Kohnert posets and polynomials of northeast diagrams

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Submitted: Jul 18, 2025; Accepted: Apr 21, 2026; Published: Jul 3, 2026

Abstract

Kohnert polynomials and their associated posets are combinatorial objects with deep geometric and representation theoretic connections, generalizing both Schubert polynomials and type-A Demazure characters. In this paper, we explore the properties of Kohnert polynomials indexed by northeast diagrams along with their associated posets. We give separate classifications of the bounded, ranked, and multiplicity-free Kohnert posets for northeast diagrams, each of which can be computed in polynomial time with respect to the number of cells in the diagram. As an initial application, we specialize these classifications to simple criteria in the case of lock diagrams.

Mathematics Subject Classifications: 05E99

1 Introduction

In [AS22], Assaf and Searles introduced *Kohnert polynomials* which simultaneously generalize two fundamental families of polynomials, namely Schubert polynomials and (type-A) Demazure characters. Assaf and Searles define the Kohnert polynomials as generating polynomials of certain diagrams in the plane. They drew inspiration from the work of Kohnert in [Koh91] that gives such a generating polynomial interpretation for the Demazure characters. In his thesis, Kohnert also conjectured that the Schubert polynomials had such an interpretation, and this was eventually proven in [Ass22a, Win02, Win99].

Schubert polynomials have their origin in the geometry of flag varieties. First introduced by Lascoux and Schützenberger, Schubert polynomials are polynomial representatives of Schubert classes in the cohomology ring of type-A flag varieties. It is a central

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https://doi.org/10.37236/14391

problem in algebraic combinatorics to give a combinatorial interpretation of the structure constants for multiplication of Schubert polynomials [Pak24, Sta00]. On the other hand, Demazure characters (also known as key polynomials) have connections to both representation theory and geometry. Demazure characters are the characters of certain finite dimensional representations, known as Demazure modules, that arise as the space of sections of a line bundle on the type-A flag variety that has been restricted to a Schubert variety. Noting these deep connections to geometry and representation theory, Assaf gave a construction in [Ass22b] of a Demazure crystal whose character is the Kohnert polynomial whenever the indexing diagram is **southwest**.

Given the discussion of the preceding paragraph, it should come as no surprise that the coefficients of Schubert polynomials and Demazure characters in the monomial basis have geometric implications. If all such coefficients are equal to 0 or 1 then we say that these polynomials are **monomial multiplicity-free**. In [FMSD21], Fink, Mészáros, and St. Dizier gave a classification of the monomial multiplicity-free Schubert polynomials in terms of permutation pattern avoidance. In [HY23], the fourth author and Yong gave a classification of the monomial multiplicity-free Demazure characters in terms of composition pattern avoidance. The monomial multiplicity-freeness of these polynomials is equivalent, via character theory, to the associated algebraic torus module being multiplicity-free as a direct sum of irreducible torus modules. This in turn has implications for the study of torus orbits in Schubert varieties [HY22]. Using these multiplicity-freeness results Gao, the fourth author, and Yong gave a classification of the Schubert varieties in the flag variety that are spherical varieties for the action of a Levi subgroup of the general linear group [GHY23].

With the wide ranging connections to fundamental problems exhibited by these Kohnert polynomials and their special cases, a better understanding of the defining combinatorial model is imperative. To discuss results in this direction, we now explicate the Kohnert polynomial construction in detail.

A **diagram** D is a finite subset of $\mathbb{N} \times \mathbb{N}$. We refer to $(r, c) \in D$ as a cell with row and column index r and c, respectively. Equivalently, we visualize a diagram as a collection of row and column coordinates of cells with rows labeled from bottom to top and columns labeled left to right (see Figure 1 below). A **Kohnert move** in a diagram D selects the rightmost cell in a row and moves that cell down to the first empty position below it, jumping over other cells if needed. A Kohnert move is called **elementary** if the cell moves to the position directly below itself, not jumping over any other cells. Otherwise, it is called a **jump** Kohnert move. The **Kohnert poset** of a diagram D, denoted by $\mathcal{P}(D)$, has an underlying set consisting of all diagrams that can be obtained from D by the application of a (possibly empty) sequence of Kohnert moves. It is the transitive

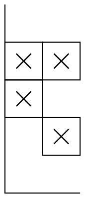


Figure 1: The diagram $D = \{(2,2), (3,1), (4,1), (4,2)\}$

closure of the relations $D_2 < D_1$ for $D_1, D_2 \in \mathcal{P}(D)$ if D_2 is the result of applying a single Kohnert move to D_1. Given two diagrams D and E, we write $D \leqslant E$ to indicate that $D \in \mathcal{P}(E)$. See Figure 2 for an example Kohnert poset.

We remark that since Kohnert moves only impact individual columns, empty columns in a diagram have no bearing on its Kohnert poset structure. Therefore, without loss of generality, in this paper we assume all diagrams have no empty columns.

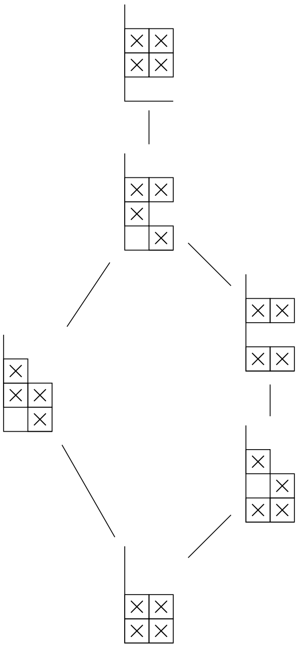


Figure 2: $\mathcal{P}(D)$ where $D = \{(2,1),(2,2),(3,1),(3,2)\}$

Towards associating polynomials with diagrams, let $\mathrm{Comp}_n := \mathbb{Z}^n_{\geqslant 0}$ be the set of weak compositions of length n. The **row weight** of a diagram D, denoted by $\mathrm{rwt}(D)$, is the weak composition $(\alpha_1, \ldots, \alpha_\ell)$ such that α_i is the number of cells in row i, and ℓ is the maximum row index such that row ℓ is nonempty. The **column weight** $\mathrm{cwt}(D)$ is the composition $(\beta_1, \ldots, \beta_m)$ such that β_i is the number of cells in column i, and column m is the rightmost nonempty column in D. In the example in Figure 1, $\mathrm{rwt}(D) = (0,1,1,2)$ and $\mathrm{cwt}(D) = (2,2)$.

Definition 1.1 ([AS22, Definition 2.2])**.** Given a diagram D, its **Kohnert polynomial** is

$$\mathfrak{K}_D = \sum_{T \in \mathcal{P}(D)} x_1^{\mathrm{rwt}(T)_1} \cdots x_n^{\mathrm{rwt}(T)_n}.$$

In [AS22], Assaf and Searles show that if $\{D_\alpha\}$ is a set of diagrams indexed by weak compositions with the property that $\mathrm{rwt}(D_\alpha) = \alpha$ for $\alpha \in \mathrm{Comp}_n$, then their Kohnert polynomials $\{\mathfrak{K}_{D_\alpha}\}$ form a basis of the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$. As remarked earlier, these bases subsume both the Schubert and Demazure character bases of the polynomial ring.

Now, moving towards the content of the present work, Colmenarejo, Hutchins, Mayers, and Phillips initiated a comprehensive study of the structural properties of $\mathcal{P}(D)$

in [CHMP25]. Therein, they determine a sufficient condition guaranteeing when such Kohnert posets are bounded and two necessary conditions for when they are ranked. They also show that the Kohnert posets associated with Demazure characters are always bounded and give a criterion for when the poset is ranked. Inspired by their work, we investigate the Kohnert posets of **northeast diagrams** (see Definition 2.1). Our initial motivation was to investigate a subclass of northeast diagrams called **lock diagrams**. Lock diagrams were first defined by Assaf and Searles [AS22] as natural analogs of the left-justified diagrams of Demazure characters. The polynomials associated to lock diagrams are called **lock polynomials**. In [Wei21], Wang showed that lock polynomials exhibit a Demazure crystal structure that is closely intertwined with the crystal structure of Demazure characters.

Here, we give a complete classification of the northeast diagrams D for which $\mathcal{P}(D)$ is ranked and, separately, bounded. Additionally, we give a complete criterion for when $\mathfrak{K}_D$ is monomial multiplicity-free for any northeast diagram D. A highlight of all three classifications is that they are given purely in terms of the diagram D; that is, they do not require any analysis of the underlying set of diagrams in $\mathcal{P}(D)$. Along the way, we develop a suite of tools that aid in the study of Kohnert posets in more general settings.

The feasibility of all three of our classifications is based on the following surprising result, which comprises our first main theorem. The underlying set of the **elementary Kohnert poset** $\mathcal{P}^{ele}(D)$ consists of all diagrams that can be obtained from D by the application of a (possibly empty) sequence of elementary Kohnert moves. It is the transitive closure of the relations $D_2 < D_1$ for $D_1, D_2 \in \mathcal{P}(D)$ if D_2 is the result of applying a single **elementary** Kohnert move to D_1.

Theorem 1.2. *Let D be a northeast diagram. Then $\mathcal{P}(D)$ is a refinement of $\mathcal{P}^{ele}(D)$; in particular, $\mathcal{P}(D)$ and $\mathcal{P}^{ele}(D)$ are equal as sets.*

In other words, any diagram $D_1 \in \mathcal{P}(D)$ may be reached from D via only elementary moves. This non-trivial fact makes the analysis of the Kohnert posets of northeast diagrams considerably more tractable. Similar types of results are found in [Gao21, Theorem 1.5] and [Wei21, Lemma 7.4] regarding the connectivity of reduced pipe dreams and reduced bumpless pipe dreams by simple ladder moves and simple droop moves, respectively.

The converse of Theorem 1.2 is false in general. For instance, any diagram with at most one cell in each column satisfies that $\mathcal{P}(D)$ is a refinement of $\mathcal{P}^{ele}(D)$, regardless of whether D is northeast. It would be very instructive to classify the diagrams D for which this phenomenon occurs. In Section 2.2, we develop the notion of tableaux, expanding on the idea of Kohnert tableaux introduced by Assaf in [Ass22b, Section 5], in a setting more general than northeast diagrams. These tableaux are instrumental in proving the subsequent theorems.

We now state our classification results. First, our classification of the monomial multiplicity-free Kohnert polynomials of northeast diagrams:

Theorem 1.3. *If D_0 is a northeast diagram, then $\mathfrak{K}_{D_0}$ is monomial multiplicity-free if and only if D_0 does not contain $x_1 = (r_1, c_1)$ and $x_2 = (r_2, c_2)$ such that:*

(a) $r_1 < r_2$,

(b) $c_1 < c_2$,

(c) there is at least one empty position (r, c_1) *where* $r < r_1$*, and*

(d) for each $c > c_1$*, there are at least two empty positions* (r, c) *where* $r \leqslant r_1$*.*

We say that a poset P is **ranked** if there exists a rank function $\rho : P \to \mathbb{Z}$ such that if y covers x, then $\rho(y) = \rho(x) + 1$. This definition does not require all minimal elements to have rank 0. An equivalent characterization of the poset P being ranked is that for all $x \leqslant y$ in P, every maximal chain from x to y has the same length.

Theorem 1.4. *If* D_0 *is a northeast diagram, then* $\mathcal{P}(D_0)$ *is ranked if and only if* D_0 *does not contain* $x_1 = (r_1, c_1), x_2 = (r_2, c_2)$*, and* $x_3 = (r_3, c_3)$ *such that:*

(a) $r_1 < r_2 \leqslant r_3$*,*

(b) $c_1 = c_2 < c_3$*,*

(c) for each $c_1 \leqslant c < c_3$*, there is at least one empty position* (r, c) *where* $r < r_1$*, and*

(d) for each $c \geqslant c_3$*, the number of* $r < r_3$ *such that* $(r, c) \in D_0$ *is strictly less than* r_1*.*

A poset P is **bounded** if P contains a unique minimal element and a unique maximal element. The Kohnert poset $\mathcal{P}(D)$ always has a unique maximal element, namely D. Hence, a Kohnert poset is bounded if and only if it has a unique minimal element.

Theorem 1.5. *If* D_0 *is a northeast diagram, then* $\mathcal{P}(D_0)$ *is bounded if and only if* D_0 *does not contain* $x_1 = (r_1, c_1), x_2 = (r_2, c_2)$*, and* $x_3 = (r_3, c_3)$ *such that:*

(a) $r_1 \leqslant r_2 < r_3$*,*

(b) $c_1 < c_2 = c_3$*,*

(c) for each $c_1 \leqslant c < c_2$*,* $\mathrm{cwt}(D_0)_c < \mathrm{cwt}(D_0)_{c_2}$*,*

(d) for each $c \geqslant c_1$*, there is at least one empty position* (r, c) *where* $r < r_1$*, and*

(e) for each $r_1 < r \leqslant r_3$*, the cell* (r, c_1) *is not in* D_0*.*

As an initial application of these results we specialize each of these classifications to lock diagrams and polynomials, yielding significantly simpler criterion. In future work, we plan to focus on lock diagrams in greater detail and study shellability, rank-unimodality, and enumerative formulas for the number of minimal elements in their Kohnert posets.

The organization of this paper is as follows. In Section 2, we define northeast diagrams, introduce labelings on northeast diagrams and the diagrams that arise in their posets, and prove several technical results that are used in the arguments for our main theorems. In Sections 3, 4, and 5, we give the proofs of Theorems 1.3, 1.4, and 1.5, respectively. Of the three, we note that the proof of Theorem 1.5 is the most complex. Finally, in Section 6 we describe and prove how these results specialize to *lock diagrams*.

2 Tools for diagrams

The purpose of this section is to define our diagrams of interest and develop the tools required to prove our three main classification theorems. Some of these notions and results will also be useful in our forthcoming work that studies combinatorial properties of the posets associated with lock diagrams in greater detail, and they may be of general interest to researchers working on Kohnert diagram combinatorics.

2.1 Classes of diagrams

We begin by defining the class of diagrams that form our primary objects of study.

Definition 2.1. A diagram D is **northeast** if for all pairs of cells $(r_1, c_1), (r_2, c_2) \in D$, we have that $(\max(r_1, r_2), \max(c_1, c_2)) \in D$. Similarly, D is **southeast** if for all pairs of cells $(r_1, c_1), (r_2, c_2) \in D$, we have that $(\min(r_1, r_2), \max(c_1, c_2)) \in D$.

A northeast diagram A southeast diagram

Figure 3: A northeast (left) and a southeast (right) diagram.

See Figure 3 for examples. The naming of these diagrams arises from intuition relating to the cardinal directions. In a northeast diagram, for every two cells where one cell (in position (r_1, c_1)) is strictly northwest of the other (in position (r_2, c_2)), there is a cell located at the "northeast corner" of the rectangle defined by the two cells, i.e., the position (r_1, c_2). Similarly, in a southeast diagram, for every two cells where one cell is strictly southwest of the other, there is a cell located at the "southeast corner" of the rectangle defined by the two cells. It is possible to define northwest and southwest diagrams in a similar manner, but we omit the definitions since they are not needed in this work.

An important family of diagrams is **lock diagrams**, which are exactly those diagrams that are both northeast and southeast.

