The number of tiles of $\mathbb{Z}^d$

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Abstract

For fixed $d \geqslant 1$, let $t_{n,d}$ be the number of subsets of $[n]^d$ that tile $\mathbb{Z}^d$ by translations. We prove that

$$t_{n,d} = (3^{1/3})^{n^d \pm o(n^d)}.$$

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1 Introduction

A set $S \subseteq \mathbb{Z}^d$ tiles $\mathbb{Z}^d$ by translations if there is a set $T \subseteq \mathbb{Z}^d$ such that

$$\mathbb{Z}^d = \dot{\bigcup}_{z \in T}(S + z).$$

Tiling $\mathbb{Z}^d$ with translations is a natural problem that has been widely studied [2, 6, 7, 4, 5, 3]. Even in dimension 1 the exact structure of tiles remains unclear in general, with a long-standing conjecture of Coven and Meyerowitz [2] aiming to shed light on this problem.

Rather than studying the *exact* structure of tiles, which is open even in dimension 1, our aim is to study their *typical* structure by counting them. Concretely, we count how many subsets of $[n]^d = \{(x_1, \ldots, x_d) \mid x_i \in \{1, \ldots, n\}\}$ can tile $\mathbb{Z}^d$, which implies bounds on the search space we are working with. We remark that our work differs from the recent work of Stern [8], which studied the problem of counting for a given $S \subseteq \mathbb{Z}^d$, how many different ways there are to tile S (S may not tile $\mathbb{Z}^d$).

Let $t_{n,d}$ be the number of subsets of $[n]^d$ that tile $\mathbb{Z}^d$. We use the notation $v_n := n^d$ for the size of $[n]^d$. We prove:

Theorem 1. *For every fixed $d \geqslant 1$,*

$$(3^{1/3})^{v_n - o(v_n)} \leqslant t_{n,d} \leqslant (3^{1/3})^{v_n + o(v_n)}.$$

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2 Proof of Theorem 1

2.1 Lower Bound

Assume first that $n = 3t$. In dimension 1, choose one point from each residue class modulo t inside $[3t]$. There are $3^t = (3^{1/3})^n$ choices, and each such set tiles $\mathbb{Z}$ by translations by t.

For general d, apply this construction independently on each line parallel to e_1. Equivalently, for every

$$(i, x_2, \ldots, x_d) \in [t] \times [3t]^{d-1},$$

choose exactly one of the three points

$$(i, x_2, \ldots, x_d), \quad (i + t, x_2, \ldots, x_d), \quad (i + 2t, x_2, \ldots, x_d).$$

The resulting set is a fundamental domain for the lattice generated by

$$te_1, \quad 3te_2, \ldots, 3te_d,$$

and hence tiles $\mathbb{Z}^d$. This gives

$$t_{3t,d} \geqslant 3^{t(3t)^{d-1}} = (3^{1/3})^{(3t)^d}.$$

For general $n \geqslant 3$, let $n' = 3\lfloor n/3 \rfloor$. Applying the construction inside $[n']^d \subseteq [n]^d$ gives

$$t_{n,d} \geqslant (3^{1/3})^{(n')^d} = (3^{1/3})^{n^d - o(n^d)}.$$

2.2 Upper bound

We now prove the upper bound. Since $H(0.1) < \frac{1}{3}\log_2 3$,

$$\sum_{r \leqslant 0.1 v_n} \binom{v_n}{r} \leqslant 2^{H(0.1)v_n} = o\big((3^{1/3})^{v_n}\big).$$

Thus it remains to count tiles $S \subseteq [n]^d$ with

$$|S| \geqslant \alpha v_n, \qquad \alpha = 0.1.$$

Let $k = k(n)$ be a positive integer with $k \to \infty$; at the end we take $k = \lfloor n^{1/(2d)} \rfloor$. Consider shifts $z_1, \ldots, z_\ell \in \mathbb{Z}^d$. We say that S tiles $[kn]^d$ with $z_1, \ldots, z_\ell$ if the translates $S + z_a$ are pairwise disjoint and

$$[kn]^d \subseteq \dot{\bigcup}_{a=1}^{\ell} (S + z_a).$$

Call the translates efficient if $([n]^d + z_a) \cap [kn]^d \neq \emptyset$ for every a. Crucially, every set S that tiles $\mathbb{Z}^d$ tiles $[kn]^d$ with a choice of efficient translates. By definition, every efficient shift lies in $\{-n,\ldots,kn\}^d$. Also, $\ell \geqslant k^d$, because $|S| \leqslant v_n$ and the translates cover $[kn]^d$, while

