A star-comb lemma for finite digraphs

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Abstract

It is well-known that for every set U of vertices in a connected graph G there is either a subdivided star in G with a large number of leaves in U, or a comb in G with a large number of teeth in U.

In this paper we extend this property to directed graphs. More precisely, we prove that for every $n \in \mathbb{N}$ and every sufficiently large set U of vertices in a strongly connected directed graph D, there exists a strongly connected butterfly minor of D with n teeth in U that is either 'shaped' by a star or 'shaped' by a comb.

Mathematics Subject Classifications: 05C20, 05C40

1 Introduction

One of the most fundamental results about unavoidable substructures is the existence of *stars*, that is graphs $K_{1,m}$ for $m \in \mathbb{N}$, or paths in connected graphs [4, Proposition 9.4.1]:

There exists a function f such that, for every $n \in \mathbb{N}$, every connected graph of order at least $f(n)$ contains either a star $K_{1,n}$ or a path P_n. (1)

In the local context, the following variant of (1) is commonly used:

There exists a function f' such that for every $n \in \mathbb{N}$ and for every set U of vertices in a connected graph G with $|U| \geqslant f'(n)$ there is either a subdivided star or a comb in G that has n teeth in U. (2)

A *comb* is the union of a path P with disjoint (possibly trivial) paths having precisely their first vertex on P and we refer to the last vertices of these paths as its *teeth*. We also refer to the leaves of a subdivided star as its *teeth*. Furthermore, we call a vertex v of a comb a *junction* if v has degree 3 or v is a tooth and has degree 2.

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The unavoidable substructures of graphs of higher connectivity are likewise well-understood. Cycles and bipartite graphs $K_{2,m}$ for $m \in \mathbb{N}$ are the unavoidable topological minors of 2-connected graphs [4, Proposition 9.4.2]. Unavoidable minors for graphs of connectivity at least 3 have been studied by Oporowski, Oxley, Thomas [10] and Geelen, Joeris [6, 8]. Furthermore, induced unavoidable structures have been investigated [1, 3] with recent progress by Chudnovsky, Gartland, Hajebi, Lokshtanov and Spirkl [2, Theorem 5.2] who proved a version of (2) for induced graphs. In all these results the size of the unavoidable substructure depends on the size of its host graph. More generally, one can relate to other parameters like for example tree-width, which has been studied in Robertson and Seymour's [12] grid theorem.

In the context of directed graphs, unavoidable substructures have been expressed in terms of butterfly minors. Seymour and Thomassen [13] discovered the notion of butterfly minor in 1987 and proved that odd bicycles are unavoidable butterfly minors of even directed graphs. Wiederrecht [16] studied the unavoidable butterfly minors of strongly 2-connected directed graphs. Moreover, large unavoidable butterfly minors with respect to the directed tree-width [9] or the directed path-width [5] of its host graph have been investigated by Kawarabayashi, Kreutzer and Erde.

The study of unavoidable substructures is tightly related to the question how certain classes of graphs can be generated from small families of base graphs. Famous results are Whitney's [15] ear-decomposition for 2-connected graphs and Tutte's [14] wheel theorem for 3-connected graphs. Whitney's ear decomposition applies also to strongly connected directed graphs. For strongly 2-connected directed graphs a decomposition was recently invented by Hatzel, Kreutzer, Protopapas, Stamoulis, Wiederrecht and the author [7].

Despite the fundamental nature of (1) and (2), so far no counterpart was known in the directed setting. We take on the task of investigating the unavoidable butterfly minors in strongly connected directed graphs. Since for every pair v, w of vertices there are two directed paths, namely a directed v–w path and a directed w–v path, the structure of strongly connected directed graphs is more complex than the structure of connected graphs. This implies that the unavoidable butterfly minors of strongly connected directed graphs are more complex than stars and combs from (2).

Given a graph G, let $\mathcal{D}(G)$ be the directed graph obtained from G by replacing each edge $uv \in E(G)$ by directed edges $(u, v), (v, u)$. We say that a directed graph D is *shaped by a star* (see Fig. 1) if either D is a directed cycle or there exists a subdivided star S such that either

(i) $D = \mathcal{D}(S)$, or

(ii) D is obtained from $\mathcal{D}(S)$ by replacing the unique vertex $w \in V(\mathcal{D}(S))$ with $d_S(w) > 2$ by a directed cycle of length $d_S(w)$ such that the strong components of $\mathcal{D}(S) - w$ become incident with distinct vertices of this directed cycle.

A directed graph D is *shaped by a comb* (see Fig. 2) if there exists a comb C such that either

(a) $D = \mathcal{D}(C)$, or

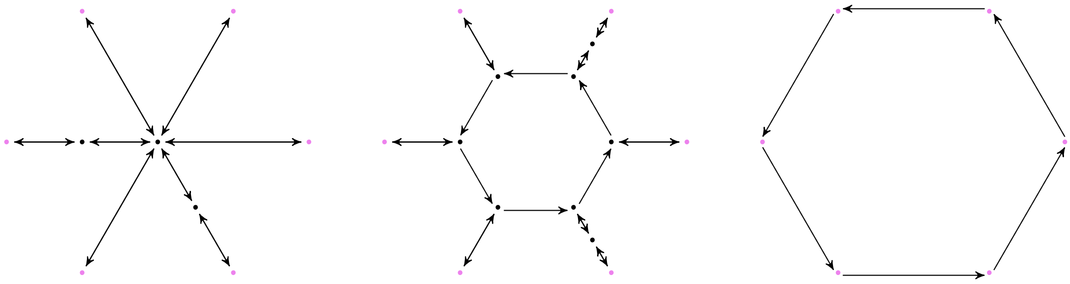


Figure 1: Three directed graphs shaped by a star.

(b) D is obtained from $\mathcal{D}(C)$ by replacing each junction j of C by a directed cycle of length 3 such that, if $d_C(j) = 3$, the three strong components of $\mathcal{D}(C) - j$ become incident with distinct vertices of this directed cycle and otherwise the tooth j is a vertex of this directed cycle and the two strong components of $\mathcal{D}(C) - j$ become incident with distinct other vertices of this directed cycle.

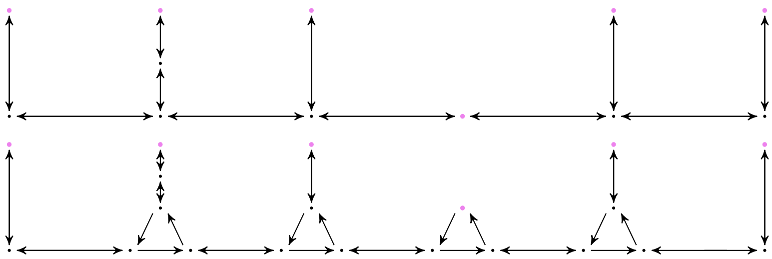


Figure 2: Two directed graphs shaped by a comb.

