On the Weisfeiler-Leman dimension of some polyhedral graphs

Haiyan Li[a] Ilia Ponomarenko[a,b] Peter Zeman[c]

Submitted: Mar 6, 2025; Accepted: Jan 21, 2026; Published: Aug 7, 2026

Abstract

Let m be a positive integer, X a graph with vertex set Ω, and $\mathrm{WL}_m(X)$ the coloring of the Cartesian m-power Ω^m, obtained by the m-dimensional Weisfeiler-Leman algorithm. The WL-dimension of the graph X is defined to be the smallest m for which the coloring $\mathrm{WL}_m(X)$ determines X up to isomorphism. It is known that the WL-dimension of any planar graph is 2 or 3, but no planar graph of WL-dimension 3 is known. We prove that the WL-dimension of a polyhedral (i.e., 3-connected planar) graph X is at most 2 if the color classes of the coloring $\mathrm{WL}_2(X)$ are the orbits of the componentwise action of the group $\mathrm{Aut}(X)$ on Ω^2.

Mathematics Subject Classifications: 05E18, 05E30, 05C60, 05C50

1 Introduction

The *WL-dimension* $\dim_{\mathrm{WL}}(X)$ of a graph X is a numerical invariant defined (implicitly) by M. Grohe in [8] in connection with the Graph Isomorphism Problem; the name refers to the Weisfeiler-Leman algorithm which constructs a canonical coloring of the pairs of vertices of X. Roughly speaking, the WL-dimension is equal to the minimal positive integer m such that the graph X can be identified up to isomorphism in a fragment of the first-order logic with counting quantifiers and m variables. In general, $1 \leqslant \dim_{\mathrm{WL}}(X) \leqslant n$, where n is the vertex number of X.

The graphs of WL-dimension 1 have completely been characterized in [1] and independently in [14]. A complete description of the graphs of WL-dimension 2 seems to be hopeless (it would imply a description of all strongly regular graphs identified by the parameters up to isomorphism); the difficulties that arise on the way to obtaining such a description are clearly visible in a very interesting paper [6]. It should be mentioned that there are graphs of arbitrary large WL-dimension, see [3].

[a] School of Science, Hainan University, Haikou, China lhy9694@163.com

[b] Steklov Institute of Mathematics at St. Petersburg, Russia inp@pdmi.ras.ru

[c] Department of Algebra, Faculty of Mathematics and Physics, Charles University, Czechia (peter.zeman@matfyz.cuni.cz)

In the last decade, quite a lot of papers have appeared in which the WL-dimension is calculated (or estimated from above by some small constant) for graphs belonging to a certain class. Not being able to give a detailed review of these papers, we will focus here only on paper [13], which is directly related to what follows. It is proved there that the WL-dimension of every planar graph is at most 3. It is quite interesting that neither [13] nor more recent paper [12] contain an example of a planar graph X with $\dim_{\text{WL}}(X) = 3$. An initial motivation of the present paper was to understand whether such examples exist among the *polyhedral* (i.e., planar and 3-connected) graphs.

With each (polyhedral) graph X with vertex set Ω, one can associate a *coherent configuration* $\text{WL}(X)$, which can be thought of as a partition S of the Cartesian square Ω^2 into the color classes of the canonical coloring obtained by the Weisfeiler-Leman algorithm applied to X (the exact definitions are given in Section 2). One of the main results of [12] shows that if the edge set E of the polyhedral graph X is a class of the partition S, then X is edge transitive, i.e., $E \in \text{Orb}(G, \Omega^2)$, where $G = \text{Aut}(X)$. All edge transitive polyhedral graphs are known and one can verify that for each of them the coherent configuration $\text{WL}(X)$ is schurian, which means that each class of S belongs to $\text{Orb}(G, \Omega^2)$.

A graph X is said to be *schurian* if the coherent configuration $\text{WL}(X)$ is schurian. Note that almost all graphs are schurian (see, e.g., [2]), and all graphs with at most 13 vertices are schurian (see [17]). In particular, this covers all the Platonic graphs except for the dodecahedral graph, for which this can be verified by a straightforward computation. We believe that every polyhedral graph is schurian, and if this is true, then by Theorem 1 below, the WL-dimension of any polyhedral graph is at most 2.

Theorem 1. *Let X be a polyhedral graph. If X is schurian, then* $\dim_{\text{WL}}(X) \leqslant 2$.

It was proved in [6, Theorem 2.5] that for an arbitrary graph X, we have $\dim_{\text{WL}}(X) \leqslant 2$ if and only if the coherent configuration $\mathcal{X} = \text{WL}(X)$ is separable, i.e., every algebraic isomorphism from $\mathcal{X}$ to another coherent configuration is induced by an isomorphism. Assume that X is schurian and $G = \text{Aut}(X)$. Then $\mathcal{X}$ coincides with the coherent configuration $\text{Inv}(G)$ corresponding to the partition of Ω^2 into the orbits of the group $G \leqslant \text{Sym}(\Omega^2)$. If, in addition, X is polyhedral, then G is a finite spherical group, i.e., a finite subgroup of the orthogonal group $O(3, \mathbb{R}^3)$, see, e.g., [15]. It follows that, in this case, $\mathcal{X}$ is the coherent configuration of a *strongly spherical* group, by which we mean a spherical group which is the full automorphism group of a polyhedral graph. Thus Theorem 1 immediately follows from the theorem below.

Theorem 2. *Let G be a strongly spherical permutation group. Then the coherent configuration* $\text{Inv}(G)$ *is separable.*

The proof of Theorem 2 is given in Section 5. The main idea of the proof is to use a recent classification of finite spherical permutation groups [15]. Based on some sufficient conditions for a coherent configuration to be separable (Section 3), we reduce the proof to the study of coherent configurations of spherical groups G that, except for finitely many groups, are isomorphic to C_n, D_{2n}, $C_n \times C_2$, and $D_{2n} \times C_2$. First, we prove in a more or less standard way that the coherent configuration $\text{Inv}(G)$ is separable if G belongs to the infinite families (Theorem 18). The same idea supplied by computer calculations works for some groups not belonging to these families (Theorem 19).

In the remaining cases, G is isomorphic to $\mathrm{Sym}(4)$, $\mathrm{Alt}(5)$, or $\mathrm{Sym}(4)\times C_2$. To manage these cases, we introduce in Section 4 a concept of spherical representation of a coherent configuration and prove that under some natural conditions, a coherent configuration admitting such a representation in dimension 3 is separable (Theorem 16). Again, using computer calculations we prove that in the considered cases, the coherent configurations $\mathrm{Inv}(G)$ satisfy the above conditions (Theorem 20) and hence are separable.

In order to make the paper as self-contained as possible, we present in Section 2 the necessary notation and facts from the theory of coherent configurations. More details about this theory can be found in monograph [4].

2 Coherent configurations

2.1 Notation

Throughout the paper, Ω denotes a finite set. For $\Delta \subseteq \Omega$, the Cartesian product $\Delta \times \Delta$ and its diagonal are denoted by $\mathbf{1}_\Delta$ and 1_Δ, respectively. For $s \subseteq \mathbf{1}_\Omega$, we set $s^* = \{(\alpha,\beta) : (\beta,\alpha) \in s\}$, and $\alpha s = \{\beta \in \Omega : \ (\alpha,\beta) \in s\}$ for all $\alpha \in \Omega$. The composition $r \cdot s$ of the relations $r, s \subseteq \mathbf{1}_\Omega$ consists of all pairs $(\alpha,\beta) \in \mathbf{1}_\Omega$ such that $(\alpha,\gamma) \in r$ and $(\gamma,\beta) \in s$ for some $\gamma \in \Omega$. For any collection S of relations, we denote by $S^\cup$ the set of all unions of elements of S, and consider $S^\cup$ as a poset with respect to inclusion.

The set of classes of an equivalence relation e on Ω is denoted by Ω/e. For $\Delta \subseteq \Omega$, we set $\Delta/e = \Delta/e_\Delta$ where $e_\Delta = \mathbf{1}_\Delta \cap e$. If the classes of e_Δ are singletons, Δ/e is identified with Δ.

We use standard notations $\mathrm{Sym}(n)$, $\mathrm{Alt}(n)$, C_n, and D_{2n} for the symmetric and alternating groups of degree n, and for cyclic and dihedral groups of orders n and $2n$, respectively.

The algebra of all square matrices with entries belonging to a field F and rows and columns indexed by the elements of Ω is denoted by $\mathrm{Mat}_\Omega(F)$. The points of Ω are treated as vectors of the F-linear space spanned by Ω. In particular, if $\alpha \in \Omega$ and $A \in \mathrm{Mat}_\Omega(F)$, then $A\alpha$ is the α-column of A.

The Hadamard product $A \circ B$ of the matrices $A, B \in \mathrm{Mat}_\Omega(\mathbb{R})$ is defined by the formula $(A \circ B)_{\alpha,\beta} = A_{\alpha,\beta}B_{\alpha,\beta}$ for all $\alpha,\beta \in \Omega$. The adjacency matrix of a relation $s \subseteq \mathbf{1}_\Omega$ is denoted by A_s. Clearly, A_{s^*} is equal to the transpose $(A_s)^*$ of the matrix A_s.

2.2 Rainbows

Let S be a partition of Ω^2. A pair $\mathcal{X} = (\Omega, S)$ is called a *rainbow* on Ω if

(C1) $1_\Omega \in S^\cup$,

(C2) $s^* \in S$ for all $s \in S$.

The rainbow $\mathcal{X}$ is said to be *symmetric* if $s^* = s$ for all $s \in S$. The numbers $|\Omega|$ and $\mathrm{rk}(\mathcal{X}) = |S|$ are called the *degree* and *rank* of $\mathcal{X}$. The elements of S and of $S^\cup$ are called *basis relations* and *relations* of $\mathcal{X}$. A unique basis relation containing the pair (α, β) is denoted by $r(\alpha, \beta)$. The set of all relations is closed with respect to intersections and unions.

Let $\mathcal{X}' = (\Omega', S')$ be a rainbow. A *combinatorial isomorphism* or, briefly, *isomorphism* from $\mathcal{X}$ to $\mathcal{X}'$ is defined to be a bijection $f : \Omega \to \Omega'$ such that the relation $s^f = \{(\alpha^f, \beta^f) : (\alpha, \beta) \in s\}$ belongs to S' for all $s \in S$. In this case, the rainbows are said to be *isomorphic*; the set of all isomorphisms from $\mathcal{X}$ to $\mathcal{X}'$ is denoted by $\mathrm{Iso}(\mathcal{X}, \mathcal{X}')$. The group of all isomorphisms from $\mathcal{X}$ to itself contains a normal subgroup

$$\mathrm{Aut}(\mathcal{X}) = \{f \in \mathrm{Sym}(\Omega) : \ s^f = s \text{ for all } s \in S\}$$

called the *automorphism group* of $\mathcal{X}$.

