Words with factor complexity $2n + 1$ and minimal critical exponent

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Abstract

Let the word $\mathbf{G}$ be the fixed point of the morphism γ sending 0 to 01, 1 to 2, and 2 to 02. In 2019, Shallit and Shur showed that $\mathbf{G}$ has factor complexity $2n+1$. They also showed that $\mathbf{G}$ has critical exponent $\mu = 2 + \frac{1}{\lambda^2-1} = 2.4808726\cdots$, where $\lambda = 1.7548777$ is the real zero of $x^3 - 2x^2 + x - 1 = 0$. They conjectured that this was the least possible critical exponent among the words with factor complexity $2n + 1$. We confirm their conjecture. We anticipate that our method, including an intricate case analysis by computer, will have wider application.

Key words: combinatorics on words, factor complexity, critical exponent, computer proof, repetition threshold

Mathematics Subject Classifications: 68R15

1 Introduction

The critical exponent of a word is the supremum of the exponents of its factors. (Definitions of basic terms in combinatorics on words, such as factors, exponents, etc., follow in the next section.) In recent years a large number of papers have been written answering questions of the following form: Given a class $\mathcal{L}$ of infinite words, what is the least critical exponent of a member of $\mathcal{L}$?

The various steps in the solution of Dejean's conjecture by several authors [15, 23, 22, 8, 4, 14, 25] answered this question for various classes of the form $\mathcal{L} = \Sigma_k^\omega$ where $\Sigma_k = \{0, 1, 2, \ldots, k-1\}$. Since the resolution of Dejean's conjecture in 2011 [14, 25] researchers have explored versions of this question where $\mathcal{L}$ is variously taken to be, e.g., the class of Sturmian words [5], rich words over certain alphabets [9, 10], balanced words [24, 17], or complementary symmetric Rote words [16].

For example, the minimum critical exponent for Sturmian words, found by Carpi and De Luca[5], is

$$2 + \varphi = 3.618033988749\cdots$$

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where φ is the golden ratio. This is realized by the Fibonacci word, the fixed point of the morphism Φ sending 0 to 01 and 1 to 0. One starting point for the study of Sturmian words and their generalizations is the relevant chapter in Allouche and Shallit's book [1].

Words of factor complexity $2n+1$ are considered in the paper of Cassaigne, Labbé, and Leroy [6]. Such words are connected to the well-studied Arnoux-Rauzy words: the Arnoux-Rauzy words over a k-letter alphabet have factor complexity $(k-1)n+1$. Note that the Arnoux-Rauzy words over a binary alphabet are just the Sturmian words. The general critical exponent for Arnoux-Rauzy words was given by Dvořáková, Opočenská, and Pelantová [18]. In the present paper, we find the minimal critical exponent for the class of words with factor complexity $2n+1$. This confirms the 2019 conjecture of Shallit and Shur [29].

To find the minimal critical exponent for binary rich words, it was necessary to prove the following structure theorem:

Theorem 1. *[9] Let $\mathbf{w} \in \Sigma_2^\omega$ be a 14/5-power-free rich word. For every $n \geqslant 1$, a suffix of $\mathbf{w}$ has the form $f(\phi^n(w_n))$ or $f(g(\phi^n(w_n)))$ for some word $w_n \in \Sigma_3^\omega$ where*

$$\begin{aligned} f(0) &= 0 & g(0) &= 011 & \phi(0) &= 01 \\ f(1) &= 01 & g(1) &= 0121 & \phi(1) &= 02 \\ f(2) &= 011 & g(2) &= 012121 & \phi(2) &= 022. \end{aligned}$$

Paraphrasing, if we let $\mathbf{H}$ be the fixed point of ϕ, then the infinite 14/5-power-free rich binary words look like one of $h_1(\mathbf{H})$ and $h_2(\mathbf{H})$, where $h_1 = f$, and $h_2 = f \circ g$.

In the present paper, we prove the analogous structure theorem for the infinite words with factor complexity $2n+1$, and not containing r-powers for $r \geqslant 5/2$. However, instead of the two possible morphisms h_1 and h_2 in the above-mentioned theorem, there are several possible morphisms. (See Figures 3 and 4.) The case analysis is therefore complex. (Interestingly, Shallit and Shur [29] prove that the minimum critical exponent for the infinite words with factor complexity $2n$ is exactly $5/2$.)

For several decades, the use of computer backtracking has been a standard tool in combinatorics on words. More recently, however, this area has seen a number of trends where computer-assisted proofs come closer to being 'computer proofs'. One such trend has been seen in the continued success of the Walnut computer package [21, 27], which proves properties of automatic sequences. Over 100 papers using Walnut are listed by Shallit [28]. Of relevance to the current paper, in many situations Walnut can check the critical exponent of the fixed point of a morphism, as in a recent example by Baranwal et al. [3].

Another instance of the intensive use of computation in proofs has been seen in papers using the 'template method' or its variations [2, 12, 13, 20, 26, 6], where one recursively generates ancestors of patterns of interest that could potentially appear in the fixed point of some morphism.

The present paper presents what is, to our knowledge, a new sort of application of computation to combinatorics on words. In analyzing the subject of this paper, a great proliferation of subcases arose, branching to a depth of, for example, subcase A.1.2.2.2

(5 levels of subcases). Some cases involved branching into as many as 7 subcases. Each subcase involved backtrack searches and might have one of several resolutions, possibly including the spawning of new subcases.

Having started with a case analysis by hand, at a certain point we realized that the spawning and resolution of subcases would be best handled by computer. In the end, we wrote a computer program which in turn wrote the desired proof.

2 Word preliminaries

An *alphabet* is a finite set with elements known as *letters*. In what follows, we use the alphabet $\Sigma_3 = \{0, 1, 2\}$. The set of finite words over Σ_3 is denoted by Σ_3^*, and can be regarded as the free semigroup generated by the letters 0, 1, and 2. The length of a word u is the number of occurrences of letters in it and is denoted by $|u|$. Thus, for example, $|01202| = 5$. For each $a \in \Sigma_3$, we denote the number of occurrences of letter a in u by $|u|_a$. Thus, for example, $|01202|_0 = 2, |01202|_1 = 1, |01202|_2 = 2$.