Let T be a subdivided star or a comb. Then the *teeth* of a directed graph D shaped by T are the vertices of D corresponding to the teeth of T. Furthermore, we call all vertices of a directed cycle *teeth*.

We remark that all directed graphs shaped by a star or shaped by a comb are strongly connected. Our main result reads as follows:

Theorem 1. *There exists a function g such that for every $n \in \mathbb{N}$ and for every set U of vertices of strongly connected directed graph D with $|U| \geqslant g(n)$ there is a butterfly minor of D that is either shaped by a star or shaped by a comb, and has n teeth in U.*

We derive a variant of (1) for directed graphs from Theorem 1:

Corollary 2. *There exists a function g such that, for every $n \in \mathbb{N}$, every strongly connected directed graph D of order at least $g(n)$ contains $\mathcal{D}(K_{1,n}), \mathcal{D}(P_n)$ or a directed cycle of length n as a butterfly minor.*

All five types of graphs in Theorem 1 are indeed necessary for a statement like Theorem 1: Let D be an arbitrary directed graph of one of these types. It suffices to show that no essential butterfly minor of D is of a different type than D. A butterfly minor is *essential* if it is strongly connected and preserves a large number of teeth of its host graph.

- If D is of type (i), every essential butterfly minor of D contains a vertex of high in-degree since the centre vertex has an in-edge for each tooth of D that it preserves.
- If D is of type (ii), every essential butterfly minor of D contains a long directed cycle whose vertices are not teeth of D as the centre cycle has a vertex for each tooth of D that is preserved.
- If D is of type (a), the essential butterfly minors of D contain a copy of $\mathcal{D}(P)$ for some long path P with the property that many vertices of $\mathcal{D}(P)$ have in-degree 3 or are teeth of D. This is due to the fact that remaining teeth of D must be connected by a copy of $\mathcal{D}(P)$.
- If D is of type (b), every essential butterfly minor of D holds a directed cycle of length 3 for each tooth (but the two outer) it preserves. Thus every essential butterfly minor of D contains many directed cycles of length 3.
- Essential butterfly minors of directed cycles are again directed cycles that contain many teeth of D.

These properties ensure that no essential butterfly minor of D is of a different type than D.

We remark that if we consider strong minors rather than butterfly minors in the statement of Theorem 1, types (ii) and (b) can be omitted since such graphs can be strongly contracted to graphs of type (i) or (a), respectively.

In the second paper of this series [11] we extend a variant of (2) for infinite graphs, which is known as the star-comb lemma and is a standard tool in infinite graph theory, to directed graphs.

This paper is organised as follows. We introduce basic notations and butterfly minors in Section 2. In Section 3 we investigate the strong connectivity of pairs of vertices: We show that every pair of directed paths (e.g. a directed a–b path and a directed b–a path for vertices a and b) can be simplified to a pair of *laced* paths. In Section 4 we show that for every large set U of vertices there exists a strongly connected butterfly minor of a certain type containing many vertices of U: These butterfly minors consist of a strongly connected 'centre' together with a family of subgraphs each ensuring that some element of U is strongly connected to the centre. Finally, we prove Theorem 1 in Section 5 by analysing the structure of this family of subgraphs.

2 Preliminaries

For standard notations we refer to Diestel's book [4]. We set $[\ell] := \{1, \dots, \ell\}$ for $\ell \in \mathbb{N}$. A *directed x_1–x_ℓ path*, or simply a *directed path*, of a directed graph is an alternating sequence

$P = x_1e_2x_2 \dots x_{\ell-1}e_{\ell-1}x_\ell$ of distinct vertices $(x_i)_{i\in[\ell]}$ and distinct edges $(e_i)_{i\in[\ell-1]}$ such that x_i is the tail of e_i and x_{i+1} is the head of e_i. We call the vertices $x_2, \dots, x_{\ell-1}$ the *internal vertices* of P. Further, we call x_1 the *startvertex* and x_ℓ the *endvertex* of P. We also refer to the directed graph induced by P as a *directed path*. Given two sets of vertices X, Y, the directed path P is a *directed X–Y path* if $x_1 \in X$, $x_\ell \in Y$ and the internal vertices of P are not contained in $X \cup Y$. Further, for subgraphs A, B, we refer to directed $V(A)$–$V(B)$ paths as *directed A–B paths*.

Given some directed path P and a vertex $x \in V(P)$ we define Px to be the initial segment of P ending in x and define xP analogously. Given further some directed path Q with $x \in V(P) \cap V(Q)$, the term PxQ refers to the directed path obtained by the concatenation of Px and xQ along x. We define an order on the vertex set of a directed path P as follows: for $v, w \in V(P)$, we say $v \leqslant_P w$ if $w \in V(vP)$.

A *double path* is the directed graph $\mathcal{D}(P)$ obtained from an undirected path P. A *directed cycle* is a sequence $C = x_1e_2x_2 \dots x_{\ell-1}e_{\ell-1}x_\ell$ of distinct vertices $(x_i)_{i\in[\ell-1]}$ with $x_1 = x_\ell$ and distinct edges $(e_i)_{i\in[\ell-1]}$ such that x_i is the tail of e_i and x_{i+1} is the head of e_i. We call a directed graph an *in-arborescence*, if its underlying undirected graph is a rooted tree and all edges are directed towards the root. The terms root and leaves transfer from the underlying undirected graph to the in-arborescence. *Out-arborescences* are defined analogously.

A directed graph D is called *strongly connected*, if for every $u, v \in V(D)$ there exists a directed u–v path in D.

Throughout this paper we implicitly make use of Ramsey's theorem:

Theorem 3. [4, Theorem 9.1.1] *There exists a function ϕ such that, for every $n \in \mathbb{N}$ and every $k \in \mathbb{N}$, every complete undirected graph on $\phi(k, n)$ vertices with a k-colouring of its edges contains a monochromatic clique of size n.*

2.1 Butterfly minors

An edge $e = (u, v) \in E(D)$ of a directed graph D is called *butterfly contractible* if e is the only outgoing edge of u or the only incoming edge of v. A directed graph H is a *butterfly minor* of a directed graph D if H can be obtained from a subgraph of D by repeatedly contracting butterfly contractible edges.

Observation 4. *Let D_1, D_2, D_3 be directed graphs such that D_1 is a butterfly minor of D_2 and D_2 is a butterfly minor of D_3. Then D_1 is butterfly-minor of D_3.*

The vertex set of a butterfly minor is in general not a subset of the vertex set of its host graph. Therefore in the statement of Theorem 1 it is not immediately clear how a butterfly minor can have its teeth in U. To overcome this problem, we identify (in each contraction step) the vertex x_e obtained by butterfly contracting an edge $e = (u, v)$ with u or v: we identify x_e with u if e is the only in-going edge of v and we identify x_e with v if e is the only out-going edge of u. Note that we allow x_e to be identified with both u and v. In this way some vertex of a butterfly minor is an element of U if we identified it with a vertex of U.