There is a natural partial order $\leqslant$ on the set of all rainbows on Ω. Namely, given two such rainbows $\mathcal{X}$ and $\mathcal{Y}$, we set

$$\mathcal{X} \leqslant \mathcal{Y} \ \Leftrightarrow \ S(\mathcal{X})^\cup \subseteq S(\mathcal{Y})^\cup.$$

The minimal and maximal elements with respect to this order are the *trivial* and *discrete* rainbows, respectively; in the former case, S consists of 1_Ω and its complement (unless Ω is not a singleton), and in the last case, S consists of singletons.

2.3 Coherent configurations

A rainbow $\mathcal{X}$ is called a *coherent configuration* if, in addition to conditions (C1) and (C2), it satisfies the condition

(C3) given $r, s, t \in S$, the number $c_{r,s}^t = |\alpha r \cap \beta s^*|$ does not depend on $(\alpha, \beta) \in t$.

We say that the c_{rs}^t are *intersection numbers* of $\mathcal{X}$ and extend the notation by admitting r and s to be arbitrary relations of $\mathcal{X}$: $c_{r,s}^t = \sum_{r',s'} c_{r',s'}^t$, where r', s' run over the basis relations contained in r and s, respectively. The coherent configuration is said to be a *scheme* if $1_\Omega \in S$, and *commutative* if $c_{r,s}^t = c_{s,r}^t$ for all $r, s, t \in S$.

Let $s \in S$. Then given a reflexive $t \in S$, the intersection number $n_s = c_{s,s^*}^t$ is either zero or is equal to $|\alpha s|$ for all α with $\alpha s \neq \varnothing$; this number is called the *valency* of s. We say that s is *thin* if $n_s = n_{s^*} = 1$. A coherent configuration is said to be *semiregular* if every basis relation is thin, and *quasi-thin* if $n_s \leqslant 2$ for all $s \in S$; a semiregular scheme is said to be *regular*.

Any $\Delta \subseteq \Omega$ such that $1_\Delta \in S$ is called a *fiber* of $\mathcal{X}$. In view of condition (C1), the set of all fibers forms a partition of Ω. Every basis relation is contained in the Cartesian product of two uniquely determined fibers. Any union of fibers is called a *homogeneity set* of $\mathcal{X}$. For any two homogeneous sets Δ and Γ, we denote by $S_{\Delta,\Gamma}$ the set of all basis relations contained in $\Delta \times \Gamma$, and abbreviate $S_\Delta = S_{\Delta,\Delta}$.

Let $G \leqslant \mathrm{Sym}(\Omega)$. Denote by S the set of all orbits $(\alpha, \beta)^G$ in the componentwise action of G on $\Omega \times \Omega$, where $\alpha, \beta \in \Omega$. Then the pair (Ω, S), which we denote by $\mathrm{Inv}(G, \Omega)$ or simply $\mathrm{Inv}(G)$ if Ω is clear from the context, is a coherent configuration. Any coherent configuration $\mathcal{X}$ associated with permutation group G in this way is said to be *schurian.* In this case every fiber Δ of $\mathcal{X}$ is an orbit of G, and the points of Δ can be identified with the right cosets of a point stabilizer G_δ with $\delta \in \Delta$, so that the action of G on Δ is induced by right multiplications. Note that a coherent configuration $\mathcal{X}$ is schurian if and only if $\mathcal{X} = \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}))$. A coherent configuration $\mathrm{Inv}(G, \Omega)$ is quasi-thin if $|G_\alpha| \leqslant 2$ for all $\alpha \in \Omega$.

A bijection $\varphi : S \to S'$ is called an *algebraic isomorphism* from $\mathcal{X}$ to $\mathcal{X}'$ if for all $r, s, t \in S$, we have

$$c_{r^\varphi, s^\varphi}^{t^\varphi} = c_{r,s}^t.$$

In this case, $|\Omega'| = |\Omega|$. The algebraic isomorphism φ is extended in a natural way to a poset isomorphism $S^\cup \to (S')^\cup$, denoted also by φ. Thus, φ respects the set theoretical operations on $S^\cup$, and $1_{\Omega'} = \varphi(1_\Omega)$.

Let Φ be a group of algebraic automorphisms of the coherent configuration $\mathcal{X}$. Denote by S^Φ the set of all relations $\cup_{\varphi\in\Phi}\varphi(s)$, $s \in S$, Then the pair $\mathcal{X}^\Phi = (\Omega, S^\Phi)$ is a coherent configuration called an *algebraic fusion* of $\mathcal{X}$.

Every isomorphism $f : \mathcal{X} \to \mathcal{X}'$ *induces* the algebraic isomorphism $\varphi_f : \mathcal{X} \to \mathcal{X}'$, $s \mapsto s^f$. Not every algebraic isomorphism is induced by an isomorphism. The coherent configuration $\mathcal{X}$ is said to be *separable* if every algebraic isomorphism from $\mathcal{X}$ is induced by an isomorphism. All trivial and discrete coherent configurations are separable.

2.4 Adjacency algebra

Let $\mathcal{X}$ be a coherent configuration on Ω. Then the linear space $\mathcal{A}(\mathcal{X})$ consisting of $\mathbb{C}$-linear combinations of the matrices A_s, $s \in S$, is a subalgebra of the full matrix algebra $\mathrm{Mat}_\Omega(\mathbb{C})$. The algebra $\mathcal{A}(\mathcal{X})$, called the *adjacency algebra* of $\mathcal{X}$, contains the all ones matrix, and is closed with respect to transpose and the Hadamard multiplication.

Every algebraic isomorphism $\varphi : \mathcal{X} \to \mathcal{X}'$ induces a matrix algebra isomorphism $\widetilde{\varphi} : \mathcal{A}(\mathcal{X}) \to \mathcal{A}(\mathcal{X}')$ such that $\varphi(A_s) = A_{\varphi(s)}$ for all $s \in S$. The isomorphism $\widetilde{\varphi}$ respects the transpose and the Hadamard product. In what follows, we simplify notation and write φ instead of $\widetilde{\varphi}$.

Lemma 3. *In the above notation, let $A \in \mathcal{A}(\mathcal{X})$ and $A' = \varphi(A)$. Then*

(1) $\{A_{\alpha,\beta} :\ \alpha, \beta \in \Omega\} = \{A'_{\alpha',\beta'} :\ \alpha', \beta' \in \Omega'\}$,

(2) *A is symmetric if and only if so is A',*

(3) *A and A' have the same eigenvalues,*

(4) *$A\alpha = A\beta$ for some $\alpha \neq \beta$ if and only if $A'\alpha' = A'\beta'$ for some $\alpha' \neq \beta'$.*

Proof. (1) Denote by W and W' the sets in the left- and right-hand sides of the required equality, respectively. For each $w \in W$, denote by s_w the relation of $\mathcal{X}$ consisting of all pairs (α,β) such that $(A)_{\alpha,\beta}=w$. Then $A_{s_w}\in\mathcal{A}(\mathcal{X})$ and

$$\sum_{w'\in W'} w'A_{s_{w'}} = A' = \varphi(A) = \sum_{w\in W} w\,\varphi(A_{s_w}) = \sum_{w\in W} wA_{\varphi(s_w)}, \tag{1}$$

which shows that $W=W'$, because $A_{\varphi(s_w)}$ and $A_{s_{w'}}$ are $\{0,1\}$ matrices for all w and w'.

(2) It is known that $\varphi(s^*)=\varphi(s)^*$ for all $s\in S$. It follows that $\varphi(A)^*=\varphi(A^*)$. Since A is symmetric if and only if $A=A^*$, we are done.

(3) Since φ is a matrix algebra isomorphism, the matrices A and A' have the same minimal polynomial. Thus A and A' have the same eigenvalues.

(4) The equality $A\alpha=A\beta$ means that $A_{\gamma,\alpha}=A_{\gamma,\beta}$ for every $\gamma\in\Omega$. The last equality holds if and only if $(\gamma,\alpha),(\gamma,\beta)\in s_w$ for some $w\in W$. However, the number of all $\gamma\in\Omega$ for which $(\gamma,\alpha),(\gamma,\beta)\in s_w$ is equal to the intersection number $c^r_{(s_w)^*,s_w}$, where $r=r(\alpha,\beta)$. Thus,

$$A\alpha=A\beta \;\Leftrightarrow\; \sum_{w\in W} c^r_{(s_w)^*,s_w}=|\Omega|.$$

Since the right-hand side equality is preserved by φ, we are done. $\square$

2.5 Parabolics and quotients

A relation of $\mathcal{X}$ that is an equivalence relation on Ω is called a *parabolic* of $\mathcal{X}$; the set of all parabolics is denoted by $E=E(\mathcal{X})$. Given $e\in E$, we put $S_{\Omega/e}=\{s_{\Omega/e}:\ s\in S\}$, and put $S_\Delta=\{s_\Delta:\ s\in S,\ s_\Delta\neq\varnothing\}$ for any $\Delta\in\Omega/e$. The pairs

$$\mathcal{X}_{\Omega/e}=(\Omega/e,S_{\Omega/e})\quad\text{and}\quad \mathcal{X}_\Delta=(\Delta,S_\Delta)$$

are coherent configurations called a *quotient* of $\mathcal{X}$ modulo e and a *restriction* of $\mathcal{X}$ to Δ, respectively. Note that the notation $\mathcal{X}_\Delta$ is correct for every homogeneity set Δ, because such Δ is a class of an obvious parabolic with two classes.

Let $\varphi:\mathcal{X}\to\mathcal{X}'$ be an algebraic isomorphism and $e\in E$. Then $\varphi(e)$ is a parabolic of $\mathcal{X}'$ with the same number of classes. Moreover, φ induces the algebraic isomorphism

$$\varphi_{\Omega/e}:\mathcal{X}_{\Omega/e}\to\mathcal{X}'_{\Omega'/e'},\ s_{\Omega/e}\mapsto s'_{\Omega'/e'}, \tag{2}$$

where $e'=\varphi(e)$ and $s'=\varphi(s)$.

2.6 Coherent closure

The *coherent closure* $\mathrm{WL}(T)$ of a set T of binary relations on Ω, is defined to be the smallest coherent configuration on Ω, for which each relation of T is the union of some basis relations. When T consists of the arc set of a graph X (respectively the basis relations of a rainbow $\mathcal{X}$), we denote $\mathrm{WL}(T)$ by $\mathrm{WL}(X)$ (respectively, $\mathrm{WL}(\mathcal{X})$). Every isomorphism from a rainbow $\mathcal{X}$ to a rainbow $\mathcal{X}'$ is also an isomorphism from $\mathrm{WL}(\mathcal{X})$ to $\mathrm{WL}(\mathcal{X}')$.

3 Separable coherent configurations

In this section, we recall and prove some facts about separable coherent configurations. The first of them is just a reformulation of [4, Theorem 3.3.19].

Lemma 4. *Let G be a permutation group. Assume that at least one of the orbits of G is faithful and regular. Then the coherent configuration* $\mathrm{Inv}(G)$ *is separable.*

Let Δ and Γ be orbits of a group $G \leqslant \mathrm{Sym}(\Omega)$. We say that Δ is *dominated* by Γ, or that Γ *dominates* Δ, if for any $\delta \in \Delta$ there is $\gamma \in \Gamma$ such that $G_\gamma \leqslant G_\delta$. In this case, $|\Delta| \leqslant |\Gamma|$, and we write $\Delta \preceq \Gamma$ or $\Gamma \succeq \Delta$. Obviously, a singleton orbit is dominated by any other orbit, and a faithful regular orbit dominates any other orbit.

