Blow-ups and extensions of trees in tournaments

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Abstract

A class of acyclic digraphs $\mathcal{C}$ is linearly unavoidable if there exists a constant c such that every digraph $D \in \mathcal{C}$ is contained in all tournaments of order $c \cdot |V(D)|$. The class of all acyclic digraphs is not linearly unavoidable, and Fox, He, and Wigderson recently showed that this is not even the case for acyclic digraphs with bounded maximum degree. On the positive side, Häggkvist and Thomason proved that the class of oriented trees is linearly unavoidable. In this work, we generalize this result to acyclic digraphs obtained from an oriented tree by adding at most k vertices, and k-blow-ups of oriented trees, for every fixed integer k.

More precisely, we show that if D is obtained from an oriented tree F of sufficiently large order n by adding k universal vertices, then D is contained in all tournaments on $2 \cdot 3^{(k+1)(2k+1)} \cdot n$ vertices; and if D is obtained from F by replacing each vertex u by an independent set X_u of size k and every arc uv by all possible arcs from X_u to X_v, then D is contained in every tournament on $2^{10+18k} k \cdot n$ vertices.

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1 Introduction

A digraph is ***p*-unavoidable**, for an integer p, if it is contained in every tournament of order p. For a digraph D, the **unavoidability** of D, denoted by $\mathrm{unvd}(D)$, is the smallest integer p such that D is p-unavoidable, if such a p exists. It is easy and well-known that a digraph is p-unavoidable for some integer p if and only if it is acyclic. Indeed, every digraph containing a directed cycle is not contained in any transitive tournament, and so is not unavoidable. Moreover, as observed by Erdős and Moser [8], an immediate

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induction and Proposition 1 shows that TT_n, the transitive tournament of order n, is 2^{n-1}-unavoidable and thus every acyclic digraph of order n is 2^{n-1}-unavoidable.

Proposition 1. *Let D be an acyclic digraph. For every source or sink x of D,*

$$\operatorname{unvd}(D) \leqslant 2 \operatorname{unvd}(D - x).$$

This is somehow tight in the sense that any upper bound on the unavoidability of acyclic digraphs as a function of their order must be exponential. Indeed, considering a random tournament and using the First Moment Method, Erdős and Moser [8] established $\operatorname{unvd}(TT_n) \geqslant 2^{(n-1)/2}$. Since this seminal paper, only little progress has been made on the unavoidability of transitive tournaments. Using the Local Lemma, one can slightly improve on the lower bound and show that $\operatorname{unvd}(TT_n) \geqslant 2^{(n+1)/2}$ when n is large enough. The unavoidability of small transitive tournaments has been established. Clearly $\operatorname{unvd}(TT_1) = 1$, $\operatorname{unvd}(TT_2) = 2$, $\operatorname{unvd}(TT_3) = 4$ (because of the directed 3-cycle), and $\operatorname{unvd}(TT_4) = 8$ (because of the Paley tournament on 7 vertices). Reid and Parker [23] showed $\operatorname{unvd}(TT_5) = 14$, $\operatorname{unvd}(TT_6) = 28$, and Sanchez-Flores [26] proved $\operatorname{unvd}(TT_7) = 54$. This result and Proposition 1 imply $\operatorname{unvd}(TT_n) \leqslant 54 \times 2^{n-7}$ for all $n \geqslant 7$.

The **average degree** of a nonempty digraph D is the average degree of its underlying graph, that is $2\frac{|A(D)|}{|V(D)|}$. The **maximum average degree** of a digraph is the maximum average degree over all its nonempty subdigraphs. Similarly to Erdős and Moser [8], one can show that the unavoidability of an acyclic digraph is exponential in its maximum average degree.

Proposition 2. *Let D be an acyclic digraph with maximum average degree α. Then $\operatorname{unvd}(D) > 2^{\alpha/2}$.*

Proof. Consider a random tournament T on $p \leqslant 2^{\alpha/2}$ vertices. Let $v_1, \ldots, v_n$ be a labelling of the vertices of a subdigraph H of D with average degree α. For each ordered n-tuple $(w_1, \ldots, w_n)$ of distinct vertices in T, the probability that $T\langle\{w_1, \ldots, w_n\}\rangle$ contains a labelled copy of H (*i.e.* such that $w_i w_j$ is an arc in T whenever $v_i v_j$ is an arc in H) is $\left(\frac{1}{2}\right)^{|A(H)|} = \left(\frac{1}{2}\right)^{\alpha n/2}$. Thus the expected number of labelled copies of H is $\frac{p!}{(p-n)!}\left(\frac{1}{2}\right)^{\alpha n/2} < \frac{p^n}{2^{\alpha n/2}} = \left(\frac{p}{2^{\alpha/2}}\right)^n \leqslant 1$. $\square$

Proposition 2 gives a non-trivial lower bound on the unavoidability of dense acyclic digraphs, that is with maximum average degree larger than logarithmic in their order. However, smaller bounds on the unavoidability are expected for sparse digraphs. In particular, one can ask which families $\mathcal{F}$ of digraphs are **linearly unavoidable**, that is such that there exists a constant C such that $\operatorname{unvd}(D) \leqslant C \cdot |V(D)|$ for every digraph D in $\mathcal{F}$, and which are **polynomially unavoidable**, that is such that there exists a polynomial P such that $\operatorname{unvd}(D) \leqslant P(|V(D)|)$ for every digraph D in $\mathcal{F}$.

Note that, by Proposition 2, the digraphs D of a linearly unavoidable family must have maximum average degree at most $2\log(|V(D)|) + \mathcal{O}(1)$. For every integer n with $n \geqslant 2$, Linial, Saks, and Sós [19] constructed an n-unavoidable digraph with n vertices and $n \log n - C \cdot n \log \log n$ arcs for some fixed constant C. Those digraphs form a linearly unavoidable family with logarithmic maximum average degree.

Several other classes of acyclic digraphs have been proved to be linearly unavoidable. Much attention has been devoted to **oriented paths**, **oriented trees**, and **oriented forests** which are orientations of paths, trees, and forests respectively. It started with Rédei's theorem [22] which states that the unavoidability of $\vec{P}_n$ is exactly n, where $\vec{P}_n$ is the **directed path** on n vertices, that is the oriented path with exactly one source and one sink.

Theorem 3 (Rédei [22])**.** *Every tournament has a Hamiltonian directed path.*

In 1971, Rosenfeld [25] conjectured that there is an integer $N > 7$ such that $\mathrm{unvd}(P) = |P|$ for every oriented path of order at least N. Several papers gave partial answers to this conjecture [11, 1, 9, 27] until Rosenfeld's conjecture was verified by Thomason, who proved in [28] Rosenfeld's conjecture for $N = 2^{128}$. Finally, Havet and Thomassé [15] showed that $\mathrm{unvd}(P) = |V(P)|$ for every oriented path P except the antidirected paths of order 3, 5, and 7.

Regarding oriented trees, Sumner (see [24]) made the following celebrated conjecture.

Conjecture 4 (Sumner)**.** Every oriented tree of order $n > 1$ is $(2n-2)$-unavoidable.

Since every forest is the subdigraph of a tree, this conjecture is equivalent to the analogue for oriented forests. If true, Sumner's conjecture would be tight. Indeed, the **out-star** S_n^+, which is the digraph on n vertices consisting of a vertex dominating the $n-1$ others, is not contained in the regular tournaments of order $2n-3$. The first linear upper bound was given by Häggkvist and Thomason [12]. Following improvements of Havet [13], Havet and Thomassé [14], and El Sahili [7], Dross and Havet [6] used the notion of median order to prove that every oriented tree of order $n \geqslant 2$ is $\lceil \frac{21}{8}n - \frac{47}{16} \rceil$-unavoidable. Kühn, Mycroft and Osthus [16] proved that Sumner's conjecture is true for all sufficiently large n.

Theorem 5 (Kühn, Mycroft, and Osthus [16])**.** *There is an integer n_0 such that for every $n \geqslant n_0$, every oriented tree of order n is $(2n-2)$-unavoidable.*

Motivated by those results and a work of Lee [18], Bucić, Letzter and Sudakov [3] asked whether the digraphs with bounded maximum average degree are linearly unavoidable. Draganić *et al.* [5] proved that this is the case for k-th powers of directed paths. The ***k*-th power** of a digraph D is the digraph, denoted by D^k, with vertex set $V(D)$ in which uv is an arc if and only if $\mathrm{dist}_D(u,v) \leqslant k$. Fox, He, and Wigderson [10] showed that this is not the case in general: they proved that for any $\Delta \geqslant 2$ and any sufficiently large n, there exists an acyclic digraph D with n vertices and maximum degree Δ such that

$$\mathrm{unvd}(D) \geqslant n^{\Omega(\Delta^{2/3}/\log^{5/3}\Delta)}.$$

This result raises the two following questions. Firstly, are the digraphs with bounded maximum average degree polynomially unavoidable?

Problem 6. Let α be a positive real number. Does there exist a polynomial P_α such that $\mathrm{unvd}(D) \leqslant P_\alpha(|V(D)|)$ for every acyclic digraph with maximum average degree at most α?

