Spectral radius, transversal Hamilton paths and cycles in bipartite graph families

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Submitted: Aug 19, 2025; Accepted: May 18, 2026; Published: Aug 7, 2026

Abstract

A theorem of Li and Ning [Linear Algebra Appl. 515 (2017)] states that, for $n \geqslant 4$, every balanced bipartite graph G on $2n$ vertices with spectral radius $\lambda(G) \geqslant \sqrt{n(n-1)}$ contains a Hamilton path unless $G \simeq K_{n,n-1} \cup K_1$. Let $[n] = \{1, 2, \dots, n\}$. We prove a generalization of this theorem in the setting of graph transversals. Namely, for $n \geqslant 4$, we show that given a family $\mathcal{G} = \{G_1, G_2, \dots, G_{2n-1}\}$ of $2n-1$ bipartite graphs on a common set V of $2n$ vertices with a common balanced bipartition, if $\lambda(G_i) \geqslant \sqrt{n(n-1)}$ for every $i \in [2n-1]$, then there exists a transversal Hamilton path on V unless $G_1 = G_2 = \cdots = G_{2n-1}$ and $G_1 \simeq K_{n,n-1} \cup K_1$.

We also show spectral analogue of transversal Hamilton cycle in bipartite graph families. Let B_n^1 be the graph obtained from $K_{n,n}$ by deleting all edges in its one subgraph $K_{n-1,1}$. Given a family $\mathcal{G} = \{G_1, G_2, \dots, G_{2n}\}$ of $2n$ bipartite graphs on a common set V of $2n$ vertices with a common balanced bipartition, for $n \geqslant 4$, we prove that if $\lambda(G_i) \geqslant \lambda(B_n^1)$ for every $i \in [2n]$, then $\mathcal{G}$ contains a transversal Hamilton cycle, unless $G_1 = G_2 = \cdots = G_{2n}$ and $G_1 \simeq B_n^1$. Our result extends a result of Li and Ning [Linear Multilinear Algebra 64 (11) (2016)].

Mathematics Subject Classifications: 05C50, 05C35

1 Introduction

For a set V, let $\binom{V}{2} = \{\{x, y\} : \{x, y\} \subseteq V\}$. A *graph* is a pair $G = (V, E)$, where V is the vertex set and $E \subseteq \binom{V}{2}$ is the edge set. For $x, y \in V$, we denote $\{x, y\}$ by xy when there is no confusion. Given a vertex subset $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. For a vertex $x \in V(G)$, let $N_G(x) := \{y \in V(G) : xy \in E(G)\}$. We denote by $d_G(x)$ the number of edges containing x in G. Denote by $G - v$ the graph obtained from G by

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deleting the vertex $v \in V$ and its incident edges, and denote by $G - uv$ the graph obtained from G by deleting the edge $uv \in E$. This notation can be naturally extended if more than one vertex or edge is deleted. Similarly, let $G + uv$ be obtained from G by adding an edge $uv \notin E$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *union* of G_1 and G_2, denoted by $G_1 \cup G_2$, is a graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. We denote $G_1 \simeq G_2$ if G_1 and G_2 are isomorphic, i.e., there exists a bijection $f : V(G_1) \to V(G_2)$ such that $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$. Moreover, we say that G_1 and G_2 are *identical*, denoted by $G_1 = G_2$, if $V(G_1) = V(G_2)$ and $E(G_1) = E(G_2)$. By the definitions, we can see that $G_1 \simeq G_2$ and $G_1 = G_2$ are slightly different. Let $K(A, B)$ be a complete bipartite graph with partite sets $\{A, B\}$. Assume G_1 and G_2 are two spanning subgraphs of $K(A, B)$. We say G_1 and G_2 are *bi-isomorphic*, if G_1 and G_2 are isomorphic such that the bijection $f : V(G_1) \to V(G_2)$ maps the vertices of $V(G_1) \cap A$ to $V(G_2) \cap A$ and maps the vertices of $V(G_1) \cap B$ to $V(G_2) \cap B$.

The *adjacency matrix* of G is $A(G) = [a_{ij}]_{i,j=1}^n$, where $a_{ij} = 1$ if $ij \in E(G)$, and $a_{ij} = 0$ otherwise. The eigenvalues of G are defined as the eigenvalues of its adjacency matrix $A(G)$. The maximum of modulus of eigenvalues of $A(G)$ is called the *spectral radius* of G and denoted by $\lambda(G)$. A path (resp. cycle) going through all vertices of the graph is called a Hamilton path (resp. cycle). The study of the existence of a Hamilton path (resp. cycle) in graphs is a basic and classical problem. Up to now, much attention has been paid to this topic. In 2010, Fiedler and Nikiforov [15] initiated the investigation on the spectral conditions for the existence of a Hamilton path (resp. cycle) in a graph, which was improved in [5, 11, 31, 40]. The problem of finding spectral conditions that guarantee the existence of Hamilton paths or cycles in graphs has been considered not only for general graphs, but also specifically for bipartite graphs. In 2017, Li and Ning [23] gave a tight condition on spectral radius of a single balanced bipartite graph for the existence of a Hamilton path.

Theorem 1 (Li and Ning, [23])**.** *Let G be a balanced bipartite graph on $2n$ vertices, where $n \geqslant 4$. If $\lambda(G) \geqslant \sqrt{n(n-1)}$, then G contains a Hamilton path unless $G \simeq K_{n,n-1} \cup K_1$.*

In 1963, Moon and Moser [29] gave an edge number condition for Hamilton cycles in balanced bipartite graphs by imposing minimum degree. Let $K_{m,n}$ denote the complete bipartite graph with partite sets of orders m and n, respectively, and B_n^1 be the graph obtained from $K_{n,n}$ by deleting all edges in its one subgraph $K_{n-1,1}$. In 2016, Li and Ning [24] obtained the spectral analogue of Moon-Moser's theorem. Especially, one of their results is as follows.

Theorem 2 (Li and Ning [24])**.** *Let G be a balanced bipartite graph of order $2n$. If $n \geqslant 4$ and $\lambda(G) \geqslant \lambda(B_n^1)$, then G contains a Hamilton cycle unless $G \simeq B_n^1$.*

In this paper, we will also seek spectral conditions that guarantee the existence of Hamilton paths (resp. cycles), but we will consider this question in the context of *graph transversals*, which are defined as follows. Let $\mathcal{G} = \{G_1, G_2, \ldots, G_t\}$ be a family of not necessarily distinct graphs with common vertex set V. Let H be a graph with t edges on vertex set $V(H) \subseteq V$. We say that $\mathcal{G}$ has a transversal isomorphic to H if there exists a bijection $\phi : E(H) \to [t]$ such that $e \in E(G_{\phi(e)})$ for each $e \in E(H)$. In other words, each edge of H is from a different G_i. In particular, if $G_1 = G_2 = \cdots = G_t$, then the transversal isomorphic to H reduces to a copy of H. In 2020, Joos and Kim [21] considered the following question:

Question 3 (Joos and Kim, [21])**.** Let H be a graph with t edges and $\mathcal{G} = \{G_1, G_2, \ldots, G_t\}$ be a collection of not necessarily distinct graphs on the same vertex set V. Which properties and constraints imposed on the family $\mathcal{G}$ can yield a transversal isomorphic to H?