One can see that $\Gamma \succeq \Delta$ if and only if there is $s \in \mathrm{Orb}(G, \Gamma \times \Delta)$ such that $|\gamma s| = 1$ for all $\gamma \in \Gamma$. It is also clear that if Δ and Γ are *equivariant* orbits of G (i.e., the permutation groups G^Δ and G^Γ are equivalent), then each of them is dominated by the other. Note that the relation "to be dominated" is transitive. The following lemma is a special case of [4, Lemma 3.3.20].

Lemma 5. *Let $G \leqslant \mathrm{Sym}(\Omega)$ and $\Delta \in \mathrm{Orb}(G)$. Assume that Δ is dominated by another orbit. Then the coherent configuration* $\mathrm{Inv}(G)$ *is separable if and only if the coherent configuration $\mathcal{X} = \mathrm{Inv}(G, \Omega \setminus \Delta)$ is separable. Moreover,* $\mathrm{Aut}(\mathcal{X}) \cong \mathrm{Aut}(\mathrm{Inv}(G))$.

The hypothesis of Lemma 5 is obviously satisfied if Δ is a singleton. Furthermore, assume that $\Delta \neq \Omega$ is of cardinality 2 and is not dominated by another orbit. Then $\alpha s = \Delta$ for all $\alpha \notin \Delta$ and all $s \in S$ such that $\alpha s \cap \Delta \neq \varnothing$. This means that the coherent configuration $\mathrm{Inv}(G)$ is the direct sum (in the sense of [4, Subsection 3.2.1]) of the coherent configurations $\mathrm{Inv}(G, \Delta)$ and $\mathrm{Inv}(G, \Omega \setminus \Delta)$. Since also any homogeneous two-point coherent configuration is trivial and hence is separable, we conclude that $\mathrm{Inv}(G, \Delta)$ is separable if so is $\mathrm{Inv}(G, \Omega \setminus \Delta)$. It follows that the coherent configuration $\mathrm{Inv}(G)$ is separable if and only if the coherent configuration $\mathrm{Inv}(G, \Omega \setminus \Delta)$ is separable, see [4, Corollary 3.2.8]. Thus we arrive to the following statement.

Corollary 6. *Let $G \leqslant \mathrm{Sym}(\Omega)$ and $\Delta \in \mathrm{Orb}(G)$. Assume that $|\Delta| = 1$ or 2. Then the coherent configuration* $\mathrm{Inv}(G)$ *is separable if and only if the coherent configuration* $\mathrm{Inv}(G, \Omega \setminus \Delta)$ *is separable.*

The group G is said to be *domination free* if no orbit Δ of G dominates an orbit other than Δ. In this case, no orbit of G is faithful regular or a singleton (Corollary 6), and no two distinct orbits of G are equivariant.

Lemma 7. *Let $G \cong D_{2n}$ be a permutation group, $n \geqslant 2$. Assume that every orbit of G is faithful of cardinality n. Then* $\mathrm{Inv}(G)$ *is separable.*

Proof. By Lemma 5, we may assume that G is domination free. Then G has no equivariant orbits and hence G has one or two orbits depending on whether n is odd or even. First, assume that G has exactly one orbit. Then $\mathrm{Inv}(G)$ is a scheme, and even a quasi-thin one, because $n_s \leqslant |G_\alpha| = 2$ for all its points α and all its basis relations s. Moreover, the number n_1 of thin basis relations of $\mathrm{Inv}(G)$ is equal to 1 or 2 depending on whether n is even or odd. On the other hand, according to [18, Theorem 1.1], a quasi-thin scheme of degree n is separable if $n_1 < 4$. Thus the scheme $\mathrm{Inv}(G)$ is separable.

Now assume that G has exactly two orbits, say Δ and Γ. Then they are faithful and non-equivariant, and n is even. Take points $\delta \in \Delta$ and $\gamma \in \Gamma$, and identify the sets Δ and Γ with the right cosets of the subgroups G_δ and G_γ respectively. Each of these cosets contains a unique element belonging to the subgroup $C_n \leqslant G$. The mappings

$$\Delta \to \Gamma,\ G_\delta c \mapsto G_\gamma c \quad \text{and} \quad \Gamma \to \Delta,\ G_\gamma c \mapsto G_\delta c,$$

where $c \in C_n$, yield a permutation $f \in N_{\mathrm{Sym}(\Omega)}(G)$ interchanging Δ and Γ. Denote by S the set of basis relations of the coherent configuration $\mathcal{X} = \mathrm{Inv}(G)$. A straightforward computation shows that f induces an involution $\varphi \in \mathrm{Sym}(S)$ that interchanges S_Δ and S_Γ, and such that

$$s^\varphi = s^* \quad \text{for all} \ \ s \in S_{\Delta,\Gamma} \cup S_{\Gamma,\Delta},$$

in particular, f and φ are respectively, an isomorphism and an algebraic isomorphism from $\mathcal{X}$ to itself, and $\Phi = \langle \varphi \rangle$ is a group of order 2.

The algebraic fusion $\mathcal{X}^\Phi$ is a schurian scheme associated with permutation group generated by G and f. Every point stabilizer of the latter group is also a point stabilizer of G, and hence is of order 2. Therefore the scheme $\mathcal{X}^\Phi$ is quasi-thin. It has exactly one thin basis relation, and using [18, Theorem 1.1] again, we conclude that $\mathcal{X}^\Phi$ is separable. This implies that so is $\mathcal{X}$ by [4, Theorem 3.1.29]. □

We conclude this section with a very special sufficient condition for a coherent configuration to be separable. To formulate the corresponding statement, we need some preparation.

Let $\mathcal{X}$ be a schurian coherent configuration on Ω and $\sigma \in \mathrm{Aut}(\mathcal{X})$ a central involution. Put $\Delta = \{\alpha \in \Omega :\ \alpha^\sigma \neq \alpha\}$ and denote by $e = e(\sigma)$ the equivalence relation with classes $\{\alpha, \alpha^\sigma\}$, $\alpha \in \Omega$. Then Δ is a homogeneity set of $\mathcal{X}$ and e is a parabolic of $\mathcal{X}$ such that $e_{\Omega\setminus\Delta} = 1_{\Omega\setminus\Delta}$. The surjection

$$\pi : \Omega \to \Omega/e, \quad \alpha \mapsto \alpha e, \tag{3}$$

defines the quotient coherent configuration $\mathcal{X}_{\Omega/e} = (\Omega/e, S_{\Omega/e})$ of $\mathcal{X}$ modulo e, where $S_{\Omega/e} = \{\pi(s) :\ s \in S\}$.

We say that a parabolic $e^\perp$ of the coherent configuration $\mathcal{X}_\Delta$ is a *complement* of e if any class of $e^\perp$ intersects any class of e_Δ in exactly one point; in this case the parabolic $e^\perp$ has exactly two classes. If the complement $e^\perp$ exists, then for every $s \in S$, we have

$$s = \begin{cases} 1_{\Delta(s)} \text{ or } e_{\Delta(s)} \setminus 1_{\Delta(s)} & \text{if } s \subseteq e_\Delta, \\ s_0 \setminus e^\perp \text{ or } s_0 \cap e^\perp & \text{if } s \in \mathbf{1}_\Delta \setminus 1_\Delta, \\ s_0 & \text{otherwise.} \end{cases} \tag{4}$$

where $\Delta(s)$ is the support of s and $s_0 = \pi^{-1}(\pi(s))$. Moreover, one can see that if $\Delta = \Omega$, then $\mathcal{X}$ is isomorphic to the tensor product of $\mathcal{X}_{\Delta/e}$ and $\mathcal{X}_{\Delta/e^\perp}$, see [5, Theorem 6], and the lemma below is a special case of [4, Corollary 3.2.24],

Lemma 8. *Let $\mathcal{X}$ be a schurian coherent configuration and $\sigma \in \mathrm{Aut}(\mathcal{X})$ a central involution. Assume that the parabolic $e = e(\sigma)$ has a complement and the coherent configuration $\mathcal{X}_{\Omega/e}$ is separable. Then $\mathcal{X}$ is also separable.*

Proof. Let $\varphi : \mathcal{X} \to \mathcal{X}'$ be an algebraic isomorphism. Then $e' = \varphi(e)$ is a parabolic of $\mathcal{X}'$ and φ induces the algebraic isomorphism $\bar{\varphi} = \varphi_{\Omega/e}$, see (2). Since the coherent configuration $\mathcal{X}_{\Omega/e}$ is separable, there is a bijection

$$\bar{f} : \Omega/e \to \Omega'/e'$$

inducing $\bar{\varphi}$, where Ω' is the point set of $\mathcal{X}'$. Put $\Delta' = \Delta^\varphi$ and denote by $e^\perp$ a complement to e. Then ${e'}^\perp = \varphi(e^\perp)$ is a complement of the parabolic e' in the coherent configuration $\mathcal{X}'_{\Delta'}$. Let Δ_1 and Δ_2 be the classes of $e^\perp$, and let Δ'_1 and Δ'_2 be the classes of ${e'}^\perp$. Then $\Delta = \Delta_1 \cup \Delta_2$ and $\Delta' = \Delta'_1 \cup \Delta'_2$.

For any point $\alpha \in \Omega$, we define a point $\alpha' \in \Omega'$ as follows. If $\alpha \notin \Delta$, then $\{\alpha\}$ is a class of e and $\{\alpha\}^{\bar{f}} = \{\alpha'\}$ for some $\alpha' \notin \Delta'$. Now, let $\alpha \in \Delta$. Then there exist $i \in \{1, 2\}$ and $\Lambda \in \Omega/e$ such that $\{\alpha\} = \Delta_i \cap \Lambda$. Denote by α' the point of Δ' for which $\{\alpha'\} = \Delta'_i \cap \Lambda'$ with $\Lambda' = \Lambda^{\bar{f}}$. Now the mapping

$$f : \Omega \to \Omega', \ \alpha \mapsto \alpha'$$

is a bijection taking Δ_1 to Δ'_1, Δ_2 to Δ'_2, and the complement of Δ to the complement of Δ'. Let us verify that f induces φ.