Definition 2.2. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathrm{Comp}_n$ with $m = \max(\alpha_1, \ldots, \alpha_n)$, the **lock diagram** is the diagram

$$\mathbb{C}\!\mathbb{I}(\alpha) \coloneqq \{(r, c) \mid 1 \leqslant r \leqslant n, m - \alpha_r \leqslant c \leqslant m\}.$$

$\mathbb{C}\!\mathbb{I}(\alpha)$ is said to be the **lock diagram of row weight $\boldsymbol{\alpha}$.**

Eqvuialently, $\mathbb{C}\!\mathbb{I}(\alpha)$ has exactly α_r cells in row r, and if $c < m$ with $(r, c) \in \mathbb{C}\!\mathbb{I}(\alpha)$, then $(r, c + 1) \in \alpha$ as well. See Figure 4 for an example of a lock diagram.

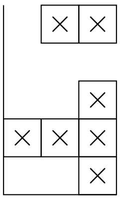


Figure 4: The lock diagram $\mathbb{L}(\alpha)$ for $\alpha = (1, 3, 1, 0, 2)$.

Lemma 2.3. *A diagram D is both northeast and southeast if and only if it is a lock diagram.*

Proof. Assume that D has m columns.

(Proof of $\Rightarrow$) Suppose D satisfies both the northeast and southeast conditions and that $(r, c) \in D$ for some $1 \leqslant c < m$. As column $c + 1$ is nonempty by assumption, there exists some r' such that $(r', c + 1) \in D$. We have three cases.

Case 1: $r' > r$. Then $(r, c + 1) \in D$ by the southeast condition.

Case 2: $r' = r$. Then $(r, c + 1) \in D$ by assumption.

Case 3: $r' < r$. Then $(r, c + 1) \in D$ by the northeast condition.

Therefore, by definition, D is a lock diagram.

(Proof of $\Leftarrow$) Suppose D is a lock diagram and $(r_1, c_1), (r_2, c_2) \in D$. By definition of a lock diagram, $(r_1, \max(c_1, c_2))$ and $(r_2, \max(c_1, c_2))$ are both in D. Thus, D satisfies both the northeast and southeast conditions. $\square$

2.2 Labelings and tableaux

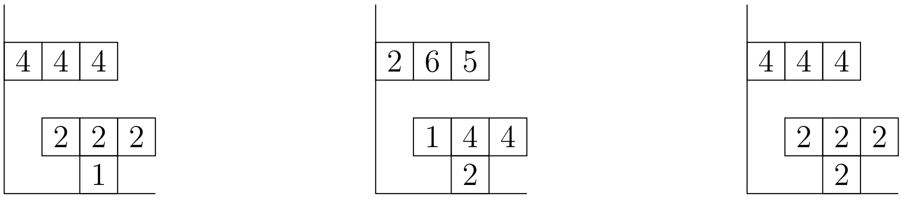
In what follows, we develop the notions of northeast labelings and tableaux. This culminates with Theorem 2.17, which yields a tableau criterion for a diagram to be contained in the Kohnert poset of a northeast diagram.

Definition 2.4. A **labeling** of a diagram D is a map $\mathcal{L} : D \to \mathbb{N}$. A labeling is **strict** if the labels within each column are strictly increasing from bottom to top. The **superstandard** labeling of a diagram labels each cell with its row index.

See Figure 5 for examples of these types of labelings. We refer to the pair of a diagram and its labeling $T = (D, \mathcal{L})$ as a **tableau** (this differs from the tableaux of [AS18]).

Definition 2.5. The **column content** $C_{\mathcal{L}} = (C_1, C_2, \ldots)$ of a strict labeling $\mathcal{L} : D \to \mathbb{N}$ is the sequence such that $C_i \subseteq \mathbb{Z}_{\geqslant 0}$ is set of labels of cells in column i. Two labelings $\mathcal{L} : D \to \mathbb{N}$ and $\mathcal{L}' : D' \to \mathbb{N}$ are said to be **column-equivalent** if $C_{\mathcal{L}} = C_{\mathcal{L}'}$. We also refer to two tableaux with column-equivalent labelings as column-equivalent in their own right.

In other words, two labelings are column-equivalent if they have the same set of labels in each fixed column.



Super-standard Strict, not super-standard Not strict

Figure 5: Three diagrams with labelings.

Definition 2.6. Given a diagram $D \in \mathcal{P}(D_0)$, the **standard labeling $\boldsymbol{\mathcal{L}}$ of $\boldsymbol{D}$ with respect to $\boldsymbol{D_0}$** is the unique strict labeling of D that is column-equivalent to the super-standard labeling of D_0. When the poset containing D is clear from context, we call $\mathcal{L}$ the standard labeling of D. In a diagram with a standard labeling, we refer to a cell with the label i as an ***i*-cell.**

Lemma 2.7. *If $D_b < D_a$ in $\mathcal{P}(D_0)$, then the standard labeling of D_b with respect to D_0 can be constructed from the standard labeling of D_a with respect to D_0. Moreover, if D_b can be obtained from D_a by a single Kohnert move, then for any $x = (r, c) \in D_a$, the cell $x' = (r', c) \in D_b$ with the same label as x has $r - 1 \leqslant r' \leqslant r$.*

Proof. In the case that D_b results from D_a by a single Kohnert move, then the standard labeling of D_b can be constructed as follows. The diagrams D_a and D_b differ by the placement of exactly one cell which we denote as x_1. We address two possible cases. If the Kohnert move that yields D_b from D_a was elementary (and therefore consisted of the cell x_1 moving one position down into an empty space), then preserve the label on x_1. If the Kohnert move was a jump move, then x_1 jumped over a string of adjacent cells $x_2, x_3, \ldots, x_\ell$, where cell x_i is directly above x_{i+1} for all i. In this case, we relabel x_2 with the label for x_1, relabel x_3 with the label for x_2, and so on, finally relabeling x_1 with the label for x_ℓ. The resulting labeling is the standard labeling of D_b. In this sense, we re-frame a Kohnert move as "pushing" all of the cells directly below it into the next available empty space, preserving the labels on all moved cells.

On the other hand, if D_b and D_a differ by more than one Kohnert move, then the standard labeling of D_b may be constructed from the standard labeling $\mathcal{L}_a$ of D_a by taking any chain $D_b = C_1 < C_2 < \cdots < C_k = D_a$ such that C_k results from C_{k+1} by a single Kohnert move for all k. Then one can repeatedly apply the process described in the preceding paragraph, starting with $\mathcal{L}_a$. The choice of chain does not affect this process since standard labelings are unique. Further, the super-standard labeling of D_0 is the standard labeling of D_0, and so this process gives a way of constructing the standard labelings of all diagrams in $\mathcal{P}(D_0)$. □

We reference Figure 6 for an example of this process.

In the setting of labelings of diagrams, it is often useful to observe which properties of the super-standard labeling are preserved by Kohnert moves. This provides a means for detecting whether $D \in \mathcal{P}(D_0)$ for some initial diagram D_0 without needing to explicitly check candidate sequences of Kohnert moves. This motivates the following definition:

Definition 2.8. Let D be a diagram. A **northeast labeling** $\mathcal{L} : D \to \mathbb{N}$ is a labeling that satisfies the following properties:

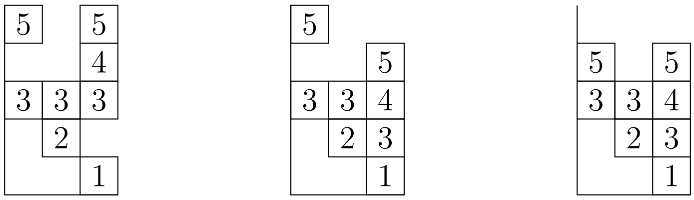


Figure 6: The standard labelings of two diagrams (center, right) with respect to a diagram with a super-standard labeling (left).

(1) $\mathcal{L}$ is strict.

(2) Each label in row i is at least i.

(3) If cell x' is to the right of cell x and $\mathcal{L}(x') < \mathcal{L}(x)$, then there is a cell x'' in the same column as x' with $\mathcal{L}(x) = \mathcal{L}(x'')$.

(4) If cell x' is to the right of cell x and $\mathcal{L}(x') = \mathcal{L}(x)$, then x' is weakly below x.

The pair $T = (D, \mathcal{L})$ is a **northeast tableau**.

Our northeast tableaux are a generalization of the lock tableaux of Assaf and Searles [AS22, Definition 6.5]. The following lemma is used in the proofs of some of our main results.

Lemma 2.9. *Let D be a diagram with a northeast labeling $\mathcal{L}$. Let $x_1 = (r_1, c_1)$ and $x_2 = (r_2, c_2)$ be cells in D such that $c_1 \leqslant c_2$ and $\mathcal{L}(x_1) > \mathcal{L}(x_2)$. Then $r_1 > r_2$.*

Proof. If x_2 and x_1 are in the same column, the result follows from the strictness of $\mathcal{L}$. Now suppose that $x_2 = (r_2, c_2)$ is strictly to the right of $x_1 = (r_1, c_1)$. By Property (3) of Definition 2.8, there is a cell $x_2' = (r_2', c_2)$ in D such that $\mathcal{L}(x_2') = \mathcal{L}(x_1)$. Then $r_2' > r_2$ by strictness, and $r_1 \geqslant r_2'$ by Property (4) of Definition 2.8. Therefore, $r_1 > r_2$. □

In other words, in a northeast tableau, the row indices of the string of cells labeled with an i are weakly descending from left to right. This also means that the labels on the cells in a given row are weakly increasing from left to right.

Northeast tableaux capture northeast diagrams in the following sense:

Lemma 2.10. *A diagram D is northeast if and only if the super-standard labeling $\mathcal{L}$ on D is a northeast labeling.*

Proof. First note that both northeast labelings and super-standard labelings always satisfy Properties (1), (2), and (4) of Definition 2.8. Therefore, it is sufficient to compare Property (3). It is straightforward to check that Property (3) being satisfied on a super-standard labeling is equivalent to the underlying diagram being northeast. □

Lemma 2.10 motivates defining the **initial northeast tableau** of a northeast tableau T:

Definition 2.11. Given a northeast tableau $T = (D, \mathcal{L})$, the **initial northeast diagram** of T is

$$D_0 := \{(\mathcal{L}(x), c) \mid x = (r, c) \in T\}.$$

The **initial northeast tableau** $T_0 = (D_0, \mathcal{L}_0)$ of T is the initial northeast diagram paired with the **initial northeast labeling** $\mathcal{L}_0 : (r, c) \mapsto r$.

Proposition 2.12. *The initial northeast tableau $T_0 = (D_0, \mathcal{L}_0)$ of a northeast tableau T is the unique northeast tableau satisfying the following properties:*

1. *$\mathcal{L}_0$ is the super-standard labeling,*
2. *T_0 is column-equivalent to T, and*
3. *D_0 is a northeast diagram.*

Proof. Certainly there is at most one tableau endowed with the super-standard labeling that is column-equivalent to T - its existence comes from the fact that $\mathcal{L}$ is strict, and thus injective when restricted to an individual column. Any super-standard labeling satisfies Properties (1), (2), and (4) of Definition 2.8, and since $\mathcal{L}_0$ is column-equivalent to $\mathcal{L}$, it must also satisfy Property (3). Thus, T_0 is a northeast tableau. Then D_0 is northeast by Lemma 2.10. $\square$

Next, we introduce the concept of **total displacement**, which quantifies how far a labeling is from being super-standard. While defined in generality for Kohnert tableaux, the total displacement is well-suited for northeast tableaux because it provides a rank function to complement our result on classifying ranked northeast posets.

Definition 2.13. Let $T = (D, \mathcal{L})$ be a tableau. The **displacement** is a function $\delta_{\mathcal{L}} : D \to \mathbb{Z}_{\geqslant 0}$ such that for $x = (r, c) \in D$,

$$\delta_{\mathcal{L}}(x) = \mathcal{L}(x) - r.$$

The **total displacement** of T is

$$\Delta_{\mathcal{L}}(T) = \sum_{x \in D} \delta_{\mathcal{L}}(x).$$

See Figure 7 for an example. If $T = (D, \mathcal{L})$ is a northeast tableau, we often assume that the labeling is standard with respect to the initial northeast diagram of T and omit the subscript $\mathcal{L}$ from Δ.

Figure 7a. A northeast tableau T with $\Delta_{\mathcal{L}}(T) = 12$.

Figure 7b. The initial tableau T_0 of T from Figure 7a.

Figure 7: Two northeast tableaux.

Lemma 2.14. *Let $T = (D, \mathcal{L})$ be a northeast tableau, and let $T_0 = (D_0, \mathcal{L}_0)$ be the initial northeast tableau of T. If D' is a diagram obtained by performing a single Kohnert move on D, then the standard labeling $\mathcal{L}'$ of D' with respect to D_0 is a northeast labeling.*

Proof. By Lemma 2.7, the standard labeling of D' may be constructed from the standard labeling for D. Observe that a cell x' in D' with label n is weakly below and at most one position below the corresponding cell x with the same label in the same column in D. We show that $\mathcal{L}'$ satisfies the four properties of Definition 2.8 and hence is northeast.

(1) $\mathcal{L}'$ is strict by construction.

(2) Each label in $\mathcal{L}'$ in row i is at least i if and only if $\delta_{\mathcal{L}'}(x') \geqslant 0$ for all $x' \in D'$. If $x' \in D'$, then there exists an $x \in D$ in the same column with the same label. Then, $\delta_{\mathcal{L}'}(x') \geqslant \delta_{\mathcal{L}}(x)$ by our observation. However, $\delta_{\mathcal{L}}(x) \geqslant 0$ since $\mathcal{L}$ is northeast. Hence, $\delta_{\mathcal{L}'}(x') \geqslant 0$.

(3) Property (3) holds for $\mathcal{L}'$ because $\mathcal{L}$ and $\mathcal{L}'$ are column-equivalent, and $\mathcal{L}$ is northeast.

(4) Let $x = (r, c)$ be the cell to which the Kohnert move on D was applied. For the sake of contradiction, suppose that $\mathcal{L}'$ does not satisfy Property (4). Then, there is a pair of cells $y'_1, y'_2 \in D'$ such that $\mathcal{L}'(y'_1) = \mathcal{L}'(y'_2)$, but y'_1 is to the left of and strictly below y'_2. Because the only difference between D and D' is in column c, it follows that column c contains y'_1 in a position weakly below x. Moreover, the cell y_1 in column c of D with the label $\mathcal{L}'(y'_1)$ is exactly one position above y'_1. Then because $\mathcal{L}$ is northeast, y_1 is in the same row as y'_2. Setting $y'_1 = (r'-1, c)$ and $y'_2 = (r', c')$ for some $r' \leqslant r$ and $c' > c$, we have that $y_1 = (r', c)$ and $y_2 = (r', c')$ are the corresponding cells in D with

$$\mathcal{L}(y_1) = \mathcal{L}'(y'_1) = \mathcal{L}'(y'_2) = \mathcal{L}(y_2).$$

Denote this label by $L := \mathcal{L}(y_1)$. Because y'_1 is below x, y_1 is weakly below x. Then by strictness, $L \leqslant \mathcal{L}(x)$. Because y_1 was affected by the Kohnert move, there are no empty positions between x and y_1 in D. In particular, there are $r - r' + 1$ cells between x and y_1 in D, including x and y_1.

If $r = r'$, then x is not rightmost in its row and thus can't be moved, which is a contradiction. Thus, we can suppose $x \neq y_1$ and let z_1 denote the cell directly above y_1 in D. Since $\mathcal{L}$ is a northeast labeling, there is a cell z_2 in column c' that is weakly below z_1 but strictly above y_2 with the same label as z_1. Then, z_2 is in the same row as z_1. Similar logic shows that for each cell in column c between x and y_1 in D, including x and y_1, there is a corresponding cell with the same label in the same row of column c' in D. This contradicts x being the cell to which the Kohnert move was applied, because x is not the rightmost cell in its row.