If u, $v \in \Sigma_3^*$, the *concatenation* uv of u and v consists of the letters of u followed by the letters of v. Thus if $u = 012$ and $v = 02$, we have $uv = 01202$. We say that u is a *prefix* of uv and v is a suffix of uv. We also say that uv is a *right extension* of u *by* v. If u, v, $w \in \Sigma_3^*$, we say that v is a *factor* of uvw. If word y is not a factor of word z we say that z *omits* the factor y. Thus 01, 120, and 02 are respectively a prefix, factor, and suffix of 01202, and 01202 is a right extension of 01 by 202. Word 01202 omits the factor 10.

Suppose $u = u_0u_1u_2 \cdots u_{n-1}$ where each $u_i \in \Sigma_3$. We say that positive integer p is a *period* of u if $u_i = u_{i+p}$ whenever $0 \leqslant i \leqslant i + p \leqslant n - 1$. In this case we say that u is an r *power*, where $r = n/p$. Thus 3 is a period of the word 01201201, which is an 8/3 power. Call a word *low* if it contains no r power for $r \geqslant 5/2$.

A *morphism* on Σ_3 is a semigroup homomorphism of Σ_3^*. A morphism is determined by its images on letters. Of particular interest is the morphism $\gamma : \Sigma_3^* \to \Sigma_3^*$ generated by

$$\begin{aligned}\gamma(0) &= 01\\ \gamma(1) &= 2\\ \gamma(2) &= 02\end{aligned}$$

For conciseness we record a morphism h as $h = [h(0), h(1), h(2)]$. Thus $\gamma = [01, 2, 02]$. If $|h(a)| > 0$ for each $a \in \Sigma_3$, we say that h is *non-erasing*. We use exponentiation to denote iteration of a morphism. Thus $h^2(u) = h(h(u))$, $h^3(u) = h(h(h(u)))$, etc.

Suppose $u = u_0u_1u_2 \cdots u_{n-1}$ where each $u_i \in \Sigma_3$. The *reverse* of u is the word $u^R = u_{n-1} \cdots u_2u_1u_0$. The reverse of the morphism $h = [h(0), h(1), h(2)]$ is defined to be $h^R = [(h(0))^R, (h(1))^R, (h(2))^R]$. Thus, for example, $\gamma^R = [10, 2, 20]$. We see that for a word u and the morphism h we have $(h(u))^R = h^R(u^R)$.

As well as the finite words, we consider the right-infinite words over Σ_3. Formally, such a word is a sequence $\mathbf{u} = \{u_n\}_{n=0}^{\infty}$ where each u_i is a single letter. We use bold-face letters to distinguish right-infinite words from finite words. For each non-negative integer

n, the length-n prefix of $\mathbf{u}$ is the finite word $u_0u_1u_2\cdots u_{n-1}$. We say that $v \in \Sigma_3^*$ is a factor of $\mathbf{u}$ if it is a factor of some finite prefix of $\mathbf{u}$. A factor of $\mathbf{u}$ which appears infinitely often is said to be *recurrent*. Call a right-infinite word *low* if each of its finite prefixes is low.

We extend the notion of suffix to right-infinite words; a suffix of $\mathbf{u}$ is a right-infinite word $\mathbf{u}' = \{u_{n+i}\}_{n=0}^{\infty}$ for some non-negative integer i. We write $\mathbf{u} = u\mathbf{u}'$ where u is the prefix of $\mathbf{u}$ of length i.

Let $\mathbf{u} \in \Sigma_3^\omega$. Let b be a non-empty finite factor of $\mathbf{u}$ which is recurrent. A word x is a *return word* for b in $\mathbf{u}$ if xb is a factor of $\mathbf{u}$ and contains exactly two occurrences of b, one as a prefix and one as a suffix.

Lemma 2. *Suppose b is a recurrent factor of $\mathbf{u}$ where $|b| \leqslant 3$ and $\mathbf{u}$ is low. Let x be a return word for b. Then $|x| \geqslant |b|$.*

Proof. Suppose that $|x| < |b|$. We thus have $(|x|, |b|) \in \{(1, 2), (1, 3), (2, 3)\}$. Since $|x|$ is a period for xb, word xb is an $|xb|/|x|$ power, i.e, a k power where $k \in \{3/1, 4/1, 3/2\}$. This contradicts the fact that $\mathbf{u}$ is low. □

It follows that if b is recurrent in $\mathbf{u}$, $|b| \leqslant 3$, and $\mathbf{u}$ is low, then a suffix of $\mathbf{u}$ can be concatenated from the return words of b. In this case we refer to the return words of b as *b-blocks* of $\mathbf{u}$.

The set of all right-infinite words over Σ_3 is denoted by Σ_3^ω. Right-infinite words arise in various ways: If $u = u_0u_1\cdots u_{p-1}$ where p is positive and the u_i are letters, then $u^\infty = \{u_i\}_{i=0}^{\infty}$ where $u_i = u_{i \bmod p}$. Thus, for example,

$$(012)^\omega = 012012012012\cdots.$$

Another way a right-infinite word arises is via certain morphisms. Let h be a morphism such that a is a prefix of $h(a)$ for some $a \in \Sigma_3$. By induction, $h^{n-1}(a)$ is a prefix of $h^n(a)$ for each n. If $|h^n(a)|$ increases without bound, we can define $h^\omega(a)$ to be the unique right-infinite word having $h^n(a)$ as a prefix for each n. For a right-infinite word $\mathbf{u}$ over Σ_3, we define $h(\mathbf{u})$ to be the right-infinite word having $h(u)$ as a prefix for each prefix u of $\mathbf{u}$; thus $h^\omega(a)$ is a fixed point of h. In this paper we are particularly interested in the fixed point $\mathbf{G} = \gamma^\omega(0)$.

A third (non-constructive) way to obtain right-infinite words is via König's lemma [19]. Here is one formulation of König's lemma in the combinatorics on words context: A subset $L \subseteq \Sigma_3^*$ is said to be a *factorial language* if whenever ℓ is in L, every factor of ℓ is also in L.