Recently, certain classical results about cycles of specified lengths in graphs have been extended to the setting of graph transversals. For example, Aharoni et al. [2] gave a transversal analogue of Mantel's theorem, showing that given a graph family $\mathcal{G} = \{G_1, G_2, G_3\}$ on a common set of n vertices in which each graph G_i has roughly at least $0.2557n^2$ edges, there must exist a transversal of $\mathcal{G}$ isomorphic to K_3. Additionally, Joos and Kim [21] showed that given a graph family $\mathcal{G} = \{G_1, \ldots, G_n\}$ on a common vertex set of n vertices, if the minimum degree of each graph G_i is at least $n/2$, then $\mathcal{G}$ has a transversal isomorphic to a Hamilton cycle, which gives a generalization of Dirac's theorem. Bradshaw [7] showed that given a family $\mathcal{G}$ of $2n$ bipartite graphs on a common set of $2n$ vertices with a common balanced bipartition, if each graph of $\mathcal{G}$ has minimum degree greater than $n/2$ in one color class and minimum degree at least $n/2$ in the other color class, then there exists a transversal isomorphic to a Hamilton cycle, which generalized the result of Moon and Moser [29]. For other interesting results on rainbow Hamilton paths (or cycles) and related problems we refer the reader to [9, 34].

Nowadays, researches on spectral extremal theory of graphs have attracted extensive attention; see, e.g., [26, 27, 36, 38, 43, 41, 33]. In 2007, Feng, Yu and Zhang [14] provided a tight condition on a graph to have a matching of size t in terms of its spectral radius. Guo, Lu, Ma and Ma [18] extended the result [14] to a family of graphs by a different method and presented a spectral condition for a family $\mathcal{G} = \{G_1, \ldots, G_t\}$ to guarantee a transversal isomorphic to a matching with t edges. Shi, Li and Chen [35] obtained the condition of spectral radius of graphs when $\mathcal{G}$ is a family of bipartite graphs and H is a matching of size t. Zhang and Zhang [46] extended the result [18] to k-factors. He, Li and Feng [19] obtained the tight lower bound on spectral radius of a family $\mathcal{G}$ of graphs to ensure that $\mathcal{G}$ admits a transversal isomorphic to a Hamilton path. Recently, Zhang and van Dam [47] gave a sufficient condition for the existence of a transversal isomorphic to a Hamilton cycle in terms of the spectral radii of the graphs in a family $\mathcal{G}$. Motivated by the above results, we shall investigate the tight lower bound on spectral radius of a family $\mathcal{G}$ of balanced bipartite graphs to ensure that $\mathcal{G}$ admits a transversal isomorphic to a Hamilton path or Hamilton cycle.

Before stating our results, we need to fix some notations. For a positive integer n, let $[n] := \{1, 2, \ldots, n\}$. Here we use $\widehat{G}$ to denote the quasi-complement of a bipartite graph G with partite sets $\{A, B\}$, i.e., one with vertex set $V(\widehat{G}) = V(G)$ and for every $a \in A$ and $b \in B$, $ab \in E(\widehat{G})$ if and only if $ab \notin E(G)$. Let $\Phi(A, B) = \widehat{K(A, B)}$. Let G_1 and G_2 be two disjoint bipartite graphs, with partite sets $\{A, B\}$ and $\{C, D\}$, respectively. We use $G_1 \sqcup G_2$ to denote the graph obtained from $G_1 \cup G_2$ by adding all possible edges between A and D and all possible edges between C and B. Let $X := \{x_1, x_2, \ldots, x_n\}$ and $Y := \{y_1, y_2, \ldots, y_n\}$. For $0 \leqslant s, t \leqslant n$, let $X_1 := \{x_1, x_2, \ldots, x_{n-s}\}$, $X_2 := X \backslash X_1$, $Y_1 := \{y_1, y_2, \ldots, y_{n-t}\}$ and $Y_2 := Y \backslash Y_1$. We set $B(n, s, t) = K(X_1, Y_1) \sqcup \Phi(X_2, Y_2)$.

In this paper, we will focus on graph transversals over bipartite graph families. First, we will prove the following theorem, which gives a generalization of Theorem 1 in the setting of graph transversals.

Theorem 4. *Let $n \geqslant 4$. Let V be a set of n red vertices and n blue vertices, and let $\mathcal{G} = \{G_1, \ldots, G_{2n-1}\}$ be a family of $2n-1$ bipartite graphs on the vertex set V. Suppose*

- *For each $i \in [2n-1]$, the red vertices of V and the blue vertices of V both form independent sets in G_i;*
- *For each $i \in [2n-1]$, $\lambda(G_i) \geqslant \sqrt{n(n-1)}$.*

Then $\mathcal{G}$ admits a transversal isomorphic to a Hamilton path, unless

$$G_1 = G_2 = \cdots = G_{2n-1} \quad \text{and} \quad G_1 \simeq K_{n,n-1} \cup K_1.$$

Our second main result is a transversal version of Theorem 2. Using a quite different method from [24], we extend the spectral condition for a Hamilton cycle to a family of balanced bipartite graphs.

Theorem 5. *Let $n \geqslant 4$. Let V be a set of n red vertices and n blue vertices, and let $\mathcal{G} = \{G_1, \ldots, G_{2n}\}$ be a family of $2n$ bipartite graphs on the vertex set V. Suppose*

- *For each $i \in [2n]$, the red vertices of V and the blue vertices of V both form independent sets in G_i;*
- *For each $i \in [2n]$, $\lambda(G_i) \geqslant \lambda(B_n^1)$.*

Then $\mathcal{G}$ admits a transversal isomorphic to a Hamilton cycle, unless

$$G_1 = G_2 = \cdots = G_{2n} \quad \text{and} \quad G_1 \simeq B_n^1.$$

If we let $G_1 = G_2 = \cdots = G_{2n-1}$ in Theorem 4, then any Hamilton path in G_1 is a transversal isomorphic to a Hamilton path of $\mathcal{G}$, so Theorem 4 is a generalization of Theorem 1. Again, if we let $G_1 = G_2 = \cdots = G_{2n}$ in Theorem 5, then any Hamilton cycle in G_1 is a transversal isomorphic to a Hamilton cycle of $\mathcal{G}$, so Theorem 5 is a generalization of Theorem 2. We stress that the proofs of both Theorems 4 and 5 could be regarded as nice applications of the shifting technique, which may be of independent interest. Over the past few years, there are various problems in extremal combinatorics, from the classical Erdős–Ko–Rado theorem and Hilton–Milner's theorem to the recent developments on Erdős' matching conjecture, where the shifting technique and its variants have been extremely useful. Moreover, the shifts on graphs have been used to treat the Turán type problems involving the matching [3, 18, 35, 37], the linear forest of given size [19, 44, 45], the Hamilton path [19] and the Hamilton cycle [47]. In addition, it seems interesting to explore other graphical properties where the shifts can be applied.

This paper is organized as follows. In Section 2, we shall introduce some basic properties of the shifting technique. In Sections 3 and 4, we shall provide the detailed proofs of Theorems 4 and 5, respectively. Some concluding remarks are given in the last section.