$$\ell \leqslant \frac{(k+2)^d}{\alpha} = O_\alpha(k^d),$$

because the efficient translates are disjoint, each has size at least αv_n, and all lie in a slightly enlarged cube of volume $O(k^d v_n)$. Therefore the number of efficient sequences is at most

$$(O(k^d v_n))^{O_\alpha(k^d)}.$$

We fix $z_1,\ldots,z_\ell$ and count the number of sets S, that tile a set which contains $[kn]^d$ with the fixed translations $z_1,\ldots,z_\ell$. Define

$$A_y = \{\, y - z_a : a \in [\ell] \,\} \cap [n]^d, \qquad y \in [kn]^d.$$

Since the translates $S + z_a$ are pairwise disjoint and cover $[kn]^d$, for every $y \in [kn]^d$ the set

$$F_y = S \cap A_y$$

has size exactly one.

Let $\mathcal{S}$ be chosen uniformly from the family of all sets $S \subseteq [n]^d$ with $|S| \geqslant \alpha v_n$ such that the fixed translates $S+z_1,\ldots,S+z_\ell$ are pairwise disjoint and cover $[kn]^d$. We bound the entropy of $\mathcal{S}$ and show $H(\mathcal{S}) \leqslant \log_2\left(3^{\frac{1}{3}} + o_k(1)\right) \cdot v_n$, which implies that for every fixed $z_1,\ldots,z_\ell$ we have at most $\left(3^{\frac{1}{3}} + o_k(1)\right)^{v_n}$ tiles. Union bounding over the efficient sequences counted above will conclude the proof [1].

Write

$$\mathcal{S} = (X_x)_{x\in[n]^d},$$

where $X_x = \mathbf{1}_{x\in\mathcal{S}}$. For each $x \in [n]^d$, the set of $y \in [kn]^d$ such that $x \in A_y$ is

$$\{x + z_a : a \in [\ell]\} \cap [kn]^d.$$

For $x \in [n]^d$, set

$$J(x) = |\{a \in [\ell] : x + z_a \notin [kn]^d\}|.$$

If $x + z_a \notin [kn]^d$, then, by efficiency, $[n]^d + z_a$ crosses the boundary of $[kn]^d$, and hence $S + z_a$ lies in the n-neighbourhood B of the boundary of $[kn]^d$. Since the corresponding translates are disjoint,

$$\alpha v_n J(x) \leqslant |B| \leqslant ((k+2)^d - (k-2)^d)v_n. \tag{1}$$

Thus, for all sufficiently large k,

$$J(x) \leqslant J := \frac{5dk^{d-1}}{\alpha}.$$

[1] For $k = \lfloor n^{1/(2d)} \rfloor$, the term coming from this union bound is $(3^{\frac{1}{3}})^{o(v_n)}$.

Therefore the family $(A_y)_{y\in[kn]^d}$ covers each coordinate $x \in [n]^d$ at least $\ell - J$ times.

To bound $H(\mathcal{S})$, we use Shearer's inequality, which gives an upper bound on the entropy of $\mathcal{S}$ in terms of entropies of its projections. Concretely:

$$H(\mathcal{S}) \leqslant \frac{1}{\ell - J} \sum_{y\in[kn]^d} H(\mathcal{S}|_{A_y}).$$

The key point is that, since $\mathcal{S}$ tiles $[kn]^d$ using the translations, for every $y \in [kn]^d$, the random vector $\mathcal{S}|_{A_y}$ has exactly one coordinate equal to 1. If no coordinate of $\mathcal{S}|_{A_y}$ is 1 then y is not covered by the translates of $\mathcal{S}$. If two coordinates of $\mathcal{S}|_{A_y}$ are 1, then y is covered by two distinct elements of $\mathcal{S}$, violating the tiling assumption. Consequently, the random variable $\mathcal{S}|_{A_y}$ can take at most $|A_y|$ possible values (which is much better than the naive $2^{|A_y|}$ bound, and is the reason for the improved size bound for tiles), hence

$$H(\mathcal{S}|_{A_y}) \leqslant \log_2 |A_y|.$$

Thus, applying Shearer's inequality we get:

$$H(\mathcal{S}) \leqslant \frac{1}{\ell - J} \sum_{y\in[kn]^d} \log_2 |A_y| \leqslant \frac{1}{\ell - J} \log_2 \prod_{y\in[kn]^d} |A_y|. \tag{2}$$