From the definition of f, it follows that $e^f = e' = \varphi(e)$ and $(e^\perp)^f = {e'}^\perp = \varphi(e^\perp)$. In view of formula (4), it therefore suffices to prove that for any $s \in S$,

$$(1_{\Delta(s)})^f = \varphi(1_{\Delta(s)}) \quad \text{and} \quad (s_0)^f = \varphi(s)_0,$$

where $\varphi(s)_0 = {\pi'}^{-1}(\pi'(\varphi(s))$. However, $\varphi(1_{\Delta(s)}) = 1_{\Delta(\varphi(s))}$ and $\Delta(\varphi(s)) = \Delta(s^f)$. Thus,

$$(1_{\Delta(s)})^f = 1_{\Delta(s^f)} = 1_{\Delta(\varphi(s))} = \varphi(1_{\Delta(s)}),$$

which proves the first equality. Finally, $\pi^{-1}(\pi(s))^f = {\pi'}^{-1}(\pi'(s^f))$, because $e^f = e'$. Since $f^{\Omega/e} = \bar{f}$ and $\bar{f}$ induces $\bar{\varphi}$, we obtain

$$(s_0)^f = \pi^{-1}(\pi(s))^f = {\pi'}^{-1}(\pi'(s^f)) = {\pi'}^{-1}(\pi(s)^{\bar{f}})) =$$

$${\pi'}^{-1}(\bar{\varphi}(\pi(s^f))) = {\pi'}^{-1}(\pi'(\varphi(s))) = \varphi(s)_0,$$

as required. ∎

4 Spherical representations of coherent configurations

Let $d \geqslant 1$ be an integer and Ω a finite set. Let $\rho : \Omega \to \mathbb{R}^{d+1}$ be an injective mapping taking a point $\alpha \in \Omega$ to a point $\rho(\alpha)$ of the real d-dimensional unit sphere $\mathbb{S}_d \subset \mathbb{R}^{d+1}$ centered at 0. Denote by $A_\rho \in \mathrm{Mat}_\Omega(\mathbb{R})$ the Gram matrix of the vectors $\rho(\alpha)$,

$$(A_\rho)_{\alpha,\beta} = \langle \rho(\alpha),\, \rho(\beta)\rangle, \quad \alpha, \beta \in \Omega,$$

where $\langle\, .\,,\, .\rangle$ is the scalar product in $\mathbb{R}^{d+1}$. Thus A_ρ is a symmetric matrix with unit diagonal.

We say that the mapping ρ is a d-dimensional *spherical representation*, or, briefly, an $\mathbb{S}_d$*-representation* of a coherent configuration $\mathcal{X}$ on Ω, if the matrix A_ρ belongs to the adjacency algebra $\mathcal{A}(\mathcal{X})$. The following lemma gives a simple sufficient condition for $\mathcal{X}$ to have an $\mathbb{S}_d$-representation. In what follows, $\|\cdot\|$ denotes the standard Euclidean norm.

Lemma 9. *Let $P \in \mathcal{A}(\mathcal{X})$ be a real symmetric idempotent matrix of rank $d+1$ without zero columns and such that for all $\alpha, \beta \in \Omega$,*

$$\alpha \neq \beta \quad \Rightarrow \quad \frac{P\alpha}{\|P\alpha\|} \neq \frac{P\beta}{\|P\beta\|}. \tag{5}$$

Then the mapping $\rho_P : \alpha \mapsto \frac{P\alpha}{\|P\alpha\|}$, $\alpha \in \Omega$, is an $\mathbb{S}_d$-representation of $\mathcal{X}$.

Proof. It suffices to verify that $A_\rho \in \mathcal{A}(\mathcal{X})$, where $\rho = \rho_P$. Since P is symmetric and $P^2 = P$, we have

$$P_{\alpha,\beta} = \|P\alpha\|\,\|P\beta\|\,\langle \rho(\alpha),\, \rho(\beta)\rangle$$

for all $\alpha, \beta \in \Omega$. It follows that $A_\rho = DPD$, where D is the diagonal matrix with entries $D_{\alpha,\alpha} = \frac{1}{\|P\alpha\|}$, $\alpha \in \Omega$. Note that $D = I \circ P$, where I is the identity matrix. Since I and P belong to $\mathcal{A}(\mathcal{X})$, this algebra contains D, and hence contains $A_\rho = DPD$, as required. $\square$

Every coherent configuration has a trivial $(n-1)$-dimensional spherical representation with $n = |\Omega|$; in this representation, $\rho(\alpha)$ is a $\{0,1\}$ vector the only nonzero entry of which is on the place α. Of particular interest are low dimensional spherical representations.

Corollary 10. *Let $\mathcal{X}$ be the coherent configuration of a polyhedral graph which has the multiplicity of the second minimal eigenvalue of the generalized Laplacian equal to 3. Then $\mathcal{X}$ admits an $\mathbb{S}_d$-representation with $d \leqslant 2$.*

Proof. Let $\mathcal{X}$ be the coherent configuration of a polyhedral graph with Laplacian L. Denote by P the orthogonal projection to the eigenspace of the matrix L, associated with the second minimal eigenvalue. Then $P \in \mathrm{Mat}_\Omega(\mathbb{R})$ is a symmetric idempotent matrix. Moreover, the rank of P is 3 and P has no zero columns, see Corollary 13.10.2 and Lemma 13.10.1 in [7]. In particular, condition (5) is satisfied.

By Lemma 9, it remains to verify that $P \in \mathcal{A}(\mathcal{X})$. However, $P = p(L)$, where $p(\cdot)$ is a polynomial with real coefficients, see e.g. [7, Sec. 8.12]. Since $L \in \mathcal{A}(\mathcal{X})$, this implies that $P \in \mathcal{A}(\mathcal{X})$, as required. $\square$

Let ρ be an $\mathbb{S}_d$-representation of a coherent configuration $\mathcal{X}$, and let $W := W(\rho)$ be the set of all entries of the matrix A_ρ. Clearly, $1 \in W$. Denote by S_ρ the set of all binary relations

$$s(w) = s_\rho(w) = \{(\alpha, \beta) \in \Omega^2 : \ (A_\rho)_{\alpha,\beta} = w\}, \tag{6}$$

where $w \in W$.

Obviously, $s(1) = 1_\Omega$ and $s(w)^* = s(w)$ for all w. Consequently, $\mathcal{X}_\rho = (\Omega, S_\rho)$ is a rainbow. Moreover, the adjacency matrix of the relation $s(w)$ belongs to the algebra $\mathcal{A}(\mathcal{X})$, because $A_\rho \in \mathcal{A}(\mathcal{X})$ and $\mathcal{A}(\mathcal{X})$ is closed with respect to the Hadamard multiplication. It follows that

$$\mathcal{X} \geqslant \mathcal{X}_\rho, \tag{7}$$

in particular, every basis relation of the coherent configuration $\mathcal{X}$ is contained in a unique basis relation of the rainbow $\mathcal{X}_\rho$. Given $s \in S$, we denote by $w(s)$ a uniquely determined element $w \in W$ such that $s \subseteq s(w)$.

The inclusion (7) implies that both $s(1)$ and $s(-1)$ are relations of $\mathcal{X}$ (the second of them can be empty). Furthermore, the union $e = s(1) \cup s(-1)$ is an equivalence relation on Ω with classes of cardinality at most two. In particular, e is a parabolic of $\mathcal{X}$; we call it the *antipodal parabolic* with respect to ρ. A pair (α, β) belongs to e if and only if $\rho(\alpha) = \pm\rho(\beta)$; any other pair is said to be *non-antipodal*.

Lemma 11. *Let $\varphi : \mathcal{X} \to \mathcal{X}'$ be an algebraic isomorphism. Assume that $\mathcal{X}$ has an $\mathbb{S}_d$-representation ρ. Then $\mathcal{X}'$ has an $\mathbb{S}_d$-representation ρ' such that $W(\rho') = W(\rho)$ and $w(\varphi(s)) = w(s)$ for all $s \in S$.*

Proof. By statements (2) and (3) of Lemma 3, the matrices A_ρ and $A' = \varphi(A_\rho)$ are symmetric and have the same real eigenvalues. These are nonnegative, because A_ρ is the Gram matrix. It follows that A' is the Gram matrix of some vectors $x(\alpha') \in \mathbb{R}^{\Omega'}$, $\alpha' \in \Omega'$, such that

$$(A')_{\alpha',\beta'} = \langle x(\alpha'),\, x(\beta')\rangle, \quad \alpha', \beta' \in \Omega'.$$

By statement (1) of Lemma 3, the set W' of all entries of the matrix A' is equal to $W(\rho)$. Furthermore, the diagonal entries of A' (coinciding with diagonal entries of A) are ones. Together with statement (4) of Lemma 3, this shows that the vectors $x(\alpha')$ are pairwise distinct points of a sphere $\mathbb{S}_{d'}$ for some d'. Moreover,

$$d' + 1 = \mathrm{rk}(A') = \mathrm{rk}(A) = d + 1.$$

Thus, the mapping $\rho' : \alpha' \mapsto x(\alpha')$ is an $\mathbb{S}_d$-representation of the coherent configuration of $\mathcal{X}'$ and $W(\rho) = W' = W(\rho')$. Finally, the equality $\varphi(s_\rho(w)) = s_{\rho'}(w)$ easily follows from formula (1). $\square$

In the sequel, we are interested in a sufficient condition for an $\mathbb{S}_2$-representation ρ to be uniquely determined by some two distinct vectors in $\mathrm{im}(\rho)$. To this end, let $u, v \in W$ and $x, y \in \mathbb{S}_2$. Put

$$\Omega_{uv}(x, y) = \{z \in \mathbb{S}_2 : \ \langle x,\, z\rangle = u, \ \langle y,\, z\rangle = v\}.$$

Assume that $\Omega_{uv}(x,y) \neq \varnothing$. The equations $\langle x,\, z\rangle = u$ and $\langle y,\, z\rangle = v$ (both with respect to z) define the circles on $\mathbb{S}_2$: one is of radius $\|u - x\|$ centered in x and the other is of radius $\|v - y\|$ centered in y. These two circles can coincide only if $x = \pm y$. Assume that these circles are distinct. Then the coordinates of the vector z can easily be found by solving an appropriate square equation. Thus we obtain the following statement.

Lemma 12. *In the above notation, assume that $x \neq \pm y$. Then $|\Omega_{uv}(x,y)| \leqslant 2$. Moreover, if π is an isometry of the sphere $\mathbb{S}_2$, then*

$$\Omega_{uv}(x,y)^\pi = \Omega_{uv}(x^\pi, y^\pi).$$

In some cases, the set $\Omega_{uv}(x,y)$ can be interpreted in terms of the original coherent configuration $\mathcal{X}$ equipped with an $\mathbb{S}_2$-representation ρ. Namely, for any $u, v \in W$ and any $\alpha, \beta \in \Omega$, we put

$$\Omega_{uv}(\alpha,\beta) = \alpha s(u) \,\cap\, \beta s(v).$$

Note that if a point γ belongs to this set, then $\langle \rho(\alpha),\, \rho(\gamma)\rangle = w(r(\alpha,\gamma)) = u$ and $\langle \rho(\beta),\, \rho(\gamma)\rangle = w(r(\beta,\gamma)) = v$. Therefore, $\rho(\gamma) \in \Omega_{uv}(\rho(\alpha), \rho(\beta)$. This shows that

$$\rho(\Omega_{uv}(\alpha,\beta)) \subseteq \Omega_{uv}(\rho(\alpha), \rho(\beta)). \tag{8}$$

Together with Lemma 12 this proves the following statement.

Corollary 13. *Let $u, v \in W$, and $t \in S$. Assume that t is not contained in the antipodal parabolic of $\mathcal{X}$ with respect to ρ. Then $c^t_{s(u),s(v)} \leqslant 2$ and equality is attained only if*

$$\rho(\Omega_{uv}(\alpha,\beta)) = \Omega_{uv}(\rho(\alpha), \rho(\beta)) \tag{9}$$

for all $(\alpha,\beta) \in t$.

