A note on Kalai's 3^d Conjecture

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Abstract

Suppose that C is a centrally symmetric d-dimensional convex polytope; in 1989 Kalai conjectured that C has at least 3^d faces. We prove this result if there are d hyperplanes with pairwise orthogonal normal vectors so that C is symmetric about all of them.

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Suppose that $C \subset \mathbb{R}^d$ is a centro-symmetric convex polytope with nonempty interior, that is,

1. $C = -C$.

2. C has nonempty interior.

3. C is compact.

4. C is the intersection of finitely many half-spaces - sets of the form

$$\{x \in \mathbb{R}^d : x \cdot n \leqslant c\}$$

for some fixed unit vector n and real number c.

Before stating the conjecture, we will need the following definition:

Definition 1. Suppose that C is as above. A d-dimensional *face* of C is C itself, and a k-dimensional *face* of C with $0 \leqslant k < d$ is a subset X of ∂C so that:

1. X has Hausdorff dimension k.

2. X is equal to $\partial C \cap H$, where H is a hyperplane not meeting the interior of C.

In 1989, Kalai made the following conjecture in [3]:

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Conjecture 2 (Kalai)**.** If C is as above, then C has at least 3^d faces (adding up all the faces of all dimensions).

It is known that Conjecture 2 is true for dimensions $\leqslant 4$ (see [4]). It is also known that it is true for all polytopes whose faces are simplices; this was proved by Stanley in [6], answering a conjecture due to Bárány and Lovász in [1]. Finally, it is known to be true for the Hansen polytopes of split graphs, see [2].

We can now state our main theorem, for which we will give a short proof:

Theorem 3. *Suppose that C is as above. Let $\mathcal{B}$ be an orthogonal basis of $\mathbb{R}^d$ and suppose that C is symmetric about each hyperplane through the origin normal to some vector in $\mathcal{B}$. Then Conjecture 2 is true.*

We would also like to point out that, at the same time that this article was completed, this result was independently proved by R. Sanyal and M. Winter in [5]. Sanyal and Winter also show that the minimal number of facets is attained exactly for Hanner polytopes. We now introduce the notion of a cone used our proof.

Definition 4. Define $\mathcal{B}' = \cup_{v\in\mathcal{B}}\{v, -v\}$. Suppose that K is any nonempty subset of $\mathcal{B}'$ so that for each $v \in \mathcal{B}$, at most one of v or $-v$ is in K. We define the *cone* X_K to be the set of non-negative linear combinations of vectors in K, that is,

$$X_K = \Big\{\sum_{v\in K} t_v\, v : \, t_v \geqslant 0 \text{ for all } v \in K\Big\}.$$

The interior of the cone is

$$X_K^\circ = \Big\{\sum_{v\in K} t_v\, v : \, t_v > 0 \text{ for all } v \in K\Big\}.$$

We also define the interior of a face of X_K° of dimension at least 1 in the obvious way, and we define the interior of a 0-dimensional face (a point) as the face (point) itself. We will use τ° to denote the interior of the face τ.

Lemma 5. *The number of cones is* $3^d - 1$.

Proof. For every subset of $\mathcal{B}$ of size k, there are 2^k cones. Thus, the total number of cones is

$$\sum_{k=1}^{d} \binom{d}{k} 2^k$$

which by the binomial theorem is $3^d - 1$. □

For every cone X_K, we will associate a face $\tau_K \subset \partial C$ to it. We will prove that they are all distinct; since there is exactly one d-dimensional face (C itself), Lemma 5 will complete the proof of Theorem 3.

Fix a cone X_K, and choose τ_K to be a face of ∂C which satisfies

$$\tau^\circ \cap X_K^\circ \neq \emptyset,$$

and which has minimal dimension among all of these. Note that such a τ_K exists because the interior of the d-dimensional face of C intersects the interior of each cone, and so ∂C intersects the interior of each cone.