We will need the following observation (which is a discrete form of Jensen's inequality):

Observation 2. *Let $x_1, \dots, x_m$ be non-negative integers with $\sum_{i=1}^m x_i = t$, with $t = am+b$ and $0 \leqslant b < m$. Then*

$$\prod_{i=1}^m x_i \leqslant (a+1)^b a^{m-b}.$$

Proof. Since for $k \geqslant 0$ and $s > 1$ we have $k(k+s) < (k + \lfloor \frac{s}{2} \rfloor)(k + \lceil \frac{s}{2} \rceil)$ any maximum contains elements x_i that differ by at most 1 from one another. The only choice (up to permutation) of $x_1, \dots, x_m$ with this property achieves the desired upper bound. $\square$

Set $x_y = |A_y|$, $y \in [kn]^d$. Then

$$\sum_{y\in[kn]^d} |A_y| \leqslant \ell v_n,$$

because each $x \in [n]^d$ contributes to at most one A_y for each shift z_a. The product is maximized, under this upper bound on the total sum, by using the full allowed sum. Applying Observation 2 with $m = (kn)^d$ and total sum at most ℓv_n, and using $\ell n^d = \lfloor \frac{\ell}{k^d} \rfloor \cdot (kn)^d + \left\{ \frac{\ell}{k^d} \right\} \cdot (kn)^d$ where $\{\}$ denotes the fractional part of a real number, we get:

$$H(\mathcal{S}) \leqslant \frac{1}{\ell - J} \log_2 \left(\left(\left\lfloor \frac{\ell}{k^d} \right\rfloor + 1 \right)^{\left\{ \frac{\ell}{k^d} \right\} \cdot k^d v_n} \left\lfloor \frac{\ell}{k^d} \right\rfloor^{(1 - \left\{ \frac{\ell}{k^d} \right\}) k^d v_n} \right) \tag{3}$$

$$= \frac{1}{\ell - J} \log_2 \left(\left(\left\lfloor \frac{\ell}{k^d} \right\rfloor + 1 \right)^{\left\{ \frac{\ell}{k^d} \right\} \cdot k^d} \left\lfloor \frac{\ell}{k^d} \right\rfloor^{(1 - \left\{ \frac{\ell}{k^d} \right\}) k^d} \right) \cdot v_n \tag{4}$$

$$= \frac{\ell}{\ell - J} \log_2 \left(\left(\left\lfloor \frac{\ell}{k^d} \right\rfloor + 1 \right)^{\left\{ \frac{\ell}{k^d} \right\} \cdot \frac{k^d}{\ell}} \left\lfloor \frac{\ell}{k^d} \right\rfloor^{(1 - \left\{ \frac{\ell}{k^d} \right\}) \cdot \frac{k^d}{\ell}} \right) \cdot v_n \tag{5}$$

To bound the term inside the log by $3^{\frac{1}{3}}$, we use the following lemma:

Lemma 3. *Let N be a positive integer and $r \in [0, 1]$. Then*

$$f(r) = (N+1)^{\frac{r}{N+r}} N^{\frac{1-r}{N+r}}$$

attains its maximum on $[0, 1]$ at an endpoint.

Proof. Let

$$g(r) = \log f(r) = \frac{r \log(N+1) + (1-r) \log N}{N + r}.$$

Then

$$g'(r) = \frac{N \log(1 + 1/N) - \log N}{(N+r)^2},$$

whose sign is independent of r. Hence g, and therefore f, is monotone on $[0, 1]$. $\square$

By the lemma, the term inside the logarithm in Equation (5) is bounded by $M^{1/M}$ for some positive integer M. Notice that since $x^{\frac{1}{x}}$ increases at $[1, e)$ and decreases at (e, ∞), together with the fact that $2^{\frac{1}{2}} < 3^{\frac{1}{3}}$ we have

$$\max_{M \in \mathbb{N}} M^{1/M} = 3^{1/3}.$$

Therefore $H(\mathcal{S}) \leqslant \frac{\ell}{\ell - J} \log_2(3^{\frac{1}{3}}) \cdot v_n$, hence the number of possible sets S for these fixed shifts and $|S| \geqslant \alpha v_n$ is at most:

$$\left(3^{\frac{1}{3}}\right)^{\frac{\ell}{\ell - 5dk^{d-1}/\alpha} v_n}$$

In conclusion, setting $k = \lfloor n^{\frac{1}{2d}} \rfloor$ concludes the proof of the upper bound.