For any basis relation s of $\mathcal{X}$, contained in the antipodal parabolic, we obviously have $n_s = n_{s^*} = 1$, i.e., s is a thin relation. By Corollary 13, this implies the following statement.

Corollary 14. *Let a coherent configuration $\mathcal{X}$ have an $\mathbb{S}_2$-representation, and let $r, s, t \in S$. Assume that t is not thin. Then $c^t_{r,s} \leqslant 2$.*

Let a set $\Delta \subseteq \Omega$ consist of at least three points. Given distinct $\alpha, \beta \in \Delta$, we denote by $S(\Delta; \alpha, \beta) = S_\rho(\Delta; \alpha, \beta)$ the set of all relations $t \in S$ contained in the intersection

$$\bigcap_{\delta \in \Delta \setminus \{\alpha,\beta\}} r_\rho(\alpha,\delta) \cdot r_\rho(\delta,\beta)$$

where $r_\rho(\alpha,\delta)$ and $r_\rho(\delta,\beta)$ are the basis relations of the rainbow $\mathcal{X}_\rho$, containing the pairs (α,δ) and (δ,β), respectively. We also define $S(\Delta; \alpha, \beta) := \{r(\alpha,\beta)\}$ for two-element subsets Δ of Ω. Thus, in any case, $S(\Delta; \alpha, \beta)$ contains $r(\alpha,\beta)$. A meaning of the set $S(\Delta; \alpha, \beta)$ lies in the fact that, roughly speaking, no two relations in it can be distinguished using only the points from Δ and basis relations of the rainbow $\mathcal{X}_\rho$.

A key point in what follows is a concept of a ρ-closed set. Informally, together with any two non-antipodal points α and β satisfying condition (9), such a set should contain each $\gamma \in \Omega_{uv}(\alpha, \beta)$. Using Corollary 13, we will show that this is true if

$$t \in S(\Delta; \alpha, \beta) \;\Rightarrow\; c^{t}_{s(u),s(v)} = 2, \tag{10}$$

where $u = w(r(\alpha, \gamma))$ and $v = w(r(\beta, \gamma))$.

To be more precise, the set Δ is said to be *ρ-closed* if it contains every point $\gamma \in \Omega$ for which there exists a non-antipodal pair $(\alpha, \beta) \in \Delta^2$ such that the condition (10) is satisfied. In this case the right-hand side is true for $t = r(\alpha, \beta)$. A subset of Ω is said to be *ρ-rigid* if the only ρ-closed set containing it coincides with Ω. Finally, the representation ρ is said to be *rigid* if there exists a ρ-rigid set of cardinality 2.

Lemma 15. *Let a coherent configuration $\mathcal{X}$ have an $\mathbb{S}_2$-representation ρ. Assume that (μ, ν) is a non-antipodal pair forming a ρ-rigid set. Then for any algebraic isomorphism $\varphi : \mathcal{X} \to \mathcal{X}'$ and any points μ', ν' for which $r(\mu', \nu') = \varphi(r(\mu, \nu))$, there exists a bijection $f : \Omega \to \Omega'$ such that*

$$(\mu, \nu)^f = (\mu', \nu') \quad \textit{and} \quad s_\rho(w)^f = \varphi(s_\rho(w)) \quad \textit{for all } w \in W(\rho). \tag{11}$$

Proof. By Lemma 11, the coherent configuration $\mathcal{X}'$ has an $\mathbb{S}_2$-representation ρ' such that $W(\rho) = W(\rho')$ and $w(s) = w(\varphi(s))$ for all $s \in S$. It follows that $w(r(\mu, \nu)) = w(\varphi(r(\mu, \nu))) = w(r(\mu', \nu'))$ and hence $\langle \rho(\mu),\, \rho(\nu) \rangle = \langle \rho'(\mu'),\, \rho'(\nu') \rangle$. Therefore there is an isometry $\pi : \mathbb{S}_2 \to \mathbb{S}_2$ such that

$$\rho(\mu)^\pi = \rho'(\mu') \quad \text{and} \quad \rho(\nu)^\pi = \rho'(\nu').$$

Denote by Δ the set of all $\gamma \in \Omega$ such that $\rho(\gamma)^\pi = \rho'(\gamma')$ for some $\gamma' \in \Omega'$. Clearly, $\mu, \nu \in \Delta$.

Claim. $\Delta = \Omega$.

Proof. Assume on the contrary that $\Delta \neq \Omega$. Since $\{\mu, \nu\}$ is s ρ-rigid subset of Δ, the set Δ is not ρ-closed by the choice of μ and ν. It follows that there are $\gamma \notin \Delta$ and a non-antipodal pair $(\alpha, \beta) \in \Delta^2$ such that the implication (10) is true. In particular $c^{t}_{s(u),s(v)} = 2$, where $t = r(\alpha, \beta)$. By Corollary 13, we obtain

$$\rho(\Omega_{uv}(\alpha, \beta)) = \Omega_{uv}(\rho(\alpha), \rho(\beta)).$$

Since $u = w(r(\alpha, \gamma))$ and $v = w(r(\beta, \gamma))$, we have $\gamma \in \Omega_{uv}(\alpha, \beta)$. Therefore, $\rho(\gamma) \in \Omega_{uv}(\rho(\alpha), \rho(\beta))$. By Lemma 12,

$$\rho(\gamma)^\pi \in \Omega_{uv}(\rho(\alpha), \rho(\beta))^\pi = \Omega_{uv}(\rho(\alpha)^\pi, \rho(\beta)^\pi).$$

Since $\alpha, \beta \in \Delta$, there are $\alpha', \beta' \in \Omega'$ such that $\rho'(\alpha') = \rho(\alpha)^\pi$ and $\rho'(\beta') = \rho(\beta)^\pi$. This shows that

$$\rho(\gamma)^\pi \in \Omega_{uv}(\rho'(\alpha'), \rho'(\beta')). \tag{12}$$

The isometry π yields the bijection from Δ to the set $\Delta' = {\rho'}^{-1}(\rho(\Delta)^\pi)$, that takes a point δ to the point ${\rho'}^{-1}(\rho(\delta)^\pi)$. This bijection is obviously an isomorphism from the rainbow $(\mathcal{X}_\rho)_\Delta$ to the rainbow $(\mathcal{X}'_{\rho'})_{\Delta'}$. Since the relation $t' = r(\alpha', \beta')$ belongs to $S(\Delta'; \alpha', \beta')$, we conclude that $\varphi^{-1}(t') \in S(\Delta; \alpha, \beta)$. Now the implication (10) yields $c^{t''}_{s(u),s(v)} = 2$, where $t'' = \varphi^{-1}(t')$. Then by Corollary 13, we have

$$\rho'(\Omega_{uv}(\alpha', \beta')) = \Omega_{uv}(\rho'(\alpha'), \rho'(\beta')).$$

But then $\rho(\gamma) \in \rho'(\Omega(u, v; \alpha', \beta')) \subseteq \Omega'$, which contradicts the maximality of Δ. $\square$

By the claim, there exists a bijection $f : \Omega \to \Omega'$ such that $\rho'(\alpha^f) = \rho(\alpha)^\pi$. Obviously $(\mu, \nu)^f = (\mu', \nu')$ and

$$w(r(\alpha, \beta)^f) = \langle \rho'(\alpha^f),\, \rho'(\beta^f) \rangle = \langle \rho(\alpha)^\pi,\, \rho(\beta)^\pi \rangle = \langle \rho(\alpha),\, \rho(\beta) \rangle = w(r(\alpha, \beta)),$$

as required. $\square$

To state the main result of this section, we need one more definition. An $\mathbb{S}_2$-representation of a coherent configuration $\mathcal{X}$ is said to be *faithful* if $\mathrm{WL}(\mathcal{X}_\rho) = \mathcal{X}$. In all calculated examples, the coherent configuration of a polyhedral graph X has a faithful $\mathbb{S}_2$-representation. However, not every $\mathbb{S}_2$-representation of an arbitrary graph X is faithful, e.g., if $\mathcal{X}$ is the coherent configuration of the Möbius–Kantor graph, then the $\mathbb{S}_2$-representations associated (as in the proof of Corollary 10) with eigenvalues -1 and 1 are not faithful.

Theorem 16. *Let a coherent configuration $\mathcal{X}$ have an $\mathbb{S}_2$-representation ρ. Assume that ρ is rigid and faithful. Then $\mathcal{X}$ is schurian and separable.*

Proof. Let $\varphi : \mathcal{X} \to \mathcal{X}'$ be an algebraic isomorphism. By Lemma 15, there exists a rainbow isomorphism $f : \mathcal{X}_\rho \to \mathcal{X}'_{\rho'}$ such that the second part of formula (11) holds. Denote by ψ the algebraic isomorphism induced by f,

$$\psi : \mathrm{WL}(\mathcal{X}_\rho) \to \mathrm{WL}(\mathcal{X}'_{\rho'}),\ s \mapsto s^f.$$

Since ρ is faithful, we have $\mathrm{WL}(\mathcal{X}_\rho) = \mathcal{X}$ and hence $\mathrm{rk}(\mathrm{WL}(\mathcal{X}'_{\rho'})) = \mathrm{rk}(\mathcal{X}')$, which implies $\mathrm{WL}(\mathcal{X}'_{\rho'}) = \mathcal{X}'$, because $\mathcal{X}' \geqslant \mathrm{WL}(\mathcal{X}'_{\rho'})$. Thus, ψ is an algebraic isomorphism from $\mathcal{X}$ to $\mathcal{X}'$.

Now if $\psi = \varphi$, then φ is induced by the bijection f. Since φ is arbitrary, the coherent configuration $\mathcal{X}$ is separable. Assume that $\psi \neq \varphi$. Then $\Phi = \langle \varphi\psi^{-1} \rangle$ is a nontrivial group of algebraic automorphisms of $\mathcal{X}$. Hence, the algebraic fusion $\mathcal{X}^\Phi$ is strictly smaller than $\mathcal{X}$. Since also Φ leaves each basis relation of the rainbow $\mathcal{X}_\rho$ fixed, we obtain

$$\mathcal{X} > \mathcal{X}^\Phi = \mathrm{WL}(\mathcal{X}^\Phi) \geqslant \mathrm{WL}(\mathcal{X}_\rho) = \mathcal{X},$$

a contradiction.

To prove the schurity of $\mathcal{X}$, it suffices to verify that given $(\mu, \nu), (\mu', \nu') \in s \in S$, one can find $f \in \mathrm{Aut}(\mathcal{X})$ such that $(\mu, \nu)^f = (\mu', \nu')$. Assume first that s is not contained in the antipodal parabolic e (with respect to ρ). Let $\varphi = \mathrm{id}_{\mathcal{X}}$ be the identical algebraic automorphism of $\mathcal{X}$. Then by Lemma 15, there exists a bijection $f : \Omega \to \Omega$ such that both parts of formula (11) hold. By the first part of the proof for $\mathcal{X} = \mathcal{X}'$, f induces $\varphi = \mathrm{id}_{\mathcal{X}}$ and hence $f \in \mathrm{Aut}(\mathcal{X})$, as required.