Let Q_K denote the dual cone to X_K. Q_K can be defined as the union of all cones $X_{K'}$ for K' containing K. Since $\mathcal{B}$ is an orthonormal basis,

$$Q_K = \{x \in \mathbb{R}^d : x \cdot v \geqslant 0, \forall v \in K\}$$

We have the following lemma:

Lemma 6. *We have the following inclusion:*

$$\tau_K^\circ \subset Q_K^\circ.$$

Proof. For contradiction, assume that the interior of τ_K is *not* a subset of the interior of Q_K. Note that since we always take K to be non-empty, we have that $X_K \neq \{0\}$, $Q_K \neq \mathbb{R}^d$ and so ∂Q_K is non-empty. Thus, as τ_K° is path-connected and contains a point in $X_K \subset Q_K$, there is some point $q \in \partial Q_K \cap \tau_K^\circ$. Note that

$$\partial Q_K \subset \bigcup_{v \in K} \{x \in \mathbb{R}^d : x \cdot v = 0\}$$

Thus, there is some $v_q \in K$ such the q lies in the hyperplane H normal to v_q. Let $R : \mathbb{R}^d \to \mathbb{R}^d$ denote the reflection about H; $R(q) = q$ and C is symmetric about H by assumption. Since q is an interior point of τ_K and no two faces share interior points, it follows that $R(\tau_K) = \tau_K$.

Now consider $p \in \tau_K^\circ \cap X_K^\circ$, which exists by the definition of τ_K. By convexity and the fact that τ_K is symmetric about H, τ_K contains the segment between $R(p)$ and p in the direction v_q. R does not fix any point in X_K°, in particular it does not fix $p \in X_K^\circ$. So the segment from $R(p)$ to p has non-zero length.

Now let L be the ray based at p and in the direction v_q. Note that $L \subset X_K^\circ$ since $v_q \in K$ and $p \in X_K^\circ$. Since τ_K is bounded, contains p in its interior, is convex, and contains a segment in the direction v_q, we see that $\partial\tau_K \cap L$ is not empty. Thus, $\partial\tau_K$ contains a face ρ_K with $\rho_K^\circ \cap X_K^\circ \neq \emptyset$. Since τ_K contains a segment of non-zero length, it has dimension at least 1 and so $\dim(\rho_K) < \dim(\tau_K)$. This contradicts the minimality of the dimension of τ_K, and so we have shown that $\tau_K^\circ \subset Q_K^\circ$. □

We can now prove the main theorem.

Proof of Theorem 3. It suffices to show that $\tau_K = \tau_{K'}$ if and only if $K = K'$. Suppose $K \neq K'$ and, without loss of generality, that $\dim(X_K) \geqslant \dim(X_{K'})$. Then there is some $v \in K$ with $v \notin K'$. So for every $p' \in X_{K'}$ we have $p' \cdot v \leqslant 0$. This means that $X_{K'}$ is disjoint from Q_K°. Therefore, by the previous lemma τ_K° is disjoint from $X_{K'}$ and so $\tau_K \neq \tau_{K'}$. This completes the proof of Theorem 3. □

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References

[1] I. Bárány and L. Lovász, Borsuk's theorem and the number of facets of centrally symmetric polytopes, *Acta Mathematica Academiae Scientiarum Hungarica.* 40(3): 323—329, 1982.

[2] R. Freij, M. Henze, M. W. Schmitt, and G. M. Ziegler, Face numbers of centrally symmetric polytopes produced from split graphs, *The Electronic Journal of Combinatorics.* 20(2), 2013.

[3] G. Kalai, The number of faces of centrally-symmetric polytopes. *Graphs and Combinatorics*, 5(1): 389—391, 1989.

[4] R. Sanyal, A. Werner, and G. M. Ziegler, On Kalai's conjectures concerning centrally symmetric polytopes. *Discrete and Computational Geometry*, 41(2): 183—198, 2009.

[5] R. Sanyal and M. Winter, Kalai's 3d-conjecture for unconditional and locally anti-blocking polytopes, *Proceedings of the AMS.* 153(1), 2024.

[6] R. P. Stanley, On the number of faces of centrally-symmetric simplicial polytopes. *Graphs and Combinatorics*, 3(1): 55—66, 1987.