Frozen-corner enumeration of Alternating Sign Matrices

Filippo Colomo[a] Andrei G. Pronko[b]

Submitted: Sep 19, 2025; Accepted: Jul 1, 2026; Published: Aug 7, 2026

Abstract

An Alternating Sign Matrix (ASM) is a square matrix with entries in $\{0, 1, -1\}$, and such that: i) in each row and columns, nonzero entries alternate in sign; ii) for any given row or column, entries sum up to 1. We define the frozen-square enumeration as the enumeration of $n \times n$ ASMs under the refinement of having, located in a corner, an $s \times s$ square of entries that are all zeroes. We state a conjectural formula for such enumeration, in terms of the determinant of some $s \times s$ matrix whose entries are given explicitly. We provide numerical support in favour of our conjecture. We also illustrate the relevance of the conjectured formula in connection with the limit shape observed in large ASMs, its fluctuations, and the Tracy–Widom distribution.

Mathematics Subject Classifications: 05A15, 82B23

1 Introduction

1.1 Alternating Sign Matrices

An Alternating Sign Matrix (ASM) is a square matrix with entries in $\{0, 1, -1\}$, and such that: i) in each row and columns, nonzero entries alternate in sign; ii) for any given row or column, entries sum up to 1. ASMs were introduced by Mills, Robbins, and Rumsey in connections with descending plane partitions [MRR1, MRR2]. There are many interesting results concerning ASMs: for an introductory review, see the nice book by Bressoud [Br], while, for recent developments, see, e.g., [BFK] and reference therein.

Let A_n denote the number of $n \times n$ ASMs. It was conjectured in [MRR2], and proved by Zeilberger [Z1], that

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}. \tag{1}$$

[a] INFN, Sezione di Firenze, Via G. Sansone 1, 50019 Sesto Fiorentino (FI), Italy (`colomo@fi.infn.it`).

[b] Steklov Mathematical Institute, Fontanka 27, 191023 Saint Petersburg, Russia (`a.g.pronko@gmail.com`).

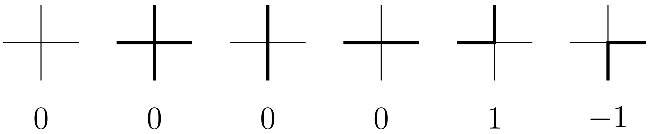
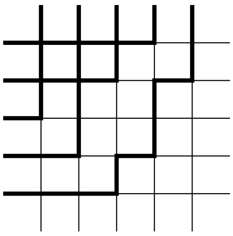


Figure 1: The six allowed vertex configurations for the square ice in the path description and the corresponding entries for ASMs



$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{array}{ccccccccc} & & & & 4 & & & & \\ & & & 3 & & 5 & & & \\ & & 1 & & 3 & & 5 & & \\ & 1 & & 2 & & 4 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

Figure 2: A path configuration on the 5×5 grid with Domain Wall boundary conditions (left), the corresponding ASM (center), and the related monotone triangle (right).

An alternative proof was provided soon after by Kuperberg [Ku]. It is based on a simple bijection, first described by Robbins and Rumsey [RR], see also [EKLP], between ASMs and configurations of square ice [L]. The proof uses Izergin–Korepin formula for the square ice partition function [Ko, I], whose derivation relies on the integrability of the model [Ba] and the Quantum Inverse Scattering Method [TF, KBI].

Let $A_{n,r}$, $r = 1, \dots, n$, denote the number of $n \times n$ ASMs whose sole nonzero entry, equal to 1, in the first (or last) row (or column) occurs at the rth position. This is referred to as the refined enumeration of ASMs. It was conjectured in [MRR2], and proven by Zeilberger [Z2], that

$$A_{n,r} = \binom{n+r-2}{r-1} \frac{(2n-r-1)!}{(n-r)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}. \tag{2}$$

The proof is built on Kuperberg's idea of using the bijection with square ice and Izergin–Korepin formula. Alternative proofs of (1) and (2) have been proposed over the years, see e.g. [CP1, F1, F2, FK1, FK2].