Secondly, which families of acyclic digraphs (with bounded maximum average degree) are linearly unavoidable? In this direction, Fox, He, and Wigderson [10] showed that for every acyclic digraph D of maximum degree Δ, if there is a partition $S_1, \dots, S_h$ of $V(D)$ such that for every arc uv of D, there exists $i \in [h-1]$ such that $u \in S_i$ and $v \in S_{i+1}$, then $\mathrm{unvd}(D) \leqslant h^{10\Delta \log \Delta} \cdot |V(D)|$. This was later improved to $\mathrm{unvd}(D) \leqslant C^\Delta \cdot |V(D)|$ for some absolute constant C by Morawski and Wigderson [20].

Our results

In this paper, we give some more partial answers to the above second question by proving some new families of acyclic digraphs to be linearly unavoidable. We are in particular interested in operations to form linearly unavoidable families from others.

Let k be a positive integer. The ***k*-blow-up** of a digraph D, denoted by $D[k]$, is obtained by blowing-up each vertex into an independent set of size k. Formally, $D[k]$ is defined by

$$\begin{aligned} V(D[k]) &= \{(v,i) \mid v \in V(D), i \in [k]\}, \text{ and} \\ A(D[k]) &= \{(u,i)(v,j) \mid uv \in A(D), i \in [k], j \in [k]\}. \end{aligned}$$

We believe that if a family $\mathcal{F}$ is linearly unavoidable, then the family $\mathcal{F}[k] = \{D[k] \mid D \in \mathcal{F}\}$ of its k-blow-ups is also linearly unavoidable.

Conjecture 7. If $\mathcal{F}$ is linearly unavoidable and has bounded maximum average degree, then $\mathcal{F}[k]$ is also linearly unavoidable.

Observe first that the bounded maximum average degree condition in the above conjecture is necessary. Indeed, for every n, let D_n be the digraph of order n which is the disjoint union of a transitive tournament of order $\lfloor \log n \rfloor + 1$ and $n - \lfloor \log n \rfloor - 1$ isolated vertices. Each D_n is clearly n-unavoidable, but $D_n[k]$ has maximum average degree $k\lfloor \log n \rfloor$ and so $\mathrm{unvd}(D_n) \geqslant 2^{k\lfloor \log n \rfloor/2} \geqslant \frac{1}{2} n^{k/2}$ by Proposition 2.

Observe that the k-blow-up of the directed path of order n is contained in the $2k$-th power of the directed path of order kn. Thus, the result of Draganić *et al.* [5] implies Conjecture 7 for directed paths. As a more general evidence to this conjecture, we show in Section 3 that it holds for oriented trees (and thus more generally oriented forests). Precisely, we prove in Theorem 18 that the k-blow-up of an oriented tree of order n is $(2^{10+18k} k \cdot n)$-unavoidable. This upper bound is certainly not tight. However, the maximum average degree of the k-blow-up of a tree of order n is $2k - 2k/n$. Therefore, by Proposition 2, the optimal bound is of the form $2^{\Theta(k)} k \cdot n$. Hence we are left with the following question.

Problem 8. What is the infimum of all the constants C such that for every large enough integer n, the k-blow-up of every oriented tree of order n is $(C^k k \cdot n)$-unavoidable?

Generalizing Proposition 1, we believe that adding a vertex cannot affect too much the unavoidability, in a sense that it can at most multiply it by a constant factor.

Conjecture 9. There exists a constant C such that $\mathrm{unvd}(D) \leqslant C \cdot \mathrm{unvd}(D-v)$ for every acyclic digraph D and for every vertex v of D.

Let A be an acyclic digraph and let k be a positive integer. An acyclic digraph D is a ***k*-extension** of A if there exists a set S of k vertices such that $D - S = A$. The set S is the **added set** and its vertices are the **added vertices**. A vertex is **universal** in a digraph D if $N^-(v) \cup N^+(v) \cup \{v\} = V(D)$. A k-extension D of A is **full**, if its added vertices are universal in D. A k-extension D of A is a ***k*-twin extension**, if all added vertices have the same in- and out-neighbourhood in $V(D-S)$. Conjecture 9, together with Conjecture 4, implies the following.

Conjecture 10. There is an absolute constant C such that for every integer k, every k-extension of an oriented tree of order n is $(C^k(2n-2))$-unavoidable.

More generally, Conjecture 9 would imply the following.

Conjecture 11. If $\mathcal{F}$ is linearly unavoidable, then the family of k-extensions of digraphs in $\mathcal{F}$ is also linearly unavoidable.

In Section 4, we prove this last conjecture for the family of oriented trees (and thus also of oriented forest). More precisely, we show in Corollary 27 that every k-extension of a sufficiently large tree F is $\left(2 \cdot 3^{\binom{2k+2}{2}} \cdot |V(F)|\right)$-unavoidable.

This upper bound on the unavoidability of extension of forests is certainly not tight. In Proposition 30, we show that the unavoidability of some k-extensions of the arcless digraph on n vertices is $2^k \cdot n - o(n)$ as $n \to +\infty$. Hence we are left with the following analogue to Problem 8.

Problem 12. What is the minimum function f such that every k-extension of every forest of order n is $(2^{f(k)} \cdot n)$-unavoidable?

The above-mentioned results show that the function f is at least linear and at most quadratic. The same question applies to any subfamily, starting with that of empty digraphs. Generally, determining the precise value or good approximations of the unavoidability of k-extensions of oriented trees would be interesting. In Section 5, we show an almost tight upper bound on the unavoidability of full k-twin extensions and the exact unavoidability of the full 1-extensions of directed paths.

2 Preliminaries

2.1 Basic properties of tournaments

We will make use of the following folklore properties of tournaments.

Lemma 13 ([2, Proposition 2.2.2])**.** *Let k be a positive integer. Every tournament has at most $2k-1$ vertices of out-degree less than k.*

The **out-section** of a vertex v in a tournament T, denoted by $S_T^+(v)$ or simply $S^+(v)$, is the set of vertices that can be reached from v by a directed path. Similarly, the **in-section** of a vertex v in a tournament T, denoted by $S_T^-(v)$ or simply $S^-(v)$ is the set of vertices from which v can be reached by a directed path. We set $s_T^+(v) = |S_T^+(v)|$ and $s_T^-(v) = |S_T^-(v)|$. An **out-generator** (resp. **in-generator**) of T is a vertex v such that $s_T^+(v) = |V(T)|$ (resp. $s_T^-(v) = |V(T)|$).

Lemma 14 ([2, Proposition 2.7.6])**.** *Let T be a tournament and v a vertex of T. In T, v is the initial (resp. terminal) vertex of a directed path of order $s^+(v)$ (resp. $s^-(v)$). In particular, if v is an out-generator (resp. in-generator) of T, then v is the initial vertex (resp. terminal vertex) of a directed Hamiltonian path.*

Let T be a tournament, and A and B be two disjoint sets of vertices of T. If every vertex of A dominates every vertex of B, then we write $A \Rightarrow B$.

Lemma 15. *Let T be a tournament and let O be the set of vertices that are initial vertices of a directed path of order k in T. Then $O \Rightarrow V(T) \setminus O$.*

Proof. Suppose for contradiction that there exists $u \in V(T) \setminus O$ and $v \in O$ such that $uv \in A(T)$. Since $v \in O$, there is a path P of order k in T with initial vertex v. Hence $s^+(u) \geqslant |V(P)| \geqslant k$ and so u is the initial vertex of a directed path of order k by Lemma 14. □

2.2 Median order

The notion of median order has been introduced in [14], see also [2, Chapter 2] and [5]. A **median order** of a digraph D is a linear order $(v_1, v_2, \dots, v_n)$ of its vertex set such that $|\{(v_i, v_j) \mid i < j\}|$, the number of **forward arcs** (*i.e.* directed from left to right), is as large as possible. Note that when D is acyclic, then a median order is an **acyclic ordering** in which all arcs are forward. Let us first note some basic properties of median orders of tournaments.

Lemma 16. *Let T be a tournament and $(v_1, v_2, \dots, v_n)$ a median order of T. Then, for any two indices i, j with $1 \leqslant i < j \leqslant n$:*

(M1) the suborder $(v_i, v_{i+1}, \dots, v_j)$ is a median order of the induced subtournament

$$T\langle\{v_i, v_{i+1}, \dots, v_j\}\rangle;$$

(M2) v_i dominates at least half of the vertices $v_{i+1}, v_{i+2}, \dots, v_j$, and v_j is dominated by at least half of the vertices $v_i, v_{i+1}, \dots, v_{j-1}$, in particular v_i dominates v_{i+1};

(M3) if $v_{i+2} \to v_i$, then $v_{i-1} \Rightarrow \{v_i, v_{i+1}, v_{i+2}\}$ when $i \geqslant 2$, and $\{v_i, v_{i+1}, v_{i+2}\} \Rightarrow v_{i+3}$ when $i \leqslant n-3$;

(M4) *if $v_n \to v_1$, then there exists ℓ such that $v_1 \to v_\ell \to v_n$; moreover, if*

$$|\{\ell \mid v_1 \to v_\ell \to v_n\}| = 1,$$

then both $(v_2, \dots, v_n, v_1)$ and $(v_n, v_1, \dots, v_{n-1})$ are median orders of T;

(M5) *if $(v'_i, \dots, v'_j)$ is a median order of $T\langle\{v_i, v_{i+1}, \dots, v_j\}\rangle$, then*

$$(v_1, \dots, v_{i-1}, v'_i, \dots, v'_j, v_{j+1}, \dots, v_n)$$

is also a median order of T.

Proof. Items (M1), (M2), and (M5) follow easily from the definition.