We show that these identifications do not create new connectivity:

Proposition 5. *Let H be a butterfly minor of a directed graph D and $v, w \in V(H)$. If there exists a directed v–w path in H, then there exists a directed v–w path in D.*

Proof. Since H is obtained from D by a sequence of butterfly contractions and edge deletions we can assume that H is obtained from D by butterfly contracting a single edge $e = (u, u')$. Let x_e be the vertex of H obtained by butterfly contracting e.

By the definition of butterfly contraction, every directed path P in H extends to a directed path P^* in D under adding the edge e whenever possible. If e is the only ingoing edge of u' and P contains x_e, then P^* contains u. Similarly, if e is the only outgoing edge of u and P contains x_e, then P^* contains u'. The way we identify vertices ensures that directed v–w paths in H turn into directed v–w paths in D. $\square$

Finally, we present two classes of substructures that can be butterfly contracted to single vertices: We call a rooted directed graph T_{in} an *in-arborescence* if the underlying undirected graph of T_{in} is a tree and all its edges are oriented towards its root. An in-arborescence T_{in} in a directed graph D can be butterfly contracted to a single vertex if all vertices of T_{in} but its root have out-degree one in D. In particular, the vertex obtained by butterfly contracting T_{in} is identified with the root of T_{in}. We define out-arborescences analogously and note that an out-arborescence in a directed graph D can be butterfly contracted to a single vertex if all its vertices but its root have in-degree one in D.

3 Laced paths

In this section we investigate the interaction of pairs of directed paths. We show that every two directed paths can be simplified to a pair of *laced paths*. This provides a simple witness for the strong connectivity of two given vertices.

Two directed paths P and Q are *laced* if either the paths P and Q are disjoint or there exist $x_1, y_1, \ldots, x_\ell, y_\ell \in V(P) \cap V(Q)$ for some $\ell \in \mathbb{N}$ such that

- $x_1 \leqslant_P y_1 <_P \cdots <_P x_\ell \leqslant_P y_\ell$,
- $x_\ell \leqslant_Q y_\ell <_Q \cdots <_Q x_1 \leqslant_Q y_1$,
- $x_i P y_i = x_i Q y_i$ for every $i \in [\ell]$,
- all segments $y_\ell Q x_{\ell-1}, \ldots, y_2 Q x_1$ are internally disjoint to P, and
- $Q x_\ell$ intersects P only in x_ℓ and $y_1 Q$ intersects P only in y_1.

See Fig. 3 for an illustration.

Proposition 6. [7, Lemma 4.3] *Let D be a directed graph and let $a, b \in V(D)$. Further, let P be some directed path and let Q be a directed a–b path in D. Then there exists a directed a–b path Q' with $Q' \subseteq P \cup Q$ such that P and Q' are laced.*

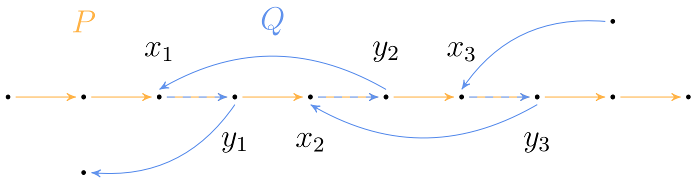


Figure 3: Two laced directed paths.

To keep the paper self-contained, we present a short proof of Theorem 6 that follows the lines of the proof given in [7]:

Proof. Let $Q' \subseteq Q \cup P$ be a directed a–b path that minimises the number of edges outside $E(P)$. We show that P and Q' are laced.

If Q' and P are disjoint, then they are laced by definition. Otherwise, let $P_1, \dots, P_\ell$ be the weakly connected components of $P \cap Q'$ in the order in which they appear on P. Note that every such component is a directed path, possibly trivial. For every $i \in [\ell]$ let x_i be the startvertex of P_i and let y_i be the endvertex of P_i. Then $x_1 \leqslant_P y_1 <_P \dots <_P x_\ell \leqslant_P y_\ell$.

Now, if Q' and P are not laced, then there must be an index $i \in [\ell - 1]$ such that y_i appears on Q' before x_{i+1}. But $P' := x_i P y_{i+1}$ is a directed subpath of P that is internally disjoint from Q' and thus the concatenation $Q'' := Q' y_i P' x_{i+1} Q'$ is a directed a–b path such that Q'' has fewer edges outside $E(P)$, a contradiction to the choice of Q'. $\square$

Observation 7. *Let D be a directed graph and let $a, b \in V(D)$. Further, let P be a directed a–b path and let Q be a directed b–a path such that P and Q are laced. Then $P \cup Q$ can be butterfly contracted to a double path starting in a and ending in b.*

We remark that in the setting of Theorem 7 there exist directed cycles $(C_j)_{j\in[p]}$ with the property that for every $j \neq k \in [p]$ $C_j \cap C_k$ is a directed path if $|j-k| = 1$ and empty otherwise, and $\bigcup_{i\in[p]} C_i = P \cup Q$.

4 Constructing the centre

In this section we establish the foundation for the proof of Theorem 1. Given a set of vertices U in a strongly connected directed graph, we prove the existence of a strongly connected butterfly minor consisting of a family of disjoint double paths that have an endpoint in U and are 'attached' to a strongly connected directed graph of certain type:

Lemma 8. *There exists a function α such that for every $n \in \mathbb{N}$ and every set U of vertices of a directed graph D with $|U| \geqslant \alpha(n)$ there is a butterfly minor that consists either*

(i) of a directed cycle containing n elements of U or of a family of directed cycles $(C_j)_{j\in[p]}$ with the property that for every $j \neq k$ $C_j \cap C_k$ is a directed path if $|j-k| = 1$ and there is an n-element subset $U' \subseteq \bigcup_{j\in[p]} V(C_j) \cap U$ such that each directed cycle of $(C_j)_{j\in[p]}$ contains at most one element of U', or

(ii) *of a family of disjoint double paths $(F_i : i \in [n])$ where F_i has endpoints $u_i \in U$ and z_i, edges $((a_i, z_i) : i \in [n])$, $((z_i, b_i) : i \in [n])$ and a directed graph A with $A \cap \bigcup_{i\in[n]} F_i = \emptyset$ such that*

(ii–a) *A is a single vertex with $A = \{a_i\} = \{b_i\}$ for every $i \in [n]$,*

(ii–b) *A is a directed cycle containing a_i, b_i for every $i \in [n]$ and the elements of $(a_i : i \in [n])$ are distinct, or*

(ii–c) *A consists of a sequence of directed cycles $(C_j)_{j\in[p]}$ such that for every $j \neq k$ $C_j \cap C_k$ is a directed path if $|j - k| = 1$, and empty otherwise, $a_i, b_i \in V(\bigcup_{j\in[p]} C_j)$ for every $i \in [n]$ and each directed cycle $(C_j)_{j\in[p]}$ contains at most one element of $(a_i : i \in [n])$.*

For proving Theorem 1 we will apply Theorem 8, and if the butterfly minor is of types (ii–b) or (ii–c), we analyse how the vertices $(a_i : i \in [n])$ and $(b_i : i \in [n])$ are distributed in the directed cycle or the union of directed cycles.