Lemma 3 (König's lemma)**.** *Let $L \subseteq \Sigma_3^*$ be an infinite factorial language. Then there is a right-infinite word $\mathbf{u} \in \Sigma_3^\infty$ such that every prefix of $\mathbf{u}$ is in L.*

If $\mathbf{u} \in \Sigma_3^\omega$, the *factor complexity* of $\mathbf{u}$ is the function counting the length n factors of $\mathbf{u}$. Thus, for example, if $\mathbf{u} = (012)^\omega$, then the factor complexity of $\mathbf{u}$ is $C(n)$ where

$$C(0) = 1$$
$$C(n) = 3, n \geqslant 1.$$

Shallit and Shur [29] have shown that $\mathbf{G}$ has factor complexity $2n+1$.

We say that $\mathbf{w} \in \Sigma_3^\omega$ is *eligible* if it is low and has factor complexity $2n+1$.

The *critical exponent* of a finite or right-infinite word is the supremum of those r such that a factor of the word is an r power. Shallit and Shur [29] have shown that $\mathbf{G}$ has critical exponent $\mu = 2 + \frac{1}{\lambda^2-1} = 2.4808726\cdots$, where $\lambda = 1.7548777$ is the real zero of $x^3 - 2x^2 + x - 1 = 0$. They conjecture that this is the least possible critical exponent among words with factor complexity $2n+1$.

We say that the right-infinite word $\mathbf{u} \in \Sigma_3^\omega$ has *letter frequencies* ρ_0, ρ_1, ρ_2, if for all $\epsilon_1 > 0$ there is an N_1 such that if u is a factor $\mathbf{u}$ of length at least N_1, then

$$\rho_a - \epsilon_1 < \frac{|u|_a}{|u|} < \rho_a + \epsilon_1, \text{ for } a \in \Sigma_3. \tag{1}$$

Using standard methods, it can be shown that $\mathbf{G}$ has letter frequencies. (For values of the ρ_a see the entry for sequence A287104 in the On-Line Encyclopedia of Integer Sequences; the word $\mathbf{G}$ is a recoding of this sequence.)

3 Main Theorem

Theorem 4 (Main Theorem). *Let $\mathbf{w} \in \Sigma_3^\omega$ have factor complexity $2n+1$. Then the critical exponent of $\mathbf{w}$ is at least $\mu = 2 + \frac{1}{\lambda^2-1} = 2.4808726\cdots$, where $\lambda = 1.7548777$ is the real zero of $x^3 - 2x^2 + x - 1 = 0$.*

A key ingredient in the proof of this main theorem is a structure theorem.

Theorem 5 (Structure Theorem). *Let $\mathbf{w} \in \Sigma_3^\omega$ be eligible. There is a non-erasing morphism h such that either*

- *for every factor g of $\mathbf{G}$, the word $h(g)$ is a factor of $\mathbf{w}$ or*
- *for every factor g of $\mathbf{G}$, the word $h(g^R)$ is a factor of $\mathbf{w}$.*

The main theorem follows from this lemma:

Lemma 6. *Let $\mathbf{u} \in \Sigma_3^\omega$ have irrational critical exponent ν, with $2 < \nu < 3$. Suppose that $\mathbf{u}$ has letter frequencies ρ_0, ρ_1, ρ_2. Let $\mathbf{w} \in \Sigma_3^\omega$ and let $h : \Sigma_3^* \to \Sigma_3^*$ be a non-erasing morphism. Suppose that for each factor g of $\mathbf{u}$, either $h(g)$ or $h(g^R)$ is a factor of $\mathbf{w}$. Then the critical exponent of $\mathbf{w}$ is at least ν.*

Remark 7. The use of the alphabet Σ_3 and the restriction that $2 < \nu < 3$ are not essential to this lemma, but simplify the notation in its proof.

Proof of main theorem. Let $\mathbf{w} \in \Sigma_3^\omega$ have factor complexity $2n+1$. Recall that $\mu = 2 + \frac{1}{\lambda^2-1} = 2.4808726\cdots$, where $\lambda = 1.7548777$ is the real zero of $x^3 - 2x^2 + x - 1 = 0$. Thus $\mu < 5/2$. If $\mathbf{w}$ is not low, then the critical exponent of $\mathbf{w}$ is at least $5/2 > \mu$, and we are done.

Suppose then that $\mathbf{w}$ is low. By the structure theorem, there is a non-erasing morphism h such that for every factor g of $\mathbf{G}$, either $h(g)$ is a factor of $\mathbf{w}$ or $h(g^R)$ is a factor of $\mathbf{w}$. (This is actually weaker than the conclusion of the structure theorem). The word $\mathbf{G}$ has letter frequencies. It is easy to show that μ is irrational. By Lemma 6, then, the critical exponent of $\mathbf{w}$ is at least μ. □

4 Eligible words and their factors

We defer the proof of the following lemma.

Lemma 8. *Suppose $\mathbf{w} \in \Sigma_3^\omega$ is eligible. Then every suffix of $\mathbf{w}$ contains letters 0, 1, and 2.*

Suppose that $\mathbf{w}$ is eligible. Consider the directed graph D with vertex set Σ_3, and with a directed edge xy for $x, y \in \Sigma_3$ exactly when xy is a factor of $\mathbf{w}$. Then there is a directed path on D labeled with the letters of $\mathbf{w}$. Since $\mathbf{w}$ has exactly 5 length-2 factors, the graph D has exactly 5 edges.

By the previous lemma, the graph D must be strongly connected; that is, there is a directed path between any two vertices of D. Therefore, D must have at least 3 non-loop edges, and if it has exactly three non-loop edges, they form a directed cycle.

It is convenient to keep track of words which are not factors of $\mathbf{w}$: Let $F = \{u \in \Sigma_3^* : u \text{ is not a factor of } \mathbf{w}\}$. To prove the structure theorem, it suffices to consider two cases involving F.

- If D has exactly three non-loop edges (forming a directed cycle) then, permuting Σ_3 if necessary, we may assume that
$$\{01, 12, 20, 00\} \subseteq F.$$
- If D has more than three non-loop edges, permuting the alphabet if necessary,we may assume that
$$\{11, 22\} \subseteq F.$$

The following technical lemmas give sufficient conditions for concluding that the structure theorem holds:

Lemma 9 (Morphism Lemma)**.** *Suppose that $\mathbf{w} \in \Sigma_3^\omega$ is low. Suppose that a suffix of $\mathbf{w}$ has the form $h(\mathbf{u})$ where $h = [a, b, c]$, some $\mathbf{u} \in \Sigma_3^\omega$ and $a, b, c \in \Sigma_3^+$ such that*

1. *The word b is a prefix of a, which is a prefix of c.*
2. *We have $|b| \geqslant |c|/2$.*
3. *The words b and c have a common suffix s such that $|s| \geqslant |b|/2$.*

If g is any factor of $\mathbf{G}$, then $h(g)$ is a factor of $\mathbf{w}$.