Figure 8 depicts what the diagram D must have looked like at the conclusion of Step (4). The red cells are the ones whose existence is inferred by the northeast property of the labeling. The cell x is the top left cell, labeled L_k. It clearly cannot be the subject of a Kohnert move, as it is not the rightmost cell in its row.

Since $\mathcal{L}'$ satisfies the four properties of Definition 2.8, it is a northeast labeling. $\square$

Remark 2.15. Using Lemma 2.14 and induction, we see that if $D \leqslant D_0$ and D_0 is a northeast diagram, then the standard labeling of D with respect to D_0 is a northeast labeling.

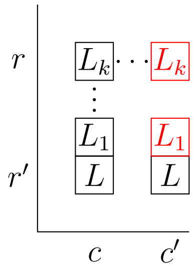


Figure 8: The diagram D described in Step (4) of the proof of Lemma 2.14.

Lemma 2.16. *Let $T = (D, \mathcal{L})$ be a northeast tableau with nonzero total displacement. Then, there exists a northeast tableau $T' = (D', \mathcal{L}')$ that is column-equivalent to T and such that $D < D'$ and $\Delta_{\mathcal{L}'}(T') = \Delta_{\mathcal{L}}(T) - 1$.*

Proof. Because $\Delta_{\mathcal{L}}(T) > 0$, D has at least one cell with nonzero displacement. Let k be the maximum label among all cells with non-zero displacement and $x = (r, c)$ be the leftmost among such cells. We claim that D has no cells $x' = (r+1, c')$ with $c' \geqslant c$. Suppose for the sake of contradiction that such a cell x' does exist. We examine this possibility in cases:

Case 1: Suppose $\mathcal{L}(x') > \mathcal{L}(x)$. Then $\delta_{\mathcal{L}}(x') = \mathcal{L}(x') - (r+1) \geqslant \mathcal{L}(x) - r = \delta_{\mathcal{L}}(x) > 0$, contradicting the maximality of the label of k among all cells with nonzero displacement.

Case 2: Suppose $\mathcal{L}(x') = \mathcal{L}(x)$. Since $\mathcal{L}$ is strict, we must have $c' > c$. This contradicts Property (4) of northeast tableaux.

Case 3: Suppose $\mathcal{L}(x') < \mathcal{L}(x)$. Again, since $\mathcal{L}$ is strict, we have $c' > c$. By Property (3) of northeast tableaux, there is a cell x'' in the same column as x' with $\mathcal{L}(x) = \mathcal{L}(x'')$. Since $\mathcal{L}$ is strict, x'' is above x', and so the pair x, x'' contradicts Property (4) of northeast tableaux.

Then D has no cells $x' = (r+1, c')$ with $c' \geqslant c$. Now say

$$D' = (D \setminus \{(r, c')\}) \cup \{(r+1, c')\}$$

and say $\tilde{x} := (r+1, c') \in D'$. By our first observation, $\tilde{x}$ has no cells to its right in D', and the position below it is empty. Therefore, $D < D'$ via a Kohnert move applied to $\tilde{x}$. Let $\mathcal{L}'$ be the unique strict labeling of D' that is column-equivalent to $\mathcal{L}$. Note that $\mathcal{L}$ agrees with $\mathcal{L}'$ on the intersection of D and D', and the cells in each diagram not in that intersection (x and $\tilde{x}$, respectively) have the same label. Thus, $\Delta_{\mathcal{L}}(T') = \Delta_{\mathcal{L}}(T) - 1$. We now check that $\mathcal{L}'$ satisfies the four properties of Definition 2.8 and hence is northeast.

(1) $\mathcal{L}'$ is strict by construction.

(2) Each label in $\mathcal{L}'$ in row i is at least i if and only if $\delta_{\mathcal{L}'}(x') \geqslant 0$ for all $x' \in D'$. If $x' \neq \tilde{x}$, then $\delta_{\mathcal{L}'}(x') = \delta_{\mathcal{L}}(x') > 0$ since $\mathcal{L}$ is northeast, while $\delta_{\mathcal{L}'}(\tilde{x}) = \delta_{\mathcal{L}}(x) - 1 \geqslant 0$ by the choice of x and the definition of $\tilde{x}$.

(3) Property (3) holds for $\mathcal{L}'$ because $\mathcal{L}$ and $\mathcal{L}'$ are column-equivalent and $\mathcal{L}$ is northeast.

(4) For the sake of contradiction, suppose that $\mathcal{L}'$ does not satisfy Property (4). Since $\mathcal{L}$ and $\mathcal{L}'$ agree on the intersection of D and D', and $\mathcal{L}$ is northeast, any violation of Property (4) must occur for a pair of cells with the rightmost cell equal to $\tilde{x}$. Let x' be the leftmost cell in the pair. Since x' and $\tilde{x}$ violate Property (4), x' is strictly below $\tilde{x}$ in D', and hence weakly below x in D. Because $\mathcal{L}$ satisfies Property (4), x' cannot be strictly below x. Then x' and x are in the same row, so $\delta_{\mathcal{L}}(x') = \delta_{\mathcal{L}}(x)$. This contradicts x being the leftmost cell in its row with nonzero displacement. We conclude that Property (4) holds for $\mathcal{L}'$. □

Theorem 2.17. *Let D_0 be a northeast diagram with super-standard labeling $\mathcal{L}_0$. Then $D \in \mathcal{P}(D_0)$ if and only if D admits a northeast labeling $\mathcal{L}$ that is column-equivalent to $\mathcal{L}_0$.*

Proof. (Proof of ⇒) Suppose $D \in \mathcal{P}(D_0)$. By Remark 2.15, we can inductively apply Lemma 2.14 to obtain a northeast labeling $\mathcal{L}$ of D that is a standard labeling with respect to D_0. Thus, $\mathcal{L}$ is column-equivalent to $\mathcal{L}_0$.

(Proof of ⇐) Let $T = (D, \mathcal{L})$ be a northeast tableau whose labeling $\mathcal{L}$ is column-equivalent to $\mathcal{L}_0$. We induct on $\Delta_{\mathcal{L}}(T)$. If $\Delta_{\mathcal{L}}(T) = 0$ then $\mathcal{L}$ is the super-standard labeling, so $D = D_0$. Now suppose that $\Delta_{\mathcal{L}}(T) > 0$ and for all $T' = (D', \mathcal{L}')$ such that T and T' are column-equivalent and $\Delta_{\mathcal{L}'}(D') = \Delta_{\mathcal{L}}(D) - 1$, we have $D' \in \mathcal{P}(D_0)$. By Lemma 2.16, there is at least one such T' with $D < D'$. Then, by transitivity $D \in \mathcal{P}(D_0)$. □

2.3 Elementary chains

We begin this section with a proof of Theorem 1.2. We then proceed to show that for any saturated chain in a Kohnert poset that is defined by the application of solely elementary moves, the moves can always be reordered so that cells labeled by smaller numbers are always moved first.

Lemma 2.18. *Let D_0 be a northeast diagram and let $D \in \mathcal{P}(D_0)$. Then D can be obtained from D_0 be a (possibly empty) sequence of elementary Kohnert moves.*

Proof. Let $\mathcal{L}_0$ be the super-standard labeling of D_0. Then $T_0 = (D_0, \mathcal{L}_0)$ is a northeast tableau by Lemma 2.10. By Theorem 2.17, there exists a labeling $\mathcal{L}$ such that $T = (D, \mathcal{L})$ is a northeast tableau that is column-equivalent to $\mathcal{L}_0$.

We induct on $\Delta_{\mathcal{L}}(T)$. If $\Delta_{\mathcal{L}}(T) = 0$, then $D = D_0$ and the desired sequence of elementary moves is the empty sequence. Now, fix a diagram D with $\Delta_{\mathcal{L}}(T) = d$ and $d > 0$. By Lemma 2.16, there is a diagram D' with column-equivalent tableau $T' = (D', \mathcal{L}')$ such that $D < D'$ and $\Delta_{\mathcal{L}'}(T') = \Delta_{\mathcal{L}}(T) - 1$. Theorem 2.17 implies that $D' \in \mathcal{P}(D_0)$. Hence, by induction, there is a sequence of elementary Kohnert moves connecting D_0 to D' in $\mathcal{P}(D_0)$. The Kohnert move that connects D and D' is elementary by construction. Thus, there is a sequence of elementary moves to get from D_0 to D in $\mathcal{P}(D_0)$. □

Our first main theorem follows from Lemma 2.18.

Theorem 1.2 *Let D be a northeast diagram. Then, $\mathcal{P}(D)$ is a refinement of $\mathcal{P}^{ele}(D)$; in particular, $\mathcal{P}(D)$ and $\mathcal{P}^{ele}(D)$ are equal as sets.*

Proof of Theorem 1.2. Let $D_1, D_2 \in \mathcal{P}(D)$ such that $D_1 \leqslant_{\mathcal{P}(D)} D_2$. Then it is clear that $D_1 \in \mathcal{P}(D_2)$. By Lemma 2.18, D_1 can be obtained from D_2 by a sequence of elementary Kohnert moves. Hence $D_1 \leqslant_{\mathcal{P}^{ele}(D)} (D_2)$. This proves that $\mathcal{P}(D)$ is a refinement of $\mathcal{P}^{ele}(D)$.

By definition, $\mathcal{P}^{ele}(D) \subseteq \mathcal{P}(D)$ as sets. Since $\mathcal{P}(D)$ is a refinement of $\mathcal{P}^{ele}(D)$, we have $\mathcal{P}^{ele}(D) = \mathcal{P}(D)$ as sets. □

The final definitions and results of this section are useful for proving our theorem on boundedness for northeast diagram posets.

Definition 2.19. Let D_0 be a diagram. A saturated chain in $\mathcal{P}(D_0)$, denoted by $D_1 \gtrdot D_2 \gtrdot \cdots \gtrdot D_M$, is an **elementary chain** if the Kohnert move connecting D_a to D_{a+1} is an elementary Kohnert move for all $1 \leqslant a < M$.

Definition 2.20. Let $T_0 = (D_0, \mathcal{L}_0)$ be a tableau, and let $\mathbf{D} = \{D_1 \gtrdot D_2 \gtrdot \cdots \gtrdot D_M\}$ be an elementary chain. For any $D_a \gtrdot D_{a+1}$ in $\mathbf{D}$, define $\operatorname{move}(D_a, D_{a+1})$ to be i when D_a and D_{a+1} are connected by an elementary Kohnert move affecting a cell with label i. Define

$$\mathcal{I}(\mathbf{D}) = \{(a,b) \mid 1 \leqslant a < b < M \text{ and } \operatorname{move}(D_a, D_{a+1}) > \operatorname{move}(D_b, D_{b+1})\}$$

and $i(\mathbf{D}) = |\mathcal{I}(\mathbf{D})|$. The pair (a,b) is said to be a **move inversion**.

The use of the word "inversion" is motivated by the fact that a move inversion (a,b) represents a pair of moves in the chain where a larger-labeled cell has moved before a smaller-labeled cell. The following lemma shows that any elementary chain with maximal element D_0 may be replaced by an elementary chain with identical maximal and minimal elements that contains no move inversions. We highlight that this result holds for all diagrams.

Lemma 2.21. *Let $T_0 = (D_0, \mathcal{L}_0)$ be a tableau with the super-standard labeling, and $\mathbf{D} := D_1 \gtrdot \cdots \gtrdot D_M$ be an elementary chain in $\mathcal{P}(D_0)$. Then, there exists an elementary chain $\mathbf{F} := F_1 \gtrdot \cdots \gtrdot F_M$ such that $F_1 = D_1$, $F_M = D_M$, and $i(\mathbf{F}) = 0$.*

Proof. If $i(\mathbf{D}) = 0$, then set $\mathbf{F} = \mathbf{D}$. Otherwise, if $i(\mathbf{D}) > 0$, we will show that a new chain $\mathbf{E} = E_1 \gtrdot \cdots \gtrdot E_M$ may be constructed such that $D_1 = E_1$, $D_M = E_M$, and $i(\mathbf{E}) < i(\mathbf{D})$. This proves our desired result since, as the number of move inversions decreases at each step of the process, the iteration of this process must terminate.

Let $i(\mathbf{D}) > 0$, and take $(a,b) \in \mathcal{I}(\mathbf{D})$ such that (a,b) is lexicographically minimal. The minimality of (a,b) implies that $\operatorname{move}(D_a, D_{a+1}) \leqslant \operatorname{move}(D_k, D_{k+1})$ for $a < k < b$ which in turn implies

$$\operatorname{move}(D_k, D_{k+1}) > \operatorname{move}(D_b, D_{b+1}) \text{ for } a < k < b. \tag{2.1}$$

For each $a \leqslant k \leqslant b$, let $x_k = (r_k, c_k)$ be the cell that is moved from D_k to D_{k+1}. Define E_{a+1} to be the diagram obtained from D_a by the elementary Kohnert move of the cell $x_b = (r_b, c_b)$. We first verify that this move is valid. The cell x_b exists in D_a by (2.1). Additionally, suppose, for contradiction, that x_b is not the rightmost cell in its row in D_a. Since x_b is the rightmost cell in its row in D_b, by (2.1), we must have that the rightmost cell x' in row r_b in D_a has label ℓ' for some $\ell' > \operatorname{move}(D_b, D_{b+1})$. Since $\mathcal{L}_0$

is super-standard, the cell x' labeled ℓ' could not have started in row r_b. So there exists some $1 \leqslant q < a$ such that $\mathrm{move}(D_q, D_{q+1}) = \ell'$. This contradicts the minimality of (a, b). Likewise, because $(r_b - 1, c_b) \notin D_a$ but $(r_b, c_b) \in D_a$, any move that would place a cell in some D_k in position $(r_b - 1, c_b)$ without affecting cell x_b could not be elementary. Thus, the move from D_a to E_{a+1} is a valid elementary Kohnert move.

For $a < k \leqslant b$, iteratively define E_{k+1} to be the diagram reached from E_k by the elementary Kohnert move of the cell (r_{k-1}, c_{k-1}). We now verify that each of these moves is valid. The cell (r_{k-1}, c_{k-1}) exists in E_k because the diagram E_k differs from D_{k-1} only by the elementary Kohnert move of the cell $x_b = (r_b, c_b)$, which was moved first. Additionally, suppose, for contradiction, that (r_{k-1}, c_{k-1}) is not the rightmost cell in its row in E_k. Since (r_{k-1}, c_{k-1}) is the rightmost cell in its row in D_{k-1}, it must be the case that the rightmost cell in E_k is the $\mathrm{move}(D_b, D_{b+1})$-cell. Thus, row r_{k-1} in E_k contains both a $\mathrm{move}(D_b, D_{b+1})$-cell and a $\mathrm{move}(D_{k-1}, D_k)$-cell. Then (2.1) implies that $\mathrm{move}(D_{k-1}, D_k) > \mathrm{move}(D_b, D_{b+1})$. Hence, there exists some $1 \leqslant q < k - 1$ such that both $\mathrm{move}(D_q, D_{q+1}) = \mathrm{move}(D_{k-1}, D_k)$ and the Kohnert move connecting D_q and D_{q+1} moves the $\mathrm{move}(D_{k-1}, D_k)$-cell into row r_{k-1}. If $1 \leqslant q < a$, then this contradicts the minimality of (a, b). Otherwise, if $a \leqslant q < k - 1$, then (r_q, c_q) is not the rightmost cell in its row in D_q; the rightmost cell is the $\mathrm{move}(D_b, D_{b+1})$-cell (r_b, c_b) (see Figure 9). This is a contradiction of the definition of $\mathbf{D}$. In either case, we have a contradiction, so we conclude that (r_{k-1}, c_{k-1}) is the rightmost cell in its row in E_k. Therefore, for each $a < k \leqslant b$, the move from E_k to E_{k+1} is a valid elementary Kohnert move.