As a first step towards the proof of Theorem 8 we show the existence of one of the following subgraphs:

Proposition 9. *There exists a function β such that for every $n \in \mathbb{N}$ and every set U of vertices of a directed graph D with $|U| \geqslant \beta(n)$ there is a strongly-connected subgraph A of D such that:*

(a) *A is a single vertex and there is a family $(P_i : i \in [n])$ of non-trivial directed A–U paths that intersect only in A, or*

(b) *A is a directed cycle and there is a family $(P_i : i \in [n])$ of (possibly trivial) disjoint directed A–U paths, or*

(c) *A is the union of a sequence of directed cycles $(C_j)_{j\in[p]}$ with the property that for every $j \neq k$ $C_j \cap C_k$ is a directed path if $|j - k| = 1$ and empty otherwise; there is a family $(P_i : i \in [n])$ of (possibly trivial) disjoint directed A–U paths such that every directed cycle in $(C_j)_{j\in[p]}$ contains at most one startvertex of the directed paths $(P_i : i \in [n])$.*

See Fig. 4 for an illustration of such subgraphs.

Proof. Let $n \in \mathbb{N}$ be arbitrary, and fix $\beta(n), m \in \mathbb{N}$ such that $\beta(n) \gg m \gg n$. Further, let U be an arbitrary set of vertices of a strongly connected directed graph D with $|U| \geqslant \beta(n)$. Let T be an out-arborescences in D that contains all vertices of U and whose leaves are contained in U. Such an out-arborescence can be constructed by depth-first search since D is strongly-connected.

If T contains a vertex x of out-degree at least n, set $A := \{x\}$ and let $(P_i : i \in [n])$ be a family of maximal directed paths in T starting in x, which witness a subgraph of type (a).

Thus we can assume that T contains only vertices of out-degree at most $n-1$. Applying (2) to the underlying undirected graph of T provides a comb with $4m$ teeth in U. Note

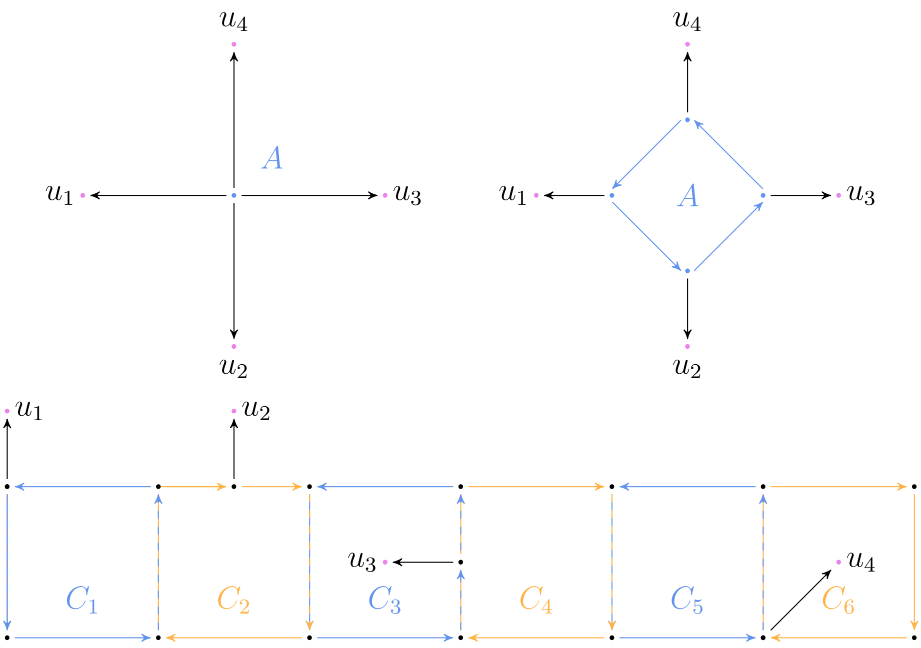


Figure 4: Directed graphs of types (a) to (c) in Theorem 9.

that the back of this comb splits into two directed paths. This implies that there exists a directed path R in T that contains either m elements of U or m vertices of out-degree at least 2 in T. These m elements of U or the m vertices of out-degree at least 2 witness that there exists a family $(\hat{P}_i : i \in [m])$ of (possibly trivial) disjoint directed R–U paths in T.

Let S be a directed path starting in the endvertex of R and ending in the startvertex of R that is laced with R, which is possible since D is strongly connected and by Theorem 6. Then $R \cup S = \bigcup_{i\in[p]} C_i$ for a sequence $(C_i)_{i\in[p]}$ of directed cycles with the property that for every $j \neq k$ $C_j \cap C_k$ is a directed path if $|j-k| = 1$ and empty otherwise. Each directed path $\hat{P}_i$ contains a terminal segment P_i that is a directed $(\bigcup_{i\in[p]} C_i)$–U path.

If there exists $j \in [p]$ such that at least n elements of $(P_i : i \in [m])$ start in the directed cycle C_j, then the union of C_j and these n elements of $(P_i : i \in [m])$ is the desired subgraph of type (b). Otherwise, by the choice of m, there exist $3n$ directed cycles in $(C_j : j \in [p])$ in which a directed path of $(P_i : i \in [m])$ starts. Therefore, we can pick a family $\mathcal{C}$ of n directed cycles in $(C_j : j \in [p])$ in which a directed path of $(P_i : i \in [m])$ starts such that the indices of the directed cycles in $\mathcal{C}$ differ pairwise by at least 3. Then the union of $\bigcup_{j\in[p]} C_j$ and one directed path of $(P_i : i \in [m])$ starting in C for each element $C \in \mathcal{C}$ is a subgraph of type (c). $\square$

Proposition 10. *Let U be a set of vertices of a directed graph D such that D is edge-minimal under the property of being strongly connected and containing U. Further, let A be a strongly-connected subgraph of D. Then for every edge e with head in $V(D) \setminus V(A)$ and tail in $V(A)$ there exists $v \in U$ for which there is no directed A–v path in $D - e$.*

Proof. By edge-minimality of D, there exist $u \neq v \in U$ such that every directed u–v path in D contains e. In particular, there exists a directed u–A path Q in $D - e$.