Lemma 10 (Dual Morphism Lemma)**.** *Suppose that* $\mathbf{z} \in \Sigma_3^\omega$ *is low. Suppose that a suffix of* $\mathbf{z}$ *has the form* $h(\mathbf{u})$ *where* $h = [a, b, c]$ *for some* $\mathbf{u} \in \Sigma_3^\omega$ *and* $a, b, c \in \Sigma_3^+$ *such that*

1. *The word* b *is a suffix of* a*, which is a suffix of* c*.*

2. *We have* $|b| \geqslant |c|/2$*.*

3. *The words* b *and* c *have a common prefix* p *such that* $|p| \geqslant |b|/2$*.*

If g *is any factor of* $\mathbf{G}$*, then* $h(g^R)$ *is a factor of* $\mathbf{z}$*.*

Lemma 11 (First Parsing Lemma)**.** *Let* $S \subseteq \Sigma_3^*$ *and let* $b \in \Sigma_3^*$*,* $|b| \leqslant 3$*. Let* d *be a non-negative integer. Suppose that every low word of length* $d + 1$ *contains a factor from* $S \cup \{b\}$*. Then if* $\mathbf{w}$ *is an eligible right-infinite word which omits the words of* S*, a suffix of* $\mathbf{w}$ *can be concatenated from* b*-blocks of the form* bu*, where* $|u| \leqslant d$*.*

Lemma 12 (Second Parsing Lemma)**.** *Let* $S, B \subseteq \Sigma_3^*$*. Let* $\mathbf{w}$ *be an eligible right-infinite word which omits the words of* S*, such that* $\mathbf{w}$ *is concatenated from words of* B*. Let* $s \in B$ *have the property that for all* $r, t \in B$*, either* rst *is not low, or* rst *contains a factor from* S*. Then a suffix of* $\mathbf{w}$ *is concatenated from words of* $B - \{s\}$*.*

5 Proof of the structure theorem

We prove the structure theorem via an intricate case analysis. Each case is determined by a set $S \subseteq F$, where F is the set of ternary words which are not factors of $\mathbf{w}$. As remarked in the previous section, it suffices to resolve the subcases $S = \{01, 12, 20, 00\}$ and $S = \{11, 22\}$.

In a given case, labelled by a set S, we proceed in one of three ways. We either

1. Show that the conditions of the morphism lemma or the dual morphism lemma hold, so that the case is resolved;

2. Show that the case cannot arise, so that the case is resolved;

3. Partition the case into subcases.

The proof is thus carried out recursively, in a depth-first way. In situation 1 or 2, a case is resolved. In situation 3, new subcases are added. If at some point all subcases have been resolved (and Hercules beats the hydra) the structure theorem has been proved.

Let a set S of forbidden factors be given. Say that a word w is *good* if it is low and has no factor in S. The concrete recipe for carrying out 1, 2, and 3 above is as follows.

A. If $T \subseteq S$, and case T has been previously resolved, then S is resolved.

B. Perform a backtrack search, looking for a good word w of length 250. Word w is a hypothetical prefix of $\mathbf{w}$. If no such word w exists, the case cannot arise.

C. If we find w of length 250, count the factors of w of each length n with $1 \leqslant n \leqslant 20$, until an n is found such that w has more than $2n+1$ factors of length n.

   a. If we find n such that w has more than $2n+1$ factors of length n, then we classify each such factor s as *needed* or *unneeded*. A factor s is needed if a backtrack shows that no low word of length 250 omits all the factors of $S \cup \{s\}$; it is unneeded otherwise.

      i. If there are more than $2n+1$ needed factors then no low word of length 250 can be a prefix of $\mathbf{w}$, since the condition on the complexity of $\mathbf{w}$ is violated. The case where $\mathbf{w}$ omits factors in S cannot arise.

      ii. Otherwise, for each unneeded factor s we form a subcase, replacing S by $S \cup \{s\}$.

   b. If no such n is found, we look for a morphism satisfying the conditions of the morphism lemma or the dual morphism lemma. To do this

      i. Consider the good words b with $|b| \leqslant 3$.

      ii. Suppose that for some positive integer $d \leqslant 200$, every good word of length $d+1$ has b as a factor. This can be checked by a finite search.

      iii. Then consider the set $B = \{bu \in \Sigma_3^* : bub$ is good, $|u| \leqslant d$, and whenever p is a prefix of bub such that b is a suffix of p, then $p = b$ or $p = bub\}$. By the first parsing lemma, $\mathbf{w}$ is concatenated from words of B.

      iv. We form a morphism using the blocks in B if either:

         - Set B contains exactly 3 b-blocks;
         - Set B contains exactly 4 b-blocks, including a b-block s such that for all $r, t \in B$, word rst is not good. In this case we replace B by $B - \{s\}$. By the second parsing lemma, a final segment of $\mathbf{w}$ is concatenated from the b-blocks of B.

   We form a morphism from the three b-blocks of B, and consider this morphism and its conjugates. If one of the considered morphisms satisfies the conditions of the morphism lemma or the dual morphism lemma the case is resolved.

Remark 13. The 'hard-wired' constants 250 (for backtrack search), 3 (for $|b|$), 20 (depth of the complexity check) are all *ad hoc*. They were originally taken to be larger, but then tuned.