To complete the proof, assume that $s \subseteq e$. Without loss of generality, we may assume that Ω contains a point λ different from ν. Then $r(\mu, \lambda) \not\subseteq e$ (recall that each class of e has at most two points). Since s is the basis relation containing both (μ, ν) and (μ', ν'), there exists a point λ' such that $r(\mu, \lambda) = r(\mu', \lambda')$. By the above paragraph, one can find $f \in \mathrm{Aut}(\mathcal{X})$ such that $(\mu, \lambda)^f = (\mu', \lambda')$. Now if $\lambda = \mu$, then $\lambda' = \mu'$ and we are done. Assuming $\lambda \neq \mu$, we have $\lambda' \neq \mu'$. On the other hand, $e^f = e$, because e is a parabolic of $\mathcal{X}$. Consequently, $\mu' = \mu^f$ and ν^f are in the same class of e. Since the class containing μ' contains ν', we conclude that $\nu^f = \nu'$, as required. □

5 Proof of Theorem 2

5.1

It is well known that every finite spherical group is isomorphic (as an abstract group) to a member of the four infinite families

$$C_n, \quad D_{2n}, \quad C_n \times C_2, \quad D_{2n} \times C_2 \qquad (n \geqslant 1), \tag{13}$$

or to one of the six groups

$$\mathrm{Alt}(4), \quad \mathrm{Sym}(4), \quad \mathrm{Alt}(5), \quad \mathrm{Alt}(4) \times C_2, \quad \mathrm{Sym}(4) \times C_2, \quad \mathrm{Alt}(5) \times C_2. \tag{14}$$

A complete classification of finite spherical groups as permutation groups was obtained in [15, Theorem 5.7]. The lemma below is just a reformulation of this result in a form convenient for subsequent application.

Lemma 17. *Let G be a finite spherical noncyclic group and $\Delta \in \mathrm{Orb}(G)$. Assume that Δ is not a faithful regular orbit.*

(1) *If $G \cong C_n$ or $G \cong C_n \times C_2$, then the group G^Δ is cyclic.*

(2) *If $G \cong D_{2n}$, $n \geqslant 3$, then $|\Delta| = 1$, 2 or n, and Δ is faithful if $|\Delta| = n$.*

(3) *If G is not as in (1) and (2), then the number $|\Delta|$ and the stabilizer G_δ, $\delta \in \Delta$, are as in the fourth column of Table 1.*

The proof of Theorem 2 is based on the three theorems below and is presented in the end of the section. The first of them states that the coherent configuration $\mathcal{X}$ associated with a spherical group G belonging to one of the families (13) or isomorphic to $\mathrm{Alt}(4) \times C_2$ is separable (Theorem 18). Then we prove that if G is isomorphic (as abstract group) to some groups in (14), then G is strongly spherical only if it has a faithful regular orbit (Theorem 19). Finally, we prove that the domination free groups among all remaining groups G are strongly spherical and the coherent configuration $\mathcal{X}$ has an $\mathbb{S}_2$-representation which is rigid and faithful (Theorem 20).

G	$\lvert G\rvert$	generators and relations of G	orbit lengths / point stabilizers
$D_{2n} \times C_2$	$4n$	$\langle t_1, t_2, t_3 :\ t_1^2 = t_2^2 = t_3^2 = (t_1t_2)^n = (t_1t_3)^2 = (t_2t_3)^2 = 1\rangle$	$2n^\infty$, $2n$, $2n$, n, n, 2 $\langle t_1\rangle$, $\langle t_2\rangle$, $\langle t_3\rangle$, $\langle t_2, t_3\rangle$, $\langle t_1, t_3\rangle$, $\langle t_1, t_2\rangle$
Alt(4)	12	$\langle r_1, r_2 :\ r_1^3 = r_2^3 = (r_1r_2)^2 = 1\rangle$	6, 4^2 $\langle r_1r_2\rangle$, $\langle r_1\rangle$
Sym(4)	24	I: $\langle t_1, t_2, t_3 :\ t_1^2 = t_2^2 = t_3^2 = (t_1t_3)^3 = (t_1t_2)^3 = (t_2t_3)^2 = 1\rangle$	12^∞, 6, 4^2 $\langle t_1\rangle$, $\langle t_2, t_3\rangle$, $\langle t_1, t_2\rangle$
		II: $\langle r_1, r_2 :\ r_1^3 = r_2^4 = (r_1r_2)^2 = 1\rangle$	12, 8, 6 $\langle r_1r_2\rangle$, $\langle r_1\rangle$, $\langle r_2\rangle$
Alt(5)	60	$\langle r_1, r_2 :\ r_1^3 = r_2^5 = (r_1r_2)^2 = 1\rangle$	30, 20, 12 $\langle r_1r_2\rangle$, $\langle r_1\rangle$, $\langle r_2\rangle$
Alt(4) $\times C_2$	24	$\langle r, t, z :\ r^3 = t^2 = z^2 = (rt)^3 = [z, r] = [z, t] = 1\rangle$	12^∞, 8, 6 $\langle z\rangle$, $\langle r\rangle$, $\langle z, t\rangle$
Sym(4) $\times C_2$	48	$\langle t_1, t_2, t_3 :\ t_1^2 = t_2^2 = t_3^2 = (t_1t_2)^4 = (t_1t_3)^3 = (t_2t_3)^2 = 1\rangle$	24^∞, 24^∞, 12, 8, 6 $\langle t_1\rangle$, $\langle t_2\rangle$, $\langle t_2, t_3\rangle$, $\langle t_1, t_3\rangle$, $\langle t_1, t_2\rangle$
Alt(5) $\times C_2$	120	$\langle t_1, t_2, t_3 :\ t_1^2 = t_2^2 = t_3^2 = (t_1t_2)^5 = (t_2t_3)^2 = (t_1t_3)^3 = 1\rangle$	60^∞, 30, 20, 12 $\langle t_1\rangle$, $\langle t_2, t_3\rangle$, $\langle t_1, t_3\rangle$, $\langle t_1, t_2\rangle$

Table 1: Orbits of a spherical permutation group G (the superscript on a number in the fourth column indicates the maximal number of G-orbits equivariant to a given one; the index 1 is omitted).

5.2

In what follows, G is a spherical permutation group on Ω, and $\mathcal{X} = \mathrm{Inv}(G)$ is the coherent configuration associated with G.

Theorem 18. *Let G belong to one of the families* (13) *or $G \cong \mathrm{Alt}(4) \times C_2$. Then the coherent configuration $\mathcal{X}$ is separable.*

Proof. Let $G \cong C_n$ or $G \cong C_n \times C_2$.[1] By Lemma 4, we may assume that G has no faithful regular orbits. By Lemma 17(1), this implies that every transitive constituent of G is a regular cyclic group. Then the lattice of the normal subgroups of the constituent is distributive. Hence the coherent configuration $\mathcal{X}$ satisfies the hypothesis of [9, Theorem 1.2], whence it follows that $\mathcal{X}$ is separable.

Let $G \cong D_{2n}$, $n \geqslant 3$. By Corollary 6, we may assume that G has no orbits of cardinalities 1 and 2. By Lemma 17(2), this implies that every orbit of G is faithful of cardinality n. Therefore, the coherent configuration $\mathcal{X}$ is separable by Lemma 7.

Let $G \cong D_{2n} \times C_2$, $n \geqslant 2$. By Lemma 5 and Corollary 6, we may assume that G is domination free and has no orbits of cardinality 2. Then by Lemma 17(3), we have $|\mathrm{Orb}(G)| \leqslant 5$ and

$$\mathrm{Orb}(G) \subseteq \{\Delta_1, \Delta_2, \Delta_3, \Gamma_1, \Gamma_2\},$$

where the notation is chosen so that the point stabilizers of the transitive constituents G^{Δ_i}, $1 \leqslant i \leqslant 3$, and G^{Γ_j}, $1 \leqslant j \leqslant 2$, are equal to $\langle t_i\rangle$ and $\langle t_j, t_3\rangle$, respectively (see the first row of Table 1). Note that $\Delta_1 = \Delta_2$ and $\Gamma_1 = \Gamma_2$ if n is odd.

First, assume that neither $\Delta_1, \Delta_2 \notin \mathrm{Orb}(G)$. Since G is domination free and Γ_1 and Γ_2 are dominated by Δ_3, it follows that if $\Delta_3 \in \mathrm{Orb}(G)$, then $\Gamma_1, \Gamma_2 \notin \mathrm{Orb}(G)$. Hence $\Omega = \Delta_3$ and the group G is regular. Thus the coherent configuration $\mathcal{X} = \mathrm{Inv}(G, \Delta_3)$ is separable by Lemma 4. On the other hand, if $\Delta_3 \notin \mathrm{Orb}(G)$, then $\Omega = \Gamma_1 \cup \Gamma_2$

[1] This includes the case $G = 1$.

and G is a permutation group isomorphic to D_{2n}. Thus, the coherent configuration $\mathcal{X} = \mathrm{Inv}(G, \Gamma_1 \cup \Gamma_2)$ is separable by Lemma 7.

Now let $\Delta := \Omega \cap (\Delta_1 \cup \Delta_2)$ be nonempty. Let $\sigma \in G$ be the permutation of Ω, corresponding to the generator t_3. Clearly, σ is a central involution of $\mathrm{Aut}(\mathcal{X})$, which acts trivially outside Δ. Put $e = e(\sigma)$, see the notation before Lemma 8.

Claim 1. *The coherent configuration $\mathcal{X}_{\Omega/e}$ is separable.*

Proof. Let π be the surjection (3). Then π is identical outside Δ, and $\pi(\Delta_i)$ and $\pi(\Gamma_j)$ are orbits of the permutation group $\bar{G} \cong G/\langle\sigma\rangle$ induced by the natural action of G on Ω/e. Inspecting the point stabilizers of $\bar{G}$, it is not difficult to verify that for $i = 1, 2$,

(i) if $\Delta_i, \Gamma_i \subseteq \Omega$, then $\pi(\Delta_i)$ and $\pi(\Gamma_i)$ are equivariant,

(2) if $\Delta_3, \Gamma_i \subseteq \Omega$, then $\pi(\Gamma_i)$ is dominated by $\pi(\Delta_3)$.

Now if $\Delta_3 \subseteq \Omega$, then $\pi(\Delta_3)$ is a faithful regular orbit of the group $\bar{G}$. Hence the coherent configuration $\mathcal{X}_{\Omega/e} = \mathrm{Inv}(\bar{G})$ is separable by Lemma 4. On the other hand, if $\Delta_3 \not\subseteq \Omega$, then $\bar{G} \cong D_{2n}$ and the coherent configuration $\mathcal{X}_{\Omega/e}$ is separable by Lemma 7. $\square$

Claim 2. *The parabolic e has a complement.*

Proof. We assume that $\Delta = \Delta_1 \cup \Delta_2$; the cases $\Delta = \Delta_1$ and $\Delta = \Delta_2$ are considered in a similar way. Let Λ be the disjoint union of two copies Λ_1 and Λ_2 of the group G, and let $H \leqslant \mathrm{Sym}(\Lambda)$ be the (semiregular) group induced by the action of G on Λ by right multiplications. Thus, $\mathrm{Orb}(H) = \{\Lambda_1, \Lambda_2\}$ and the transitive constituents of H are regular.

