \geqslant \kappa_2(3)$, Theorem 4 with $\lambda = 3$ implies

$$c \leqslant 3^2(3-1) = 18,$$

because its alternative $n-k-1 \leqslant 2$ does not hold. As $k \geqslant 106$, we have

$$c \leqslant 18 \leqslant 31 \leqslant k-75.$$

Hence Theorem 37 shows that $\mathfrak{h}$ has an integral reduced representation of norm 3. By Lemma 34, $\mathfrak{h}$ also has an integral representation of norm 3, and therefore its slim graph Γ is 1-integrable. □

Proof of Theorem 1. Let κ_1 be the integer κ_5 which is such that Theorem 8 holds. It follows from Theorem 7, Theorem 9 and Theorem 8 immediately. □

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References

[1] A. E. Brouwer and W. H. Haemers. *Spectra of graphs.* Springer, New York, 2012.

[2] A. E. Brouwer and H. Van Maldeghem. *Strongly regular graphs.* Cambridge University Press, Cambridge, 2022.

[3] P. J. Cameron, J. M. Goethals, J. J. Seidel, and E. E. Shult. Line graphs, root systems, and elliptic geometry. *J. Algebra*, 43:305–327, 1976.

[4] S. M. Cioabă, J. H. Koolen, and W. Li. Disconnecting strongly regular graphs. *European J. Combin.*, 38:1–11, 2014.

[5] J. H. Conway and N. J. A. Sloane. Low-dimensional lattices V. integral coordinates for integral lattices. *Proc. Roy. Soc. London Ser. A*, 426:211–232, 1989.

[6] D. Cvetković, P. Rowlinson, and S. K. Simić. *Spectral Generalizations of Line Graphs: On Graphs with Least Eigenvalue* -2. Cambridge Univ. Press, Cambridge, 2004.

[7] H.-J. Ge and J. H. Koolen. On co-edge-regular graphs with 4 distinct eigenvalues. *Electron. J. Combin.*, 32(3):#P3.55, 2025.

[8] N. I. Gillespie. Equiangular lines, incoherent sets and quasi-symmetric designs. arXiv:1809.05739, 2018.

[9] C. D. Godsil and G. Royle. *Algebraic Graph Theory*. Springer-Verlag, New York, 2001.

[10] J. M. Goethals and J. J. Seidel. The regular two-graph on 276 vertices. *Discrete Math.*, 12:143–158, 1975.

[11] A. J. Hoffman and D. K. Ray Chaudhuri. On a spectral characterization of regular line graphs. unpublished manuscript.

[12] H. J. Jang, J. H. Koolen, A. Munemasa, and T. Taniguchi. On fat Hoffman graphs with smallest eigenvalue at least -3. *Ars Math. Contemp.*, 7:105–121, 2014.

[13] T. P. Kirkman. On a problem in combinations. *Cambridge Dublin Math. J.*, 2:191–204, 1847.

[14] J. Koolen, C. H. Lv, G. Markowsky, and J. Y. Park. An improved bound for strongly regular graphs with smallest eigenvalue $-m$. arXiv:2506.04964, 2025.

[15] J. H. Koolen, M.-Y. Cao, and Q. Yang. Recent progress on graphs with fixed smallest adjacency eigenvalue: A survey. *Graphs Combin.*, 37:1139–1178, 2021.

[16] J. H. Koolen, H.-J. Ge, C. H. Lv, and Q. Yang. A Bose-Laskar-Hoffman theory for μ-bounded graphs with fixed smallest eigenvalue. *European J. Combin.*, 136:104378, 2026.

[17] J. H. Koolen and B. Gebremichel. There does not exist a strongly regular graph with parameters $(1911, 270, 105, 27)$. *Electron. J. Combin.*, 29(2):#P2.3, 2022.

[18] J. H. Koolen, B. Gebremichel, J. Y. Yang, and Q. Yang. Sesqui-regular graphs with fixed smallest eigenvalue. *Linear Algebra Appl.*, 668:1–10, 2023.

[19] J. H. Koolen, J. Y. Yang, and Q. Yang. On graphs with smallest eigenvalue at least -3 and their lattices. *Adv. Math.*, 338:847–864, 2018.

[20] J. H. Koolen, Q. Yang, and J. Y. Yang. A generalization of a theorem of Hoffman. *J. Combin. Theory Ser. B*, 135:75–95, 2019.

[21] A. W. Marcus, D. A. Spielman, and N. Srivastava. Interlacing families I: Bipartite Ramanujan graphs of all degrees. *Ann. of Math. (2)*, 182(1):307–325, 2015.

[22] A. Neumaier. Strongly regular graphs with smallest eigenvalue $-m$. *Arch. Math. (Basel)*, 33(4):392–400, 1979.

[23] Z. Qiao and Y. Pan. A note on optimistic strongly regular graphs. *J. University of Science and Technology of China*, 47(3):197–203, 2017.

[24] E. R. van Dam, H.-J. Ge, and J. H. Koolen. Co-edge-regular graphs with four eigenvalues and unbounded coherent rank. arXiv:2606.19981, 2026.

[25] R. Woo and A. Neumaier. On graphs whose smallest eigenvalue is at least $-1-\sqrt{2}$. *Linear Algebra Appl.*, 226–228:577–591, 1995.