Proof of the structure theorem. A python implementation of the above scheme in SageMath 9.3 (on a Microsoft Surface with Intel(R) Core(TM) i5-1035G4 CPU at 1.10GHz) resolved the case $S = \{11, 22\}$ in about 36 seconds, and the case $S = \{01, 12, 20, 00\}$ in about 9 seconds. The code is available at https://github.com/drjamescurrie-ai/RACE-3/tree/main. □

Remark 14. In fact, the implementation mentioned above kept track of sets S and the reasoning at each of its steps, and produced 'human readable' proofs. We give the reasoning that was output for the case $S = \{01, 12, 20, 00\}$ in an appendix. The implementation

$h(0)$	$h(1)$	$h(2)$
1210202	12102	1210202102
1210202102	1210202	121020210202
021201212	0212012	021201212012
0201212012	0201212	020121201212
0201212	02012	0201212012
201021010	2010210	201021010210
0102121021	0102121	010212102121
0102121	01021	0102121021
102012020	1020120	102012020120
21021102211021102	2102110221102	2102110221102110221102

Figure 1: Morphisms h satisfying the Morphism Lemma arising in the computer implementation.

also kept track of the morphisms h arising. (See Figurse 1 and 2.) It may be the case that for some of these morphisms, the word $h(\mathbf{G})$ is eligible and has critical exponent μ.

6 Proofs of Lemmas

Proof of Lemma 6. The critical exponent of $\mathbf{u}$ is irrational, and therefore is not realized by any of its finite factors. Thus for all $\epsilon_2 > 0$ and any positive integer N, the word $\mathbf{u}$ has a factor uuu' such that u' is a prefix of u with $|u'| > N$ and

$$\frac{|uuu'|}{|u|} > \nu - \epsilon_2. \tag{2}$$

Since h is non-erasing, we have

$$\sum_{a=0}^{2} |h(a)|\rho_a \geqslant \sum_{a=0}^{2} \rho_a = 1$$

and is non-zero. Then

$$\lim_{\epsilon \to 0^+} \frac{\sum_{a=0}^{2} |h(a)|(\rho_a - \epsilon)}{\sum_{a=0}^{2} |h(a)|(\rho_a + \epsilon)} = 1$$

Given $\epsilon > 0$, choose $\epsilon_1 > 0$ such that

$$\frac{\sum_{a=0}^{2} |h(a)|(\rho_a - \epsilon_1)}{\sum_{a=0}^{2} |h(a)|(\rho_a + \epsilon_1)} > 1 - \epsilon.$$

Choose N_1 such that when $|u'| > N_1$, then

$$\rho_a - \epsilon_1 < \frac{|u|_a}{|u|} < \rho_a + \epsilon_1, \text{ for } a \in \Sigma_3.$$

10212	0212	0210212
2120210	20210	202120210
121201210	1201210	120121201210
1201212010	1212010	121201212010
1212010	12010	1201212010
1210120	10120	101210120
202012021	2012021	201202012021
21020	1020	1021020
0210202101	0202101	020210202101
0202101	02101	0210202101
1012102	12102	121012102
101021012	1021012	102101021012
0101202	01202	0120101202
0120101202	0101202	010120101202
010120102	0120102	012010120102
221102210	21102210	2110221102210

Figure 2: Morphisms h satisfying the Dual Morphism Lemma arising in the computer implementation.

Given $\epsilon_2 > 0$, choose a factor $g = uuu'$ of $\mathbf{u}$ with $|u'| > N_1$ so that (2) holds. (If $h(g^R)$ is a factor of $\mathbf{w}$, we replace g by g^R, redefining u and u' accordingly.)

The word $h(\mathbf{u})$ contains the factor $h(uuu') = h(u)h(u)h(u')$, and $h(u')$ is a prefix of $h(u)$.

Also

$$\begin{aligned}
\frac{|h(uuu')|}{|h(u)|} &= \frac{\sum_{a=0}^{2} |h(a)||uuu'|_a}{\sum_{a=0}^{2} |h(a)||u|_a} \\
&\geqslant \frac{\sum_{a=0}^{2} |h(a)|(\rho_a - \epsilon)|uuu'|}{\sum_{a=0}^{2} |h(a)|(\rho_a + \epsilon)|u|} \\
&= \frac{|uuu'|}{|u|} \frac{\sum_{a=0}^{2} |h(a)|(\rho_a - \epsilon)}{\sum_{a=0}^{2} |h(a)|(\rho_a + \epsilon)} \\
&> (\nu - \epsilon_2)(1 - \epsilon).
\end{aligned}$$

We see that $\mathbf{w}$ contains factors with exponent arbitrarily close to ν. The result follows. $\square$

Proof of Lemma 8. Let $\mathbf{s}$ be a suffix of $\mathbf{w}$. It suffices to show that $\mathbf{s}$ contains letter 0.

Suppose $\mathbf{s}$ contains only letters 1 and 2. It must contain factors 12 and 21; otherwise $\mathbf{s}$ would end with 1^ω or 2^ω and therefore contain 5/2 powers.

Suffix $\mathbf{s}$ must also contain factor 11 (and, symmetrically, factor 22): If not, assume $\mathbf{s}$ starts with 1, replacing $\mathbf{s}$ with one of its suffixes if necessary. Then $\mathbf{s}$ is concatenated from the 1-blocks of $\mathbf{w}$. Since 222 is a 3 power, the only possible 1-blocks of $\mathbf{w}$ are 12 and

122. The 1-block 12 can only be used once in this concatenation, or else it appears in the context $2\underline{12}12$, which is a $5/2$ power. Then a suffix is $(122)^\omega$, which contains $5/2$ powers. This is a contradiction. Thus 11 must indeed occur.

So far we have shown that $\mathbf{s}$ contains factors 11, 12, 21, and 22. Since $\mathbf{w}$ contains $2(2)+1=5$ factors of length 2, there is exactly one length-2 factor of $\mathbf{w}$ containing a 0. It follows that the only 0 in $\mathbf{w}$ is its first letter. We thus see that $\mathbf{w}$ contains no factor of the form $a0$ where $a \in \Sigma_3$.

Consider the following set of 10 length-4 words:

$$S = \{1121, 1122, 1211, 1212, 1221, 2112, 2121, 2122, 2211, 2212\}.$$

Let s and t be distinct words of S. Suppose u is a low word over Σ_3 such that the set of factors of u does not include any of 00, 10, 20, s, or t. For each of the $\binom{10}{2}$ possibilities for s and t, a backtrack search shows that the length of u is at most 39. It then follows that $\mathbf{w}$ contains 9 out of 10 words of S as factors. In addition,the word $\mathbf{w}$ contains a length-4 factor starting with 0. However, the word $\mathbf{w}$ is eligible, and cannot contain 10 length-4 factors. This is a contradiction. ☐

We use two preliminary lemmas in the proof of the morphism lemma (Lemma 9).