Denote by e_0 the equivalence relation on Λ, the classes of which are the right cosets of $\langle t_1\rangle$ in Λ_1 and the right cosets of $\langle t_2\rangle$ in Λ_2. Then e_0 is a parabolic of the coherent configuration $\mathrm{Inv}(H)$, and identifying in a natural way the sets Δ and Λ/e_0, we have

$$\mathcal{X}_\Delta = \mathrm{Inv}(H)_{\Lambda/e_0}.$$

The cosets of $\langle t_3\rangle$ in Λ_1 and in Λ_2 are the classes of an equivalence relation e_1 on Λ that is again a parabolic of $\mathrm{Inv}(H)$. Moreover, if $\pi_0 : \Lambda \to \Lambda/e_0$, $\lambda \mapsto \lambda e_0$ is the natural surjection, then $\pi_0(e_1) = e$.

Denote by Λ_1' the union of the group $\langle t_1, t_2\rangle \leqslant G$ contained in Λ_1 and the same group contained in Λ_2. Denote by Λ_2' the set theoretical complement of Λ_1' in Λ. This complement is equal to the union of the coset $\langle t_1, t_2\rangle t_3$ contained in Λ_1 and the same coset in Λ_2. In these notations, Λ is split into the disjoint union of the two sets Λ_1' and Λ_2', and the equivalence relation e_2 corresponding to this partition is a parabolic of $\mathrm{Inv}(H)$. It is easily seen that

$$e_1 \cap e_2 = e_0$$

implying that the intersection of any class of e_2 with any class of e_1 is a class of e_0. Thus, the parabolic $\pi_0(e_2)$ is a complement of the parabolic $\pi_0(e_1) = e$. $\square$

By Claims 1 and 2, the coherent configuration $\mathcal{X}$ satisfies the condition of Lemma 8, and hence is separable, which completes the case $G \cong D_{2n} \times C_2$.

Finally, let $G \cong \mathrm{Alt}(4) \times C_2$. By Lemma 5 and Corollary 6, we may assume that G is domination free. By Lemma 17(3), this implies that $|\operatorname{Orb}(G)| \leqslant 3$ and the set

$$N(G) = \{|\Delta| : \ \Delta \in \operatorname{Orb}(G)\}$$

is contained in the set $\{6, 8, 12\}$. The rest of the proof is based on computations in the GAP-package COCO2P [16]. First, we have

$$\mathrm{Aut}(\mathcal{X}) \cong \begin{cases} \mathrm{Alt}(4) & \text{if } N(G) \subseteq \{6, 12\}, \\ \mathrm{Sym}(4) \times C_2 & \text{if } N(G) = \{8\}. \end{cases}$$

This leaves us with three cases: $N(G) = \{6, 8\}$, $\{8, 12\}$, and $\{6, 8, 12\}$. Furthermore, the orbit of cardinality 6 is dominated by the orbit of cardinality 12. Thus either $N(G) = \{6, 8\}$ or $\{8, 12\}$.

In each of the two cases, the group $\mathrm{Aut}(\mathcal{X})$ has a unique central involution σ, and (in the notation of Lemma 8) $|\Delta| = 8$ and the parabolic $e = e(\sigma)$ of $\mathcal{X}$ has a complement. Finally, the quotient coherent configuration $\mathcal{X}_{\Omega/e}$ is separable: if $N(G) = \{8, 12\}$, then this follows from Lemma 4 (the group $\mathrm{Aut}(\mathcal{X}_{\Omega/e})$ has faithful regular orbit), whereas if $N(G) = \{6, 8\}$, then this follows by inspecting the database of small coherent configurations [19]. Thus the coherent configuration $\mathcal{X}$ satisfies the condition of Lemma 8, and hence is separable. This completes the proof of Theorem 18. □

5.3

The proof of Theorems 19 and 20 is based on case-by-case computer computations. They use the GAP packages COCO2P [16] for computations with coherent configurations and WLFast which is a faster implementation of the Weisfeiler-Lehman algorithm provided to us by S. V. Skresanov, and computer systems Mathematica [11] and Maple [10]. The protocols of the computations can be found at https://zemanpeter.github.io/assets/files/comp.zip.

Theorem 19. *Let $G \cong \mathrm{Alt}(4)$, $\mathrm{Sym}(4)$ in the representation II, or $\mathrm{Alt}(5)$. Assume that G has no faithful regular orbit. Then G is not strongly spherical unless G is induced by the action of $\mathrm{Sym}(4)$ of degree 12.*[2]

Proof. The assumption of the theorem implies by Lemma 17(3) that G has at most three orbits. All possible cases are given in Table 2, the second column of which represents the set $N(G)$. The computation starts with constructing the coherent configuration $\mathcal{X} = \mathrm{Inv}(G)$; the rank and the automorphism group of $\mathcal{X}$ are indicated in the third and fourth columns, respectively. Note that G is strongly spherical only if

$$G = \mathrm{Aut}(\mathcal{X}). \tag{15}$$

[2]The permutation groups induced by the action of $\mathrm{Sym}(4)$ of degree 12 in the representations I and II are permutation isomorphic.

type	Orb(G)	rk($\mathcal{X}$)	Aut($\mathcal{X}$)	$\lvert\mathfrak{X}'(G)\rvert$	$\lvert\mathfrak{X}(G)\rvert$
Alt(4)	6	4	$C_2 \times$ Alt(4)		
	4	2	Sym(4)		
	$4+4$	8	Sym(4)		
	$6+4$	10	Alt(4)	19	0
	$6+4+4$	20	Alt(4)	294	0
Sym(4)II	6	3	$C_2 \times$ Sym(4)		
	8	4	$C_2 \times$ Sym(4)		
	$12+8$	19	Sym(4)	330	14
	$12+6$	16	Sym(4)	148	6
	$8+6$	11	$C_2 \times$ Sym(4)		
	$12+8+6$	32	Sym(4)	4827	36
Alt(5)	30	16	Alt(5)	52	5
	20	8	Alt(5)	10	0
	12	4	$C_2 \times$ Alt(5)		
	$30+20$	44	Alt(5)	2385	148
	$30+12$	32	Alt(5)	601	25
	$20+12$	20	Alt(5)	80	1
	$30+20+12$	68	Alt(5)	46631	685

Table 2: The groups $G \cong$ Alt(4), Sym(4)II, or Alt(5).

Therefore comparing G and $\mathrm{Aut}(\mathcal{X})$ reduces the number of possibilities for G, e.g., the degree of a strongly spherical permutation group isomorphic to Alt(4) can be equal to 10 or 20 only (see the first five rows of Table 2).

For each remaining group G, we need to verify that there is no polyhedral graph X with vertex set Ω, such that $\mathrm{WL}(X) = \mathcal{X}$. Note that every such X satisfies the following conditions:

$$E \in S^{\cup}, \quad \mathrm{WL}(X) = \mathcal{X}, \quad |E| \leqslant 3|\Omega| - 6,$$

where S is the set of basis relations of $\mathcal{X}$. The first condition allows us to search for possible E among the subsets of S. Since every subgraph of X is planar, we can strengthen the third inequality by requiring that

$$|E \cap (\Delta \times \Gamma)| \leqslant |3|\Delta| - 6$$

for all $\Delta, \Gamma \in \mathrm{Orb}(G)$. Denote $\mathfrak{X}' = \mathfrak{X}'(G)$ the set of all such graphs X.

The action of the group Φ of algebraic automorphisms of the coherent configuration $\mathcal{X}$ on $S^{\cup}$ induces a natural action of this group on the set $\mathfrak{X}'$. Let $\mathfrak{X}(G)$ be the set of distinct representatives X of $\mathrm{Orb}(\Phi, \mathfrak{X}')$, such that $\mathrm{WL}(X) = \mathcal{X}$. Since every algebraic isomorphism of $\mathcal{X}$ is induced by an isomorphism (this can also be checked with the help of COCO2P), we conclude that the group G is strongly spherical if and only if the set $\mathfrak{X}(G)$ contains a polyhedral graph.

To complete the proof, we compute the set $\mathfrak{X}(G)$ for all groups G in Table 2, that satisfy (15); the number $|\mathfrak{X}(G)|$ is given in the fifth column of Table 2. Then using the functions "IsPlanar" and "VertexConnectivity" included in Maple, we verified that $\mathfrak{X}(G)$ contains no polyhedral graphs. □

type	Orb(G)	$\|S\|/\|S_\rho\|$	X	deg(X_ρ)	A_ρ	λ_ρ
Sym(4)I	12	7/6	Truncated Tetrahedral	12[3]	L	1
	4	2/2	Tetrahedral	4[3]	L	4
	$6+4$	9/9	X_{10}, polyhedral	6[6] + 4[3]	A	$\sqrt{2}$
Sym(4)× C_2	24I	14/12	Small Rhombicuboctah.	24[4]	L	1
	24II	16/11	Truncated Octahedral	24[3]	L	$4-\sqrt{2}$
	12	5/5	Cuboctahedral	12[4]	L	2
	8	4/4	Cubical	8[3]	L	2
	6	3/3	Octahedral	6[4]	L	4
	24I+24II	54/49	X_{48}, polyhedral	24[6] + 24[4]	A	$(3+\sqrt{33})/2$
	24II+8	28/25	X_{32}, polyhedral	24[5] + 8[6]	A	$1-\sqrt{10}$
	$12+8$	15/14	X_{20}, polyhedral	12[6] + 8[3]	A	$1+\sqrt{5}$
	$12+6$	14/12	X_{18}, polyhedral	12[6] + 6[4]	A	$1-\sqrt{5}$
	$8+6$	11/11	Rhombic Dodecahedral	8[3] + 6[4]	L	$(7-\sqrt{17})/2$
	$12+8+6$	28/25	Disdyakis Dodecahedral	12[4] + 8[6] + 6[8]	A	4
Alt(5)× C_2	60	32/22	Small Rhombicosidodecah.	60[4]	L	$(3-\sqrt{5})/2$
	30	10/9	Icosidodecahedral	30[4]	L	$3-\sqrt{5}$
	20	6/6	Dodecahedral	20[3]	L	$3-\sqrt{5}$
	12	4/4	Icosahedral	12[5]	L	$5-\sqrt{5}$
	$30+20$	30/28	X_{50}, nonplanar	30[2] + 20[3]	A	$(\sqrt{10}-\sqrt{2})/2$
	$30+12$	24/22	X_{42}, planar	30[2] + 12[5]	A	$\sqrt{5-\sqrt{5}}$
	$20+12$	18/18	Rhombic Triacotahedral	20[3] + 12[5]	L	$3-\sqrt{5}$
	$30+20+12$	52/49	Disdyakis Triacontahedral	30[4] + 20[6] + 12[10]	A	$\sqrt{5}$

Table 3: The groups $G \cong \mathrm{Sym}(4)\mathrm{I}$, $\mathrm{Sym}(4)\times C_2$, $\mathrm{Alt}(5)\times C_2$

5.4

In this subsection, we consider the remaining groups G and prove Theorem 20. A key role here is played by the concept of $\mathbb{S}_2$-representation studied in Section 4.