Suppose for a contradiction that there exist a directed A–v path P in $D - e$. Since A is strongly-connected there exists a directed u–v path in $Q \cup A \cup P \subseteq D - e$, contradicting the choice of u and v. $\square$

Proof of Theorem 8. Let $n \in \mathbb{N}$ be arbitrary. We set $\alpha(n) := \beta(2n)$, where β is the function from Theorem 9. Let U be an arbitrary set of vertices of a strongly connected directed graph D with $|U| \geqslant \alpha(n)$. We assume that D is edge-minimal with the property that U is strongly-connected and apply Theorem 9 to U to obtain a strongly connected subgraph A and a family $(P_i : i \in [2n])$ of directed paths as described in Theorem 9.

If n directed paths of $(P_i : i \in [2n])$ are trivial, then A is of type (b) or (c), i.e. A is either a directed cycle or a union of directed cycles $(C_j)_{j\in[p]}$ with the property that for every $j \neq k$ $C_j \cap C_k$ is a directed path if $|j - k| = 1$ and empty otherwise, and thus A is the desired butterfly minor of type (i). Otherwise, up to relabelling, we can assume that the directed paths $(P_i : i \in [n])$ are non-trivial. Let e_i be the first edge of P_i and let $\tilde{P}_i$ be the terminal segment of P_i that starts in the second vertex of P_i for every $i \in [n]$.

Claim 11. *We can assume that every directed path P_i for $i \in [n]$ has the property that there is no directed A–$\tilde{P}_i$ path in $D - e_i$.*

Proof. We assume that the number of directed paths in $(P_i : i \in [n])$ without this property is minimal under all possible choices for such directed paths $(P_i)_{i\in[n]}$. Suppose for a contradiction that there exists some $k \in [n]$ such that there is a directed A–$\tilde{P}_k$ path in $D - e_k$.

We apply Theorem 10 to the edge e_k to obtain a vertex $v \in U$ such that there is no directed A–v path in $D - e_k$. Let P'_k be a directed A–v path in D. Note that e_k is the first edge of P'_k by choice of e_k. Furthermore, every P_j for $j \in [n] \setminus \{k\}$ intersects P'_k only in A since otherwise there is a directed A–v path in $P_j \cup (P'_k - e_k) \subseteq D - e_k$ as $P_j \subseteq D - e_k$. Additionally, if P_k is disjoint to $\bigcup_{j\in[n]\setminus\{k\}} P_j$, then P'_k is disjoint to $\bigcup_{j\in[n]\setminus\{k\}} P_j$. Thus the family consisting of $(P_i : i \in [n]\setminus\{k\})$ and P'_k contradicts the minimality of $(P_i)_{i\in[n]}$. $\blacksquare$

Let $v_i \in U$ be the last vertex of P_i and let Q_i be some directed v_i–A path that is laced with P_i for every $i \in [n]$, which exists by Theorem 6. Further let z_i be the last vertex of Q_i that is contained in $\tilde{P}_i$.

Claim 12. *The directed path Q_i avoids $\tilde{P}_j \cup Q_j z_j$ for every $i \neq j \in [n]$.*

Proof. Suppose for a contradiction that Q_i intersects $\tilde{P}_j \cup Q_j z_j$. Then $Q_i \cup \tilde{P}_j \cup Q_j z_j$ contains a directed v_i–v_j path that avoids e_j. Since P_i avoids $e_j \in E(P_j)$, there exists a directed A–v_j path in $(P_i \cup Q_i) \cup (\tilde{P}_j \cup Q_j z_j) \subseteq D - e_j$, a contradiction to Theorem 11. $\blacksquare$

By Theorem 12, Q_i and Q_j intersect only in $(V(z_iQ_i) \cap V(z_jQ_j)) \setminus \{z_i, z_j\}$ for every $i \neq j \in [n]$. In particular, if $x \in Q_i \cap Q_j$ for some $i \neq j \in [n]$, then xQ_i avoids $\bigcup_{k\in[n]} \tilde{P}_k$. Up to re-routing of the directed paths $(z_iQ_i)_{i\in[n]}$ we can assume that $\bigcup_{i\in[n]}(z_iQ_i)$ is a disjoint union of in-arborescences. See Fig. 5.

Finally we obtain the desired butterfly minor D' by

- butterfly contracting $\tilde{P}_i \cup Q_i z_i$ to a double path F_i for every $i \in [n]$,
- butterfly contracting the common terminal segment of Q_i and Q_j to a single vertex for every $i \neq j \in [n]$, and
- butterfly contracting the remaining part of $z_i Q_i$ to an edge (z_i, b_i) for every $i \in [n]$.

If A is of type (a), D' is of type (ii–a) and if A is of type (b), D' is of type (ii–b). Further, if A is of type (c), D' is of type (ii–c). This completes the proof. $\square$

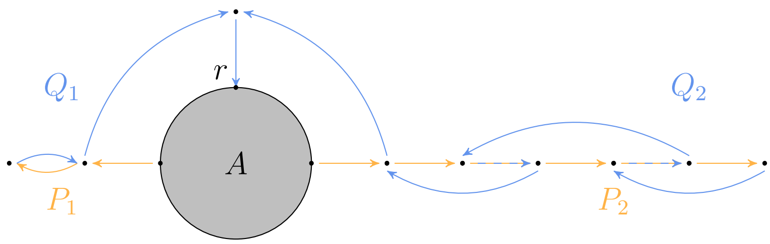


Figure 5: The setup in the proof of Theorem 8: A strongly connected subgraph A and two disjoint directed A–U paths P_1 and P_2. The directed paths Q_1 and Q_2 are laced with the directed paths P_1 and P_2, respectively, and share a common terminal segment. Furthermore, Q_1 avoids P_2 and Q_2 avoids P_1.

5 Proof of Theorem 1

We turn our attention to the main theorem:

Theorem 1. *There exists a function g such that for every $n \in \mathbb{N}$ and for every set U of vertices of strongly connected directed graph D with $|U| \geqslant g(n)$ there is a butterfly minor of D that is either shaped by a star or shaped by a comb, and has n teeth in U.*

Before proving Theorem 1, we show the existence of certain in-arborescences and out-arborescences in every subgraph A of type (c), and prove that for every three vertices in a strongly connected directed graph there exists a simple butterfly minor connecting them.