Lemma 15. *Suppose that $\mathbf{w} \in \Sigma_3^\omega$ is low. Let $a, b, c \in \Sigma_3^+$, and suppose that a suffix of $\mathbf{w}$ has the form $h(\mathbf{u})$, some $\mathbf{u} \in \Sigma_3^\omega$, where $h = [a, b, c]$. Suppose further that*

1. *The word b is a prefix of a, which is a prefix of c.*
2. *We have $|b| \geqslant |c|/2$.*
3. *The words b and c have a common suffix s such that $|s| \geqslant |b|/2$.*

Then $\mathbf{u}$ doesn't contain any of the factors 00, 11, 21, and 22.

Proof. We treat the factors one at a time. For each factor we rule out 1-letter right extensions of that factor.

Factor 11: If 11 is a factor of $\mathbf{u}$, then so is a factor v, where $v = 110$, $v = 111$, or $v = 112$. In each case, $h(v)$ has the prefix bbb which is a 3 power. Since $\mathbf{w}$ doesn't contain 3 powers, the factor 11 doesn't occur in $\mathbf{u}$.

Factor 00: If 00 is a factor of $\mathbf{u}$, then so is a factor v, where $v = 000$, $v = 001$, or $v = 002$. In each case $h(v)$ has the prefix aab since b is a prefix of a and c. However, aab is an r power with $r \geqslant 5/2$, since b is a prefix of a and $|b| \geqslant |c|/2 \geqslant |a|/2$.

Factor 21: If 21 is a factor of $\mathbf{u}$, then so is a factor v, where $v = 211$, $v = 210$, or $v = 212$. In each case, $h(v)$ has the prefix cbb, since b is a prefix of a and c. This has the suffix sbb, since s is a suffix of c. But sbb is an r power with $r \geqslant 5/2$.

Factor 22: If 22 is a factor of $\mathbf{u}$, then so is a factor v, where $v = 220$, $v = 221$, or $v = 222$. In each case, $h(v)$ has the prefix ccb, which is an r power with $r \geqslant 5/2$, since b is a prefix of c and $|b| \geqslant |c|/2$. ☐

It is convenient to work with $\gamma^2 = [012, 02, 0102]$ rather than simply γ. If $h = [a, b, c]$, then $h \circ \gamma^2 = [abc, ac, abac]$, and each image of a letter under $h \circ \gamma^2$ ends in c. This allows us to define $h' = c(h \circ \gamma^2)c^{-1} = [cab, ca, caba]$.

Lemma 16. *Suppose that* $\mathbf{w} \in \Sigma_3^\omega$ *is low. Suppose that* $\mathbf{w}$ *has the form* $h(\mathbf{u})$ *where* $h = [a, b, c]$*, some* $\mathbf{u} \in \Sigma_3^\omega$ *and* $a, b, c \in \Sigma_3^+$ *such that*

i. The word b *is a prefix of* a*, which is a prefix of* c*.*

ii. We have $|b| \geqslant |c|/2$*.*

iii. The words b *and* c *have a common suffix* s *such that* $|s| \geqslant |b|/2$*.*

Then a suffix of $\mathbf{w}$ *has the form* $h(\gamma^2(\mathbf{v}))$*, some* $\mathbf{v} \in \Sigma_3^\omega$*; equivalently, a suffix of* $\mathbf{w}$ *has the form* $h'(\mathbf{v})$ *some* $\mathbf{v} \in \Sigma_3^\omega$*, where* $h' = c(h \circ \gamma^2)c^{-1} = [cab, ca, caba]$*. Letting* $A = h'(0)$*,* $B = h'(1)$*, and* $C = h'(2)$*, we have*

1. *The word* B *is a prefix of* A*, which is a prefix of* C*.*

2. *We have* $|B| \geqslant |C|/2$*.*

3. *The words* B *and* C *have a common suffix* S *such that* $|S| \geqslant |B|/2$*.*

Proof. Suppose that $\mathbf{w}$ has the form $h(\mathbf{u})$. By Lemma 15, words 00 and 11 are not factors of $\mathbf{u}$. Therefore, if 2 is not a factor of some suffix of $\mathbf{u}$, then that suffix is $(01)^\omega$ or $(10)^\omega$. This is impossible since $\mathbf{w}$ is low. We conclude that every suffix of $\mathbf{u}$ contains a 2.

Parse a suffix of $\mathbf{u}$ into blocks starting with 2, and containing a single 2. Words 21 and 22 are not factors of $\mathbf{u}$; thus the blocks begin with 20. Since 00 and 11 are not factors, 0's and 1's alternate in the blocks. The block 20101 is impossible; it would imply a block $z = 2010101$ or $z = 201012$. In both cases, $h(z)$ begins with $cababa$, since a is a prefix of c. However, $ababa$ is an r power with $r \geqslant 5/2$. It follows that a suffix of $\mathbf{u}$ is concatenated from blocks 20, 201, and 2010. Therefore, a suffix of $\mathbf{w}$ is concatenated from blocks $ca = A$, $cab = B$, and $caba = C$, and has the form $h'(\mathbf{v})$ for some $\mathbf{v} \in \Sigma_3^\omega$.

We see that property 1 certainly holds. Also,

$$|C| = |caba| \leqslant 2|ca| = 2|B|$$

since $|c| \geqslant |b|$, establishing 2.

Finally, the words C and B have the common suffix $S = sa$. However,

$$2|s| + |a| \geqslant |b| + |a| \geqslant 2|b| \geqslant |c|.$$

Thus

$$|S| = |sa| \geqslant |c| - |s| = |ca| - |sa| = |B| - |S|,$$

so that $2|S| \geqslant |B|$, establishing 3. $\square$

Proof of morphism lemma. Define morphisms h_n recursively by $h_1 = h$, and $h_{n+1} = c(h_n \circ \gamma^2)c^{-1}$. By induction on the previous lemma, for each positive integer n, a suffix of $\mathbf{w}$ has the form $h_n(\mathbf{v})$, some $\mathbf{v} \in \Sigma_3^\omega$.