2 The shifting technique

In dealing with extremal problems for finite sets, a very important technique is *shifting* (or *compression*). This operation on families of sets turns out to be very useful in establishing various upper bounds [17]. The formal definition goes back to the original paper by Erdős, Ko and Rado [13]. Let G be a graph with vertex set $[n]$. For $1 \leqslant i < j \leqslant n$, the (i, j)-*shift* S_{ij} for G is defined as follows:

$$S_{ij}(G) := \{S_{ij}(e) : e \in E(G)\},$$

where

$$S_{ij}(e) := \begin{cases} (e \setminus \{j\}) \cup \{i\}, & \text{if } j \in e,\ i \notin e \text{ and } (e \setminus \{j\}) \cup \{i\} \notin E(G); \\ e, & \text{otherwise.} \end{cases}$$

In words, (i, j)-shift only try to make changes the edges of G which do not contain i but contain j. In this case, we check whether the edge obtained by replacing j with i is already in $E(G)$. Only if *not* do we then replace the edge with the new one. Clearly, by definition, we have $|E(S_{ij}(G))| = |E(G)|$. Now assume G is a spanning subgraph of $K(X, Y)$. If $S_{x_ix_j}(G) = G$ and $S_{y_ky_\ell}(G) = G$ for every pairs (i, j) and (k, ℓ) satisfying $1 \leqslant i < j \leqslant n$ and $1 \leqslant k < \ell \leqslant n$, respectively, then G is said to be *shifted.* The following observation shows a basic property of (x_i, x_j)-shift and (y_k, y_ℓ)-shift for bipartite graph G. Since we can obtain it easily from the definition of shift, here we omit the proof of it.

Observation 6. If G is a shifted bipartite graph with partite sets $\{X, Y\}$, then for every $\{x_i, y_k\}, \{x_j, y_\ell\} \subseteq X \cup Y$ such that $i \leqslant j$ and $k \leqslant \ell$, $\{x_j, y_\ell\} \in E(G)$ implies $\{x_i, y_k\} \in E(G)$.

Iterating the (x_i, x_j)-shift and (y_k, y_ℓ)-shift for all pairs (i, j) and (k, ℓ) satisfying $1 \leqslant i < j \leqslant n$ and $1 \leqslant k < \ell \leqslant n$, respectively, will eventually produce a shifted bipartite graph. Roughly speaking, the shifting operation moves the edges forward. Indeed, after a shifting operation $S_{x_ix_j}$ or $S_{y_ky_\ell}$, we will get a new graph $S_{x_ix_j}(G)$ or $S_{y_ky_\ell}(G)$, where the sum of all subscripts of edges will decrease strictly since $i < j$ and $k < \ell$. For a bipartite graph G, let $S(G)$ be a bipartite graph which is obtained from G by a series of (x_i, x_j)-shifts and (y_k, y_ℓ)-shifts with $i < j$ and $k < \ell$, and is invariant under all such possible shifts. Consequently, we obtain that $S(G)$ is shifted.

In 2009, Csikvári [10] proved that the spectral radius of a graph does not decrease after the shifting operation.

Theorem 7 (Csikvári, [10])**.** *Let u, v be two vertices of G. Then*

$$\lambda(S_{uv}(G)) \geqslant \lambda(G).$$

The following lemma is well-known and the proof can be found in [8].

Lemma 8. *If u, v are two nonadjacent vertices of a graph G, then*

$$\lambda(G + uv) \geqslant \lambda(G).$$

Moreover, if G is connected, then the above inequality holds strictly.

Theorem 9 (Wu, Xiao and Hong, [39])**.** *Let u, v be two vertices of the connected graph G. Suppose $v_1, v_2, \ldots, v_s \in N_G(v) \setminus (N_G(u) \cup \{u\})$ with $s \geqslant 1$ and $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ is the Perron vector of $A(G)$, where $\mathbf{x}_i$ corresponds to the vertex v_i for $1 \leqslant i \leqslant n$. Let G^* be the graph obtained from G by deleting the edges $\{v, v_i\}$ and adding the edges $\{u, v_i\}$ for $1 \leqslant i \leqslant s$. If $\mathbf{x}_u \geqslant \mathbf{x}_v$, then $\lambda(G^*) > \lambda(G)$.*

The following lemma for connected graphs is an immediate consequence of Corollary 4.1 of Liu, Lai and Das [25]. The detailed proof can be found in [18] by applying Theorem 9.

Lemma 10 (Guo, Lu, Ma and Ma, [18])**.** *Let G be a connected graph on vertex set $[n]$. Let u, v be two vertices of G. Then $\lambda(S_{uv}(G)) > \lambda(G)$ unless $G \simeq S_{uv}(G)$.*

Given a collection of graphs $\mathcal{G} = \{G_1, \ldots, G_t\}$, let $S_{ij}(\mathcal{G}) := \{S_{ij}(G_1), \ldots, S_{ij}(G_t)\}$. The following lemma is inspired by a result of Huang, Loh and Sudakov [20], in which they proved a similar result for a rainbow matching of size t.

Lemma 11 (He, Li and Feng, [19])**.** *Given a family of graphs $\mathcal{G} = \{G_1, \ldots, G_{n-1}\}$ on vertex set $[n]$. Let $i < j$ be two vertices in $[n]$. If $S_{ij}(\mathcal{G})$ admits a transversal isomorphic to a Hamilton path, then so does $\mathcal{G}$.*

Lemma 12 (Zhang, van Dam, [47])**.** *Given a family of graphs $\mathcal{G} = \{G_1, \ldots, G_n\}$ on vertex set $[n]$. Let $i < j$ be two vertices in $[n]$. If $S_{ij}(\mathcal{G})$ admits a transversal isomorphic to a Hamilton cycle, then so does $\mathcal{G}$.*

3 Proof of Theorem 4

In this section, we shall give the proof of Theorem 4. Our proof has its root in [18].

Lemma 13. *Let $n \geqslant 3$ and G be a balanced bipartite graph of order $2n$ with partite sets $\{X, Y\}$. If $\lambda(G) = \sqrt{n(n-1)}$ and $1 \leqslant i < j \leqslant n$, then the following statements hold:*

(i) if $S_{x_ix_j}(G)$ is bi-isomorphic to $B(n, n, 1)$, then G is bi-isomorphic to $B(n, n, 1)$;
(ii) if $S_{y_iy_j}(G)$ is bi-isomorphic to $B(n, n, 1)$, then G is bi-isomorphic to $B(n, n, 1)$;
(iii) if $S_{x_ix_j}(G)$ is bi-isomorphic to $B(n, 1, n)$, then G is bi-isomorphic to $B(n, 1, n)$;
(iv) if $S_{y_iy_j}(G)$ is bi-isomorphic to $B(n, 1, n)$, then G is bi-isomorphic to $B(n, 1, n)$.

Proof. If $S_{x_ix_j}(G)$ is bi-isomorphic to $B(n, n, 1)$, then there is a set $U \subseteq Y$ of size $n-1$ such that $S_{x_ix_j}(G)[X \cup U] \simeq K_{n,n-1}$. By the definition of (x_i, x_j)-shift, we have $G = S_{x_ix_j}(G)$. Thus (i) holds. In the following, assume $S_{y_iy_j}(G)$ is bi-isomorphic to $B(n, n, 1)$. Similar to (i), there is a set $U \subseteq Y$ of size $n-1$ such that $S_{y_iy_j}(G)[X \cup U] \simeq K_{n,n-1}$. If $|\{y_i, y_j\} \cap U| \neq 1$, then we have $G = S_{y_iy_j}(G)$. Otherwise, by the definition of (y_i, y_j)-shift, we have $y_i \in U$ and $y_j \notin U$. Note that $\lambda(G) = \sqrt{n(n-1)} = \lambda(S_{y_iy_j}(G))$. If $d_G(y_i) > 0$ and $d_G(y_j) > 0$, then G is connected and $G, S_{y_iy_j}(G)$ are not isomorphic. By Lemma 10, we have $\lambda(S_{y_iy_j}(G)) > \lambda(G)$, a contradiction. Thus we have $N_G(y_i) = \emptyset$ or $N_G(y_j) = \emptyset$. It follows that G is bi-isomorphic to $S_{y_iy_j}(G)$. Hence, G is bi-isomorphic to $B(n,n,1)$ and (ii) holds.