(M3) Assume for a contradiction that $v_{i+2} \to v_i$ but there is an arc from v_i, v_{i+1}, v_{i+2} to v_{i-1}. By (M2) there is exactly one such arc and its tail u belongs to $\{v_{i+1}, v_{i+2}\}$. If $u = v_{i+2}$, then v_{i+2} has two out-neighbours in $\{v_{i-1}, v_i, v_{i+1}\}$, contradicting (M2). If $u = v_{i+1}$, then $(v_{i+1}, v_{i-1}, v_{i+2}, v_i)$ has less forward arcs than $(v_{i-1}, v_i, v_{i+1}, v_{i+2})$, a contradiction to (M1). By directional duality, we also have $\{v_i, v_{i+1}, v_{i+2}\} \Rightarrow v_{i+3}$.

(M4) Suppose that $v_n \to v_1$. By (M2), v_1 dominates at least half of $v_2, \dots, v_n$, and so more than half of $v_2, \dots, v_{n-1}$. Symmetrically, v_n is dominated by more than half of $v_2, \dots, v_{n-1}$. Hence there exists $\ell \in \{2, \dots, n-1\}$ such that $v_1 \to v_\ell \to v_n$. Now suppose that this ℓ is unique. Hence v_1 dominates exactly $\frac{n-1}{2}$ vertices among $v_2, \dots, v_n$. It follows that the number of forward arcs in $(v_2, \dots, v_n, v_1)$ is

$$|\{(v_i, v_j) \mid 1 \leqslant i < j \leqslant n\}| - d^+(v_1) + d^-(v_1) = |\{(v_i, v_j) \mid 1 \leqslant i < j \leqslant n\}|,$$

which is maximum since $(v_1, \dots, v_n)$ is a median order. This proves that $(v_2, \dots, v_n, v_1)$ is a median order. Symmetrically, $(v_n, v_1, \dots, v_{n-1})$ is a median order too. $\square$

2.3 A Kővári-Sós-Turán-like lemma

The following lemma is inspired by the classical result by Kővári, Sós, and Turán [17], and a similar one was recently used in [5].

Lemma 17. *Let α, ϵ be two positive reals and k a positive integer. Let (A, B, E) be a bipartite graph with $|A| \geqslant \frac{k}{\frac{1}{2}-\epsilon}$ and $\alpha \leqslant \frac{\binom{(\frac{1}{2}-\epsilon)|A|}{k}}{\binom{|A|}{k}}$, such that $d(v) \geqslant (\frac{1}{2} - \epsilon)|B|$ for every $v \in A$. Then there exist $A^\star \subseteq A$ and $B^\star \subseteq B$ such that*

1. *$|A^\star| \geqslant k$,*
2. *$|B^\star| \geqslant \alpha|B|$, and*
3. *$A^\star$ is complete to $B^\star$.*

Proof. Suppose for contradiction that the result does not hold. We will count by two different methods the number M of pairs (X, b) with $X \subseteq A$ of size k and $b \in B$ such that $X \subseteq N(b)$. On the one hand, for every $X \subseteq A$ of size k, there are less than $\alpha|B|$ vertices b in B such that $X \subseteq N(b)$. We thus have

$$M < \binom{|A|}{k}\alpha|B|.$$

On the other hand, by Jensen's inequality, we have

$$\begin{aligned} M = \sum_{b\in B}\binom{d(b)}{k} &\geqslant |B|\binom{\frac{1}{|B|}\sum_{b\in B} d(b)}{k} = |B|\binom{\frac{1}{|B|}\sum_{a\in A} d(a)}{k} \\ &\geqslant |B|\binom{\frac{|A|}{|B|}(\frac{1}{2}-\epsilon)|B|}{k} \geqslant \binom{|A|}{k}\alpha|B|. \end{aligned}$$

This contradiction proves the lemma. $\square$

Note that $\alpha = 2^{-\frac{k}{\frac{1}{2}-\epsilon}}$ satisfies the hypothesis of Lemma 17. Indeed, simple computations show that $\frac{\binom{(\frac{1}{2}-\epsilon)|A|}{k}}{\binom{|A|}{k}}$ is a non-increasing function of $|A|$, and as $|A| \geqslant \frac{k}{\frac{1}{2}-\epsilon}$ we have

$$\frac{\binom{(\frac{1}{2}-\epsilon)|A|}{k}}{\binom{|A|}{k}} \geqslant \frac{\binom{k}{k}}{\binom{\frac{k}{\frac{1}{2}-\epsilon}}{k}} \geqslant 2^{-\frac{k}{\frac{1}{2}-\epsilon}}.$$

3 Blow-ups of oriented trees

In this section, we show that for every fixed integer k, the class of k-blow-ups of oriented trees is linearly unavoidable.

Theorem 18. *Let F be an oriented tree of order n and k a positive integer. Every tournament on at least $2^{10+18k}k \cdot n$ vertices contains $F[k]$.*

Proof. We root F on an arbitrary vertex s. Let $q = 2^{1+9k}$, $p = \lceil 2^{5+3^2k}k \rceil$ and $\epsilon = \frac{1}{2q}$. Observe that we have $\epsilon \leqslant \frac{1}{6}$, $q \geqslant 2^{1+\frac{k}{(\frac{1}{2}-\epsilon)^2}}$, and $p \geqslant q\frac{k}{(\frac{1}{2}-\epsilon)^2}$. Let

$$N = \frac{1}{2}(4qn+1)p \leqslant \frac{1}{2}2^{10+18k}k \cdot n$$

(using $4qn+1 \leqslant 2\times 4qn$ and $p \leqslant 2\times 2^{5+3^2k}k$) and let T be a tournament of order $2N$. We will show that T has a subdigraph isomorphic to $F[k]$. Let $(v_{-N}, \dots, v_{N-1})$ be a median order of T. We partition $\{v_{-N}, \dots, v_{N-1}\}$ into $4qn+1$ intervals $V_i = \{v_{pi}, \dots, v_{p(i+1)-1}\}$ for $i \in \{-2qn, \dots, 2qn\}$. We will find a copy of $F[k]$ in T iteratively, by adding at each step the **out-children** (*i.e.* its children which are out-neighbours) or the **in-children** (*i.e.* its children which are in-neighbours) of a node in F. Let $F_1, \dots, F_r$ be a sequence of subtrees of F such that (*i*) F_1 is the singleton $\{s\}$,

(*ii*) for every $i \leqslant \frac{r-1}{2}$, F_{2i+1} is obtained from F_{2i} by adding all the out-children in F of a vertex $x \in V(F_{2i})$ without any out-children in F_{2i},

(*iii*) for every $i \leqslant \frac{r-2}{2}$, F_{2i+2} is obtained from F_{2i+1} by adding all the in-children in F of a vertex $x \in V(F_{2i+1})$ without any in-children in F_{2i+1}, and

(*iv*) $F_r = F$.

Observe that such a sequence exists with $r \leqslant 2n-1$. Moreover, we say that a vertex x in F_i is **fully processed** if all its children in F are also in F_i, **half processed** if all its out-children in F are in F_i. Otherwise we say that x is **not processed**. We prove the following property by induction:

For every $j \in [r]$, there is an injection $\iota\colon V(F_j) \to \{-2q|V(F_j)|, \dots, 2q|V(F_j)|\}$ and for every $x \in V(F_j)$ there exists a set $U_x \subseteq V_{\iota(x)}$ such that

(*i*) $|U_x| \geqslant \frac{k}{\left(\frac{1}{2}-\epsilon\right)^2}$ *if x is not processed,*

(*ii*) $|U_x| \geqslant \frac{k}{\frac{1}{2}-\epsilon}$ *if x is half processed,*

(*iii*) $|U_x| \geqslant k$ *if x is fully processed,*

(*iv*) *if xy is an arc in F_j, then every vertex in U_x dominates U_y, and*

(*v*) *for every $i \in \{-2q|V(F_j)|, \dots, 2q|V(F_j)|\}$ a fraction of at most $\frac{1}{q}$ of the elements of both intervals $\{i, \dots, 2q|V(F_j)|\}$ and $\{-2q|V(F_j)|, \dots, i\}$ is in $\{\iota(x) \mid x \in V(F_j)\}$. More precisely,*

$$|\{i, \dots, 2q|V(F_j)|\} \cap \{\iota(x) \mid x \in V(F_j)\}| \leqslant \frac{1}{q}\left(2q|V(F_j)| - i + 1\right),$$
$$\text{and} \quad |\{-2q|V(F_j)|, \dots, i\} \cap \{\iota(x) \mid x \in V(F_j)\}| \leqslant \frac{1}{q}\left(i + 2q|V(F_j)| + 1\right).$$

Observe that, for $j = 1$, setting $\iota(s) = 0$ and taking for U_s any set in V_0 of size $p \geqslant \frac{k}{\left(\frac{1}{2}-\epsilon\right)^2}$ satisfies the invariant. Note also that, for $j = r$, the invariant implies the theorem. Now suppose that the property holds for some $j < r$. Let ι and $(U_x)_{x \in F_j}$ witnessing that the property holds at step j. Let x be the vertex in F_j such that F_{j+1} is obtained from F_j by adding the out-children of x in F, or the in-children of x in F. Our goal is to find a new value U'_x for U_x, and values for $\iota(y), U_y$ for every out-children (respectively in-children) y of x.

Case 1: $\boldsymbol{F_{j+1}}$ is obtained from $\boldsymbol{F_j}$ by adding the $\boldsymbol{\ell}$ out-children of $\boldsymbol{x}$ in $\boldsymbol{F}$.