Proposition 13. *Let $(C_j)_{j\in[p]}$ be a sequence of directed cycles with the property that for every $j \neq k$ $C_j \cap C_k$ is a directed path if $|j-k| = 1$ and empty otherwise. Then for every $k \in [p-1]$ there exist an in-arborescence $T^{\text{in}} \subseteq \bigcup_{j\in[p]} C_j$ and an out-arborescence $T^{\text{out}} \subseteq \bigcup_{j\in[p]} C_j$ that intersect only in their common root such that $\bigcup_{i\in[p]\setminus[k]} V(C_i) \subseteq V(T^{\text{in}})$ and $\bigcup_{i\in[k]} V(C_i) \setminus V(C_{k+1}) \subseteq V(T^{\text{out}})$.*

Proof. Let P be the directed path in $C_{k+1} \cap C_k$ and let v the last vertex of P. Since the union $\bigcup_{j\in[p]\setminus[k]} C_j$ is strongly connected, there exists an in-arborescence spanning $\bigcup_{j\in[p]\setminus[k]} V(C_j)$ with root v. Since every vertex of $\bigcup_{i\in[k]} V(C_i)\setminus V(C_{k+1})$ can be reached from v in $\bigcup_{j\in[k]} C_j - (V(P) \setminus \{v\})$, there exists an out-arborescence T^{out} rooted at v spanning $\bigcup_{i\in[k]} V(C_i)\setminus V(C_{k+1})$ that hits T^{in} only in its root. Then T^{in} and T^{out} are as desired. $\square$

Proposition 14. *Let D be a strongly connected directed graph and let u_1, u_2, u_3 be distinct vertices of D. Then there exists a butterfly minor of D that is*

- *a double path containing u_1, u_2, u_3,*
- *the union of three non-trivial double paths that intersect only in a common endpoint and whose other endpoints are u_1, u_2, u_3, or*
- *the union of a directed cycle C of length 3 and three disjoint (possibly trivial) double paths having one endpoint in $V(C)$ and whose other endpoints are u_1, u_2, u_3.*

Proof. Let D' be an edge-minimal butterfly minor of D that is strongly connected and contains u_1, u_2, u_3. By Theorem 6, there exist a directed u_1–u_2 path P and a directed u_2–u_1 path Q in D' that are laced. Further, there exist a directed u_3–$(P \cup Q)$ path R and a directed $(P \cup Q)$–u_3 path S such that R and S are laced by Theorem 6.

Note that $D' = P \cup Q \cup R \cup S$ by edge-minimality. Further, note that every vertex of $V(D') \setminus \{u_1, u_2, u_3\}$ has in-degree at least 2 and out-degree at least 2 since otherwise some edge incident to it can be butterfly contracted, a contradiction to edge-minimality of D'.

First, we assume that $P \cup Q$ is a double path. Note that $R \cup S$ contains a trail connecting the startvertex of S and the endvertex of R. Thus the startvertex of S and the endvertex of R coincide since otherwise some edge of $P \cup Q$ with tail in the startvertex of S can be deleted without disconnecting u_1, u_2, u_3, a contradiction to the edge-minimality of D'. This implies that $R \cup S$ is a double path as all vertices of $R \cup S$ but u_3 have in- and out-degree at least 2 in D'. Therefore D' is a double path containing u_1, u_2, u_3, if the startvertex of S and the endvertex of R is either u_1 or u_2, and otherwise D' is a union of three double paths intersecting only in a common endpoint whose other endpoints are u_1, u_2, u_3, as desired.

Now we assume that $P \cup Q$ is not a double path. Since all vertices of $P \cup Q$ but u_1, u_2, u_3 have in- and out-degree at least 2 in D', $P \cup Q$ is obtained from a double path with endpoints u_1, u_2 by subdividing one edge and the subdivision vertex of $P \cup Q$ is the startvertex of S and the endvertex of R. This implies that $R \cup S$ is a double path. Then D' is the union of a directed cycle C of length 3 and disjoint (possibly trivial) double paths F_1, F_2, F_3 having one endpoint in $V(C)$ and whose other endpoints are u_1, u_2, u_3. This completes the proof. $\square$

Proof of Theorem 1. Let $n \in \mathbb{N}$ be arbitrary. Further, let $m, m', m'' \in \mathbb{N}$ with $m \gg m' \gg m'' \gg n$. We set $g(n) := \alpha(m)$, where α is the function from Theorem 8. Let U be an arbitrary set of vertices in a strongly connected directed graph D with $|U| \geqslant g(n)$. We apply Theorem 8 to D and U to obtain a butterfly minor D' as stated in Theorem 8 with m elements in U.

***D'* is of type (i)** If some cycle directed C contains n elements of U, we restrict to C and butterfly contract some edges of C to obtain a directed cycle containing precisely these n vertices, which provides the desired butterfly minor shaped by a star.

Otherwise, there is a family of directed cycles $(C_j)_{j\in[p]}$ with the property that for every $j \neq k$ $C_j \cap C_k$ is a directed path if $|j-k| = 1$ and there is an m-element subset $U' \subseteq \bigcup_{j\in[p]} V(C_j) \cap U$ such that each directed cycle of $(C_j)_{j\in[p]}$ contains at most one element of U'. Then there is an n-element subset $U'' \subseteq U'$ such that either each element of U'' is contained in the intersection of two elements of $(C_j)_{j\in[p]}$ or each element of U'' is contained in precisely one element of $(C_j)_{j\in[p]}$. In both cases we construct a butterfly minor shaped by a comb. In the former case, the union $\bigcup_{j\in[p]} C_j$ can be butterfly contracted to a double path containing the vertices of U'' as in Theorem 7. Note that double paths are shaped by a comb. In the latter case, we butterfly contract all directed cycles of $\bigcup_{j\in[p]} C_j$ as in Theorem 7 but butterfly contract the directed cycles containing a vertex of U'' to directed cycles of length 3. This ensures that the vertices of U'' are still present in the butterfly minor. Note that also this graph is shaped by a comb.

***D'* is of type (ii–a)** As A is a single vertex, D' is a union of double paths that intersect only in A. Thus D' is shaped by a star by definition.

***D'* is of type (ii–b)** We set $[x, y] := V(xCy)$ for every $x, y \in V(C)$. Furthermore, we write $x \leqslant_C y \leqslant_C z$ if $x, y, z \in V(C)$ appear in this order on C. For every $i \neq i' \in [m]$ precisely one of the following holds:

(1) $[b_i, a_i] \cap [b_{i'}, a_{i'}] = \emptyset$

(2) $([b_i, a_i] \subseteq [b_{i'}, a_{i'}]) \vee ([b_{i'}, a_{i'}] \subseteq [b_i, a_i])$

(3) $([b_i, a_i] \cap [b_{i'}, a_{i'}] \neq \emptyset) \wedge ([b_i, a_i] \nsubseteq [b_{i'}, a_{i'}]) \wedge ([b_{i'}, a_{i'}] \nsubseteq [b_i, a_i])$

By Ramsey's theorem, Theorem 3, we can assume, up to relabelling, that for every $i \neq i' \in [m']$ the same property applies. Further, we assume that $a_1 <_C a_2 <_C \cdots <_C a_{m'}$. From now on we consider the subgraph $C \cup \bigcup_{i\in[m']}(F_i + (a_i, z_i) + (z_i, b_i))$.