Since $\ell \geqslant k^d$ and $J \leqslant 5dk^{d-1}/\alpha$,

$$\frac{\ell}{\ell - J} \leqslant \frac{k^d}{k^d - 5dk^{d-1}/\alpha} = 1 + O_\alpha(1/k).$$

$$t_{n,d} \leqslant \sum_{r \leqslant \alpha v_n} \binom{v_n}{r} + (O(k^d v_n))^{O_\alpha(k^d)} \cdot \left(3^{1/3}\right)^{\frac{k^d}{k^d - 5dk^{d-1/\alpha}} v_n} \leqslant (3^{1/3})^{v_n + o(v_n)}.$$

With $k = \lfloor n^{1/(2d)} \rfloor$, the sequence-counting factor is $(3^{1/3})^{o(v_n)}$, and the exponent factor is $1 + o(1)$.

Remark 4. It also follows from the proof that all translation sequences for which $\frac{\ell}{k^d} \notin (3-\varepsilon, 3+\varepsilon)$ yield a stronger bound on the entropy. This implies that as n goes to ∞ the size of a uniform random tile divided by n^d converges to $\frac{1}{3}$ a.a.s.

3 Concluding remarks

The upper-bound argument uses only two coarse facts about boxes in $\mathbb{Z}^d$: efficient translates lie in a slightly enlarged region, and the n-neighbourhood of the boundary of $[kn]^d$ has $o(k^d n^d)$ points when $k \to \infty$. Thus the same proof applies, with minor changes, to other Følner-type regions in $\mathbb{Z}^d$, such as lattice balls for a fixed norm. In spaces of exponential growth this boundary layer may have volume comparable to the whole ball, and the constant $3^{1/3}$ does not seem to be the right one.

We conclude with several conjectures and questions:

Conjecture 5. Fix $r \geqslant 1$. Let $x \in [n]^d$ be chosen uniformly, and let $\mathcal{S}_n$ be a uniformly random tile contained in $[n]^d$, independently of x. Does the law of

$$\left(\mathbf{1}_{x+u \in \mathcal{S}_n}\right)_{u \in B_r(0)}$$

converge, as $n \to \infty$, to the product measure of Bernoulli-1/3 random variables?

Conjecture 6. For each d there is a constant $C(d)$ that depends only on d such that: $t_{n,d} = O(n^{C(d)} \cdot (3^{\frac{1}{3}})^{v_n})$. Can one take $C(d) = d - 1$?

Question 7. Let $\mathcal{S}$ be a tile contained in $[n]^d$ chosen uniformly at random. What is the size of the largest connected component in the subgraph induced by $\mathcal{S}$? The lower-bound family resembles site percolation on $[n]^{d-1} \times [n/3]$ with $p = 1/3$. Since site percolation on $\mathbb{Z}^d$ with $p = 1/3$ is supercritical for $d \geqslant 3$ and subcritical for $d \leqslant 2$[2], this suggests asking whether a similar phase transition occurs for the largest connected component of a random tile.

Question 8. Let $M \subseteq \mathbb{Z}$ be any subset of size n. Are there at most $\left(3^{1/3} + o(1)\right)^n$ tiles inside M? Is the number of tiles contained in M maximized when M consists of n consecutive integers?

Question 9. How many tiles are contained in the set $\{1, 2, 4, \ldots, 2^n\}$? Are the only such tiles the sets of size 1 and 2? [3]

[2] See https://en.wikipedia.org/wiki/Percolation_threshold

[3] Notice that if one considers $0, 1, 2, 4, \ldots, 2^n$, then for values of n admitting a prime $p \approx n/3$ for which 2 generates $\mathbb{F}_p^\times$, one can construct many tiles by picking one from each mod p class to be in the tile.

Question 10. Call $S \subseteq \mathbb{Z}^d$ *perfect* if all non-empty subsets of S tile $\mathbb{Z}^d$. E.g. it follows immediately by the work of [7] that the largest size of a perfect subset of $\mathbb{Z}$ is 3. What is the size of the largest perfect subset in $\mathbb{Z}^d$ for $d \geqslant 2$?

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