Let $A = U_x$ and $B = \bigcup_{\iota(x) < i \leqslant 2q|V(F_j)| + q\ell} V_i$. Note that $|B| \geqslant p(2q|V(F_j)| + \ell q - \iota(x)) \geqslant \ell qp$. We first justify that every vertex in $V_{\iota(x)}$ dominates at least $(\frac{1}{2} - \epsilon)|B|$ vertices in

B. To this purpose, let v_i be any vertex in $V_{\iota(x)}$, so $i \in \{p\iota(x), \dots, p(\iota(x)+1)-1\}$. Let X be the set of vertices $\{v_{i+1}, \dots, v_{p(\iota(x)+1)-1}\}$, which is disjoint from B (X is empty if $i = p(\iota(x)+1)-1$). Observe that

$$N^+(v_i) \cap B = \Big(N^+(v_i) \cap (X \cup B)\Big) \setminus \Big(N^+(v_i) \cap X\Big).$$

Since $X \cup B = \{v_{i+1}, \dots, v_{p(\iota(x)+1)-1+|B|}\}$, and because $(v_{-N}, \dots, v_{N-1})$ is a median order, by (M2) we have $|N^+(v_i) \cap (X \cup B)| \geqslant \frac{1}{2}|X \cup B|$. Therefore, we obtain

$$\begin{aligned}
|N^+(v_i) \cap B| &\geqslant \frac{1}{2}|X \cup B| - |X| \\
&= \frac{1}{2}|B| - \frac{1}{2}|X| \\
&\geqslant \frac{1}{2}(|B| - p) \\
&\geqslant \left(\frac{1}{2} - \frac{1}{2q}\right)|B| \\
&= \left(\frac{1}{2} - \epsilon\right)|B|.
\end{aligned}$$

This shows that every vertex in $V_{\iota(x)}$ dominates at least $(\frac{1}{2} - \epsilon)|B|$ vertices in B, and so does every vertex in A. By Lemma 17 applied for $\alpha = 2^{-\frac{k}{(\frac{1}{2}-\epsilon)^2}}$, there exist $A^\star \subseteq U_x$ and $B^\star \subseteq B$ such that every vertex in $A^\star$ dominates $B^\star$, $|B^\star| \geqslant 2^{-\frac{k}{(\frac{1}{2}-\epsilon)^2}}|B|$, and $|A^\star| \geqslant \frac{k}{\frac{1}{2}-\epsilon}$. We set $U'_x = A^\star$.

Let I be the set of indices $i \in \{\iota(x)+1, \dots, 2q|V(F_j)|+q\ell\}$ such that $|B^\star \cap V_i| \geqslant \frac{k}{(\frac{1}{2}-\epsilon)^2}$ and $i \notin \{\iota(z) \mid z \in V(F_j)\}$. In what follows we show that I is large enough, which will enable us to find U_y for every out-child y of x in F.

Claim 19. $|I| \geqslant \ell$

Proof of claim. Suppose that this does not hold, so $|I| \leqslant \ell - 1$. We define $J, I^\star$, and I^ι as follows:

$$\begin{aligned}
J &= \{\iota(x)+1, \dots, 2q|V(F_j)| + q\ell\}, \\
I^\star &= \{i \in J \mid |V_i \cap B^\star| \geqslant \tfrac{k}{(\frac{1}{2}-\epsilon)^2}\}, \text{ and} \\
I^\iota &= \{\iota(z) \mid z \in V(j)\} \cap J.
\end{aligned}$$

Observe that $I = I^\star \setminus I^\iota$ by definition. By the induction hypothesis, $|I^\iota| \leqslant \frac{1}{q}(|J| - q\ell)$. Since $|I| \leqslant \ell - 1$ by assumption, we obtain $|I^\star| \leqslant \ell - 1 + \frac{1}{q}(|J| - q\ell) = \frac{|J|}{q} - 1$. This leads to the following inequalities, in which we also use $|B| = p|J|$

$$\begin{aligned}
|B^\star| &= \sum_{i\in J\cap I^\star} |V_i \cap B^\star| + \sum_{i\in J\setminus I^\star} |V_i \cap B^\star| \\
&\leqslant \left(\frac{|J|}{q} - 1\right) p + \left(\frac{q-1}{q}|J| + 1\right) \frac{k}{(\frac{1}{2}-\epsilon)^2} \\
&= \left(\frac{|B|}{q}\right) + \left(\frac{q-1}{q}\frac{|B|}{p}\frac{k}{(\frac{1}{2}-\epsilon)^2}\right) + \left(\frac{k}{(\frac{1}{2}-\epsilon)^2} - p\right) \\
&< \left(\frac{1}{q} + \frac{k}{p(\frac{1}{2}-\epsilon)^2}\right)|B| \\
&\leqslant \frac{2}{q}|B| \\
&\leqslant 2^{-\frac{k}{(\frac{1}{2}-\epsilon)^2}}|B|.
\end{aligned}$$

This is a contradiction since $|B^\star| \geqslant 2^{-\frac{k}{(\frac{1}{2}-\epsilon)^2}}|B|$. ♢

We can now extend ι to the out-children of x by taking values in I, and then we can arbitrarily take $U_z \subseteq B^\star \cap V_{\iota(z)}$ for every out-child z of x. Recall that $|V(F_{j+1})| = |V(F_j)| + \ell$. The invariant still holds because we add $2\ell qp$ vertices at the end and at the beginning of the median order:

(*i*) For every $i \in \{-2q|V(F_j)| - 2q\ell, \dots, 2q|V(F_j)|\}$, $|\{i, \dots, 2q|V(F_j)| + 2q\ell\} \cap \{\iota(z) \mid z \in V(F_{j+1})\}| \leqslant \frac{1}{q}(2q|V(F_j)| - i + 1) + \ell \leqslant \frac{1}{q}(2q|V(F_{j+1})| - i + 1)$.

(*ii*) For every $i \in \{2q|V(F_j)| + 1, \dots, 2q|V(F_j)| + q\ell\}$, $|\{i, \dots, 2q|V(F_j)| + 2q\ell\} \cap \{\iota(z) \mid z \in V(F_{j+1})\}| \leqslant \ell \leqslant \frac{1}{q}(q\ell) \leqslant \frac{1}{q}(2q|V(F_{j+1})| - i + 1)$.

(*iii*) For every $i \in \{2q|V(F_j)| + q\ell + 1, \dots, 2q|V(F_j)| + 2q\ell\}$, $|\{i, \dots, 2q|V(F_j)| + 2q\ell\} \cap \{\iota(z) \mid z \in V(F_{j+1})\}| = 0 \leqslant \frac{1}{q}(2q|V(F_{j+1})| - i + 1)$.

(*iv*) For every $i \in \{-2q|V(F_j)|, \dots, 2q|V(F_j)| + 2q\ell\}$, $|\{-2q|V(F_j)| - 2q\ell, \dots, i\} \cap \{\iota(z) \mid z \in V(F_{j+1})\}| \leqslant \frac{1}{q}(i + 2q|V(F_j)| + 1) + \ell \leqslant \frac{1}{q}(i + 2q|V(F_{j+1})| + 1)$.

(*v*) For every $i \in \{-2q\ell - 2q|V(F_j)|, \dots, -2q|V(F_j)| - 1\}$, $|\{-2q|V(F_j)| - 2q\ell, \dots, i\} \cap \{\iota(z) \mid z \in V(F_{j+1})\}| = 0 \leqslant \frac{1}{q}(i + 2q|V(F_{j+1})| + 1)$.

Case 2: F_{j+1} is obtained from F_j by adding the ℓ in-children of x in F.

Let $A = U_{\iota(x)}$ and $B = \bigcup_{-2q|V(F_j)|-q\ell\leqslant i<\iota(x)} V_i$. Since x is half processed, we have $|A| \geqslant \frac{k}{\frac{1}{2}-\epsilon}$. Moreover $|B| \geqslant q\ell p$ and every vertex in A is dominated by at least $\frac{1}{2}|B| - |A| \geqslant (\frac{1}{2}-\epsilon)|B|$ vertices in B. Using Lemma 17, we deduce that there are sets $A^\star \subseteq A, B^\star \subseteq B$ with $B^\star \Rightarrow A^\star$, $|A^\star| \geqslant k$ and $|B^\star| \geqslant 2^{-\frac{k}{\frac{1}{2}-\epsilon}}|B| \geqslant 2^{-\frac{k}{(\frac{1}{2}-\epsilon)^2}}|B|$. We set $U'_x = A^\star$. We now find U_y for every y in-children of x in F. With a proof almost identical to the one of Claim 19 we deduce the following claim.

Claim 20. *There is a set $I \subseteq \{-2q|V(F_j)| - q\ell, \dots, \iota(x) - 1\}$ of size at least ℓ such that $\iota(z) \notin I$ for every $z \in V(F_j)$ and $|B^\star \cap V_i| \geqslant \frac{k}{(\frac{1}{2}-\epsilon)^2}$ for every $i \in I$.*

Then we build $U_y, \iota(y)$ for every in-children y of x as in the first case. This concludes the proof of the theorem. □

4 Extensions of oriented trees

In this section, we show that for every fixed integer k, the class of k-extensions of oriented trees is linearly unavoidable. Our method is based on the fact that the number of copies of TT_{2k+1} in any N-vertex tournaments is in $\Omega_k(N^{2k+1})$. We start with the case $k = 1$, which is a good warm-up for the general case.