- If (1) applies to every $i \neq j \in [m']$: Then $b_1 \leqslant_C a_1 <_C b_2 \leqslant_C a_2 <_C b_3 \leqslant_C a_3 <_C \ldots$. See Fig. 6.

  By butterfly contracting $C[[b_i, a_i]]$ to a single vertex and butterfly contracting $C[[a_i, b_{i+1}]]$ to a single edge for every $i \in [m']$ ($b_{m'+1} := b_1$) we obtain a butterfly minor that is shaped by a star with m' teeth in U.

- If (2) applies to every $i \neq j \in [m']$: There exist one interval $[b_i, a_i]$ that is contained in all intervals of this form. We assume without loss of generality that

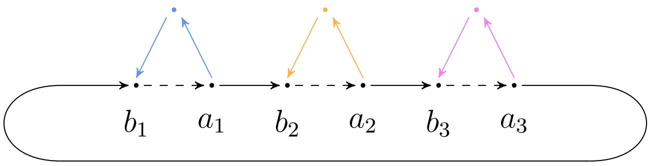
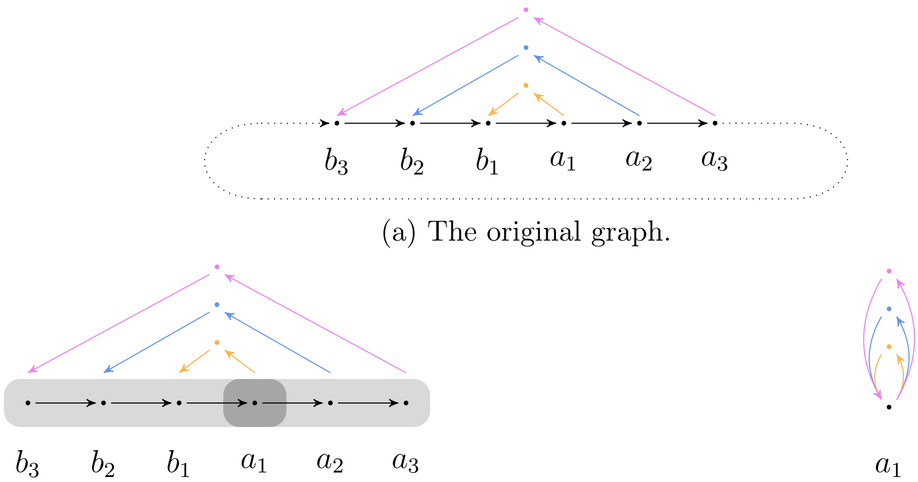


Figure 6: A butterfly minor D' of type (ii–b) where (1) applies to all pairs of $[m']$ in the proof of Theorem 1.



(b) The graph after deletion of the edges in $C[[a_{m'}, b_{m'}]]$. The gray marked edges can be butterfly contracted.

(c) The butterfly minor obtained by butterfly contracting the gray marked edges.

Figure 7: A butterfly minor D' of type (ii–b) where (2) applies to all pairs of $[m']$ in the proof of Theorem 1.

$[b_1, a_1]$ is this interval. Then $\ldots \leqslant_C b_3 \leqslant_C b_2 \leqslant_C b_1 \leqslant_C a_1 <_C a_2 <_C a_3 <_C \ldots$. We remark that elements of $(b_i : i \in [n])$ can coincide. See Fig. 7.

We construct the desired butterfly minor in the following way: Firstly, we delete the edges of $C[[a_{m'}, b_{m'}]]$. Now every vertex in $[b_{m'}, a_1)$ has out-degree precisely 1 and we butterfly contract $C[[b_{m'}, a_1]]$ to the single vertex a_1. Similarly, each vertex in $(a_1, a_{m'}]$ has in-degree precisely 1 and we butterfly contract $C[[a_1, a_{m'}]]$ to the single vertex a_1. This butterfly minor is the union of disjoint double paths $(F_i : i \in [m'])$ each connected to a single vertex a_1 via edges (a_i, z_i) and (z_i, b_i). Thus this butterfly minor is shaped by a star with m' teeth in U.

- If (3) applies to every $i \neq j \in [m']$: The intervals $[b_i, a_i], [b_j, a_j]$ intersect either in one or in two intervals. By Ramsey's theorem, Theorem 3, we can assume, up to relabelling, that either $[b_i, a_i] \cap [b_j, a_j]$ is one interval for every $i \neq j \in [2n]$ or $[b_i, a_i] \cap [b_j, a_j]$ are two intervals for every $i \neq j \in [2n]$. Note that (3) implies $a_i \neq b_i$ for every $i \neq j \in [m']$.

  If these intervals intersect pairwise in one interval, we can assume, up to relabelling, that the intervals $\{[b_i, a_i] : i \in [n] \setminus \{1\}\}$ contain either a_1 but not b_1 or b_1 but not a_1. In the former case, $a_1 \leqslant_C a_i <_C b_1 \leqslant_C b_i \leqslant_C a_1$ for every $i \in [n] \setminus \{1\}$. We construct the desired butterfly minor in the following way: See Fig. 8. Firstly, we consider the subgraph $C \cup \bigcup_{i\in[n]}(F_i + (a_i, z_i) + (z_i, b_i))$ and delete the edges of $C[[a_n, b_1]]$. Note that every vertex in $[b_1, a_1)$ has out-degree precisely 1 and we butterfly contract $C[[b_1, a_1]]$ to the vertex a_1. Similarly, each vertex in $(a_1, a_n]$ has in-degree precisely 1 and we butterfly contract $C[[a_1, a_n]]$ to the single vertex a_1. This butterfly minor is the union of disjoint double paths $(F_i : i \in [n])$ that are connected to a_1 via edges (a_i, z_i) and (z_i, b_i). Thus this butterfly minor is shaped by a star with n teeth in U. In the latter case, $a_1 <_C b_i \leqslant_C b_1 \leqslant_C a_i \leqslant_C a_1$ for every $i \in [n] \setminus \{1\}$ and we apply a similar argument to obtain a butterfly minor shaped by a star with n teeth in U.

If these intervals intersect pairwise in two intervals, then $a_i <_C b_i \leqslant_C a_j <_C b_j \leqslant_C a_i$ for every $i \neq j \in [2n]$. See Fig. 9. We consider the directed cycle C' consisting of the directed paths $b_1Ca_2, \ldots, b_{2n}Ca_1$ and the edges $(a_i, z_i), (z_i, a_i)$ for every $i \in [2n]$. The union of C' and $(F_i : i \in [2n])$ is a subdivision of a directed graph shaped by a star with $2n$ teeth in U, as desired.

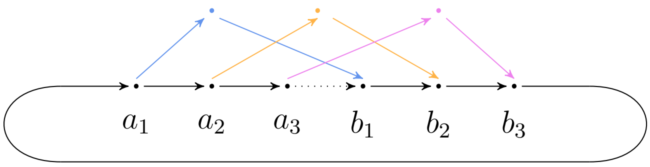


Figure 8: A butterfly minor D' of type (ii–b) where (3) applies to all pairs of $[m']$ and the intervals intersect pairwise in one interval in the proof of Theorem 1.