Similar to the proof of (i) and (ii), we can prove (iii) and (iv). $\square$

Recall that for a bipartite graph G with partite sets $\{X,Y\}$, $S(G)$ is a graph which is obtained from G by applying a series of (x_i,x_j)-shifts and (y_k,y_ℓ)-shifts with $i<j$ and $k<\ell$. The following result is an immediate consequence of Theorem 7 and Lemma 13, which will be used in the proof of Theorem 4.

Corollary 14. Let $n \geqslant 3$ and G be a balanced bipartite graph of order $2n$ with partite sets $\{X,Y\}$. Assume $\lambda(G)=\sqrt{n(n-1)}$. If $S(G)$ is bi-isomorphic to $B(n,n,1)$, then G is bi-isomorphic to $B(n,n,1)$. If $S(G)$ is bi-isomorphic to $B(n,1,n)$, then G is bi-isomorphic to $B(n,1,n)$.

Lemma 15. *Let $n \geqslant 4$ and $\mathcal{G}=\{G_1,\ldots,G_{2n-1}\}$ be a family of bipartite graphs with partite sets $\{X,Y\}$. Assume for each $i\in[2n-1]$, G_i is bi-isomorphic to $B(n,n,1)$. If there exist $p,q\in[2n-1]$ such that $U_p\neq U_q$, where $U_i=\{y\in Y: d_{G_i}(y)=n\}$, then $\mathcal{G}$ admits a transversal isomorphic to a Hamilton path.*

Proof. Without loss of generality, we may assume that $p=1$ and $q=2n-1$. Since $G_1\neq G_{2n-1}$, there exist vertices $b_1\in U_1\setminus U_{2n-1}$ and $b_n\in U_{2n-1}\setminus U_1$ such that $d_{G_1}(b_1)=n$, $d_{G_{2n-1}}(b_1)=0$, $d_{G_1}(b_n)=0$ and $d_{G_{2n-1}}(b_n)=n$. For notational convenience, in the following, we reorder the vertices of Y by $b_1,b_2,\ldots,b_n$. So we may find an edge $e_1=b_1a_1\in E(G_1)$, where $a_1\in X$. Since $d_{G_2}(a_1)=n-1\geqslant 3$, there is an edge $a_1b_2\in E(G_2)$ such that $b_2\in Y\backslash\{b_1,b_n\}$. Since at least one of $d_{G_3}(b_1)$ and $d_{G_3}(b_2)$ is equal to n, without loss of generality, we may assume that $d_{G_3}(b_2)=n>1$. So we may find an edge $e_3=b_2a_2\in E(G_3)$ satisfying $a_2\in X\backslash\{a_1\}$. Now suppose we have found a path $P_{r+1}=b_1a_1b_2a_2\cdots b_{\frac{r+1}{2}}a_{\frac{r+1}{2}}$ of odd length r such that $b_1a_1\in E(G_1), a_1b_2\in E(G_2), b_2a_2\in E(G_3),\ldots,b_{\frac{r+1}{2}}a_{\frac{r+1}{2}}\in E(G_r)$ or $P_{r+1}=b_1a_1b_2\cdots a_{\frac{r}{2}}b_{\frac{r+2}{2}}$ of even length r such that $b_1a_1\in E(G_1), a_1b_2\in E(G_2),\ldots,a_{\frac{r}{2}}b_{\frac{r+2}{2}}\in E(G_r)$. If $r=2n-1$, then P_{2n} is a desired transversal isomorphic to a Hamilton path of $\mathcal{G}$. So we assume that $r<2n-1$. If $r=2n-2$, since $d_{G_{2n-1}}(b_n)=n$, we have $b_na_n\in E(G_{2n-1})$ and $P_{r+1}+b_na_n$ is a transversal isomorphic to a Hamilton path of $\mathcal{G}$, where $a_n\neq a_i$ for $1\leqslant i\leqslant n-1$. If $r<2n-2$, then the end-vertex of P_{r+1} different from b_1 is $a_{\frac{r+1}{2}}$ for odd r or $b_{\frac{r+2}{2}}$ for even r. In what follows, we distinguish the proof into two cases.

Case 1. r is odd.

In this case,

$$|N_{G_{r+1}}(a_{\frac{r+1}{2}})\backslash V(P_{r+1})| \geqslant n-1-\frac{r+1}{2}=\frac{2n-3-r}{2}\geqslant 0.$$

If $|N_{G_{r+1}}(a_{\frac{r+1}{2}})\backslash V(P_{r+1})|>1$, then there exists an edge $a_{\frac{r+1}{2}}b_{\frac{r+3}{2}}\in E(G_{r+1})$ such that $b_{\frac{r+3}{2}}\notin V(P_{r+1})\cup\{b_n\}$. If $|N_{G_{r+1}}(a_{\frac{r+1}{2}})\backslash V(P_{r+1})|=1$ and $a_{\frac{r+1}{2}}b_n\notin E(G_{r+1})$, then there exists an edge $a_{\frac{r+1}{2}}b_{\frac{r+3}{2}}\in E(G_{r+1})$ such that $b_{\frac{r+3}{2}}\notin V(P_{r+1})\cup\{b_n\}$. Hence, $P_{r+1}+a_{\frac{r+1}{2}}b_{\frac{r+3}{2}}$ is a transversal (isomorphic to a path) of $\{G_1,G_2,\ldots,G_{r+1}\}$. If $|N_{G_{r+1}}(a_{\frac{r+1}{2}})\backslash V(P_{r+1})|=$

1 and $a_{\frac{r+1}{2}}b_n \in E(G_{r+1})$, then $N_{G_{r+1}}(a_{\frac{r+1}{2}})\backslash V(P_{r+1}) = \{b_n\}$ and $n-1 = d_{G_{r+1}}(a_{\frac{r+1}{2}}) \leqslant |V(P_{r+1}) \cap Y| + 1 = \frac{r+1}{2} + 1$. Hence, $r \in \{2n-5, 2n-3\}$.

If $r = 2n-5$ and $N_{G_{r+1}}(a_{\frac{r+1}{2}})\backslash V(P_{r+1}) = \{b_n\}$, then $|N_{G_{r+1}}(a_{\frac{r+1}{2}})\backslash V(P_{r+1})| = 1 = n-1-\frac{r+1}{2}$ and $P_{r+1} - b_1a_1 + b_1a_{\frac{r+1}{2}} + b_1a_{\frac{r+3}{2}}$ is a path of length $r+1$ such that $b_1a_{\frac{r+3}{2}} \in E(G_1), a_1b_2 \in E(G_2), \dots, b_{\frac{r+1}{2}}a_{\frac{r+1}{2}} \in E(G_r), b_1a_{\frac{r+1}{2}} \in E(G_{r+1})$, where $a_{\frac{r+3}{2}} \in X\backslash\{a_1, \dots, a_{\frac{r+1}{2}}\}$. Note that one of $d_{G_{r+2}}(b_n)$ and $d_{G_{r+2}}(b_{n-1})$ is n. Without loss of generality, assume $d_{G_{r+2}}(b_{n-1}) = n$. Similarly, assume $d_{G_{r+3}}(b_{n-1}) = n$. Then $P_{r+1} - b_1a_1 + b_1a_{\frac{r+1}{2}} + b_1a_{\frac{r+3}{2}} + a_{\frac{r+3}{2}}b_{n-1} + b_{n-1}a_n + a_nb_n$ is a transversal isomorphic to a Hamilton path of $\mathcal{G}$.