Lemma 21 (Folklore)**.** *Every tournament of order N contains at least $\frac{1}{8}N(N-1)(N-3)$ copies of TT_3.*

Proof. Let T be a tournament of order N. Let c_3 be the number of directed 3-cycles in T and let tt_3 be the number of TT_3 in T. The number of copies of $\vec{P}_3$ (the directed path of order 3) in T is $\sum_{v \in V(T)} d^-(v)d^+(v) \leqslant N\left(\frac{N-1}{2}\right)^2 = \frac{1}{4}N(N-1)^2$. But the number of $\vec{P}_3$ is also $tt_3 + 3c_3 = tt_3 + 3(\binom{N}{3} - tt_3) = 3\binom{N}{3} - 2tt_3$. We deduce that $3\binom{N}{3} - 2tt_3 \leqslant \frac{1}{4}N(N-1)^2$ and so $tt_3 \geqslant \frac{1}{2}\left(\frac{N(N-1)(N-2)}{2} - \frac{N(N-1)^2}{4}\right) = \frac{1}{8}N(N-1)(N-3)$. □

We will also need the following well-known lemma. Recall that a **hypergraph** $\mathcal{H}$ is a pair (V, E) where V is a finite set and E is a subset of 2^V. We call the elements of V the **vertices** of $\mathcal{H}$, and the elements of E the **hyperedges** of $\mathcal{H}$. An hypergraph $\mathcal{H}' = (V', E')$ is a **subhypergraph** of $\mathcal{H}$, and we write $\mathcal{H}' \subseteq \mathcal{H}$, if $V' \subseteq V$ and $E' \subseteq E$.

Lemma 22 (Folklore)**.** *Let $\mathcal{H}$ be a hypergraph on n vertices and with m hyperedges. There is a hypergraph $\mathcal{H}' \subseteq \mathcal{H}$ such that every vertex of $\mathcal{H}'$ belongs to at least $\frac{m}{n}$ hyperedges of $\mathcal{H}'$.*

Proof. We proceed by induction on n. If every vertex in $\mathcal{H}$ is in at least $\frac{m}{n}$ hyperedges we are done. Otherwise there is a vertex u in less than $\frac{m}{n}$ hyperedges, then $\mathcal{H} - u$ has $m' \geqslant m - \frac{m}{n} = \frac{m}{n}(n-1)$ hyperedges and $n' = n - 1$ vertices. Hence, by the induction hypothesis, $\mathcal{H} - u$ has a subhypergraph with minimum degree at least $\frac{m'}{n'} \geqslant \frac{m}{n}$ as wanted. □

Given an acyclic digraph D, we denote by $\mathrm{unvd}^*(D)$ the maximum of $\mathrm{unvd}(H) + |V(D) \setminus V(H)|$ over all the subdigraphs H of D.

Theorem 23. *Let D be a 1-extension of an oriented tree F on n vertices, then $\mathrm{unvd}(D) \leqslant 8\,\mathrm{unvd}^*(F) + 3$.*

By applying Theorem 5 to bound $\mathrm{unvd}^*(F)$, we obtain the following.

Corollary 24. *There exists a constant n_0 such that the following holds. Let D be a 1-extension of an oriented tree F on at least n_0 vertices, then* $\mathrm{unvd}(D) \leqslant 16|V(F)| - 13$.

An **embedding** $\phi\colon H \to D$ of a digraph H in a digraph D is an injective function $\phi\colon V(H) \to V(D)$ such that $\phi(u)\phi(v) \in A(D)$ for every $uv \in A(H)$. Observe that D contains a copy of H if and only if there is an embedding of H in D.

Proof of Theorem 23. Let D be a 1-extension of F with added vertex s. We assume that D is a full 1-extension of F, for otherwise we just add missing arcs. Let T be a tournament on $N = 8\,\mathrm{unvd}^*(F) + 3$ vertices. By Lemma 21, the number of TT_3 in T is at least $\frac{1}{8}N(N-1)(N-3)$. We deduce that there is a vertex t which is the middle vertex (*i.e.* the vertex with in- and out-degree 1) of at least $\frac{1}{8}(N-1)(N-3)$ copies of TT_3 in T. In other words there are at least $\frac{1}{8}(N-1)(N-3)$ arcs from $N^-(t)$ to $N^+(t)$. Consider the bipartite graph $H_0 = (N_T^-(t), N_T^+(t), E)$ where $E = \{uw \in A(T) \mid u \in N_T^-(t), w \in N_T^+(t)\}$. By Lemma 22, H_0 has a nonempty subgraph H with minimum degree at least $\frac{(N-1)(N-3)}{8(N-1)} = \frac{1}{8}(N-3) \geqslant \mathrm{unvd}^*(F)$. Let $A = V(H) \cap N_T^-(v), B = V(H) \cap N_T^+(t)$. Since H is bipartite, note that in particular $|A|, |B| \geqslant \mathrm{unvd}^*(F)$.

We will now inductively build an embedding $\phi\colon F \to T\langle V(H)\rangle$ such that

- for every $y \in N_D^-(s)$, $\phi(y) \in A$, and
- for every $y \in N_D^+(s)$, $\phi(y) \in B$.

Extending such an embedding of F by setting $\phi(s) = t$ yields the desired embedding of D in T.

If F is empty, then the result is clear. Suppose now that F is not empty and consider the tree F' obtained from F by contracting every connected component C of the underlying graphs of $D\langle N_D^-(s)\rangle$ and $D\langle N_D^+(s)\rangle$ into a single vertex $u_{V(C)}$. Consider a leaf u_Z of F', and without loss of generality, assume that $Z \subseteq N_D^-(s)$. If $Z = V(F)$, then $V(F) \subseteq N_D^-(s)$, and we can embed F in $T\langle A\rangle$ since $|A| \geqslant \mathrm{unvd}^*(F) \geqslant \mathrm{unvd}(F)$. Otherwise, assume by induction that we have an embedding $\phi\colon F - Z \to T\langle V(H)\rangle$ with $\phi(N_D^-(s) \setminus Z) \subseteq A$ and $\phi(N_D^+(s)) \subseteq B$. Let uv be the arc of F whose extremities belong to Z and $V(F) \setminus Z$ respectively. Observe that we have $u \in Z$ and $v \in N_D^+(s)$, for otherwise $\{u, v, s\}$ induces a directed cycle on D, a contradiction.

As H has minimum degree at least $\mathrm{unvd}^*(F)$, and because $\phi(v) \in B$, the size of $A \cap N_T^-(\phi(v))$ is at least $\mathrm{unvd}^*(F)$, hence there are at least $\mathrm{unvd}^*(F) - |V(F) \setminus Z| \geqslant \mathrm{unvd}(F\langle Z\rangle)$ vertices in $A \cap N_T^-(\phi(v)) \setminus \phi(V(F - Z))$. Thus we can extend ϕ to the vertices in $F\langle Z\rangle$ in $A \cap N_T^-(\phi(v)) \setminus \phi(V(F - Z))$, and this gives the desired embedding of D. See Figure 1 for an illustration. $\square$

We now move to the general case. We first prove an analogue of Lemma 21 for larger values of k.

Lemma 25. *Let k be a positive integer and let T be a tournament on $N \geqslant 3^{\binom{k+1}{2}}$ vertices. Then the number of distinct copies of TT_k in T is at least $3^{-\binom{k+1}{2}} \cdot N^k$.*

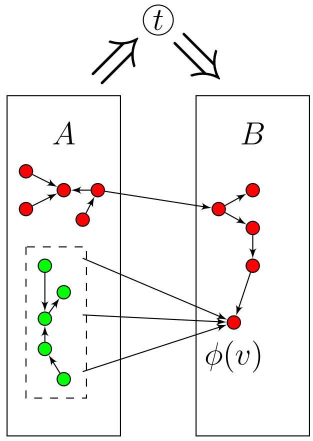

Figure 1: Illustration of the proof of Theorem 23. The red vertices are the image of $V(F-Z)$ under ϕ. The green vertices represent an embedding of $F\langle Z\rangle$ in the set $A \cap N_T^-(\phi(v)) \setminus \phi(V(F-Z))$, which is dashed in the figure.

Proof. We proceed by induction on k. For $k=1$ the result is clear. Now suppose $k>1$ and that the result holds for smaller values of k. By Proposition 13, in T there are at least $\frac{N}{3}$ vertices of out-degree at least $\frac{N}{3}$. For every such vertex u, there are by induction at least $3^{-\binom{k}{2}}\left(\frac{N}{3}\right)^{k-1}$ copies of TT_{k-1} in the subtournament induced by $N^+(u)$. Together with u, this gives as many copies of TT_k in T. We conclude that the number of copies of TT_k in T is at least $\frac{N}{3}\cdot 3^{-\binom{k}{2}}\left(\frac{N}{3}\right)^{k-1} = 3^{-\binom{k+1}{2}}\cdot N^k$. $\square$

This bound could be improved to $(2^{-\binom{k}{2}}+o(1))N^k$, see [4].

Theorem 26. *Let k,n be integers with $k \geqslant 1, n \geqslant 2$ and let F be an oriented tree of order n. Let D be a k-extension of F, then* $\operatorname{unvd}(D) \leqslant 3^{\binom{2k+2}{2}}\cdot \operatorname{unvd}^*(F)$.

Again, applying Theorem 5 to bound $\operatorname{unvd}^*(F)$, we deduce the following corollary.