For $1 \leqslant k \leqslant a$ or $b + 1 < k \leqslant M$, we define $E_k = D_k$. Note that $E_a = D_a$ and $E_{b+1} = D_{b+1}$. Then $\mathbf{E} = E_1 \gtrdot E_2 \gtrdot \cdots \gtrdot E_M$ is a new chain such that $D_1 = E_1$, $D_M = E_M$, and $i(\mathbf{E}) = i(\mathbf{D}) - (b - a) < i(\mathbf{D})$. □

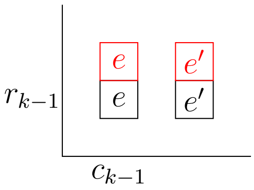


Figure 9: If (r_{k-1}, c_{k-1}) with label $e = \mathrm{move}(D_{k-1}, D_k)$ were not rightmost in E_k (black boxes), then it would need to be blocked by a box with label $e' = \mathrm{move}(D_b, D_{b+1})$. But then this cell would not be rightmost when it is at $(r_{k-1} + 1, c_{k-1})$ in D_q (red boxes) in the case of $a \leqslant q \leqslant k - 1$.

Combining Lemmas 2.18 and 2.21, we have the following proposition, which plays a key role in our proof of Theorem 1.5.

Proposition 2.22. *If D_0 is northeast and $D \in \mathcal{P}(D_0)$, then there exists an elementary chain with maximal element D_0 and minimal element D that does not contain any move inversions.*

3 Monomial multiplicity-free polynomials

We now prove our classification of monomial multiplicity-free Kohnert polynomials of northeast diagrams. We start by restating the theorem for convenience. We refer the reader to Figure 10a for a visualization of conditions (a) - (d) and to Figure 10b for a northeast diagram whose Kohnert poset is not multiplicity-free.

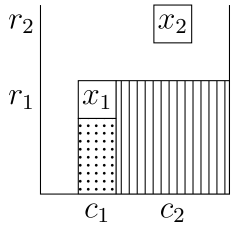


Figure 10a. The arrangement of cells x_1 and x_2 satisfying (a) and (b). Condition (c) states that there is an empty position in the dotted region. Condition (d) states that there are at least two empty positions in each column of the striped region.

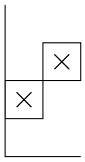


Figure 10b. A minimal example of a northeast diagram whose Kohnert poset does not yield a monomial multiplicity-free polynomial.

Theorem 1.3 *If D_0 is a northeast diagram, then $\mathfrak{K}_{D_0}$ is monomial multiplicity-free if and only if D_0 does not contain $x_1 = (r_1, c_1)$ and $x_2 = (r_2, c_2)$ such that:*

(a) $r_1 < r_2$,

(b) $c_1 < c_2$,

(c) there is at least one empty position (r, c_1) where $r < r_1$, and

(d) for each $c > c_1$, there are at least two empty positions (r, c) where $r \leqslant r_1$.

Proof of Theorem 1.3. (Proof of $\Rightarrow$) We prove the contrapositive. Suppose D_0 has a pair of cells $x_1 = (r_1, c_1)$ and $x_2 = (r_2, c_2)$ with the described properties. If D_0 has multiple such pairs, choose the pair such that the word $(c_2, -r_2, c_1, r_1)$ is maximal in the lexicographic order on $\mathbb{Z}^4$. By this choice of x_1 and x_2, the following are true:

- The only cell (r, c) such that $r_1 \leqslant r < r_2$ and $c_1 \leqslant c < c_2$ is x_1.
- The only cell (r, c) such that $r = r_2$ and $c \geqslant c_2$ is x_2.
- If present, (r_1, c_2) must have at least two empty positions below it.
- The only possible cell (r, c) such that $r_1 \leqslant r < r_2$ and $c \geqslant c_2$ is (r_1, c_2).
- Column c_2 must be the rightmost nonempty column in D_0.

For the last two bullet points, the violation of either property, combined with the fact that D_0 is northeast, would imply the presence of a cell that contradicts our choice of x_2.

We perform the following sequence of operations on D_0 to obtain a diagram D_1:

1. If $(r_1, c_2) \in D_0$ or $(r_1 - 1, c_2) \in D_0$, we perform Kohnert moves on D_0 to obtain a diagram where (r_1, c_2) and $(r_1, c_2 - 1)$ are empty.

2. If $(r_1 - 1, c) \in D_0$ for any $c_1 < c \leqslant c_2$, we perform Kohnert moves to obtain a diagram where $(r_1 - 1, c)$ is empty.
3. By condition (c), if there is a cell directly below x_1, we can move it as well, so that the position $(r_1 - 1, c_1)$ is empty.

This creates the diagram D_1, where x_1 and x_2 are unmoved and the following are satisfied:

- The only cell (r, c) such that $r_1 - 1 \leqslant r < r_2$ and $c \geqslant c_1$ is x_1.
- The only cell (r, c) such that $r = r_2$ and $c \geqslant c_2$ is x_2.

In other words, x_1 is rightmost in r_1, and x_2 will remain rightmost in its row if it moves down to any row of index $\geqslant r_1 - 1$. Then, the diagram D_2 obtained from D_1 by performing a single Kohnert move on x_1 and $r_2 - r_1$ Kohnert moves on x_2 has the same row weight as the diagram D_2' obtained from D_1 by performing $r_2 - r_1 + 1$ Kohnert moves on x_2. Thus, $\mathcal{P}(D_0)$ is not monomial multiplicity-free.

(Proof of $\Leftarrow$) Suppose that $\mathfrak{K}_{D_0}$ is not monomial multiplicity-free. In other words, there exist distinct $D, D' \in \mathcal{P}(D_0)$ such that $\mathrm{rwt}(D) = \mathrm{rwt}(D')$. Let $\mathcal{L}_0, \mathcal{L}$, and $\mathcal{L}'$ denote the standard labelings on D_0, D, and D', respectively.

Let row r denote the highest row where D and D' differ. Choose c_1 and c_2 to be the two maximum column indices such that $y_1 = (r, c_1)$ is a cell in $D \setminus D'$ and $y_2' = (r, c_2)$ is a cell in $D' \setminus D$. Without loss of generality, assume that $c_1 < c_2$. Define $L_1 = \mathcal{L}(y_1)$ and $L_2 = \mathcal{L}'(y_2')$.

Since D and D' are identical above row r, the column-equivalence and strictness of $\mathcal{L}$ and $\mathcal{L}'$ imply that all of the cells above row r in D have the same labels as the corresponding cells in D'. Thus, the cells $y_1' = (r_1, c_1) \in D'$ and $y_2 = (r_2, c_2) \in D$ such that $\mathcal{L}'(y_1') = L_1$ and $\mathcal{L}(y_2) = L_2$ must have $r_1, r_2 < r$. See Figure 11.

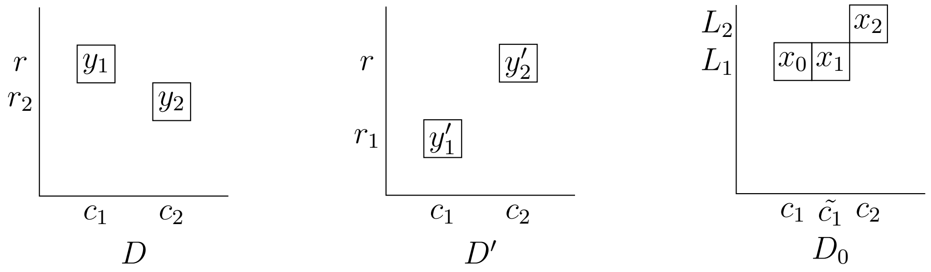


Figure 11: The relative positions of cells y_1, y_2 in D and y_1', y_2' in D', respectively, as well as the cells in D_0 with labels L_1 and L_2.

Note that y_1' is strictly below and to the left of y_2'. By Property (4) of Definition 2.8, $L_1 \neq L_2$. Moreover, $L_1 > L_2$ would contradict Lemma 2.9. Therefore, $L_1 < L_2$.

Define $x_2 = (L_2, c_2) \in D_0$. Let $x_1 = (L_1, \tilde{c}_1)$ be the cell in D_0 such that $\tilde{c}_1 < c_2$ and $\tilde{c}_1$ is maximal. Such a choice of cell exists because $x_0 := (L_1, c_1) \in D_0$ and $c_1 < c_2$. Because $L_1 < L_2$ and $\tilde{c}_1 < c_2$, x_1 and x_2 satisfy Properties (a) and (b) of theorem statement 1.3. We now show that x_1 and x_2 satisfy Properties (c) and (d) as well.

(c) Observe that $\mathcal{L}_0(x_1) = L_1$. In D', the cell y_1' has nonzero displacement since it occurs below the cell with label L_1 in the same column in D. Then, by Property (4) of Definition 2.8, the cell labeled L_1 in column $\tilde{c}_1$ of D' must also have nonzero displacement. Thus, there exists at least one empty space below x_1 in D_0.

(d) If there are no cells to the right of and weakly below x_1 in D_0, then because x_1 is not in the first row, this property is satisfied. Now suppose that there is some cell $x_3 = (r_3, c_3)$ in D_0 that is to the right of and weakly below x_1. Since D_0 is northeast, there is a cell $x_4 = (L_1, c_3)$ in D_0 that may or may not be distinct from x_3. By our choice of $\tilde{c}_1$, we have $c_3 \geqslant c_2$. Let $y_4 \in D$ denote the cell in column c_3 with label L_1. In other words, y_4 is where x_4 ended up after the sequence of Kohnert moves that produced D from D_0. The cell y_2 is strictly below y_1 in D. And since $\mathcal{L}(y_2) = L_2 > L_1 = \mathcal{L}(y_4)$, Lemma 2.9 implies that y_4 is strictly below y_2 in D. Then $\delta_{\mathcal{L}}(y_4) \geqslant \delta_{\mathcal{L}}(y_1) + 2 \geqslant 2$, so y_4 moved at least twice in the sequence of Kohnert moves used to obtain D from D_0. Thus, in D_0, x_4 has at least two empty spaces below it. Therefore, all columns $c > \tilde{c}_1$ in D_0 must have at least two empty positions in rows below or equal to L_1 as desired. □

4 Ranked posets

Before proving our criterion for rankedness of a Kohnert poset of a northeast diagram, we first note that the total displacement function Δ described in Section 2.2 can be used to construct a rank function when applicable.

Lemma 4.1. *Let D_0 be a northeast diagram with the super-standard labeling $\mathcal{L}_0$, and let $T_0 = (D_0, \mathcal{L}_0)$. Assign all $D \in \mathcal{P}(D_0)$ the standard labeling. Then $\mathcal{P}(D_0)$ is ranked if and only if it has $-\Delta$ as a rank function. Equivalently, every covering relation in $\mathcal{P}(D_0)$ is given by an elementary Kohnert move.*

Proof. First, if $\mathcal{P}(D_0)$ has $-\Delta$ as a rank function, then by definition, $\mathcal{P}(D_0)$ is ranked.

Next, suppose that $\mathcal{P}(D_0)$ is ranked. By Lemma 2.16, if $D \in \mathcal{P}(D_0)$ has nonzero displacement, then there is some $D' \in \mathcal{P}(D_0)$ with $D < D'$ and $\Delta(T) = \Delta(T') + 1$. Note that two diagrams with equal displacement are not comparable, since comparable diagrams differ by a sequence of Kohnert moves, each of which increase displacement. So, there is no D'' such that $D < D'' < D'$. Thus, D is covered by D'.

By induction, there is a maximal chain of length $\Delta(T)$ in $\mathcal{P}(D_0)$ that connects D_0 to D. Since $\mathcal{P}(D_0)$ is ranked, all maximal chains from D_0 to D have length $\Delta(T)$. As applying a Kohnert move always increases displacement, each covering relation in each maximal chain changes displacement by 1. Thus, $\mathcal{P}(D_0)$ is ranked if and only if it has $-\Delta$ as a rank function. □

Next, we record a result of Colmenarejo, Hutchins, Mayers, and Phillips [CHMP25] on general ranked Kohnert posets which we apply in the proof of Theorem 1.4.

Lemma 4.2 ([CHMP25, Theorem 3.4])**.** *Let D_0 be a diagram. Suppose that there exists a diagram $D \in \mathcal{P}(D_0)$ and natural numbers $r, c_1, c_3 \in \mathbb{N}$ (with $1 \leqslant c_1 < c_3$) satisfying either conditions (a)(i) through (a)(iv) or conditions (b)(i) through (b)(iv):*

(a) (i) $(r+1, c_1), (r+2, c_1), (r+1, c_3) \in D$.

(ii) $(r+2,c') \notin D$ *for all* $c' > c_1$.

(iii) $(r+1,c') \notin D$ *for all* $c' > c_1$ *not equal to* c_3.

(iv) $(r,c_1),(r,c_3) \notin D$.

(b) *(i)* $(r+1,c_1),(r+2,c_1),(r,c_3),(r+2,c_3) \in D$.

(ii) $(r+2,c') \notin D$ *for all* $c' > c_2$.

(iii) $(r+1,c') \notin D$ *for all* $c' > c_1$.

(iv) $(r,c_1) \notin D$.

Then $\mathcal{P}(D_0)$ *is not ranked.*

We can now prove our classification of northeast diagrams with ranked Kohnert posets. For the reader's convenience, we restate the theorem here. See Figure 12a for a visualization of conditions (a) - (d) and Figure 12b for an example of a northeast diagram whose Kohnert poset is not ranked.

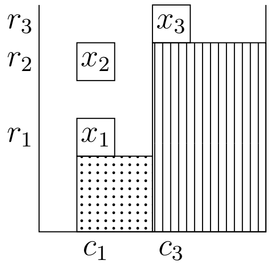


Figure 12a. The arrangement of cells x_1, x_2, and x_3 satisfying (a) and (b). Condition (c) states that there is an empty position in each column of the dotted region. Condition (d) states that the number of cells in each column of the striped region is less than r_1.

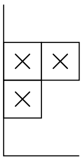


Figure 12b. A minimal example of a northeast diagram whose Kohnert poset is not ranked.

Theorem 1.4 *If* D_0 *is a northeast diagram, then* $\mathcal{P}(D_0)$ *is ranked if and only if* D_0 *does not contain* $x_1=(r_1,c_1), x_2=(r_2,c_2)$*, and* $x_3=(r_3,c_3)$ *such that:*

(a) $r_1 < r_2 \leqslant r_3$,

(b) $c_1 = c_2 < c_3$,

(c) for each $c_1 \leqslant c < c_3$*, there is at least one empty position* (r,c) *where* $r < r_1$*, and*

(d) for each $c \geqslant c_3$*, the number of* $r < r_3$ *such that* $(r,c) \in D_0$ *is strictly less than* r_1.