Having a suffix of the form $h_{n+1}(\mathbf{v})$ is equivalent to having a suffix of the form $h_n(\gamma^2(\mathbf{v}))$. By induction on n, having a suffix of the form $h_n(\mathbf{v})$ is equivalent to having a suffix of the form $h(\gamma^{2n}(\mathbf{v}))$. Thus a suffix of $\mathbf{w}$ has the form $h(\gamma^{2n}(\mathbf{v}))$. It follows that whenever g is a prefix of $\mathbf{G}$, then $h(g)$ is a factor of $\mathbf{w}$, and the lemma follows. $\square$

Proof of dual morphism lemma. Replacing $\mathbf{z}$ by a suffix if necessary, suppose that $\mathbf{z}$ has the form $h(\mathbf{u})$. Let $T = \{z \in \Sigma_3^* : h(z) \text{ is a factor of } \mathbf{z}\}$. Let $T^R = \{z^R : z \in S\}$. Then T^R is a right factorial language. Using König's lemma, choose a word $\mathbf{z}' \in \Sigma_3^\omega$ such that every prefix of $\mathbf{z}'$ is in T^R.

Define $h^R = [h(0)^R, h(1)^R, h(2)^R]$. Let $\mathbf{Z} = h^R(\mathbf{z}')$. Then $\mathbf{w} = \mathbf{Z}$ and $h' = h^R$ satisfy the conditions of the morphism lemma, so that for every $g \in \mathbf{G}$, word $h^R(g)$ is a factor of $\mathbf{Z} = h^R(\mathbf{z}')$.

Let p be a prefix of $\mathbf{z}'$ such that $h^R(g)$ is a factor of $h^R(p)$. Since every prefix of $\mathbf{z}'$ is in T^R, write $p = z^R$ where $z \in T$. Then $h^R(g)$ is a factor of $h^R(z^R)$. Thus $h(g^R)$ is a factor of $h(z)$. By the definition of T, $h(z)$ is a factor of $\mathbf{z}$, so that $h(g^R)$ is a factor of $\mathbf{z}$. $\square$

Proof of first parsing lemma. Let $\mathbf{w}$ be an eligible right-infinite word which omits the words of S. Every factor of $\mathbf{w}$ of length $d+1$ is low, and omits words of S. By the condition of the lemma it cannot omit b. Thus b appears infinitely often in $\mathbf{w}$, so that a suffix can be concatenated from the b-blocks of $\mathbf{w}$. These blocks have the form bu where

- The word bub is a factor of $\mathbf{w}$;
- Whenever p is a prefix of bub such that b is a suffix of p, then $p = b$ or $p = bub$.

Such a word u is a factor of $\mathbf{w}$ and is therefore low and omits words of S. If $|u| > d$ it must therefore have b as a factor. In this case, write $u = vbz$. Then $p = bvb$ is a prefix of bub distinct from b and bub. This is a contradiction. $\square$

Proof of second parsing lemma. Write $\mathbf{w} = b_1 b_2 b_3 b_4 \cdots$ such that the $b_i \in B$. Then for $j \geqslant 2$, if $b_j = s$ let $b_{j-1} = r$ and $b_{j+1} = t$. Since rst is a factor of $\mathbf{w}$ it is low and cannot contain a factor from S. This is a contradiction. We conclude that the suffix $b_2 b_3 b_4 \cdots$ of $\mathbf{w}$ is concatenated from words of $B - \{s\}$. $\square$

7 A sharpened structure theorem and open problems

Consider the morphism which is the reverse of γ, namely $\gamma^R = [10, 2, 20]$. Let $\mathbf{G}'$ be its fixed point

$$\mathbf{G}' = (\gamma^R)^\infty(2) = 20102102010\cdots .$$

(An anonymous referee points out that $\mathbf{G}' = \pi(\mathbf{p})$ where $\mathbf{p}$ is the sequence studied in [11] and π is the permutation of $[2, 0, 1]$ of Σ_3. See the entry for sequence A287072 in the On-Line Encyclopedia of Integer Sequences.) Then the factors of $\mathbf{G}'$ are precisely the

$h(0)$	$h(1)$	$h(2)$
210	20	21
02	021	02121
2	0	1
102	2	1

Figure 3: Possible morphisms h such that for each factor g of $\mathbf{G}$, word $\pi(h(g))$ is a factor of $\mathbf{w}$.

reverses of the factors of $\mathbf{G}$. One can easily adjust the proof of the morphism lemma to prove

Lemma 17 (Alternate Dual Morphism Lemma). *Suppose that $\mathbf{z} \in \Sigma_3^\omega$ is low. Suppose that a suffix of $\mathbf{z}$ has the form $h(\mathbf{u})$ where $h = [a, b, c]$, some $\mathbf{u} \in \Sigma_3^\omega$ and $a, b, c \in \Sigma_3^+$ such that*

1. *The word b is a suffix of a, which is a suffix of c.*
2. *We have $|b| \geqslant |c|/2$.*
3. *The words b and c have a common prefix p such that $|p| \geqslant |b|/2$.*

Then if g is any factor of $\mathbf{G}'$, then $h(g)$ is a factor of $\mathbf{z}$.

As remarked earlier, it is possible to keep track of the morphisms used in the proof of the structure theorem. Doing so gives the following:

Theorem 18 (Sharpened Structure Theorem). *Let $\mathbf{w} \in \Sigma_3^\omega$ be eligible. One of the following holds.*

- *For some non-erasing morphism h in Figure 3 and a permutation π of Σ_3, the word $\pi(h(g))$ is a factor of $\mathbf{w}$ whenever g is a factor of $\mathbf{G}$;*
- *For some non-erasing morphism h in Figure 4 and some permutation π of Σ_3, the word $\pi(h(g))$ is a factor of $\mathbf{w}$ whenever g is a factor of $\mathbf{G}'$.*

To get from the morphisms of Figures 1 and 2 to those of Figures 3 and 4, we first notice that some morphisms (for example, the first and the fifth in Figure 1) are the same up to a permutation of the alphabet. After eliminating these duplications, we factor out powers of γ (or γ^R for Figure 2) from the morphisms. For example, consider the third morphism in Figure 1, namely

$$h = [021201212, 0212012, 021201212012].$$

This is a conjugate of

$$h_1 = [201212021, 2012021, 201212012021].$$

$h(0)$	$h(1)$	$h(2)$
2	0	1
0	2	21
1	0	2
12	012	01012
201	21	01
102	1	2

Figure 4: Possible morphisms h such that for each factor g of $\mathbf{G}'$, word $\pi(h(g))$ is a factor of $\mathbf{w}$.