Corollary 27. *There is a constant n_0 such that the following holds. Let D be a k-extension of an oriented tree F on at least n_0 vertices, then* $\operatorname{unvd}(D) \leqslant 2\cdot 3^{\binom{2k+2}{2}}\cdot |V(F)|$.

Proof of Theorem 26. Let T be a tournament on $N = 3^{\binom{2k+2}{2}}\cdot \operatorname{unvd}^*(F)$ vertices.

Let H be a transitive tournament on $2k+1$ vertices with acyclic ordering

$$(u_0', u_1, u_1', \dots, u_k, u_k').$$

By Lemma 25, T contains at least $3^{2k\binom{2k+2}{2}}\cdot \operatorname{unvd}^*(F)^{2k+1}$ copies of H. By the Pigeon-hole Principle, there are k vertices $v_1,\dots,v_k$ in T such that there are at least $\frac{1}{N^k}\cdot 3^{2k\binom{2k+2}{2}}\cdot \operatorname{unvd}^*(F)^{2k+1} = 3^{k\binom{2k+2}{2}}\cdot \operatorname{unvd}^*(F)^{k+1}$ embeddings of H in T with u_i mapped to v_i for every $i\in\{1,\dots,k\}$. From now on, let us say that an embedding $\phi\colon H\to T$ is **valid** if $\phi(u_i)=v_i$ for every $i\in\{1,\dots,k\}$.

We consider the hypergraph $\mathcal{H}$ with vertex set $V(T) \setminus \{v_1, \dots, v_k\}$ which contains as a hyperedge $\{\phi(u_i') \mid 0 \leqslant i \leqslant k\}$ for every valid embedding $\phi\colon H \to T$. By Lemma 22 applied to $\mathcal{H}$, there is a set of vertices $X \subseteq V(\mathcal{H})$ such that every vertex in X has degree at least $\frac{1}{N} \cdot 3^{k\binom{2k+2}{2}} \cdot \mathrm{unvd}^*(F)^{k+1}$ in $\mathcal{H}$. Let $(X_0, \dots, X_k)$ be the partition of X where X_i contains all the vertices $x \in X$ such that $x = \phi(u_i')$ for some valid embedding $\phi\colon H \to T$.

For all $i, j \in \{0, \dots, k\}$ with $i < j$ and for every vertex $x \in X_i$, we claim that x has at least $\frac{1}{N^k} \cdot 3^{k\binom{2k+2}{2}} \cdot \mathrm{unvd}^*(F)^{k+1} = \mathrm{unvd}^*(F)$ out-neighbours in X_j. Assume that this is not the case, then the degree of x in $\mathcal{H}\langle X\rangle$, which is exactly the number of valid embedding $\phi\colon H \to T$ with the extra conditions $\phi(u_i') = x$ and $\phi(u_\ell') \in X_\ell$ for every $\ell \in \{0, \dots, k\}$, is at most

$$|N^+(x) \cap X_j| \cdot \prod_{\ell \in \{0,\dots,k\} \setminus \{i,j\}} |X_\ell| < N^{k-1} \cdot \frac{1}{N^k} \cdot 3^{k\binom{2k+2}{2}} \cdot \mathrm{unvd}^*(F)^{k+1},$$

a contradiction to the choice of X. Similarly, if $j < i$ then x has at least $\mathrm{unvd}^*(F)$ in-neighbours in X_j. Since $k \geqslant 1$, this implies in particular that $|X_i| \geqslant \mathrm{unvd}^*(F)$ for every $i \in \{0, \dots, k\}$.

Let D be a full k-extension of F. As D is acyclic, let σ be an acyclic ordering of $V(D)$. Let $W \subseteq V(D)$ be such that $F = D - W$ and $(w_1, \dots, w_k)$ the ordering of W with respect to σ. We let $Y_0, \dots, Y_k$ be the partition of $V(F)$ where Y_i contains every vertex between w_i and w_{i+1} in σ, for every $i \in \{1, \dots, k-1\}$, Y_0 contains every vertex before w_1 in σ, and Y_k contains every vertex after w_k in σ. By construction, every arc between Y_i and Y_j in F is from Y_i to Y_j if $i < j$. We will now construct by induction on $|V(F)|$ an embedding $\phi\colon F \to T\langle X\rangle$ such that $\phi(Y_i) \subseteq X_i$ for every $i \in \{0, \dots, k\}$. By setting $\phi(w_i) = v_i$ for every $i \in [k]$, we will get the desired embedding of D in T.

If F is empty, then the result is clear. Now assume $|V(F)| > 0$. Consider the tree F' obtained from F by contracting every connected component C of the underlying graph of $D\langle Y_i\rangle$ into a single vertex $x_{V(C)}$, for every $i \in \{0, \dots, k\}$. Let x_Z be a leaf of F' and $i \in \{0, \dots, k\}$ be the index such that $Z \subseteq Y_i$. If $Z = V(F)$, then, as $|X_i| \geqslant \mathrm{unvd}^*(F) \geqslant \mathrm{unvd}(F)$, there is an embedding ϕ of F into $T\langle X_i\rangle$ and we are done. Otherwise let p be the unique neighbour in F of a vertex in Z which does not belong to Z. Assume that p is the tail of the (unique) arc between $F - Z$ and Z, the other case being symmetric. Let $j \in \{0, \dots, k\}$ be the index such that $p \in Y_j$. Recall that, by construction, we have $j < i$. By the induction hypothesis, there is an embedding $\phi\colon F - Z \to T\langle X_0 \cup \dots \cup X_k\rangle$ such that $\phi(Y_\ell \setminus Z) \subseteq X_\ell$ for every $\ell \in \{0, \dots, k\}$. We know that $\phi(p)$ has at least $\mathrm{unvd}^*(F)$ in-neighbours in X_i. Moreover $\mathrm{unvd}^*(F) - |V(F) \setminus Z| \geqslant \mathrm{unvd}(F\langle Z\rangle)$ and so $|X_i \cap N^-(\phi(p)) \setminus \phi(V(F - Z))| \geqslant \mathrm{unvd}(F\langle Z\rangle)$ (by definition of $\mathrm{unvd}^*(F)$). It follows that $F\langle Z\rangle$ can be embedded in $X_i \cap N^-(\phi(p)) \setminus \phi(V(F-Z))$. This gives the desired embedding of F and concludes the proof of the theorem. □

5 Particular extensions of some particular forests

5.1 Twin extensions of arcless digraphs

For every positive integers k, n_1, n_2, let $D_k(n_1, n_2)$ be full k-twin-extension of the arcless digraph D of order n_1+n_2 in which the k added vertices have the same in-neighbourhood V_1 of size n_1 and the same out-neighbourhood V_2 of size n_2 (with V_1, V_2 a partition of $V(D)$). Note that $D_1(n_1, n_2)$ is simply the oriented star made of a vertex dominated by n_1 vertices and dominating n_2 vertices.

Proposition 28. *For every positive integers n, n_1, n_2 with $n = n_1 + n_2 + 1$, each of the following holds:*

1. $\mathrm{unvd}(D_1(0, n-1)) = \mathrm{unvd}(D_1(n-1, 0)) = 2n-2$, *and*
2. $\mathrm{unvd}(D_1(n_1, n_2)) = 2n-3$.

Proof. The facts that

- $\mathrm{unvd}(D_1(0, n-1)) = \mathrm{unvd}(D_1(n-1, 0)) \leqslant 2n-2$ and that
- $\mathrm{unvd}(D_1(n_1, n_2)) \leqslant 2n-3$

follow from Lemma 13. For the lower bounds, consider first any $(n-2)$-regular tournament T. Then T has $2n-3$ vertices and does not contain $D_1(0, n-1)$. Thus $\mathrm{unvd}(D_1(0, n-1)) = \mathrm{unvd}(D_1(n-1, 0)) \geqslant 2n-2$. Similarly, if T_1 is an (n_1-1)-regular tournament and T_2 an (n_2-1)-regular tournament, then $T_1 \Rightarrow T_2$ is a tournament of order $2n-4$ which does not contain $D_1(n_1, n_2)$, implying that $\mathrm{unvd}(D_1(n_1, n_2)) \geqslant 2n-3$. □

To prove a lower bound on $\mathrm{unvd}(D_k(0, n))$, we will use a probabilistic argument relying on Chernoff's bound.

Proposition 29 (Chernoff's Bound)**.** *If X is a random variable following a binomial law with parameters $p \in [0,1]$ and $n \geqslant 0$, then for every $\epsilon \in [0,1]$*

$$\mathbf{P}[X \geqslant (1+\epsilon)pn] \leqslant \exp\left(-\frac{\epsilon^2}{3}pn\right).$$

Proposition 30. *For every fixed $k \geqslant 1$, $\mathrm{unvd}(D_k(0, n)) = 2^k \cdot n - o(n)$ as $n \to +\infty$.*

Proof. The added vertices of $D_k(0, n)$ are sources, so $\mathrm{unvd}(D_k(0, n)) \leqslant 2^k \cdot n$ by Proposition 1.