Proof of Theorem 1.4. (Proof of ⇒) We show the contrapositive. Suppose D_0 contains cells x_1, x_2, and x_3 satisfying Properties (a) - (d). In the case that there are multiple such triples, order them lexicographically as tuples $(c_3,r_3,c_2,r_2,c_1,r_1)$ and choose the maximal tuple. This yields a choice of x_1,x_2,x_3 such that x_3 is right- and upper-most, then x_2 is the right- and upper-most choice of cell that works with x_3, then x_1 is the right- and upper-most choice of cell that works with x_3 and x_2. We now perform a sequence of Kohnert moves to obtain a diagram to which we may apply Lemma 4.2. This sequence is broken down into five subsequences, and we denote by D_i the result after applying the ith subsequence of Kohnert moves.

1. First we argue that there is at least one empty position (r, c) with $r < r_1$ in every column c with $c \geqslant c_3$. Supposing otherwise, by the northeast condition we would also have a cell at (r_1, c), but this violates Property (d). Together with Property (c), this implies that we can perform Kohnert moves on D_0 to the cells strictly below and weakly to the right of x_1 to obtain a diagram D_1 that does not contain any cells in positions $(r_1 - 1, c)$ for $c \geqslant c_1$.

2. By Property (d), we can perform Kohnert moves on D_1 to the cells strictly below and weakly to the right of x_3 to obtain a diagram D_2 that does not contain any cells in positions (r, c) for $r_1 \leqslant r < r_3$ and $c \geqslant c_3$. Furthermore we observe that there are no cells to the right of x_3 in row r_3 of D_2. If such a cell x_3' existed, the triple (x_1, x_2, x_3') would certainly satisfy Properties (a), (b), and (d), and Property (d) applied to the triple (x_1, x_2, x_3) would imply Property (c) for (x_1, x_2, x_3'). This would contradict the maximality of (x_1, x_2, x_3).

3. By the maximality of our choices of x_1, x_2, and x_3, D_2 has at most one cell (r, c) in each column $c_1 < c < c_3$ with $r_1 \leqslant r \leqslant r_3$. By our previous set of moves, the only cell to the right of this rectangle is x_3. Moreover, by our first set of moves, there are no cells in positions $(r_1 - 1, c)$ for any $c_1 \leqslant c < c_3$. Then we can perform Kohnert moves on the cells in this rectangle of D_2 to obtain a diagram D_3 such that the only remaining cells (r, c) with $r_1 \leqslant r < r_3$ and $c_1 \leqslant c$ are x_1 and x_2.

We now have two cases, dependent on whether or not $(r_1 - 1, c_3) \in D_3$. We describe the rest of the subsequences and finish the proof separately for each case.

Case (a): $(r_1 - 1, c_3) \notin D_3$.

(4a) By the result of our second set of moves, we can perform $r_3 - r_1 + 1$ elementary moves on every cell $(r_3, c) \in D_3$ for $c_1 < c \leqslant c_3$, *except for the leftmost such cell*, to which we apply $r_3 - r_1$ moves. We refer to this leftmost cell, now in row r_1, as $x_3' = (r_1, c_3')$. If $(r_3, c) \in D_3$, then the maximality of our tuple implies that in the previous step there was no cell to move into position $(r_1 - 1, c)$. Thus, the moves described here are valid and yield a diagram D_4.

(5a) As there are no cells (r, c) in D_4 with $r_1 < r \leqslant r_3$ and $c_1 \leqslant c$ other than x_2, we can perform $r_2 - r_1 - 1$ elementary moves on x_2 to obtain a diagram D_5 such that the cell in column c_1 with label x_2 is in row $r_1 + 1$. We call this cell x_2'.

We claim that D_5 satisfies the conditions (a)(i) through (a)(iv) of Lemma 4.2(a), choosing cells x_1, x_2', and x_3' of D_5 (in that order) to be those described in condition (i) of the lemma. By our second set of moves, D_5 satisfies condition (ii) of the lemma. By our second, third, and final sets of moves, D_5 satisfies condition (iii) of the lemma. Finally, by our first set of moves and the hypothesis of this case, D_5 satisfies condition (iv) of the lemma. Then in this case, $\mathcal{P}(D_0)$ is not ranked.

Case (b): $(r_1 - 1, c_3) \in D_3$.

If this cell has an empty space beneath it, we can again move all but the leftmost cell to the right of column c_1 in row r_3 down to row $r_1 - 1$, take this leftmost cell to the same x_3' position, and apply Lemma 4.2(a) as above. So instead suppose there are no empty spaces beneath $(r_1 - 1, c_3)$.

(4b) We perform $r_3 - r_1 - 1$ elementary moves on every cell (r_3, c) for $c_1 < c \leqslant c_3$, so that they all end up in row $r_1 + 1$. Call the resulting diagram D_4 and its cell $x'_3 = (r_1 + 1, c_3)$.

(5b) We perform $r_2 - r_1 - 1$ elementary moves on x_2 in D_4 to obtain a diagram D_5 such that the cell in column c_1 with label r_2 is in row $r_1 + 1$ as well. We call this cell x'_2.

This diagram then satisfies the conditions (b)(i) through (b)(iv) in the statement of Lemma 4.2(b), choosing x_1, x'_2, $(r_1 - 1, c_3)$, and x'_3 of D_5 (in that order) to be the cells described in condition (b)(i). This completes the proof of the forward implication.

(Proof of $\Leftarrow$) We show the contrapositive. Suppose D_0 is northeast and $\mathcal{P}(D_0)$ is not ranked. By Lemma 4.1, there are $D, D' \in \mathcal{P}(D_0)$ such that $D' \lessdot D$, but D' is obtained from D by a Kohnert move applied to a cell $x'_2 = (r, c_1)$ such that $x'_1 = (r-1, c_1) \in D$. If there were no cells of the form $x'_3 = (r-1, c_3) \in D$ for some $c_3 > c_1$, then D' could be obtained from D by first applying a Kohnert move to x'_1 and then to x'_2. Thus, D does contain such a cell x'_3.

Let $\mathcal{L}$ be the standard labeling of D relative to D_0. For brevity, we say $L_1 = \mathcal{L}(x'_1)$, $L_2 = \mathcal{L}(x'_2)$, and $L_3 = \mathcal{L}(x'_3)$. By strictness, $L_1 < L_2$. If $L_2 > L_3$, then by strictness and Property (3) of Definition 2.8, there is a cell $y = (r', c_3)$ in D such that $r' > r - 1$ and $\mathcal{L}(y) = L_2$. By Property (4) of Definition 2.8, $r' \leqslant r$. Thus, $r' = r$. In other words, y is in the same row as x'_2 and to its right. This contradicts the fact that x'_2 is the rightmost cell in its row, which is necessary for it to be moved. Thus, $L_2 \leqslant L_3$.

By Theorem 2.17, D_0 has cells $x_1 = (L_1, c_1)$, $x_2 = (L_2, c_1)$, and $x_3 = (L_3, c_3)$. From our construction of c_1 and c_3 and the derived conditions on L_1, L_2, and L_3, we have that x_1, x_2, and x_3 satisfy Properties (a) and (b) in the theorem statement. We now show that they satisfy Properties (c) and (d) and thus are the desired cells.

(c) A Kohnert move can be applied to $x'_2 \in D$, so there is at least one empty space below $x'_2 \in D$. Because x'_1 is one space below x'_2, that empty space is also below x'_1. By strictness and the column-equivalence of $\mathcal{L}$ to the super-standard labeling on D_0, $x_1 \in D_0$ and $x'_1 \in D$ have the same number of cells below them in the same column. Then because $L_1 \geqslant r - 1$, x_1 must also have an empty space below it in the same column.

Now fix a column c with $c_1 < c < c_3$ and suppose that $(r', c) \in D_0$ for all $r' < L_1$. By the northeast property, $(L_1, c) \in D_0$ and $(L_2, c) \in D_0$. In particular, the cell (L_1, c) has no empty spaces below it, so it can never move. Then x_1 cannot move either. In other words, $x_1 = x'_1$. By Property (4) of Definition 2.8, the cell labeled L_2 in column c of D is weakly below x'_2, which is in the row directly above x'_1. But the only such available cell is to the right of x'_2, contradicting x'_2 being the rightmost cell in its row. Therefore, in each column c with $c_1 < c < c_3$, there is at least one empty space in D_0 in a row below row L_1.

(d) For the sake of contradiction, suppose that this condition is not satisfied. Then there is a column $c \geqslant c_3$ such that the total number of cells (r', c) with $r' < L_3$ is at least L_1. By the northeast property, there is a cell $(L_3, c) \in D_0$. Then by strictness, the cell labeled L_3 in column c of D is above row L_1. Because $L_1 \geqslant r - 1$, the same cell is also strictly above row $r-1$. But by Property (4) of Definition 2.8, x_3' would also need to be above row $r-1$, contradicting the definition of x_3' as $(r-1, c_3)$.

Then the initial diagram D_0 contains cells x_1, x_2, and x_3 satisfying Properties (a) - (d) as described in the statement of the theorem. □

5 Bounded posets

In this section, we prove Theorem 1.5, repeated here for the reader's convenience. We refer the reader to Figure 13a for a visualization of conditions (a) - (e) and to Figure 13b for an example of a northeast diagram whose Kohnert poset is not bounded.

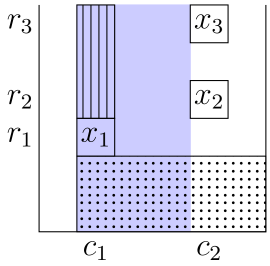


Figure 13a. The arrangement of cells x_1, x_2, and x_3 satisfying (a) and (b). Condition (c) states that the number of cells in each column of the blue region is less than the number of cells in the column containing x_2 and x_3. Condition (d) states that there is an empty position in each column of the dotted region. Condition (e) states that there are no cells in the striped region.

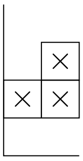


Figure 13b. A minimal example of a northeast diagram whose Kohnert poset is not bounded.

Theorem 1.5 *If D_0 is a northeast diagram, then $\mathcal{P}(D_0)$ is bounded if and only if D_0 does not contain $x_1 = (r_1, c_1), x_2 = (r_2, c_2)$, and $x_3 = (r_3, c_3)$ such that:*

(a) $r_1 \leqslant r_2 < r_3$,

(b) $c_1 < c_2 = c_3$,

(c) for each $c_1 \leqslant c < c_2$, $\mathrm{cwt}(D_0)_c < \mathrm{cwt}(D_0)_{c_2}$,

(d) for each $c \geqslant c_1$, *there is at least one empty position* (r, c) *where* $r < r_1$, *and*

(e) for each $r_1 < r \leqslant r_3$, *the cell* (r, c_1) *is not in* D_0.

By Proposition 2.22, every minimal element can be reached by a sequence of elementary moves with no move inversions, meaning that within the sequence, all elementary moves of i-cells occur before all elementary moves of $(i+1)$-cells for each $i \geqslant 1$. We claim that we can always construct a minimal element of $\mathcal{P}(D_0)$ for northeast diagram D_0 by applying the following "greedy" sequence of elementary moves with no move inversions to D_0. We assign the super-standard labeling to D_0. Then, for each i, starting with $i = 2$, first move the rightmost i-cell downward as far as possible using only elementary moves. During this part of the process we call this i-cell the **moving cell** and say that this moving cell is taking its **turn**. Now, going leftward along the row, do the same for each subsequent i-cell. Then repeat the process for the $(i+1)$-cells. We refer to this greedy, inversion-free sequence of elementary moves as the **canonical procedure**. In our next lemma, we prove our claim that this procedure produces a minimal element when D_0 is northeast. We also show that at no point during the canonical procedure is it ever possible to apply a jump Kohnert move to the cell that we are moving.

For the remainder of this section, we let D_{can} denote the element of $\mathcal{P}(D_0)$ that is obtained from the canonical procedure (see Figure 14 for an example). Throughout, $\mathcal{L}_{can}$ is the standard labeling of D_{can} with respect to D_0. We refer to D_{can} as the **canonical minimal element**, since it follows from Lemma 5.1 that when $\mathcal{P}(D_0)$ is bounded, the diagram obtained in this way is its only minimal element. Note however that the minimality of D_{can} is not immediate from the description of the canonical procedure and requires proof in Lemma 5.1.

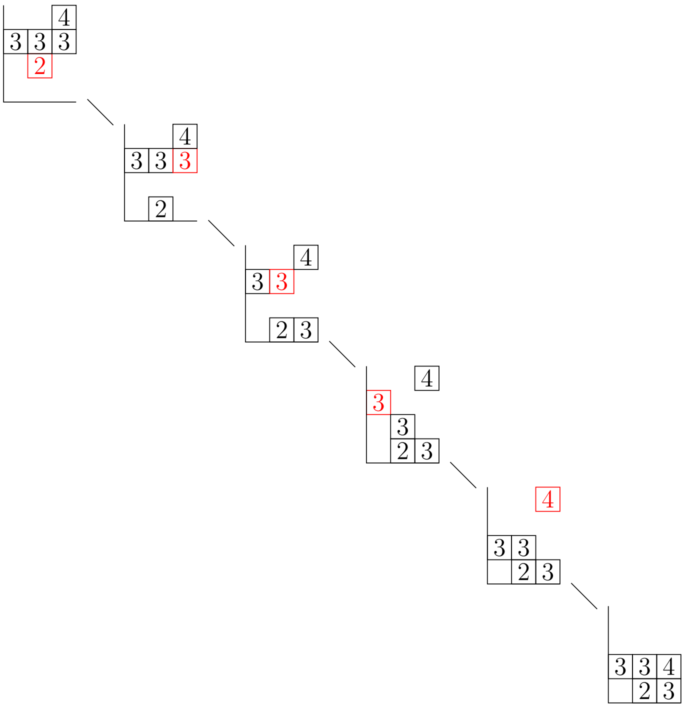


Figure 14: An example of constructing the canonical minimal element D_{can} from an initial northeast diagram D_0 with the super-standard labeling. Note that the edges do not necessarily represent covering relations, for example in the last step.

Lemma 5.1. *When applying the canonical procedure to a northeast diagram D_0, the following hold.*

(i) At no point is it possible to apply a jump Kohnert move to the moving cell. In particular, once a cell ends its turn in the canonical procedure, it remains in that position in D_{can}.

(ii) The diagram D_{can} resulting from the canonical procedure is a minimal element of $\mathcal{P}(D_0)$.

Proof. We begin with a proof of (i). Assume to the contrary that there is a moving cell to which a jump Kohnert move can be applied during the canonical procedure. Let x be the first such cell. Let (r, c) be the position of x in some diagram, say D', when its first jump Kohnert move is possible. There must be an empty position (r', c) where $r' < r-1$, and there must be cells $x_1, \ldots, x_n$ that occupy all positions $(r'+1, c), \ldots, (r-1, c)$, respectively (with $n = r - r' - 1$). Further, x must be the rightmost cell in row r of D'.

For each x_i with $1 \leqslant i \leqslant n$, consider the diagram D_i obtained immediately after x_i has ended its turn as the moving cell. By the minimality of x, a jump Kohnert move could not have been applied to move x_i into the position $(r'+i, c)$. Thus, it must have been the case that x_i had a cell to its right in the same row in D_i. Let y_i be the leftmost such cell in D_i. Let $\mathcal{L}_i$ be the standard labeling of D_i with respect to D_0. Then Lemma 2.9 gives $\mathcal{L}_i(y_i) \geqslant \mathcal{L}_i(x_i)$, while by the definition of the canonical procedure, $\mathcal{L}_i(y_i) \leqslant \mathcal{L}_i(x_i)$ as we have not yet moved any cells above row $\mathcal{L}_i(x_i)$. Thus, $\mathcal{L}_i(y_i) = \mathcal{L}_i(x_i)$ for each i.