If $r = 2n-3$ and $N_{G_{r+1}}(a_{\frac{r+1}{2}})\backslash V(P_{r+1}) = \{b_n\}$, then $P_{r+1} + a_{\frac{r+1}{2}}b_n + b_na_n$ is a transversal isomorphic to a Hamilton path of $\mathcal{G}$.

If $|N_{G_{r+1}}(a_{\frac{r+1}{2}})\backslash V(P_{r+1})| = 0$, then $n-1 = d_{G_{r+1}}(a_{\frac{r+1}{2}}) \leqslant |V(P_{r+1}) \cap Y| = \frac{r+1}{2}$. Hence, $r = 2n-3$. It is easy to see that $P_{r+1} - b_1a_1 + b_1a_{\frac{r+1}{2}} + b_1a_{\frac{r+3}{2}} + a_{\frac{r+3}{2}}b_n$ is a transversal (isomorphic to a Hamilton path) of $\mathcal{G}$, where $b_1a_{\frac{r+3}{2}} \in E(G_1), b_1a_{\frac{r+1}{2}} \in E(G_{r+1})$ and $a_{\frac{r+3}{2}}b_n \in E(G_{2n-1})$.

Case 2. r is even.

In this case, either $d_{G_{r+1}}(b_1) = n$ or $d_{G_{r+1}}(b_{\frac{r+2}{2}}) = n$. Without loss of generality, we may assume that $d_{G_{r+1}}(b_{\frac{r+2}{2}}) = n$. Then we choose $a_{\frac{r+2}{2}} \in X\backslash V(P_{r+1})$ such that $b_{\frac{r+2}{2}}a_{\frac{r+2}{2}} \in E(G_{r+1})$. Note that $P_{r+1} + b_{\frac{r+2}{2}}a_{\frac{r+2}{2}}$ is a path of odd length $r+1$. Continuing this process as Case 1 for at most $2n-1$ steps, we may obtain the desired transversal isomorphic to a Hamilton path. □

Similarly, we can prove the following lemma.

Lemma 16. *Let $n \geqslant 4$ and $\mathcal{G} = \{G_1, \dots, G_{2n-1}\}$ be a family of bipartite graphs with partite sets $\{X, Y\}$. Assume for each $i \in [2n-1]$, G_i is bi-isomorphic to $B(n,1,n)$. If there exist $p, q \in [2n-1]$ such that $W_p \neq W_q$, where $W_i = \{x \in X : d_{G_i}(x) = n\}$, then $\mathcal{G}$ admits a transversal isomorphic to a Hamilton path.*

Now, we are ready to prove Theorem 4.

Proof of Theorem 4. Assume X is the set of n red vertices and Y is the set of n blue vertices. For each $i \in [2n-1]$, $\{X, Y\}$ are partite sets of G_i. Suppose that $\{G_1, \dots, G_{2n-1}\}$ dose not admit a transversal isomorphic to a Hamilton path. By Lemma 11, the shifted family $\{S(G_1), \dots, S(G_{2n-1})\}$ dose not admit a transversal isomorphic to a Hamilton path. For notational convenience, we denote $H_i := S(G_i)$ for each $i \in [2n-1]$. By Theorem 7 and the spectral conditions, for $i \in [2n-1]$,

$$\lambda(H_i) \geqslant \lambda(G_i) \geqslant \sqrt{n(n-1)}. \tag{1}$$

Let $e_j := \{x_{n-j}, y_{j+1}\}$ for every $\lfloor \frac{n}{2} \rfloor \leqslant j \leqslant n-1$. First of all, we shall prove that for each $i \in [2n-1]$,

$$\{e_{\lfloor \frac{n}{2} \rfloor}, e_{\lfloor \frac{n}{2} \rfloor+1}, \dots, e_{n-2}\} \subseteq E(H_i). \tag{2}$$

Otherwise, assume that there exist $\lfloor \frac{n}{2} \rfloor \leqslant r \leqslant n-2$ and $t \in [2n-1]$ such that $e_r = \{x_{n-r}, y_{r+1}\} \notin E(H_t)$. Since H_t is shifted, by Observation 6, every edge $\{x_i, y_j\} \in E(H_t)$ satisfies $i < n-r$ or $j < r+1$. Thus H_t is a subgraph of $B(n, r+1, n-r)$. Note that $B(n, r+1, n-r)$ contains no Hamilton path and $B(n, r+1, n-r)$ is not isomorphic to $K_{n,n-1} \cup K_1$. By Theorem 1 and Lemma 8, we have

$$\lambda(H_t) \leqslant \lambda(B(n, r+1, n-r)) < \lambda(K_{n,n-1} \cup K_1) = \sqrt{n(n-1)},$$

a contradiction. Hence, $\{e_{\lfloor \frac{n}{2} \rfloor}, e_{\lfloor \frac{n}{2} \rfloor+1}, \ldots, e_{n-2}\} \subseteq E(H_i)$ for each $i \in [2n-1]$.

Let $e'_j := \{x_{j+1}, y_{n-j}\}$ for every $\lfloor \frac{n}{2} \rfloor \leqslant j \leqslant n-1$. Similar to the above, we can prove that for each $i \in [2n-1]$,

$$\{e'_{\lfloor \frac{n}{2} \rfloor}, e'_{\lfloor \frac{n}{2} \rfloor+1}, \ldots, e'_{n-2}\} \subseteq E(H_i). \tag{3}$$

Claim 1. Either $e_{n-1} = \{x_1, y_n\} \in E(H_i)$ or $e'_{n-1} = \{x_n, y_1\} \in E(H_i)$ for each $i \in [2n-1]$.

Proof of Claim 1. Otherwise, assume there is an H_t satisfying $\{x_1, y_n\} \notin E(H_t)$ and $\{x_n, y_1\} \notin E(H_t)$. By Observation 6, every edge $\{x_i, y_j\} \in E(H_t)$ satisfies $i < n$ and $j < n$. Thus H_t is a subgraph of $K_{n-1,n-1} \cup 2K_1$. Hence, $\lambda(H_t) \leqslant \lambda(K_{n-1,n-1} \cup K_1) = n-1 < \sqrt{n(n-1)}$, a contradiction. □

Claim 2. Either $H_1 = \cdots = H_{2n-1} = B(n, n, 1)$ or $H_1 = \cdots = H_{2n-1} = B(n, 1, n)$.

Proof of Claim 2. In what follows, we distinguish the proof into four cases.

Case 1. There exist $p, q \in [2n-1]$ such that $p \neq q$, $e_{n-1} \in E(H_p)$ and $e'_{n-1} \in E(H_q)$.

Without loss of generality, we may assume that $p = 1$ and $q = 2n-1$. Then (2), (3) and Observation 6 implies that $P_{2n-1} = y_n x_1 y_{n-1} x_2 y_{n-2} x_3 \ldots x_{n-1} y_1 x_n$ is a transversal isomorphic to a Hamilton path, a contradiction.

Case 2. There exists $p \in [2n-1]$ such that $e_{n-1} \in E(H_p)$ and $e'_{n-1} \in E(H_p)$.

Without loss of generality, we assume that $p = 1$. By Claim 1, at least one of e_{n-1} and e'_{n-1} is an edge of H_2. Hence, according to Case 1, $\mathcal{G}$ has a transversal isomorphic to a Hamilton path, a contradiction.

Case 3. $e_{n-1} \notin \bigcup_{i=1}^{2n-1} E(H_i)$.