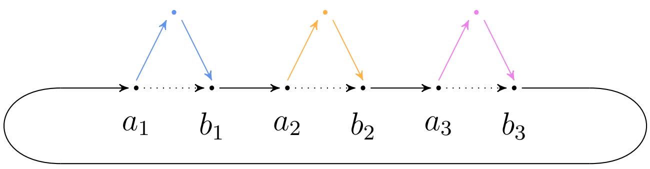


Figure 9: A butterfly minor D' of type (ii–b) where (3) applies to all pairs of $[m']$ and the intervals intersect pairwise in two intervals in the proof of Theorem 1.

***D′* is of type (ii–c)** For every $v \in \bigcup_{j\in[p]} V(C_j)$ let $\gamma(v) \in [p]$ be the minimal index with $v \in V(C_{\gamma(v)})$. Then either $\gamma(a_i) \leqslant \gamma(b_i)$ or $\gamma(b_i) \leqslant \gamma(a_i)$. By Ramsey's theorem, Theorem 3, we can assume, up to relabelling of the elements in $[m]$ and up to reversing the labelling of the directed cycles $(C_j)_{j\in[p]}$, that $\gamma(a_i) \leqslant \gamma(b_i)$ for every $i \in [m']$.

As provided by Theorem 8, $\gamma(a_i) \neq \gamma(a_{i'})$ holds for every $i \neq i' \in [m']$. We assume, up to relabelling of $[m']$, that $\gamma(a_i) < \gamma(a_{i'})$ for every $i < i' \in [m']$.

For every $i < i' \in [m']$ either $\gamma(b_i) < \gamma(a_{i'})$ or $\gamma(b_i) \geqslant \gamma(a_{i'})$ holds. By Ramsey's theorem, Theorem 3, we can assume, up to relabelling, that one property holds for every pair of distinct elements in $[m'']$.

- In the former case, i.e. $\gamma(b_i) < \gamma(a_{i'})$ for every $i < i' \in [m'']$: See Fig. 10. Then $\gamma(a_1) \leqslant \gamma(b_1) < \gamma(a_2) \leqslant \gamma(b_2) < \dots$. We consider the subgraph

$$\bigcup_{j\in[p]} C_j \cup \bigcup_{i\in[m'']\cap 2\mathbb{N}} (F_i + (a_i, z_i) + (z_i, b_i))$$

and butterfly contract $C_{\gamma(a_i)-1} \cap C_{\gamma(a_i)}$ to a single vertex z_i for every even $i \in [m'']$. Note that $C_{\gamma(a_i)-1} \cap C_{\gamma(a_i)}$ neither contains a_i nor b_i for every even $i \in [m'']$.

For even $i \in [m'']$, the subgraph $H_i := F_i \cup \bigcup_{\gamma(a_i)\leqslant j<\gamma(a_{i+2})} C_j$ is strongly connected and contains z_i, z_{i+2} and v_i. We apply Theorem 14 to obtain a butterfly minor D_i of H_i that is a double path containing z_i, z_{i+2}, v_i, the union of non-trivial double paths that intersect only in a common endpoint and whose other endpoints are z_i, z_{i+2}, v_i, or the union of a directed cycle C of length 3 and disjoint (possibly trivial) double paths having one endpoint in $V(C)$ and whose other endpoints are z_i, z_{i+2}, v_i.

Then there exists an n-element subset $J \subseteq [m'']$ such that all $(D_j)_{j\in J}$ are of the same type and the elements of J differ pairwise by at least two. We set $D_i' := D_i$ for every $i \in J$ and let D_i' be the butterfly minor of D_i that is a double path with endpoints z_i, z_{i+2} for every $i \in [m''] \setminus J$. Then $\bigcup_{i\in[m'']} D_i'$ the desired butterfly minor shaped by a comb with n teeth in U.

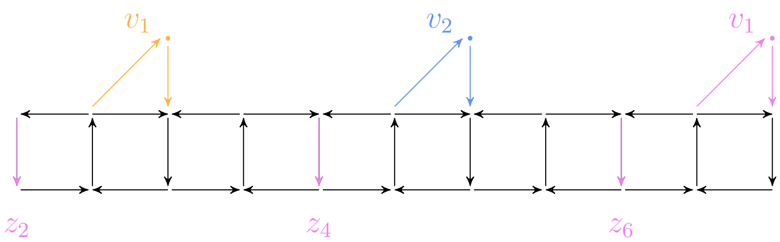


Figure 10: A butterfly minor D' of type (ii–c) where $\gamma(b_i) < \gamma(a_{i'})$ for every $i < i' \in [m'']$ in the proof of Theorem 1.

- In the latter case, i.e. $\gamma(b_i) \geqslant \gamma(a_{i'})$ for every $i < i' \in [m'']$: See Fig. 11. Then $\gamma(a_{m''}) \leqslant \gamma(b_i)$ for every $i \in [m'']$ and thus $\gamma(a_{m''-1}) < \gamma(b_i)$ for every $i \in [m'']$. In particular, all b_i for $i \in [m''-1]$ are contained in $\bigcup_{i>\gamma(a_{m''-1})} V(C_i)$ and all a_i for $i \in [m''-1]$ are contained in $\bigcup_{i\leqslant\gamma(a_{m''-1})} V(C_i)$.

By Theorem 13, there are T^{in} and T^{out} as stated such that all b_i are in $V(T^{\text{in}})$ and all a_i are in $V(T^{\text{out}})$ for $i \in [m''-1]$. We consider the subgraph

$$T^{\text{in}} \cup T^{\text{out}} \cup \bigcup_{i\in[m''-1]} (F_i + (a_i, z_i) + (z_i, b_i))$$

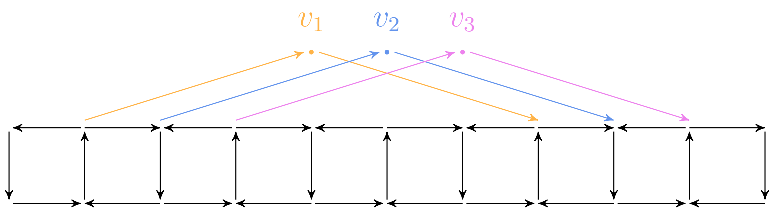


Figure 11: A butterfly minor D' of type (ii–c) where $\gamma(b_i) \geqslant \gamma(a_{i'})$ for every $i < i' \in [m'']$ in the proof of Theorem 1.

and butterfly contract T^{in} and T^{out} to their common root. This butterfly minor is shaped by a star with $m'' - 1$ teeth in U.

This completes the proof. □

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