In particular, $\mathcal{L}_n(y_n) = \mathcal{L}_n(x_n) < \mathcal{L}_n(x)$. Thus, by Property (3) of Definition 2.8, there must be a cell y in the same column as y_n in D' with $\mathcal{L}_n(y) = \mathcal{L}_n(x)$. By Property (4) of Definition 2.8, y must be located in row r to the right of x in D'. This contradicts our assumption that x can move and completes our proof of (i).

Now we show that D_{can} is minimal. We know that D_{can} is a minimal element of $\mathcal{P}(D_0)$ if and only if no cells in D_{can} are movable. By part (i), during the canonical procedure, each moving cell ends its turn immovable. This implies that each moving cell ends its turn with either a cell to the right of it in its row or with no empty spaces beneath it in its column.

Suppose there exists a cell $x \in D_{can}$ that is movable. Then there is an empty space somewhere below x. By (i), this empty space must have existed when x ended its turn as the moving cell, yielding a diagram D'. Thus, there must have been some cell y in the same row to the right of x in D'. But then by (i), the cell y must be in the same position in D_{can} as it is in D'. This contradicts that x is movable and completes our proof of (ii). □

Since $\mathcal{P}(D_0)$ is not bounded if and only if it has two or more distinct minimal elements, to prove Theorem 1.5, it is sufficient to prove the following proposition.

Proposition 5.2. *Let D_0 be a northeast diagram. Then D_0 contains cells x_1, x_2, x_3 satisfying conditions (a) - (e) of Theorem 1.5 if and only if $\mathcal{P}(D_0)$ contains a minimal element $\tilde{D}$ distinct from the canonical minimal element D_{can}.*

The proof of Proposition 5.2 requires a few additional lemmas. Our first is a general result about minimal diagrams.

Lemma 5.3. *Let $\tilde{D}$ be a minimal element of $\mathcal{P}(D_0)$. Suppose there exists a column c that contains an empty position below a cell. Let $r_{\max}$ be the maximum row index such that $(r_{\max}, c) \in \tilde{D}$. Then there exists a column to the right of column c in $\tilde{D}$ in which all rows 1 through $r_{\max}$ are occupied by cells.*

Proof. Since $\tilde{D}$ is a minimal element, there must be at least one column $c' > c$ such that $(r_{\max}, c') \in \tilde{D}$. Let $c_{\max}$ equal the maximum such column. As $(r_{\max}, c_{\max})$ is the rightmost cell in its row, the minimality of $\tilde{D}$ implies that in column $c_{\max}$, all rows 1 through $r_{\max}-1$ are occupied by cells. This completes our proof, since $(r_{\max}, c_{\max}) \in \tilde{D}$. □

The next lemma specializes Lemma 5.3 to the canonical minimal element to deduce further information about D_0. We introduce the notation $\mathrm{cwt}_{\leqslant r}(D)_c$ (respectively, $\mathrm{cwt}_{<r}(D)_c$) to denote the number of cells in column c of diagram D that lie weakly (respectively, strictly) below row r. Likewise, let $\mathrm{cwt}_{\geqslant r}(D)_c$ (respectively, $\mathrm{cwt}_{>r}(D)_c$) denote the number of cells in column c of diagram D that lie weakly (respectively, strictly) above row r.

Lemma 5.4. *Let $x = (r, c)$ be a cell in the northeast diagram D_0 that reaches position $x' = (r', c)$ at the end of the canonical procedure in the canonical minimal element D_{can}. If x' has an empty position below it in column c of D_{can}, then:*

(i) there is a column $c' > c$ that satisfies $\mathrm{cwt}_{\leqslant r}(D_0)_{c'} > \mathrm{cwt}_{\leqslant r}(D_0)_c$ and

(ii) if $\hat{c}$ is the leftmost such column, then the cell $\hat{x} = (r', \hat{c})$ exists in D_{can} and $\mathcal{L}_{can}(\hat{x}) = r$.

Proof. First we show (i). Let D' be the diagram reached after x has ended its turn as the moving cell in position (r', c). Since x' has an empty position below it in column c of D_{can}, it must have an empty position below it in column c of D' by part (i) of Lemma 5.1. This, combined with the fact that x' is not movable by part (i) of Lemma 5.1 implies that x' must have a cell in the same row to the right of it in D'. Let y' be the rightmost such cell and let $\mathcal{L}'$ be the standard labeling on D' with respect to D_0. Then Lemma 2.9 gives $\mathcal{L}'(y') \geqslant \mathcal{L}'(x') = r$, while by the definition of the canonical procedure, $\mathcal{L}'(y') \leqslant r$ as we have not yet moved any cells above row r. Hence, y' is an r-cell.

Suppose y' is in column c'. There must not be any empty spaces below y' since y' is not movable and it is the rightmost cell in its row in D'. Thus, D' has cells labeled by exactly r' values less than or equal to r in column c' while D' has strictly fewer than r' cells labeled less than or equal to r in column c. We conclude that $\mathrm{cwt}_{\leqslant r}(D_0)_{c'} > \mathrm{cwt}_{\leqslant r}(D_0)_c$, which proves (i).

Next, we show (ii). Let $\hat{c}$ be the leftmost column satisfying $\mathrm{cwt}_{\leqslant r}(D_0)_{\hat{c}} > \mathrm{cwt}_{\leqslant r}(D_0)_c$ such that $\hat{c} > c$. From (i), we know that such a column exists, and is either c' or to the left of c', so $c < \hat{c} \leqslant c'$. Because of the column-weight condition, there is at least one cell in column $\hat{c}$ in a row strictly lower than r in D_0. So, since D_0 is northeast, $(r, \hat{c}) \in D_0$, or equivalently, there is an r-cell in column $\hat{c}$. Then, in D_{can}, the equality $\mathcal{L}_{can}(x') = r = \mathcal{L}_{can}(y')$ forces the r-cell in column $\hat{c}$ to be at $(r', \hat{c})$ by Property (4) of Definition 2.8. Thus, $\hat{x} = (r', \hat{c}) \in D_{can}$ and $\mathcal{L}_{can}(\hat{x}) = r$, completing the proof of (ii). $\square$

Our next lemma consists of the most technical parts of the proof for the ($\Rightarrow$) direction of Proposition 5.2.

Lemma 5.5. *Let x_1, x_2, and x_3 satisfy conditions (a) - (e) of Theorem 1.5 and be chosen so that the tuple $(c_1, r_1, -c_2, r_2, -c_3, r_3)$ is maximal in the lexicographic order on $\mathbb{Z}^6$. Indicate the positions of the x_i in D_{can} at the end of the canonical procedure by $x'_i = (r'_i, c_i)$. Then:*

(i) for $r > r_3$, $(r, c_1) \in D_0$ if and only if $(r, c_2) \in D_0$, and

(ii) $r'_1 \leqslant r'_2$.

Proof. Notice that the choice of x_1, x_2, and x_3 forces x_1 to have the maximum possible c_1, then maximum possible r_1. With respect to this x_1, this then forces the minimum possible c_2, and then the maximum possible r_2. Finally, with respect to the chosen x_1 and x_2, this forces the the maximum possible r_3.

We start with part (i). If $r > r_3$ and $(r, c_1) \in D_0$, then since D_0 is northeast, x_2 and x_3 force $(r, c_2) \in D_0$. For the converse, let $y = (r_y, c_2)$ be a cell with $r_y > r_3$. If $(r_y, c_1) \notin D_0$, then there are two cases. If $(r, c_1) \notin D_0$ for all $r_3 < r < r_y$, then Properties (a) - (e) hold for x_1, x_2, y. As $y = (r_y, c_2)$ has $r_y > r_3$, this contradicts the maximality of the chosen x_1, x_2, and x_3. If there does exist $(r, c_1) \in D_0$ with $r_3 < r < r_y$, then let $z = (r_z, c_1)$ be the upper-most such cell. Since D_0 is northeast, x_2 and x_3 force $(r_z, c_2) \in D_0$. Then z, (r_z, c_2), and y form a triple that satisfies (a) - (e), again contradicting the maximality of the choice x_1, x_2, x_3. So we must have $(r_y, c_1) \in D_0$, and thus part (i) follows.

We now prove part (ii), continuing with x_1, x_2, and x_3, chosen so that the tuple $(c_1, r_1, -c_2, r_2, -c_3, r_3)$ is maximal in the lexicographic ordering. Part (i) and condition (e) imply that all of the cells in column c_2 above row r_3 have a corresponding cell in column c_1 above row r_3. This means that x_3 is the only cell in column c_2 above row r_2 that does not have a cell in the same row in column c_1.

In other words, $\mathrm{cwt}(D_0)_{c_1} - \mathrm{cwt}_{\leqslant r_1}(D_0)_{c_1} + 1 = \mathrm{cwt}(D_0)_{c_2} - \mathrm{cwt}_{\leqslant r_2}(D_0)_{c_2}$. Rewriting this yields

$$(\mathrm{cwt}_{\leqslant r_2}(D_0)_{c_2} + 1) - \mathrm{cwt}_{\leqslant r_1}(D_0)_{c_1} = \mathrm{cwt}(D_0)_{c_2} - \mathrm{cwt}(D_0)_{c_1} > 0,$$

where the fact that this quantity is greater than zero is due to condition (c). Thus, we have that

$$\mathrm{cwt}_{\leqslant r_1}(D_0)_{c_1} \leqslant \mathrm{cwt}_{\leqslant r_2}(D_0)_{c_2}. \tag{5.1}$$

If x_1' has no empty positions beneath it, then $r_1' \leqslant r_2'$ follows directly from (5.1). Hence we may now assume x_1' has a space beneath it in its column. By Lemma 5.4, this means that there is some column $\hat{c} > c_1$ with

$$\mathrm{cwt}_{\leqslant r_1}(D_0)_{\hat{c}} > \mathrm{cwt}_{\leqslant r_1}(D_0)_{c_1}. \tag{5.2}$$

We choose the leftmost such column, and we claim that such a $\hat{c}$ must satisfy $\hat{c} \geqslant c_2$. Suppose, for contradiction, that $\hat{c}$ satisfies $c_1 < \hat{c} < c_2$.

Case 1: $(r_3, \hat{c}) \notin D_0$. Let $z = (r_z, \hat{c})$ be the uppermost cell in column $\hat{c}$ that is below r_3; such a cell must exist by (5.2). Then either the triple z, x_2, x_3 satisfies (a) - (e) if $r_z \leqslant r_2$, or by the northeast property of D_0 there is a triple z, (r_z, c_2), x_3 that satisfies (a) - (e). In either situation, lexicographic maximality of our choice of triple is violated since $\hat{c} > c_1$, which produces a contradiction.

Case 2: $(r_3, \hat{c}) \in D_0$. Since $\mathrm{cwt}_{\leqslant r_1}(D_0)_{\hat{c}} > \mathrm{cwt}_{\leqslant r_1}(D_0)_{c_1} > 0$, by the northeast property of D_0 we have that $(r_1, \hat{c}) \in D_0$ so there is at least one cell of the form $(s, \hat{c})$ with $r_1 \leqslant s < r_3$. Let $y = (r_y, \hat{c})$ be the cell of this form with maximal row index r_y. Then the triple x_1, y, $(r_3, \hat{c})$ straightforwardly satisfies conditions (a), (b), (d), and (e). We now show that condition (c) is satisfied as well.

The combination of $(r_1, \hat{c}) \in D_0$ and the northeast property of D_0 imply that if $(s, c_1) \in D_0$, then $(s, \hat{c}) \in D_0$ for all $s > r_1$. Thus, $\mathrm{cwt}_{>r_1}(D_0)_{\hat{c}} \geqslant \mathrm{cwt}_{>r_1}(D_0)_{c_1}$. Adding this inequality to (5.2) gives

$$\mathrm{cwt}(D_0)_{c_1} < \mathrm{cwt}(D_0)_{\hat{c}}. \tag{5.3}$$

To handle the other columns, suppose for sake of contradiction that the inequality

$$\mathrm{cwt}(D_0)_{c'} = \mathrm{cwt}_{\leqslant r_1}(D_0)_{c'} + \mathrm{cwt}_{>r_1}(D_0)_{c'} \geqslant \mathrm{cwt}_{\leqslant r_1}(D_0)_{\hat{c}} + \mathrm{cwt}_{>r_1}(D_0)_{\hat{c}} = \mathrm{cwt}(D_0)_{\hat{c}}, \quad (5.4)$$

holds for some $c_1 < c' < \hat{c}$. Once again, the combination of $(r_1, \hat{c}) \in D_0$ and the northeast property of D_0 imply

$$\mathrm{cwt}_{>r_1}(D_0)_{c'} \leqslant \mathrm{cwt}_{>r_1}(D_0)_{\hat{c}}$$

Combined with (5.4), this implies that

$$\mathrm{cwt}_{\leqslant r_1}(D_0)_{c'} \geqslant \mathrm{cwt}_{\leqslant r_1}(D_0)_{\hat{c}}.$$

But since $c' < \hat{c}$, this contradicts our choice of $\hat{c}$ as the least column index $c > c_1$ satisfying $\mathrm{cwt}_{\leqslant r_1}(D_0)_c \geqslant \mathrm{cwt}_{\leqslant r_1}(D_0)_{c_1}$. We must then conclude that $\mathrm{cwt}(D_0)_{c'} < \mathrm{cwt}(D_0)_{\hat{c}}$ for each c' such that $c_1 < c' < \hat{c}$. Together with (5.3), this shows that condition (c) applies to the triple x_1, y, $(r_3, \hat{c})$ as well, contradicting lexicographic maximality of our original triple and completing this case.

Having dealt with both cases, we have now shown that $\hat{c} \geqslant c_2$. Applying Lemma 5.4(ii) we know that there is an r_1-cell $(r_1', \hat{c}) \in D_{can}$. If $\hat{c} = c_2$, then the fact that $r_2' \geqslant r_1'$ follows since x_2 starts above this r_1-cell in D_0. Otherwise if $\hat{c} > c_2$, the fact $r_2' \geqslant r_1'$ follows from Lemma 2.9, comparing the labels on x_2' and $(r_1', \hat{c})$. This completes the proof of (ii). □

We are now ready to prove Theorem 1.5 by proving Proposition 5.2.

Proof of Theorem 1.5. We show both ⇒ and ⇐ directions of Proposition 5.2.

(Proof of ⇒) Assume D_0 contains cells x_1, x_2, x_3 satisfying conditions (a) - (e). As in Lemma 5.5, we choose the triple such that $(c_1, r_1, -c_2, r_2, -c_3, r_3)$ is maximal in the lexicographic order. Assign to D_0 the super-standard labeling, and to each diagram in $\mathcal{P}(D_0)$ the standard labeling with respect to D_0. We recall the notation $x_i' = (r_i', c_i)$ for the cell in column c_i of D_{can} with label r_i. We will construct an element of $\mathcal{P}(D_0)$ called $\tilde{D}$, and show that $\tilde{D}$ is a minimal element of $\mathcal{P}(D_0)$ distinct from the canonical minimal element D_{can}. We let $\tilde{x}_i = (\tilde{r}_i, c_i)$ denote the cell in $\tilde{D}$ that originated as x_i in D_0.