By Observation 6, every edge $\{x_i, y_j\} \in E(H_t)$ satisfies $j < n$. Thus H_t is a subgraph of $B(n, n, 1)$. By Lemma 8, one can see that $\lambda(H_t) \leqslant \lambda(B(n, n, 1)) = \sqrt{n(n-1)}$. Combining with (1), we have $\lambda(H_t) = \lambda(B(n, n, 1))$. Recall that H_t is a subgraph of $B(n, n, 1)$. It follows that $H_t = B(n, n, 1)$ for each $t \in [2n-1]$.

Case 4. $e'_{n-1} \notin \bigcup_{i=1}^{2n-1} E(H_i)$.

Similar to the proof of Case 3, we have $H_1 = \cdots = H_{2n-1} = B(n, 1, n)$. This completes the proof of Claim 2. □

By (1), Theorem 7 and Claim 2, we have

$$\lambda(G_1) = \cdots = \lambda(G_{2n-1}) = \sqrt{n(n-1)}.$$

Thus by Claim 2 and Corollary 14, we have either all G_i are bi-isomorphic to $B(n,n,1)$ or all G_i are bi-isomorphic to $B(n,1,n)$. If there exist $i,j \in [2n-1]$ such that $G_i \neq G_j$, by Lemmas 15 and 16, $\{G_1,\dots,G_{2n-1}\}$ has a transversal isomorphic to a Hamilton path, a contradiction. Therefore, $G_1 = \cdots = G_{2n-1}$ and $G_1 \simeq K_{n,n-1} \cup K_1$. This completes the proof. $\square$

4 Proof of Theorem 5

In what follows, we shall prove Theorem 5 by using the shifting technique. First of all, we present some lemmas and corollaries for our purpose.

Lemma 17. *Let $n \geqslant 3$ and G be a balanced bipartite graph with partite sets $\{X,Y\}$. Assume $\lambda(G) = \lambda(B_n^1)$ and $1 \leqslant i < j \leqslant n$. Then the following statements hold:*

(i) if $S_{x_ix_j}(G)$ is bi-isomorphic to $B(n,n-1,1)$, then G is bi-isomorphic to $B(n,n-1,1)$;
(ii) if $S_{y_iy_j}(G)$ is bi-isomorphic to $B(n,n-1,1)$, then G is bi-isomorphic to $B(n,n-1,1)$;
(iii) if $S_{x_ix_j}(G)$ is bi-isomorphic to $B(n,1,n-1)$, then G is bi-isomorphic to $B(n,1,n-1)$;
(iv) if $S_{y_iy_j}(G)$ is bi-isomorphic to $B(n,1,n-1)$, then G is bi-isomorphic to $B(n,1,n-1)$.

Proof. If $S_{x_ix_j}(G)$ is bi-isomorphic to $B(n,n-1,1)$, there is a set $U \subset X$ of size $n-1$ such that $d_{S_{x_ix_j}(G)}(u) = n-1$ for every $u \in U$. If $\{x_i,x_j\} \subset U$, then we have $G = S_{x_ix_j}(G)$. In the following, we may assume that $|\{x_i,x_j\} \cap U| = 1$. Then by the definition of (x_i,x_j)-shift, we have $x_j \in U$ and $x_i \notin U$. Note that $N_{S_{x_ix_j}(G)}(x_j) \subset N_{S_{x_ix_j}(G)}(x_i)$ and $d_{S_{x_ix_j}(G)}(x_i) = n = d_{S_{x_ix_j}(G)}(x_j) + 1$. By the definition of (x_i,x_j)-shift, we have G is bi-isomorphic to $S_{x_ix_j}(G)$. Hence, G is bi-isomorphic to $B(n,n-1,1)$. Thus (i) holds.

If $S_{y_iy_j}(G)$ is bi-isomorphic to $B(n,n-1,1)$, there is a set $W \subset Y$ of size $n-1$ such that $d_{S_{y_iy_j}(G)}(w) = n$ for every $w \in W$. If $\{y_i,y_j\} \subset W$, then we have $G = S_{y_iy_j}(G)$. In the following, we may assume that $|\{y_i,y_j\} \cap W| = 1$. Then by the definition of (y_i,y_j)-shift, we have $y_i \in W$ and $y_j \notin W$. Note that $\lambda(G) = \lambda(B_n^1) = \lambda(B(n,n-1,1)) = \lambda(S_{y_iy_j}(G))$ and G is connected. By Lemma 10, we have G is bi-isomorphic to $S_{y_iy_j}(G)$. Hence, G is bi-isomorphic to $B(n,n-1,1)$. Thus (ii) holds.

Similar to the proof of (i) and (ii), we can prove (iii) and (iv). $\square$

Recall that $S(G)$ is a graph which is obtained from G by a series of (x_i,x_j)-shifts with $i<j$ and (y_k,y_ℓ)-shifts with $k<\ell$. The following result is an immediate consequence of Theorem 7 and Lemma 17 which will be used in the proof of Theorem 5.

Corollary 18. Let $n \geqslant 3$ and G be a bipartite graph with partite sets $\{X,Y\}$. Assume $\lambda(G) = \lambda(B_n^1)$. If $S(G)$ is bi-isomorphic to $B(n,n-1,1)$, then G is bi-isomorphic to $B(n,n-1,1)$. If $S(G)$ is bi-isomorphic to $B(n,1,n-1)$, then G is bi-isomorphic to $B(n,1,n-1)$.

Lemma 19. *Let $n \geqslant 4$ and $\mathcal{G} = \{G_1,G_2,\dots,G_{2n}\}$ be a family of balanced bipartite graphs of order $2n$ with partite sets $\{X,Y\}$. If for each $i \in [2n]$, G_i is bi-isomorphic to $B(n,n-1,1)$, then $\mathcal{G}$ admits a transversal isomorphic to a Hamilton cycle unless $G_1 = G_2 = \cdots = G_{2n}$.*

Proof. Note that $B(n,n-1,1) \simeq B_n^1$. If $G_1 = G_2 = \cdots = G_{2n}$, by Theorem 2, $\mathcal{G}$ has no transversal isomorphic to a Hamilton cycle. In the following, we assume that the graphs in $\mathcal{G}$ are not all identical. Suppose that $G_1, G_2, \ldots, G_{2n}$ have k distinct vertices of degree 1 contained in Y and denote these k vertices by $b_1, b_2, \ldots, b_k$. We distinguish the cases $k = 1$ and $k \geqslant 2$.

If $k = 1$, all graphs in $\mathcal{G}$ have vertex b_1 of degree 1. Without loss of generality, assume that G_1 and G_2 are distinct. In this case, the vertex adjacent to b_1 in G_1, say a_1, and the one adjacent to b_1 in G_2, say a_2, are different. Since the induced subgraphs of $G_1, G_2, \ldots, G_{2n}$ on vertex subset $X \cup (Y \backslash \{b_1\})$ are identical (isomorphic to $K_{n,n-1}$), it is clear that by using edges b_1a_1 in G_1 and b_1a_2 in G_2, we can find a transversal isomorphic to a Hamilton cycle of $\mathcal{G}$.