To prove the lower bound, let $0 < \epsilon < 1$ and let T be a tournament of order $\left\lfloor \frac{2^k}{1+\epsilon} n + k \right\rfloor$ taken uniformly at random. For every $X \subseteq V(T)$ of size k, let d_X^+ be the number of common out-neighbours of the vertices in X, that is $d_X^+ = |\bigcap_{x \in X} N^+(x)|$. Then d_X^+

follows a binomial law with parameters 2^{-k} and $\left\lfloor \frac{2^k}{1+\epsilon}n \right\rfloor \leqslant \frac{2^k}{1+\epsilon}n$. By Chernoff's bound (Proposition 29),

$$\begin{aligned}\mathbf{P}[d_X^+ \geqslant n] &\leqslant \mathbf{P}\left[d_X^+ \geqslant (1+\epsilon)\mathbf{E}[d_X^+]\right] \\ &\leqslant \exp\left(-\frac{\epsilon^2}{3}\mathbf{E}[d_X^+]\right) \\ &\leqslant \exp\left(-\frac{\epsilon^2}{3}2^{-k}\left\lfloor \frac{2^k}{1+\epsilon}n \right\rfloor\right) \\ &\leqslant \exp\left(-\frac{\epsilon^2}{3(1+\epsilon)}n+1\right).\end{aligned}$$

It follows by the Union Bound that $\mathbf{P}[T \text{ contains } D_k(0,n)] \leqslant \binom{\lfloor \frac{2^k}{1+\epsilon}n \rfloor}{k} \mathrm{e}^{-\frac{\epsilon^2}{3(1+\epsilon)}n+1} < 1$ for n large enough. This proves that $\mathrm{unvd}(D_k(0,n)) > \left\lfloor \frac{2^k}{1+\epsilon}n + k \right\rfloor$ for n large enough. ☐

We will now prove a similar upper bound for $D_k(n_1,n_2)$. To do so, we will use Ramsey's theorem, that we first recall here.

Theorem 31 (Ramsey's theorem [21])**.** *For every positive integers k, a and b, there exists $R_k(a,b)$ such that for every 2-colouring of the k-subsets of $[R_k(a,b)]$, either there is a set $X \subseteq [R_k(a,b)]$ of size a such that every k-subset of X is coloured 1, or there is a set $Y \subseteq [R_k(a,b)]$ of size b such that every k-subset of Y is coloured 2.*

Theorem 32. *Let $\epsilon > 0$ and let k be a positive integer. There is a constant C depending only on k and ϵ such that* $\mathrm{unvd}(D_k(n_1,n_2)) \leqslant (2^k+\epsilon)\cdot(n_1+n_2)+C$.

Proof. For fixed constants k, ϵ, simple computations show the existence of constants $C_0 \in \mathbb{N}, \eta > 0$ such that $\frac{\binom{C}{k}}{\binom{C(\frac{1}{2}-\eta)}{k}} \leqslant 2^k+\epsilon$ for every integer $C \geqslant C_0$. We let C be the maximum between C_0 and $R_k\left(\left\lceil \frac{k}{\frac{1}{2}-\eta} \right\rceil, \left\lceil \frac{k}{\frac{1}{2}-\eta} \right\rceil\right)$.

We first assume that $n_1, n_2 \geqslant \frac{C}{\eta(2^k+\epsilon)}$. Let T be a tournament on $\lceil (2^k+\epsilon)n_1 \rceil + \lceil (2^k+\epsilon)n_2 \rceil + C$ vertices. Consider a median order of T and let B_1 be the first $\lceil (2^k+\epsilon)n_1 \rceil$ vertices in this order, B_2 be the last $\lceil (2^k+\epsilon)n_2 \rceil$ vertices, and A be set of the remaining C vertices in the middle.

By (M2), every vertex $v \in A$ is dominated by at least $\frac{1}{2}|B_1| - |A| \geqslant (\frac{1}{2}-\eta)|B_1|$ vertices in B_1 (using $|B_1| = \lceil (2^k+\epsilon)n_1 \rceil \geqslant \frac{C}{\eta} = \frac{|A|}{\eta}$). Let $A' \subseteq A$ be any set of size $\left\lceil \frac{k}{\frac{1}{2}-\eta} \right\rceil$. By Lemma 17, there is a subset $A^\star$ of A' of size k such that $A^\star$ has at least n_1 common in-neighbours in B_1. Since this holds for every subset A' of size $\left\lceil \frac{k}{\frac{1}{2}-\eta} \right\rceil$, and by choice of C, by Ramsey's Theorem there is a subset $\tilde{A}$ of A of size $\left\lceil \frac{k}{\frac{1}{2}-\eta} \right\rceil$ such that every subset of k vertices in $\tilde{A}$ have at least k common in-neighbours in B_1. By repeating the same reasoning, because every vertex in A dominates at least $\frac{1}{2}|B_2| - |A| \geqslant (\frac{1}{2}-\eta)|B_2|$

vertices in B_2, we deduce that $\tilde{A}$ contains a set X of k vertices with at least n_2 common out-neighbours in B_2. Since every k-subset of $\tilde{A}$ has at least n_1 common in-neighbours in B_1, this gives the desired copy of $D_k(n_1, n_2)$.

We now consider the case $n_1 < \frac{C}{\eta(2^k+\epsilon)}$ or $n_2 < \frac{C}{\eta(2^k+\epsilon)}$. We apply the previous case for n_1, n_2 replaced respectively by $n_1 + \left\lceil \frac{C}{\eta(2^k+\epsilon)} \right\rceil$ and $n_2 + \left\lceil \frac{C}{\eta(2^k+\epsilon)} \right\rceil$. This shows that $\text{unvd}(D_k(n_1, n_2)) \leqslant (2^k + \epsilon)\left(n_1 + n_2 + 2\left(\frac{C}{\eta(2^k+\epsilon)} + 1\right)\right) + C \leqslant (2^k + \epsilon) \cdot (n_1 + n_2) + C'$ for $C' = C + 2(2^k + \epsilon)\left(\frac{C}{\eta(2^k+\epsilon)} + 1\right)$. $\square$

While Proposition 30 shows that this upper is tight when n_1 or n_2 is in $o(n)$, we do not know the precise asymptotic of $\text{unvd}(D_k(n_1, n_2))$ when $n_1 \approx n_2$.

5.2 1-extensions of directed paths

In this section, we give the exact value of $\text{unvd}(D)$ when D is a full 1-extension of a directed path.

Theorem 33. *Let D be a full 1-extension of a directed path with added vertex s. If s is a source or a sink, or if D is TT_3, then* $\text{unvd}(D) = 2|V(D)| - 2$. *Otherwise,* $\text{unvd}(D) = 2|V(D)| - 3$.

Proof. Observe that every orientation of K_4 contains TT_3, and that the directed cycle on three vertices does not contain TT_3, so $\text{unvd}(TT_3) = 4$. We now assume that D is not TT_3.

Set $n = |V(D)|$, and let s be a vertex of D such that $D - s$ is a directed path P. Let $V_1 = N_D^-(s)$ and $V_2 = N_D^+(s)$ and set $n_1 = |V_1|$ and $n_2 = |V_2|$. Observe that D contains $D_1(n_1, n_2)$ as a subdigraph. By Proposition 28 we obtain $\text{unvd}(D) \geqslant 2n - 3$, and $\text{unvd}(D) \geqslant 2n - 2$ if s is a sink or a source. If $n_1 = 0$ or $n_2 = 0$, then s is a source or a sink. Thus, by Proposition 1 and Theorem 3, we have $\text{unvd}(D) \leqslant 2\,\text{unvd}(P) = 2n - 2$.

It remains to show that $\text{unvd}(D) \leqslant 2n - 3$, assuming $n_1 \geqslant 1$ and $n_2 \geqslant 1$, and $\max\{n_1, n_2\} \geqslant 2$. By directional duality, assume without loss of generality that $n_1 \geqslant 2$. Let T be a tournament of order $2n - 3$, and let

$$\sigma_0 = (v_{-2n_1+1}, \dots, v_0, \dots, v_{2n_2-1})$$

be a median order of T. Set $A_1 = \{v_{-2n_1+1}, \dots, v_{-1}\}$ and $A_2 = \{v_0, \dots, v_{2n_2-1}\}$. Let $B_1 = N^-(v_{-1}) \cap A_1$ and $B_2 = N^+(v_{-1}) \cap A_2$, see Figure 2 for an illustration.

Observe that, by (M2), $|B_2| \geqslant n_2$ and $|B_1| \geqslant n_1 - 1$. We first consider the case $|B_1| \geqslant n_1$.

Claim 34. *If $|B_1| \geqslant n_1$, then T contains a copy of D with $s = v_{-1}$.*

Proof of claim. Let I^- be the set of in-generators of $T\langle B_1\rangle$ and O^+ be the set of vertices which are initial vertices of a directed path of order n_2 in $T\langle B_2\rangle$. Note that $I^- \neq \emptyset$ and $O^+ \neq \emptyset$. Moreover, by Lemma 15, $(B_1 \setminus I^-) \Rightarrow I^-$ and $O^+ \Rightarrow (B_2 \setminus O^+)$. Let v_{i^-} be

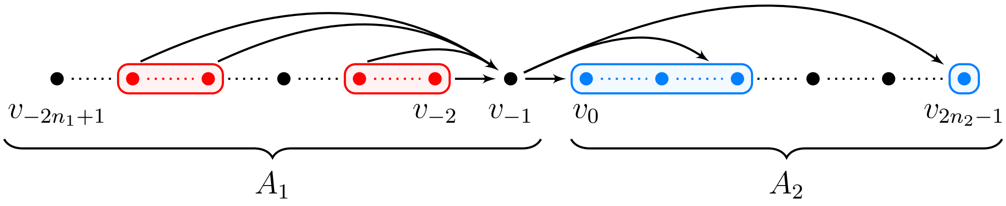

Figure 2: An illustration of the sets A_1, A_2, B_1, and B_2. The sets B_1 and B_2 are highlighted in red and blue, respectively.

the vertex in I^- with largest index and v_{i^+} be the vertex in O^+ with smallest index. By Lemma 14, v_{i^-} is the terminal vertex of a directed path P^- of order n_1 in $T\langle B_1\rangle$ and v_{i^+} is the initial vertex of a directed path P^+ of order n_2 in $T\langle B_2\rangle$. Hence we may assume $v_{i^+} \to v_{i^-}$, for otherwise the union of v_{-1}, P^-, P^+, the arcs between those paths and v_{-1}, and $v_{i^-}v_{i^+}$ is a copy of D in T, see Figure 3 for an illustration.