We first show that during the canonical procedure, x_1 moves down at least one position. Assume to the contrary that it does not move, and recall that by condition (d), each column $c \geqslant c_1$ has at least one empty position (r, c) for $r < r_1$ in D_0. Thus, there is an empty position somewhere below x_1 in D_0 and also in the diagram D' reached at the start of x_1's turn as the moving cell. By Lemma 5.1, if a cell does not move in the canonical procedure, it is not because there is only a jump move available. So, in D', the cell x_1 is not rightmost in its row. Let y_1 be the rightmost cell in row r_1 of D'. Let $\mathcal{L}'$ be the standard labeling of D'. Then Lemma 2.9 gives $\mathcal{L}'(y_1) \geqslant \mathcal{L}'(x_1) = r_1$, while by the definition of the canonical procedure, $\mathcal{L}'(y_1) \leqslant r_1$ as we have not yet moved any cells above row r_1. Hence, y_1 is an r_1-cell. However, since y_1 is in row r_1, this means that y_1 did not move in the canonical procedure, and yet has an empty position below it. Again by Lemma 5.1, at no point is it possible to apply a jump Kohnert move to a cell, so there is an empty position immediately below y_1. This contradicts that y_1 is the rightmost cell in its row, and therefore x_1 must have moved at least once during the canonical procedure.

To construct $\tilde{D}$, we begin applying elementary moves to D_0 according to the canonical procedure, but refrain from making the final possible elementary move on the cell in column c_1 with label r_1, so that it instead rests in position $(r_1'+1, c_1)$, leaving an empty space in position (r_1', c_1). Denote the diagram obtained immediately after this cell with label r_1 stops moving at row $r_1'+1$ as D''. From D'', we continue to move all subsequent cells greedily down as far as possible using only elementary moves, continuing from the cell immediately left of x_1 as in the canonical procedure. At the end of this procedure, if necessary, in column c_1 apply Kohnert moves to the cells strictly above row $r_1'+1$ to move these cells down as far as possible until they are no longer movable. We will see that none of these cells can jump below $(r_1'+1, c_1)$, and thus position (r_1', c_1) remains empty. Then, if any cells in columns strictly to the left of column c_1 are movable, apply Kohnert moves in these columns until this is no longer the case. After all such moves are made, we call the resulting diagram $\tilde{D}$ and denote the position of the r_i-cell in column c_i (the cell that started as x_i) by $\tilde{x}_i = (\tilde{r}_i, c_i) \in \tilde{D}$ for $i \in \{1,2,3\}$. It remains to verify that D_{can} and $\tilde{D}$ differ in column c_1, and that $\tilde{D}$ is minimal.

First we show that D_{can} and $\tilde{D}$ differ in column c_1 by showing that no cell above $(r_1'+1, c_1)$ in D'' can jump over $(r_1'+1, c_1)$ into the empty position (r_1', c_1). Each column strictly to the right of column c_1 is identical in D_{can} and $\tilde{D}$, as these columns are unaffected by changes in column c_1. We use this fact repeatedly in what follows. By Lemma 5.5, $r_1' \leqslant r_2'$, so

$$r_1' + 1 \leqslant r_2' + 1 = \tilde{r}_2 + 1 \leqslant \tilde{r}_3 = r_3'. \tag{5.5}$$

Suppose there is another cell above x_1 in D_0, and let $y_1 = (r_y, c_1)$ be such a cell with minimal r_y, noting that $r_y > r_3$ by condition (e). By Lemma 5.5(i), there is a corresponding $y_2 = (r_y, c_2) \in D_0$. In $\tilde{D}$ (and D_{can}), y_2 must land at some row $r_y' > r_3'$. But then by Lemma 2.9, y_1 must land in a row with index at least r_y', which is then strictly greater than $r_1'+1$ by (5.5).

Thus, in applying additional Kohnert moves to the cells in D'' that lie above $(r_1'+1, c_1)$, none of these cells land in position (r_1', c_1). Therefore, position (r_1', c_1) is occupied in D_{can} but empty in $\tilde{D}$. This proves that D_{can} and $\tilde{D}$ differ in column c_1.

Next we show that $\tilde{D}$ is minimal. Our construction of $\tilde{D}$ ensures that no cells in column c_1 strictly above $\tilde{x}_1$ or in any column strictly to the left of column c_1 are movable. Furthermore, the fact that $\tilde{D}$ and D_{can} are identical in columns to the right of column c_1 as well as below row r_1' of column c_1 means that none of those cells are movable either. Thus, to show that $\tilde{D}$ is minimal, we only need to show that $\tilde{x}_1$ is not the rightmost cell in its row.

If all rows $1 \leqslant r \leqslant \tilde{r}_3$ of column c_2 are occupied in $\tilde{D}$, then $\tilde{x}_1$ is not rightmost in its row and is immovable. Otherwise, the fact that $\tilde{x}_3$ has a space somewhere below it in $\tilde{D}$ implies that x_3' has a space somewhere below it in D_{can}. Lemma 5.3 then implies that there is some column $c' > c_2$ in D_{can} that has rows 1 through r_3' occupied by cells. Thus, $\tilde{D}$ also has rows 1 through $\tilde{r}_3 = r_3'$ occupied by cells in column c'. Hence, in this case it is also true that $\tilde{x}_1$ is not rightmost in its row. This completes our proof of $(\Rightarrow)$.

(Proof of $\Leftarrow$) Assume that $\mathcal{P}(D_0)$ has a second minimal element, $\tilde{D}$, distinct from D_{can}. This $\tilde{D}$ can be produced by some sequence of elementary Kohnert moves with no move inversions by Proposition 2.22. We show that there exist cells x_1, x_2, x_3 in D_0 satisfying conditions (a) - (e).

Let $\mathcal{L}_{can}$ and $\tilde{\mathcal{L}}$ denote the standard labelings on D_{can} and $\tilde{D}$, respectively. Let column c_1 be the rightmost column where D_{can} and $\tilde{D}$ differ. Let s_1' be the smallest row index where D_{can} and $\tilde{D}$ differ in column c_1.

We claim that the position (s'_1, c_1) is empty in $\tilde{D}$ and occupied in D_{can}. For sake of contradiction, suppose otherwise that $x = (s'_1, c_1) \in \tilde{D}$ and is empty in D_{can}. Then, since both diagrams have the same column weights, there is a cell $x' = (r', c_1)$ in D_{can} that is the first occupied space above row s'_1. Since x' has an empty position below it, by Lemma 5.4(ii), there is a column $\hat{c}$ to the right of column c_1 and a cell $\hat{x} = (r', \hat{c}) \in D_{can}$ with $\mathcal{L}_{can}(\hat{x}) = \mathcal{L}_{can}(x')$. We also have $\tilde{\mathcal{L}}(x) = \mathcal{L}_{can}(x')$, and since D_{can} and $\tilde{D}$ are identical to the right of column c_1, $\tilde{\mathcal{L}}(\hat{x}) = \mathcal{L}_{can}(\hat{x})$ as well. But this implies that x and $\hat{x}$ are cells of $\tilde{D}$ with the same label and for which $\hat{x}$ is above and to the right of x, contradicting Lemma 2.9. Thus, (s'_1, c_1) must be occupied in D_{can} and empty in $\tilde{D}$.

Let $s_1 = \mathcal{L}_{can}((s'_1, c_1))$. Then the first cell in column c_1 of $\tilde{D}$ above row s'_1 is precisely the cell with label s_1. Call this cell $\tilde{y}_1 = (\tilde{s}_1, c_1)$, let $y_1 = (s_1, c_1) \in D_0$, and denote by $y'_1 = (s'_1, c_1)$ the s_1-cell in D_{can}.

For each nonnegative integer j, let $C_j := \{c_1, c_1 + 1, \ldots, c_1 + j\}$ be a set of consecutive column indices. Let (r_{C_j}, c_{C_j}) be the lexicographically maximum cell in $\tilde{D}$ such that $c_{C_j} \in C_j$ and the cell has an empty position somewhere below it; this maximum exists since $\tilde{y}_1$ is in column c_1 and has an empty position below it. Notice that by this lexicographic maximality, as j increases, the position of the cell (r_{C_j}, c_{C_j}) moves weakly rightward and upward. For each j, by applying Lemma 5.3, there is a leftmost column $m_j > c_{C_j}$ in which all rows 1 through r_{C_j} contain cells. Then as j increases, m_j weakly increases. Let j' be the minimum index such that $m_{j'} = c_1 + j'$. Such an index always exists since the sequence $(m_0, m_1, m_2, \ldots)$ is weakly increasing and bounded above, with $c_1 < m_j \leqslant m$ for all j, where m is the last column. Further let $c_2 = m_{j'}$ and $r_{\max} = r_{C_{j'}}$.

We point out a useful property of this choice of c_2. Notice that column c_2 of $\tilde{D}$ contains cells in rows 1 through $r_{\max}$. Furthermore, for each column c such that $c_1 \leqslant c < c_2$, we claim that the maximum row index of any cell in column c of $\tilde{D}$ is also $r_{\max}$. To show this, suppose for contradiction that there is a cell $(r, c) \in \tilde{D}$ with $r > r_{\max}$ and $c_1 \leqslant c < c_2$. If there are no empty positions below (r, c), then consider columns c_1 through $c = c_1 + d$. We have that $m_0, \ldots, m_d$ are each at most c, and since this sequence is weakly increasing, there exists some $m_{j''} = c_1 + j''$ further left than j', which contradicts the definition of j'. If there is an empty position below (r, c), then (r, c) would be a lexicographically larger cell than $(r_{C_{j'}}, c_{C_{j'}})$, contradicting our choice when identifying this cell.

Therefore, all cells in each column $c_1 \leqslant c < c_2$ are contained in rows 1 through $r_{\max}$. By a similar argument to the previous paragraph, if in one of these columns, every such row contains a cell, we would contradict the choice of $m_{j'}$. Thus,

$$\mathrm{cwt}_{\leqslant r_{\max}}(\tilde{D})_c < r_{\max} = \mathrm{cwt}_{\leqslant r_{\max}}(\tilde{D}_{c_2}) \text{ for all } c_1 \leqslant c < c_2. \tag{5.6}$$

Since $(r_{\max}, c_{C_{j'}})$ is a cell with a space beneath it, $r_{\max} \geqslant 2$. Hence, column c_2 contains at least two cells. We claim that D_0 contains two cells $y_2 = (s_2, c_2)$ and $y_3 = (s_3, c_3)$ where $s_1 \leqslant s_2 < s_3$ and $c_1 < c_2 = c_3$. As the canonical procedure that yields D_{can} moves y_1 down at least one row below the position of $\tilde{y}_1$ in $\tilde{D}$, the cell $(\tilde{s}_1 - 1, c_2)$ in $\tilde{D}$ has label greater than or equal to $\tilde{\mathcal{L}}(\tilde{y}_1) = s_1$ by Lemma 2.9. Call this cell $\tilde{y}_2$ and let $s_2 = \tilde{\mathcal{L}}(\tilde{y}_2)$ Then by the strictness on the labels in each column, the cell directly above $\tilde{y}_2$, which we call $\tilde{y}_3$, has $s_3 = \tilde{\mathcal{L}}(\tilde{y}_3)$ where $s_3 > s_2 \geqslant s_1$. As these labels are equal to the initial row indices of the corresponding cells in D_0 (which we refer to as y_2 and y_3, respectively), D_0

contains cells $y_2 = (s_2, c_2)$, $y_3 = (s_3, c_2)$ with $s_1 \leqslant s_2 < s_3$ and $c_1 < c_2 = c_3$, proving our claim.

Having shown that there exists in D_0 a triple of cells y_1, y_2, y_3 that satisfies (a), (b), and the inequality $\tilde{s}_1 > s'_1$, we now choose a particular such triple in the same columns. Assign $x_1 = (r_1, c_1)$, $x_2 = (r_2, c_2)$, and $x_3 = (r_3, c_3)$ to be the cells in columns c_1 and $c_2 = c_3$ of D_0 such that the tuple $(r_1, -r_2, -r_3)$ is lexicographically maximal, conditions (a) and (b) are satisfied, and $\tilde{r}_1 > r'_1$, where $\tilde{r}_1$ and r'_1 are the row indices of the r_1-cells in column c_1 of $\tilde{D}$ and D_{can}, respectively. We now show that this choice of x_1, x_2, and x_3 either satisfies conditions (a) - (e) or can be used to produce a related triple that does, particularly in the event that (e) is not satisfied.

(a),(b) These inequalities are satisfied by x_1, x_2, and x_3 by construction.

(c) We saw in (5.6) that $\mathrm{cwt}_{\leqslant r_{\max}}(\tilde{D})_c < r_{\max} = \mathrm{cwt}_{\leqslant r_{\max}}(\tilde{D})_{c_2}$ for all $c_1 \leqslant c < c_2$. Further, we have seen that all cells in $\tilde{D}$ in columns c with $c_1 \leqslant c < c_2$ are located in rows 1 through $r_{\max}$. So $\mathrm{cwt}_{\leqslant r_{\max}}(\tilde{D})_c = \mathrm{cwt}(D_0)_c$ for each $c_1 \leqslant c < c_2$. Thus,

$$\mathrm{cwt}(D_0)_c < \mathrm{cwt}_{\leqslant r_{\max}}(\tilde{D})_{c_2} \leqslant \mathrm{cwt}(D_0)_{c_2}$$

for all $c_1 \leqslant c < c_2$, so condition (c) is satisfied.

(d) For both D_{can} and $\tilde{D}$ to be reachable from D_0, by Lemma 2.9, it must be the case that all r_1-cells in columns $c \geqslant c_1$ in D_0 move down during the canonical procedure. This means that each column $c \geqslant c_1$ that contains an r_1-cell must have an empty position in some row $r < r_1$ in D_0. Otherwise, if a column $c \geqslant c_1$ does not contain an r_1-cell, then assume for the sake of contradiction that there is no empty position (r, c) in D_0 for some $r < r_1$. Then as every (r, c) with $r < r_1$ is a cell in D_0, the presence of x_1 implies that $(r_1, c) \in D_0$ by the northeast property of D_0. This contradicts our assumption that the column had no r_1-cell and hence this case can not occur. Thus, condition (d) is satisfied.

(e) Finally, we consider cells in positions (r, c_1) in D_0 such that $r > r_1$. We must show that D_0 does not contain any cells (r, c_1) with $r_1 < r \leqslant r_3$. If there was a cell $(r, c_1) \in D_0$ with $r_2 < r < r_3$, then the northeast property of D_0 would imply the presence of a cell in column c_2 that would contradict our choice of x_3. If there was a cell (r, c_1) with $r_1 < r < r_2$, then the northeast property of D_0 would imply the presence of a cell in column c_2 that would contradict our choice of x_2. This leaves us to consider position (r, c_1) in the following three cases: $r_1 = r_2$ with $r = r_3$ or $r_1 < r_2$ with $r = r_2,\ r_3$.

Case 1: $r_1 = r_2$, $(r_3, c_1) \in D_0$. Let $x_4 := (r_3, c_1)$. If there were any cell (r, c) with $r_1 < r < r_3$ and $c < c_2$, by the northeast property we would have $(r, c_2) \in D_0$, contradicting our choice of x_3 as having minimal row index. Thus, this rectangular region with opposite corners $(r_1 + 1, c_1)$ and $(r_3 - 1, c_2)$ is empty.