Theorem 20. *Let $G \cong \mathrm{Sym}(4)$ in the representation I, $\mathrm{Sym}(4)\times C_2$, or $\mathrm{Alt}(5)\times C_2$. Assume that G is domination free. Then G is strongly spherical and the coherent configuration $\mathcal{X}$ has an $\mathbb{S}_2$-representation which is rigid and faithful.*

Proof. The assumption of the theorem implies by Lemma 17(3) that G has at most three orbits. All possible cases are given in Table 3, the second column of which represents the set $N(G)$.

To prove that the group G is strongly spherical, one needs to find a polyhedral graph X such that the coherent configurations $\mathrm{WL}(X)$ and $\mathcal{X}$ are isomorphic. Except for the seven cases, X can be found among the known polyhedral graphs the commonly used names of which are given in the fourth column of Table 3; for the other seven cases we take the graphs shown in Fig. 1 (the indices at X and Y are equal to the number of vertices of the graph).

Using the Mathematica package, we first verified that each graph in fourth column of Table 3 is polyhedral[3] and construct its the adjacency matrix $A = A(X)$. Then using the COCO2P function ColorGraphByMatrix and the WLFast function WLStabilizationByHash, we construct the coherent configuration $\mathrm{WL}(X)$. Finally, we verified that $\mathrm{WL}(X) \cong \mathcal{X}$ with the help of the COCO2P function IsColorIsomorphicColorGraph. This completes the proof of the first statement.

To prove the second statement, we make use of Lemma 9 to construct $\mathbb{S}_2$-representation of the coherent configuration $\mathcal{X}$. As the matrix P, we take the normalized orthogonal

[3]Certainly, this is not necessary to do for the known graphs.

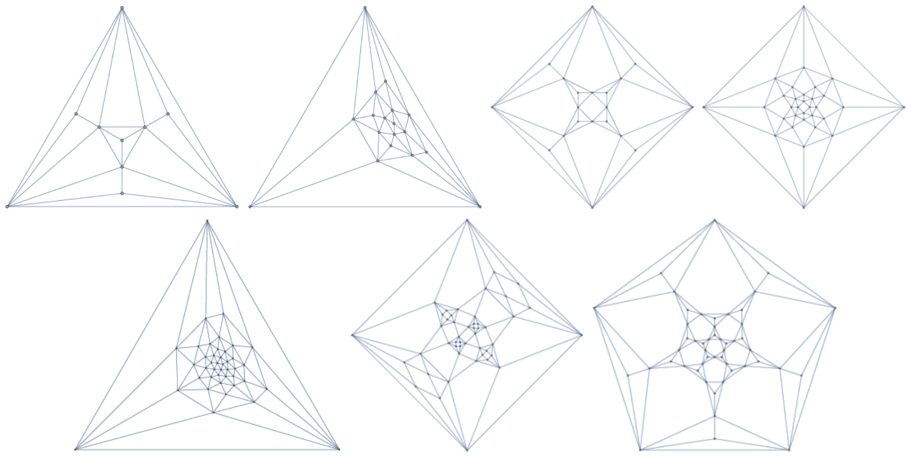

Figure 1: The polyhedral graphs $X_{10}, X_{18}, X_{20}, X_{32}, Y_{42}, X_{48}, Y_{50}$.

projection P_ρ to the eigenspace of a matrix A_ρ which is the Laplacian ("L" in the sixth column of Table 3) or adjacency matrix ("A" in the sixth column of Table 3) of the graph X, corresponding to an eigenvalue λ_ρ (the seventh columns of Table 3) of multiplicity 3. For the polyhedral graphs Y_{42} and Y_{50}, the computation of P_ρ was not successful.[4] Because of this, these graphs were replaced by the graphs X_{42} and X_{50}, see Fig. 2. For each of them, $\mathrm{WL}(X)$ and $\mathcal{X}$ are still isomorphic, and the adjacency matrix still has an eigenvalue λ_ρ of multiplicity 3. The distributions of vertex degrees of all the graphs X are indicated in

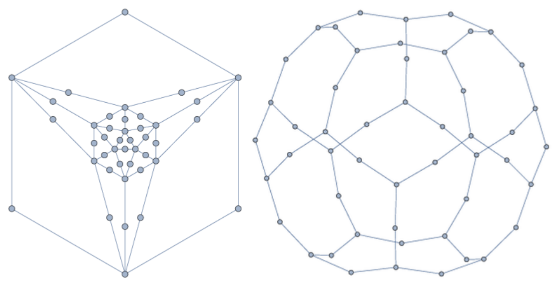

Figure 2: The nonpolyhedral graphs X_{42}, X_{50}.

the fifth column of Table 3.

The remaining calculations are organized as follows. In view of the isomorphism $\mathrm{WL}(X) \cong \mathcal{X}$, we may assume that $\mathcal{X} = \mathrm{WL}(X)$. Inspecting the entries of the matrix P_ρ (they are algebraic numbers and therefore we need symbolic linear algebra in Maple), we compute the relations (6) in the form of unions of some basis relations of $\mathcal{X}$, and then the rainbow $\mathcal{X}_\rho \leqslant \mathcal{X}$, see Section 4. The ranks of $\mathcal{X}$ and $\mathcal{X}_\rho$ are presented in the third column of Table 3. If these ranks coincide, then $\mathcal{X} = \mathcal{X}_\rho$ and the $\mathbb{S}_2$-representation ρ of the coherent configuration $\mathcal{X}$, corresponding to P_ρ, is obviously faithful; otherwise, we

[4]The symbolic linear algebra in the Maple package was unable to calculate the projection due to too much calculation.

compute $\mathrm{WL}(\mathcal{X}_\rho)$ by using the package WLFast and test that in all cases, $\mathrm{WL}(\mathcal{X}_\rho) = \mathrm{WL}(X)$. Thus the $\mathbb{S}_2$-representation ρ is faithful.

Finally, to verify that ρ is rigid, denote by T_ρ the set of all $s \in S$ contained in the antipodal parabolic with respect to ρ, and by Π_ρ the partition of S such that r and s belong to the same class if and only if $w(r) = w(s)$. We defined a new COCO2P function which given a coherent configuration $\mathcal{X}$, a relation $s \in S$, a partition Π_ρ, and the set T_ρ, tests whether a set $\Delta = \{\alpha, \beta\}$ for some $(\alpha, \beta) \in s$ is ρ-rigid (see Section 4). The computations shows that for each graph X, there is at least one $s \in S$, for which Δ is ρ-rigid. This shows that the $\mathbb{S}_2$-representation ρ is rigid. This completes the proof. $\square$

5.5 Proof of Theorem 2.

Let G be a strongly spherical permutation group and $\mathcal{X} = \mathrm{Inv}(G)$. Then G belongs to one of the families (13), or is isomorphic to one of the groups (14). By Theorem 18, we may assume that neither G belong to one of the families (13) nor $G \cong \mathrm{Alt}(4) \times C_2$.

Now if $G \cong \mathrm{Alt}(4)$, $\mathrm{Sym}(4)$ in the representation II, or $\mathrm{Alt}(5)$, then G has a faithful regular orbit by Theorem 19 and hence the coherent configuration $\mathcal{X}$ is separable by Lemma 4.

Finally, in the remaining case, we may assume by Lemma 5 that G is domination free. By Theorem 20, this implies that the coherent configuration $\mathcal{X}$ has an $\mathbb{S}_2$-representation which is both rigid and faithful. Thus, $\mathcal{X}$ is separable by Theorem 16. $\square$

Acknowledgements

Peter Zeman was funded by the European Union (ERC, POCOCOP, 101071674). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

References

[1] V. Arvind, J. Köbler, G. Rattan, and O. Verbitsky, *On the Power of Color Refinement*, Lecture Notes in Computer Science, **9210**, 339–350 (2015).

[2] L. Babai, P. Erdős, and S. M. Selkow, *Random graph isomorphism*, SIAM J. Comput. **9** (1980), 628–635.

[3] J. Y. Cai, M. Fürer, and N. Immerman, *An optimal lower bound on the number of variables for graph identification*, Combinatorica, **12**, 389–410 (1992).

[4] G. Chen and I. Ponomarenko, *Coherent configurations*, Central China Normal University Press, 2019, the updated version is available at http://www.pdmi.ras.ru/~inp/ccNOTES.pdf.

[5] G. Chen and I. Ponomarenko, *Tensor products of coherent configurations*, Front. Math. China, **17**, no. 5, 829–852 (2021).

[6] F. Fuhlbrück, J. Köbler, and O. Verbitsky, *Identifiability of Graphs with Small Color Classes by the Weisfeiler–Leman Algorithm*, SIAM J. Discrete Math., **35**, no. 3, 1792–1853 (2021).

[7] C. Godsil and G. Royle, *Algebraic graph theory*, Graduate Texts in Mathematics. 207, New York, NY: Springer (2001).

[8] M. Grohe, *Descriptive complexity, canonisation, and definable graph structure theory*, Cambridge University Press (2017).

[9] M. Hirasaka, K. Kim, and I. Ponomarenko, *Schurity and separability of quasiregular coherent configurations*, Journal of Algebra, **510**, 180–204 (2018).

[10] Maple Inc., *Maple 2022.0*, Maplesoft, a division of Waterloo Maple Inc., Waterloo, Ontario (2022).

[11] Wolfram Research, Inc., *Mathematica, Version 11.3*, https://www.wolfram.com/mathematica, Champaign, IL (2018).

[12] S. Kiefer and D. Neuen, *A Study of Weisfeiler–Leman Colorings on Planar Graphs*, 49th International Colloquium on Automata, Languages, and Programming (ICALP 2022), pp. 81:1–81:20 (2022).

[13] S. Kiefer, I. Ponomarenko, and P. Schweitzer, *The Weisfeiler–Leman Dimension of Planar Graphs Is at Most 3*, Journal of the ACM, **66**, no. 6, 1–31 (2019).

[14] S. Kiefer, P. Schweitzer, and E. Selman, *Graphs identified by logics with counting*, Lecture Notes in Computer Science, **9234**, 319–330 (2015).

[15] P. Klavík, R. Nedela, and P. Zeman, *Jordan-like characterization of automorphism groups of planar graphs*, J. Combin. Theory, Series B, **157**, 1–39 (2022).

[16] M. Klin, C. Pech, and S. Reichard, *Coco2p – a GAP4 package, ver.* 0.18, https://github.com/chpech/COCO2P/archive/coco2p-0.18.tar.gz (2020).

[17] M. Klin and M. Ziv-Av, *A non-Schurian coherent configuration on* 14 *points exists*, Designs, Codes and Cryptography, **84**, no. 1-2, 203–221 (2017).

[18] M. Muzychuk and I. Ponomarenko, *On quasi-thin association schemes*, J. Algebra, **351**, 467–489 (2012).

[19] M. Ziv-av, *Enumeration of coherent configurations of order at most* 15, Acta Univ. M. Belii Ser. Math., **26**, 65–75 (2018).