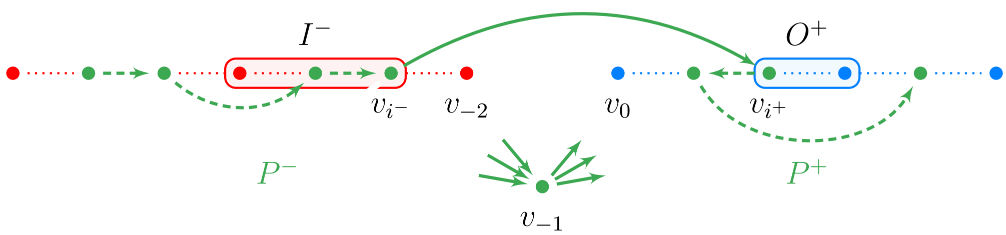

Figure 3: A copy of D when $v_{i^-} \to v_{i^+}$. For better readability, only the vertices of $B_1 \cup B_2 \cup \{v_{-1}\}$ are illustrated. The copy of D is highlighted in green.

Assume first that $|B_2| = n_2$. Then, since $|A_2| = 2n_2$, v_{-1} has exactly n_2 out-neighbours and n_2 in-neighbours in A_2. Consequently,

$$\sigma' = (v_{-2n_1+1}, \ldots v_{-2}, v_0, \ldots, v_{2n_2-1}, v_{-1})$$

is also a median order of T. By (M4) for σ', since $v_{i^+} \to v_{i^-}$, there is a vertex v_ℓ in $\{v_{i^-}, \ldots, v_{i^+}\} \setminus \{v_{-1}\}$ such that $v_{i^-} \to v_\ell \to v_{i^+}$. Observe that $v_\ell \notin I^-$ by maximality of i^- and $v_\ell \notin (B_1 \setminus I^-)$ because $(B_1 \setminus I^-) \Rightarrow \{v_{i^-}\}$. Thus $v_\ell \notin B_1$, and similarly $v_\ell \notin B_2$. In particular, $v_\ell \notin (V(P^-) \cup V(P^+))$. If $v_{-1} \to v_\ell$, remove the end-vertex from P^+, otherwise remove the initial vertex of P^-. In both cases, the resulting union of v_{-1}, P^-, P^+ and the arcs between those paths and v_{-1} is a copy of D in T (with $s = v_{-1}$) as desired, see Figure 4 for an illustration.

Henceforth, we may assume that $|B_2| > n_2$. In particular it implies $|O^+| \geqslant 2$: to see this, take any Hamiltonian directed path of $T\langle B_2\rangle$ (which exists by Theorem 3), then the two first vertices of this directed path are initial vertices of directed paths of order n_2 in $T\langle B_2\rangle$. If there is a vertex v_ℓ in $\{v_{i^-}, \ldots, v_{i^+}\} \setminus \{v_{-1}\}$ such that $v_{i^-} \to v_\ell \to v_{i^+}$, then as above there is a copy of D in T (with $s = v_{-1}$), so assume that there is no such v_ℓ. Hence,

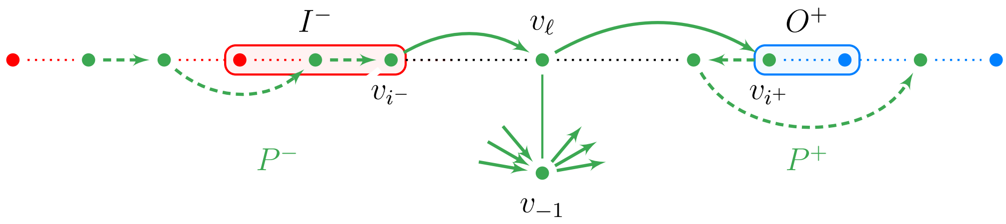

Figure 4: A copy of D when $|B_2| = n_2$ and $v_{i^+} \to v_{i^-}$. The copy of D is highlighted in green.

by (M4) and (M5),

$$\tilde{\sigma} = (v_{-2n_1+1}, \dots, v_{i^--1}, v_{i^-+1}, \dots, v_{i^+}, v_{i^-}, v_{i^++1}, \dots, v_{2n_2-1})$$

is a median order of T. Let v_{j^+} be the vertex of smallest index in $O^+ \setminus \{v_{i^+}\}$. Let Q^+ be a directed path of order n_2 in $T\langle B_2\rangle$ with initial vertex v_{j^+}. If $v_{i^-} \to v_{j^+}$, then the union of v_{-1}, P^-, Q^+, the arcs between those paths and v_{-1}, and $v_{i^-}v_{j^+}$ is a copy of D in T. Henceforth we may assume $v_{j^+} \to v_{i^-}$. By (M4) for $\tilde{\sigma}$, there is a vertex v_p with $i^+ + 1 \leqslant p \leqslant j^+ - 1$ such that $v_{i^-} \to v_p \to v_{j^+}$. Note that $v_p \neq v_{-1}$ because $i^+ \geqslant 0$, and as above $v_p \notin B_1 \cup B_2$. Again, by removing either the end-vertex of Q^+ or the initial vertex of P^-, we get that the union of v_{-1}, P^-, Q^+ and the arcs between those paths and v_{-1} is a copy of D in T. $\Diamond$

Since Claim 34 settles the case $|B_1| \geqslant n_1$, and because $|B_1| \geqslant n_1 - 1$, we now assume that $|B_1| = n_1 - 1$. Then, by (M4) and (M5) applied on σ_0, we have that

$$\sigma_1 = (v_{-1}, v_{-2n_1+1}, \dots, v_{-2}, v_0, \dots, v_{2n_2-1})$$

is a median order of T. In particular, this implies $v_{-2} \to v_0$ and $v_{-1} \to v_{-2n_1+1}$. Observe that $v_{-1}, v_{-2n_1+1}, \dots, v_{-2}, v_{-1}$ is then a Hamiltonian directed cycle of $T\langle A_1\rangle$. Moreover, note that if $|N^-(v_2) \cap A_1| \geqslant n_1$ then we can apply Claim 34 with σ_1, v_{-2} playing the roles of σ_0, v_{-1} respectively. By repeating this process, either we eventually find a copy of D in T with added vertex $s \in A_1$ or we ensure that $A_1 \Rightarrow \{v_0\}$. Henceforth, we assume that $A_1 \Rightarrow \{v_0\}$.

The end of the proof is similar to the proof of Claim 34, with v_0 playing the role of v_{-1}. Let $B_1' = A_1$ and $B_2' = N^+(v_0) \cap A_2$. We have $|B_1'| = 2n_1 - 1$ and, by (M2) on σ_0, $|B_2'| \geqslant n_2$. Since $v_{-1}, v_{-2n_1+1}, \dots, v_{-1}$ is a directed cycle, in particular both v_{-1} and v_{-2} are the terminal vertices of a directed path of order n_1 in B_1'. Let $O^{+\prime}$ be the set of vertices which are initial vertex of a directed path of order n_2 in $T\langle B_2'\rangle$. Note that $O^{+\prime} \neq \emptyset$ because $|B_2'| \geqslant n_2$ and, by Lemma 15, $O^{+\prime} \Rightarrow (B_2 \setminus O^{+\prime})$. Let v_{k^+} be the vertex in $O^{+\prime}$ with smallest index. Let R^- be the directed path $v_{-n_1}, \dots, v_{-1}$ and R^+ be a directed path of order n_2 in $T\langle B_2\rangle$ starting on v_{k^+}.

We may assume that $v_{k^+} \to v_{-1}$, for otherwise v_0, R^-, R^+, the arcs between those paths and v_0, and $v_{-1}v_{k^+}$ is a copy of D in T. If there is a vertex v_ℓ in $\{v_1, \dots, v_{k^+-1}\}$

such that $v_{-1} \to v_\ell \to v_{k^+}$, then by removing either the end-vertex of R^+ or the initial vertex of R^-, we get that the union of v_0, R^-, R^+ and the arcs between those paths and v_0 is a copy of D in T. This holds because $v_\ell \notin B'_2$ by choice of v_{k^+} and R^+ is a directed path in B'_2. Henceforth, we assume that there is no such v_ℓ. Therefore, by (M4) and (M5) applied on σ_0,

$$\hat{\sigma} = (v_{-2n_1+1}, \ldots, v_{-2}, v_{k^+}, v_{-1}, v_0 \ldots, v_{k^+-1}, v_{k^++1}, \ldots, v_{2n_2-1})$$

is a median order of T. In particular, $v_{-2} \to v_{k^+}$. Let $R^{-'}$ be $v_{-n_1-1}, \ldots, v_{-2}$. We thus obtain that the union of v_0, $R^{-'}$, R^+ and the arcs between those paths and v_0 is a copy of D in T. □

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