Applying the permutation $\sigma = [1, 2, 0]$ we find

$$\sigma(h_1) = [012020102, 0120102, 012020120102] = \gamma^4.$$

Morphism h is thus replaced in Figure 3 by $\sigma^{-1} = [2, 0, 1]$.

The sharpened theorem says that the set of factors of $\mathbf{w}$ includes (up to a permutation) the set of factors of $h(\mathbf{G})$ for some morphism h in Figure 3, or of $h(\mathbf{G}')$ for some morphism h in Figure 4. It is not known for which of the listed h the words $h(\mathbf{G})$ and $h(\mathbf{G}')$ have factor complexity $2n+1$, or what the critical exponents of those words are. We make the following conjecture:

Conjecture 19. For each morphism h in Figure 3, the word $h(\mathbf{G})$ has factor complexity $2n + 1$ and critical exponent μ. For each morphism h in Figure 4, the word $h(\mathbf{G}')$ has factor complexity $2n + 1$ and critical exponent μ.

Appendix: 'Human readable' proof of the resolution of the case $S = \{01, 12, 20, 00\}$ of the structure theorem

The Python implementation outputs the following proof:

We assume F includes [01, 12, 20, 00].

Every good word longer than 25 must include factor(s)

[0221, 1021, 1022, 1102, 2102, 2110].

Author's comment: In step B of the algorithm, we found a length-250 low word w omitting $\{01, 12, 20, 00\}$. In step C of the algorithm, for $n = 4$, we find that w has more than $2n+1$ length-4 factors. Of these, 6 (announced above) are found to be needed factors in step C.b of the algorithm.

The word w must omit a factor from

[0210, 0211, 2210, 2211].

Author's comment: These are the 4 unneeded factors found in step C.b of the algorithm. They cannot all be included in $\mathbf{w}$*, because* $6+4=10$ *would be too many length-4 factors in* $\mathbf{w}$*.*

This gives rise to 4 cases.

Case 1: F includes [01, 12, 20, 00, 0210]
Case 2: F includes [01, 12, 20, 00, 0211]
Case 3: F includes [01, 12, 20, 00, 2210]
Case 4: F includes [01, 12, 20, 00, 2211]

Author's comment: This is the case division of step C.b.ii.

Case 1: F includes [01, 12, 20, 00, 0210].

Every good word longer than 68 must include factor(s)

[02110, 02210, 02211, 10211, 10221, 11022, 21021, 21102, 22102, 22110].
The word w must omit a factor from

[11021, 21022].
This gives rise to 2 cases.

Case 1.1: F includes [01, 12, 20, 00, 0210, 11021]
Case 1.2: F includes [01, 12, 20, 00, 0210, 21022]

Case 1.1: F includes [01, 12, 20, 00, 0210, 11021].

Every good word longer than 81 must include factor(s)

[02110221022, 02110221102, 02210211022, 02210221102, 02211022102, 10211022102, 10211022110, 10221021102, 10221022110, 10221102210, 11022102110, 11022102211, 11022110221, 21021102210, 21021102211, 21022110221, 21102210211, 21102210221, 21102211022, 22102110221, 22102211022, 22110221021, 22110221022].
In this case w must omit the factor 02110221021.

Author's comment: Here, because only 1 unneeded factor was found in Step C.b, we don't make a new subcase.

The 210-blocks are among [2102, 21021102, 210221102, 2102110221102].

Author's comment: It turns out that when we find a length-250 low word w omitting the current list of missing factors, the complexity is $2n + 1$ for $n = 1, 2, \ldots, 20$. We therefore, in step C.a, search for a morphism, considering factors b of w with $|b| \leqslant 3$ and finding the b-blocks of w.

The word 2102 cannot occur twice in a concatenation of these

Author's comment: If 2102 occurs twice, it occurs in the context $s2102t$, where s, t are two of the 210-blocks. However, one checks that each such word $s2102t$ contains an r power with $r \geqslant 5/2$.

Thus a suffix of w is concatenated from [21021102, 210221102, 2102110221102].

Taking conjugates, a suffix is the image under morphism

[221102210, 21102210, 2110221102210].
which satisfies the dual morphism lemma.

(This concludes Case 1.1).
Author's comment: Since $\mathbf{w}$ has a suffix of the form $h(\mathbf{u})$, the dual morphism lemma shows that $\mathbf{w}$ has critical exponent at least μ, and the current subcase is resolved.

Case 1.2: F includes [01, 12, 20, 00, 0210, 21022].

The 210-blocks are

[21021102, 2102110221102, 21021102211021102, 2102110221102110221102].
The word 21021102 cannot occur twice in a concatenation of these.

Thus a suffix of w is concatenated from

[2102110221102, 21021102211021102, 2102110221102110221102]
Taking conjugates, a suffix is the image under morphism

[21021102211021102, 2102110221102, 2102110221102110221102].
which satisfies the morphism lemma.

(This concludes Case 1.2).

(This concludes Case 1).

Case 2: F includes [01, 12, 20, 00, 0211].

Every good word longer than 54 must include factor(s)

[02102, 02210, 02211, 10210, 10221, 11021, 11022, 21021, 21022, 21102, 22102, 22110]. The word w has $12 > 2 \times 5 + 1$ factors of length 5. This is impossible.

(This concludes Case 2).

Author's comment: We found too many needed factors in step C.b.

Case 3: F includes [01, 12, 20, 00, 2210].

Every good word longer than 56 must include factor(s)

[02102, 02110, 02211, 10210, 10211, 10221, 11021, 11022, 21021, 21022, 21102, 22110]. The word w has $12 > 2 \times 5 + 1$ factors of length 5. This is impossible.

(This concludes Case 3).

Case 4: F includes [01, 12, 20, 00, 2211].

Every good word longer than 41 must include factor(s)

[02102, 02110, 02210, 10210, 10211, 10221, 11021, 11022, 21021, 21022, 21102, 22102]. The word w has $12 > 2 \times 5 + 1$ factors of length 5. This is impossible.

(This concludes Case 4).

Author's comment: The proof in the present case is much shorter than for the resolution of $S = \{11, 22\}$. In that longer resolution the computer frequently invokes step A of the algorithm (not used in the present case) to avoid repeating case analysis which was already carried out.

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