Let $\tilde{x}_4 = (\tilde{r}_4, c_1)$ and $x'_4 = (r'_4, c_1)$ denote the positions of the r_3-cells in column c_1 of the standard labeling of $\tilde{D}$ and D_{can}, respectively. Note that $r_{\max} \geqslant \tilde{r}_1 > r'_1 \geqslant r'_2$ since we have assumed $r_1 = r_2$. Since column c_2 has rows 1 through $r_{\max}$ occupied in D_{can} (and $\tilde{D}$), it must be the case that $r'_3 = r'_2 + 1$, so this implies that $r'_3 \leqslant r_{\max}$. By the emptiness of the rectangle described above, applying the canonical procedure to D_0 yields $r'_4 = r'_1 + 1$. Thus, if $\tilde{r}_1 > r'_1$, we must have $\tilde{r}_4 > r'_4$ as well. In this case, $\tilde{r}_4 \leqslant r_{\max}$ means $\tilde{x}_4$ must have a cell $\tilde{y}$ in $\tilde{D}$ in column c_2 to its right, where $\tilde{y}$ would have started as cell y above x_3 in D_0. Thus, we have shown that the triple x_4, x_3, y satisfies (a) and (b) as well as the property $\tilde{r}_4 > r'_4$, contradicting the maximality of our original triple choice.

Case 2(a): $r_1 < r_2$, $(r_2, c_1) \in D_0$. If D_0 had a cell in column c_2 strictly below row r_1, then the northeast property of D_0 gives $(r_1, c_2) \in D_0$ which contradicts the maximality of the triple x_1, x_2, and x_3. If D_0 had a cell in column c_2 strictly below row r_2 and weakly above row r_1, then again maximality is contradicted. Thus, column c_2 in D_0 has no cells below row r_2. Let $x_4 = (r_2, c_1) \in D_0$.

In this case, instead of showing that the presence of x_4 leads to contradiction, we show that its presence can be used to deduce the existence of a different triple satisfying (a) - (e). Observe that combining $\mathrm{cwt}(D_0)_{c_2} > \mathrm{cwt}(D_0)_{c_1}$ and $\mathrm{cwt}_{\leqslant r_2}(D_0)_{c_1} \geqslant 2$, we have

$$\mathrm{cwt}(D_0)_{c_2} \geqslant 3 + \mathrm{cwt}_{>r_2}(D_0)_{c_1}.$$

But since $\mathrm{cwt}(D_0)_{c_2} = 1 + \mathrm{cwt}_{>r_2}(D_0)_{c_2}$, this equation becomes

$$\mathrm{cwt}_{>r_2}(D_0)_{c_2} \geqslant 2 + \mathrm{cwt}_{>r_2}(D_0)_{c_1}. \tag{5.7}$$

This implies that there is some row index $t_3 > r_2$ such that $z_3 := (t_3, c_2) \in D_0$ while $(t_3, c_1) \notin D_0$. Let t_1 be the largest row index satisfying $r_2 \leqslant t_1 < t_3$ and $z_1 := (t_1, c_1) \in D_0$. Then by the northeast property of D_0, $z_2 := (t_1, c_2) \in D_0$ as well since $t_1 \geqslant r_2$. Now, z_1, z_2, z_3 is a triple that satisfies (a), (b) and (e) by construction, as well as (c) and (d) since the columns have not changed from our original triple.

Case 2(b): $r_1 < r_2$, $(r_3, c_1) \in D_0$. As in the previous case, we know that column c_2 is empty below row r_2, since otherwise the minimality of our triple x_1, x_2, and x_3 is violated. We give a sketch of this case since it is nearly identical to the previous case.

Here, we instead have $\mathrm{cwt}_{\leqslant r_3}(D_0)_{c_1} \geqslant 2$ and $\mathrm{cwt}(D_0)_{c_2} = 2 + \mathrm{cwt}_{>r_3}(D_0)_{c_2}$, so the analog of (5.7) becomes

$$\mathrm{cwt}_{>r_3}(D_0)_{c_2} \geqslant 1 + \mathrm{cwt}_{>r_3}(D_0)_{c_1}.$$

This can similarly be used to deduce the existence of a cell $z_3 \in D_0$ somewhere above row r_3, from which we obtain z_1 and z_2 in an identical way as in the previous case. The Properties (a) - (e) apply to the triple z_1, z_2, z_3 for the exact same reasons as the previous case.

Having shown that all conditions (a) - (e) are satisfied by either the triple x_1, x_2, x_3 or a related triple in the same columns, this concludes the proof of Proposition 5.2 and then Theorem 1.5.

□

6 Applications to lock diagrams

We are now equipped to apply our main theorems to lock diagrams, a special class of northeast diagrams. First, we specialize Theorem 1.3 to lock polynomials. To do this, we give one additional definition.

Definition 6.1. A **subcomposition** of a weak composition $\alpha \in \mathrm{Comp}_n$ is a weak composition $\beta \in \mathrm{Comp}_k$ for $k \leqslant n$ such that there exist indices $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ with $\beta_j = \alpha_{i_j}$ for all $1 \leqslant j \leqslant k$.

Corollary 6.2. *The lock polynomial $\mathfrak{K}_{\mathbb{L}(\alpha)}$ is monomial multiplicity-free if and only if α has no subcomposition of the form $(0, 0, \alpha_i, \alpha_j)$ for $\alpha_i > 1$ and $\alpha_j > 0$.*

Proof. (Proof of $\Rightarrow$) Suppose α has a subcomposition of the form $(0, 0, \alpha_i, \alpha_j)$ with $\alpha_i > 1$ and $\alpha_j > 0$. Then take x_1 to be the leftmost cell in row i and x_2 the rightmost cell in row j of $\mathbb{L}(\alpha)$. We will show that x_1 and x_2 are the cells described in Theorem 1.3 by verifying each of the required conditions:

(a) This is a result of the fact that $i < j$.

(b) Because $\alpha_i > 1$, x_1 is not the rightmost cell in its row. Then x_1 is strictly to the left of x_2.

(c) There are at least two empty rows below row i, so this condition is satisfied.

(d) Again there are at least two empty rows below row i, so this condition is satisfied.

(Proof of $\Leftarrow$) Suppose $\mathfrak{K}_{\mathbb{L}(\alpha)}$ is not monomial multiplicity-free. Then the cells x_1 and x_2 as described in Theorem 1.3 exist in $\mathbb{L}(\alpha)$. Let i be the row of x_1 and j the row of x_2. Because of condition (a), $i < j$. Because x_2 is a cell in row j, row j is certainly nonempty, so $\alpha_j > 0$. Furthermore, because x_1 is to the left of x_2, $\alpha_i > 1$. Finally, because of condition (d), there are at least two empty positions below the rightmost cell in row i of $\mathbb{L}(\alpha)$. As $\mathbb{L}(\alpha)$ is a southeast diagram, the corresponding rows must be entirely empty. Then α has the desired subcomposition. $\square$

Next we specialize Theorem 1.4 to lock diagrams.

Corollary 6.3. *$\mathcal{P}(\mathbb{L}(\alpha))$ is ranked if and only if for every pair $\alpha_i, \alpha_{i+k} \geqslant 2$ with $\alpha_{i+j} \in \{0, 1\}$ for all $1 \leqslant j < k$, we have $\#\{j : 1 \leqslant j < k$ and $\alpha_{i+j} = 1\} \geqslant \#\{j : j < i$ and $\alpha_j = 0\}$.*

Proof. As lock diagrams are a subset of northeast diagrams, we know that $\mathcal{P}(\mathbb{L}(\alpha))$ is ranked if and only if it does not contain x_1, x_2, x_3 satisfying the conditions in Theorem 1.4. We show that in a lock diagram, having this forbidden configuration is equivalent to having a pair of row weights $\alpha_i, \alpha_{i+k} \geqslant 2$ with $\alpha_{i+j} \in \{0, 1\}$ for all $1 \leqslant j < k$ such that $\#\{j : 1 \leqslant j < k$ and $\alpha_{i+j} = 1\} < \#\{j : j < i$ and $\alpha_j = 0\}$.

First assume that $\mathbb{L}(\alpha)$ has x_1, x_2, and x_3 satisfying conditions (a), (b), (c), and (d) of Theorem 1.4. In particular, consider such a configuration where x_2 and x_3 are chosen to have the minimum possible row index with respect to the position of x_1. As $\mathbb{L}(\alpha)$ is right-justified, x_2 and x_3 lie in the same row. Set $x_1 = (r_1, c_1)$, $x_2 = (r_2, c_1)$, and

$x_3 = (r_2, c_3)$ as in the statement of the theorem. Then all positions to the right of cells x_1 and x_2 in rows r_1 and r_2, respectively, are occupied by cells.

If c_n is the rightmost column of $\mathbb{L}(\alpha)$, then we know c_n satisfies condition (d), so there is at least one entirely empty row below row r_1. Thus, all columns $c \geqslant c_1$ satisfy condition (c). Therefore, we can instead consider the triple $x_1 = (r_1, c_{n-1})$, $x_2 = (r_2, c_{n-1})$, $x_3 = (r_2, c_n)$, which satisfies all of the conditions as well. If r_2 is not the first row of length at least 2 above row r_1, we can now reassign r_2 to be this row, and the conditions remain satisfied.

With this final choice of x_1, x_2, and x_3, we see that condition (d) implies that the number of length 1 rows lying strictly between rows r_1 and r_2 is less than the number of empty rows below row r_1. Setting $r_1 = i$ and $r_2 = i + k$, we conclude that in a lock diagram, having the forbidden configuration described in Theorem 1.4 implies that we have a pair of row weights $\alpha_i, \alpha_{i+k} \geqslant 2$ with $\alpha_{i+j} \in \{0, 1\}$ for all $1 \leqslant j < k$ such that $\#\{j : 1 \leqslant j < k \text{ and } \alpha_{i+j} = 1\} < \#\{j : j < i \text{ and } \alpha_j = 0\}$.

For the converse, assume $\mathbb{L}(\alpha)$ has rows i and $i+k$ such that $\alpha_i, \alpha_{i+k} \geqslant 2$, $\alpha_{i+j} \in \{0, 1\}$ for all $1 \leqslant j < k$, and $\#\{j : 1 \leqslant j < k \text{ and } \alpha_{i+j} = 1\} < \#\{j : j < i \text{ and } \alpha_j = 0\}$. Let c_n be the rightmost column of $\mathbb{L}(\alpha)$ and choose $x_1 = (r_i, c_{n-1})$, $x_2 = (r_{i+k}, c_{n-1})$, and $x_3 = (r_{i+k}, c_n)$. Then this triple of cells satisfies conditions (a) (b), (c), and (d) by construction. □

Specializing Theorem 1.5 to lock diagrams, we have the following result.

Corollary 6.4. *$\mathcal{P}(\mathbb{L}(\alpha))$ is bounded if and only if in the row weight vector α, the nonzero entries after the first 0 are weakly increasing.*

Proof. Every lock diagram is northeast, so $\mathcal{P}(\mathbb{L}(\alpha))$ is bounded if and only if it does not contain x_1, x_2, x_3 satisfying the conditions (a) - (e) outlined in Theorem 1.5. We show that in a lock diagram, having this forbidden configuration is equivalent to having some $i < j < k$ such that $\alpha_i = 0$ and $\alpha_j > \alpha_k > 0$.

As $\mathbb{L}(\alpha)$ is right-justified, we observe that the set of rows occupied by cells in a column c_k is always a subset of the set of rows occupied by cells in column c_{k+1}. Thus, condition (c) is true of any lock diagram where we choose columns $c_1 < c_2$ such that column $c_2 - 1$ is not identical to column c_2. Thus, condition (c) holds exactly when there is some pair of indices j, k with $\alpha_j \neq \alpha_k$.

Furthermore, in a lock diagram, the existence of x_1, x_2, x_3 satisfying (a) and (b) is equivalent to the existence of an $\alpha_j > 2$ and an $\alpha_k > 1$ for some $k > j$, since we can always take $r_2 = r_1 = j$ and $r_3 = k$. Adding condition (e) to this configuration, we see it is satisfied if and only if (r_3, c_1) is empty, meaning that row r_1 has a greater weight than row r_3 or $\alpha_j > \alpha_k$.

Finally, condition (d) in a lock diagram is equivalent to saying that there is an empty row below row $r_1 = j$. That is, there is i with $i < j$ and $\alpha_i = 0$. Thus, we conclude that for lock diagrams, the forbidden configuration for boundedness in northeast Kohnert posets is equivalent to having some $i < j < k$ such that $\alpha_i = 0$ and $\alpha_j > \alpha_k > 0$. □

We can now combine Corollaries 6.4 and 6.3 to give a classification of lock diagrams that yield ranked and bounded Kohnert posets.

Corollary 6.5. *$\mathcal{P}(\mathbb{D}(\alpha))$ is ranked and bounded if and only if α satisfies the following: If i is the smallest index such that $\alpha_i = 0$, then for all $j > i$, we have $\alpha_j > 1$ if and only if α_j is the last nonzero entry in α.*

In other words, $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ has the following form:

- α_1 through α_{i-1} are greater than 0,
- $\alpha_i = 0$,
- α_{i+1} through $\alpha_{\ell-1}$ are either 0 or 1, and
- $\alpha_\ell > 0$.

An example of such a diagram can be found in Figure 15.

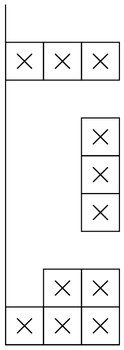


Figure 15: An example of a lock diagram that yields a ranked and bounded Kohnert poset.

Proof. For $\mathcal{P}(\mathbb{D}(\alpha))$ to be bounded, recall that α must be such that the nonzero entries that follow the first $\alpha_i = 0$ are weakly increasing. Thus, α is such that α_1 through α_{i-1} are greater than 0, $\alpha_i = 0$, and for each $\alpha_j \neq 0$ with $j > i$ and α_k being the next nonzero entry following α_j, we have $\alpha_j \leqslant \alpha_k$.

Recall also that for $\mathcal{P}(\mathbb{D}(\alpha))$ to be ranked, we must have that for every pair $\alpha_j, \alpha_{j+k} \geqslant 2$, the number of 1s between them must be greater than or equal to the number of 0s below α_j. Considering α for which $\mathcal{P}(\mathbb{D}(\alpha))$ is bounded, we observe that this condition holds by default for $j < i$. For $j > i$, if $\alpha_j, \alpha_{j+k} \geqslant 2$, then since $\mathcal{P}(\mathbb{D}(\alpha))$ is bounded, there cannot be any 1s between α_j and α_{j+k}. But we know that $\alpha_i = 0$ and $i < j$, so we conclude that if $\mathcal{P}(\mathbb{D}(\alpha))$ is both ranked and bounded, then we cannot have two entries $\alpha_j, \alpha_{j+k} \geqslant 2$ if $j > i$.

Thus, at most one nonzero entry after α_i is greater than 1, and by the weakly increasing condition for boundedness, this entry must be the final one, α_ℓ. We conclude that if $\mathcal{P}(\mathbb{D}(\alpha))$ is both ranked and bounded, then it has the aforementioned form. ☐

Acknowledgements

This project began during the 2024 Graduate Research Workshop in Combinatorics, which was supported by the University of Wisconsin-Milwaukee, the Combinatorics Foundation, and the National Science Foundation (NSF Grant DMS-1953445). B. A. Castellano was supported by an NSF Graduate Research Fellowship. A. Bingham was partially supported by the grant DST/INT/RUS/RSF/P-41/2021 from the Department of Science & Technology, Govt. of India, and FONDECYT-ANID grant 3250472. We would like to thank Kim Harry, Joakim Jakovleski, and Chelsea Sato for their early contributions to this project. We would also like to thank Nick Mayers for helpful discussions related to this work.

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