Next, we consider the case $k \geqslant 2$. Recall that $b_1, b_2, \ldots, b_k$ are all the k distinct vertices of degree 1 in $G_1, G_2, \ldots, G_{2n}$. Let $\mathcal{M}_j = \{G_i : d_{G_i}(b_j) = 1\}$ and $m_j = |\mathcal{M}_j|$ for $1 \leqslant j \leqslant k$. Let $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_k$ by reordering vertices $b_1, b_2, \ldots, b_k$ if necessary. Without loss of generality, we may assume that $\mathcal{M}_1 = \{G_1, G_2, \ldots, G_{m_1}\}, \mathcal{M}_2 = \{G_{m_1+1}, \ldots, G_{m_1+m_2}\}, \ldots, \mathcal{M}_k = \{G_{2n-m_k+1}, G_{2n-m_k+2}, \ldots, G_{2n}\}$.

By reordering vertices, assume $Y = \{b_1, b_2, \ldots, b_n\}$. By the definition of $\mathcal{M}_1, \ldots, \mathcal{M}_k$, we have $d_{G_1}(b_2) = d_{G_2}(b_3) = \cdots = d_{G_{n-1}}(b_n) = n$. Hence, we can choose $n-1$ edges $e_i = b_{i+1}a_{i+1}$ for $1 \leqslant i \leqslant n-1$ such that $e_i \in E(G_i)$ and $a_{i+1} \neq a_{j+1}$ for arbitrary $1 \leqslant i \neq j \leqslant n-1$, where $\{a_2, a_3, \ldots, a_n\} \subseteq X$. If $a_{i+1}b_{i+2} \in E(G_{n+i-1})$ for each $1 \leqslant i \leqslant n-2$, then $b_2a_2b_3a_3 \cdots b_na_n$ is a transversal (isomorphic to a path) of $\{G_1, G_2, \ldots, G_{2n-3}\}$. If $a_{j+1}b_{j+2} \notin E(G_{n+j-1})$ for some $1 \leqslant j \leqslant n-2$, then b_{j+2} is a vertex of degree 1 in G_{n+j-1}. Hence, b_{j+1} is a vertex of degree n in G_{n+j-1}. Hence, $a_{j+1}b_{j+1} \in E(G_{n+j-1})$. Note that b_{j+2} is a vertex of degree n in G_j. We have $a_{j+1}b_{j+2} \in E(G_j)$. Hence, $b_2a_2b_3a_3 \cdots b_na_n$ is a transversal (isomorphic to a path) of $\{G_1, G_2, \ldots, G_{2n-3}\}$ such that either $b_{i+1}a_{i+1} \in E(G_i)$ and $a_{i+1}b_{i+2} \in E(G_{n+i-1})$ or $b_{i+1}a_{i+1} \in E(G_{n+i-1})$ and $a_{i+1}b_{i+2} \in E(G_i)$. Let $P := b_2a_2b_3a_3 \cdots b_na_n$. We aim to construct a transversal isomorphic to a Hamilton cycle from P by adding vertices b_1 and a_1 and replacing some edges, where $\{a_1\} = X \backslash \{a_2, a_3, \ldots, a_n\}$.

We first claim that $a_nb_1 \in E(G_{2n-2})$. Otherwise, $d_{G_{2n-2}}(b_1) = 1$. Hence, $m_1 \geqslant 2n-2 > 2 \geqslant m_2$, a contradiction. Similarly, we have $b_1a_1 \in E(G_{2n-1})$. If $a_1b_2 \in E(G_{2n})$, then adding edges a_nb_1, b_1a_1 and a_1b_2 to path P results in a transversal isomorphic to a Hamilton cycle of $\mathcal{G}$. Otherwise, we have $a_1b_2 \notin E(G_{2n})$, which implies $d_{G_{2n}}(b_2) = 1$. Thus, $k = 2$ and $m_1 + m_2 = 2n$. Since $m_1 \leqslant m_2$, we have $m_1 \leqslant n$. Note that $d_{G_{m_1}}(b_1) = 1$. If $m_1 = n$, adding $b_2a_n, a_nb_1, a_1b_1, a_1b_n$, replacing a_nb_n by b_2a_n and keeping all the other edges in P results in a transversal isomorphic to a Hamilton cycle of $\mathcal{G}$ such that $b_2a_n \in E(G_{n-1}), b_1a_n \in E(G_{2n-2}), b_1a_1 \in E(G_{2n-1})$ and $b_na_1 \in E(G_{2n})$. If $m_1 < n$, then we have $b_2a_{m_1+1} \in E(G_{m_1})$. Thus, except for the case $m_1 = 1$, we have $a_1b_{m_1+1} \in E(G_{2n})$ since $d_{G_{2n}}(b_2) = 1$. Adding $b_2a_{m_1+1}, a_1b_{m_1+1}, b_1a_1$ and b_1a_n, replacing $b_{m_1+1}a_{m_1+1}$ by $b_2a_{m_1+1}$ and keeping all the other edges in P results in a transversal isomorphic to a Hamilton cycle of $\mathcal{G}$, where $a_{m_1+1}b_{m_1+2} \in E(G_{n+m_1-1})$. For the remaining case ($m_1 = 1$ and $a_1b_2 \notin E(G_{2n})$), suppose that $b_2a_j \in E(G_{2n})$, where $j \neq 1$. Adding $a_nb_1, b_1a_1, b_2a_j, a_1b_j$ and replacing a_jb_j by a_1b_j and keeping all the other edges in P results in a transversal isomorphic to a Hamilton cycle of $\mathcal{G}$ such that $a_1b_j \in E(G_{j-1}), b_1a_n \in E(G_{2n-2}), b_1a_1 \in$

$E(G_{2n-1})$ and $b_2a_j \in E(G_{2n})$. $\square$

Similarly, we can prove the following lemma.

Lemma 20. *Let $n \geqslant 4$ and $\mathcal{G} = \{G_1, G_2, \ldots, G_{2n}\}$ be a family of balanced bipartite graphs of order $2n$ with partite sets $\{X, Y\}$. If for each $i \in [2n]$, G_i is bi-isomorphic to $B(n, 1, n-1)$, then $\mathcal{G}$ admits a transversal isomorphic to a Hamilton cycle unless $G_1 = G_2 = \cdots = G_{2n}$.*

Next, we shall present the proof of Theorem 5.

Proof of Theorem 5. Assume X is the set of n red vertices and Y is the set of n blue vertices. For each $i \in [2n]$, $\{X, Y\}$ are partite sets of G_i. Suppose that $\{G_1, G_2, \ldots, G_{2n}\}$ dose not admit a transversal isomorphic to a Hamilton cycle. Then by Lemma 12, $\{S(G_1), S(G_2), \ldots, S(G_{2n})\}$ dose not admit a transversal isomorphic to a Hamilton cycle. Let $H_i := S(G_i)$ for $i \in [2n]$. By the definition of shifting, Theorem 7 and the spectral conditions, for $i \in [2n]$,

$$\lambda(B_n^1) \leqslant \lambda(G_i) \leqslant \lambda(H_i). \tag{4}$$

Let $e_j^X := \{x_j, y_{n+2-j}\}$ for $2 \leqslant j \leqslant \lfloor \frac{n}{2} \rfloor + 1$. First of all, we shall prove that for each $i \in [2n]$,

$$\{e_3^X, e_4^X, \ldots, e_{\lfloor \frac{n}{2} \rfloor + 1}^X\} \subseteq E(H_i). \tag{5}$$

Otherwise, assume that there exist $3 \leqslant r \leqslant \lfloor \frac{n}{2} \rfloor + 1$ and $t \in [2n]$ such that $e_r^X = \{x_r, y_{n+2-r}\} \notin E(H_t)$. Since H_t is shifted, by Observation 6, every edge $\{x_i, y_j\} \in E(H_t)$ satisfies $i \leqslant r-1$ or $j \leqslant n+1-r$. Thus H_t is a subgraph of $B(n, n-(r-1), r-1)$. Note that $B(n, n-(r-1), r-1)$ contains no Hamilton cycle and $B(n, n-(r-1), r-1)$ is not isomorphic to B_n^1. By Theorem 2 and Lemma 8, we have

$$\lambda(H_t) \leqslant \lambda(B(n, n-(r-1), r-1)) < \lambda(B_n^1),$$

a contradiction. Hence, $\{e_3^X, e_4^X, \ldots, e_{\lfloor \frac{n}{2} \rfloor + 1}^X\} \subseteq E(H_i)$ for each $i \in [2n]$.

Let $e_j^Y := \{x_{n+2-j}, y_j\}$ for $2 \leqslant j \leqslant \lfloor \frac{n}{2} \rfloor + 1$. Similarly, we can prove that, for each $i \in [2n]$,

$$\{e_3^Y, e_4^Y, \ldots, e_{\lfloor \frac{n}{2} \rfloor + 1}^Y\} \subseteq E(H_i). \tag{6}$$

It is easy to see that, for each $i \in [2n]$,

$$\{x_1, y_n\}, \{x_n, y_1\} \in E(H_i). \tag{7}$$

Otherwise, by the definition of shifting, y_n or x_n is an isolated vertex of H_i and $\lambda(H_i) < \lambda(B_n^1)$, a contradiction.

Let $e_{j'}^X := \{x_{j-1}, y_{n+2-j}\}$ and $e_{j'}^Y := \{x_{n+2-j}, y_{j-1}\}$ for $3 \leqslant j \leqslant \lfloor \frac{n}{2} \rfloor + 1$. By Observation 6, we have, for each $i \in [2n]$,

$$\{e_{3'}^X, e_{3'}^Y, e_{4'}^X, e_{4'}^Y, \ldots, e_{(\lfloor \frac{n}{2} \rfloor + 1)'}^X, e_{(\lfloor \frac{n}{2} \rfloor + 1)'}^Y\} \subseteq E(H_i). \tag{8}$$

Similarly,

$$\{x_1, y_1\}, \{x_{\lceil \frac{n}{2} \rceil}, y_{\lceil \frac{n}{2} \rceil}\} \in E(H_i) \tag{9}$$

for each $i \in [2n]$.

Claim 3. Either $e_2^X \in E(H_i)$ or $e_2^Y \in E(H_i)$ for $i \in [2n]$.

Proof of Claim 3. Otherwise, H_i is a subgraph of

$$H := K(X, Y) - \{x_n y_2, x_n y_3, \ldots, x_n y_n, y_n x_2, y_n x_3, \ldots, y_n x_{n-1}\}.$$

Since H contains no Hamilton cycle, by Theorem 2 and Lemma 8, we have $\lambda(G_i) \leqslant \lambda(H_i) \leqslant \lambda(H) < \lambda(B_n^1)$, a contradiction. □

Claim 4. Either $H_1 = H_2 = \cdots = H_{2n} = B(n, n-1, 1)$ or $H_1 = H_2 = \cdots = H_{2n} = B(n, 1, n-1)$.

Proof of Claim 4. According to Claim 3, we distinguish the proof into three cases.

Case 1. There exist $p, q \in [2n]$ such that $p \neq q$, $e_2^X \in E(H_p)$ and $e_2^Y \in E(H_q)$.

Note that (4)-(9) holds. For odd n, the following $2n$ edges

$$\{x_1, y_n\}, e_2^X, e_{3'}^X, e_3^X, \ldots, e^X_{\lceil \frac{n}{2} \rceil'}, e^X_{\lceil \frac{n}{2} \rceil}, \{x_{\lceil \frac{n}{2} \rceil}, y_{\lceil \frac{n}{2} \rceil}\}, e^Y_{\lceil \frac{n}{2} \rceil}, e^Y_{\lceil \frac{n}{2} \rceil'}, \ldots, e_3^Y, e_{3'}^Y, e_2^Y, \{x_n, y_1\}, \{y_1, x_1\}$$

induces a transversal isomorphic to a Hamilton cycle. For even n, the following $2n$ edges

$$\{x_1, y_n\}, e_2^X, e_{3'}^X, e_3^X, \ldots, e^X_{(\frac{n+2}{2})'}, e^X_{\frac{n+2}{2}}, e^Y_{(\frac{n+2}{2})'}, e^Y_{\frac{n}{2}}, e^Y_{(\frac{n}{2})'}, \ldots, e_3^Y, e_{3'}^Y, e_2^Y, \{x_n, y_1\}, \{y_1, x_1\}$$

induces a transversal isomorphic to a Hamilton cycle. Thus $\{H_1, H_2, \ldots, H_{2n}\}$ has a transversal isomorphic to a Hamilton cycle, a contradiction.

Case 2. $e_2^X \in E(H_i)$ and $e_2^Y \notin E(H_i)$ for each $i \in [2n]$.

In this case, H_i is a subgraph of $B(n, 1, n-1)$. Hence, by Lemma 8, we have $\lambda(H_i) \leqslant \lambda(B(n, 1, n-1)) = \lambda(B_n^1)$. By (4), we have $H_i = B(n, 1, n-1)$.

Case 3. $e_2^X \notin E(H_i)$ and $e_2^Y \in E(H_i)$ for each $i \in [2n]$.

In this case, H_i is a subgraph of $B(n, n-1, 1)$. Hence, by Lemma 8, we have $\lambda(H_i) \leqslant \lambda(B(n, n-1, 1)) = \lambda(B_n^1)$. By (4), we have $H_i = B(n, n-1, 1)$.

This completes the proof of Claim 4. □

Note that $B(n, n-1, 1) \simeq B(n, 1, n-1) \simeq B_n^1$. By (4), Theorem 7 and Claim 4, we have $\lambda(G_1) = \cdots = \lambda(G_{2n}) = \lambda(B_n^1)$. By Corollary 18 and Claim 4, we have either all G_i are bi-isomorphic to $B(n, n-1, 1)$ or all G_i are bi-isomorphic to $B(n, 1, n-1)$. By Lemmas 19 and 20, we get $G_1 = G_2 = \cdots = G_{2n}$ and $G_1 \simeq B_n^1$, as required. □

Concluding remarks

The condition that we give in Theorem 4 for a transversal isomorphic to a Hamilton path is a bipartite analogue to a result of He, Li and Feng [19], which generalizes Fiedler and Nikiforov's result [15] into the language of graph transversals. Similarly, the condition that we give in Theorem 5 for a transversal isomorphic to a Hamilton cycle is a bipartite analogue to a result of Zhang and van Dam [47], which generalizes Fiedler and Nikiforov's result [15] into the language of graph transversals.

Li and Ning [23, 24] gave a sufficient condition for Hamilton paths and cycles, respectively, in terms of the spectral radius and minimum degree of graphs which generalized Moon and Moser's [29] theorem. A question that remains open is whether these spectral conditions on bipartite graphs with prescribed minimum degree for Hamilton paths and cycles can be translated into the language of bipartite graph transversals.

Declaration of competing interest

The authors declare that they have no conflicts of interest to this work.

Acknowledgments

The authors are grateful to the anonymous referees for valuable suggestions and comments, which improved the presentation of the original manuscript. This work was supported by Hebei Provincial Department of Education Science and Technology Project - Young Top-notch Talent Program (No. BJ2026380), the National Natural Science Foundation of China (No. 12501477), the Hebei Natural Science Foundation (No. A2025205045) and the Science Foundation of Hebei Normal University (No. L2025B02).

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