ASMs, and square ice, can conveniently be described in terms of paths on a square grid. More precisely, let us consider n paths on the $n \times n$ square grid. Allowed path configurations are such that each vertex of the grid may be in one out of six allowed configurations, that we call of type $1, \dots, 6$, respectively, see Fig. 1. It follows that the vertices of type 5 and 6 (turning vertices) must alternate along each horizontal or vertical line of the grid. We require the n paths to be densely packed on the top and left boundary of the grid, see Fig. 2; this is known as the 'Domain Wall' boundary condition. The imposed boundary condition implies that the first and last 'turning' vertices on each horizontal or vertical line must all be of type 5. The bijection with $n \times n$ ASMs follows.

$$\begin{pmatrix} * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 1 \\ * & * & * & * & 0 \end{pmatrix}$$

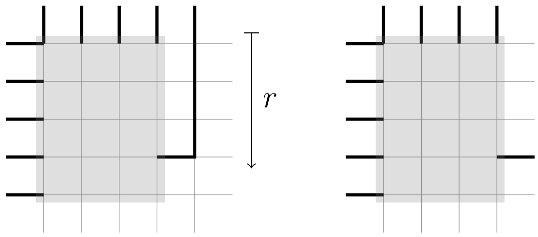


Figure 3: Refined enumeration of ASMs: A generic ASM with the sole nonzero entry of the last column at position r (left), the corresponding path configuration on the square grid (center), and the modified grid with its last line removed (right). Path configurations within the shaded area are generic.

Under this bijection, the refined enumeration of $n \times n$ ASMs becomes the enumeration of path configurations on the $n \times n$ square grid, conditioned to have their sole vertex of type 5 at the rth position in the first (or last) horizontal (or vertical) line. Focussing of the last column for definiteness, see Fig. 3, it is clear that the refined enumeration is also equivalent to the enumeration of path configurations on the $n \times (n-1)$ square grid, with $n-1$ paths flowing vertically from the top boundary, and one horizontally, from the right boundary, at the rth position.

1.2 Frozen square enumeration of ASMs

In the present paper we are interested in the enumeration of ASMs under a different refinement, namely that of having in, say, its top-left corner, a square of entries that are all zeroes (a frozen square). More precisely, a given ASM $(a_{ij})_{1\leqslant i,j\leqslant n}$ has an $s \times s$ frozen square in its top-left corner if $a_{ij} = 0$ for all $i, j = 1, \ldots, s$. Let $B_{n,s}$, $s = 1, \ldots, n$, be the number of such matrices. We call the quantity $B_{n,s}$ *frozen-corner* enumeration. Clearly, by symmetry, $B_{n,s}$ does not depend on which of the four corners we choose to freeze.

Let us briefly mention some simple properties of the quantity $B_{n,s}$. When $s = 1$, a zero is required to be in the top left corner. Hence, the sole 1 of the first row must be at position $r = 2, \ldots, n$. It follows that

$$B_{n,1} = \sum_{r=2}^{n} A_{n,r} = \sum_{r=1}^{n-1} A_{n,r}, \tag{3}$$

where the second equality follows from the set of identities $A_{n,r} = A_{n,n-r+1}$. Further properties of the frozen-corner enumeration may be easily derived by using the path description of square ice. For instance, we have

$$B_{n,s} = 0, \qquad s > \lfloor n/2 \rfloor. \tag{4}$$

We also have, for ASMs of even size:

$$B_{2s,s} = (A_s)^2, \tag{5}$$

and for ASMs of odd size:

$$B_{2s+1,s} = \sum_{j,k=1}^{s+1} \binom{j+k-2}{j-1} A_{s+1,j} A_{s+1,k}. \tag{6}$$

The derivation of this last identity is slightly more involved. We sketch it in Appendix A.

1.3 A conjecture

Before stating our conjecture for the numbers $B_{n,s}$, it is convenient to introduce some notations. Let us introduce the (normalized) generating function for the refined enumeration of ASMs (2):

$$h_n(z) := \frac{1}{A_n} \sum_{r=1}^{n} A_{n,r} z^{r-1}. \tag{7}$$

For $i = 1, 2, \ldots$, let

$$f_i^{\pm}(z) := \left[1 \pm (-1)^i z\right] \frac{(1-z)^{i-1}}{z^i} h_{n-s+i}(z). \tag{8}$$

For any given n and s, let us introduce the $s \times s$ matrix M, with entries

$$M_{ij} := \frac{A_{n-s+j}}{A_{n-s+j-1}} \oint_{C_0} \oint_{C_0} \frac{f_i^{+}(z) f_j^{-}(w)}{1-z-w} \frac{\mathrm{d}z \mathrm{d}w}{(2\pi \mathrm{i})^2}, \qquad i, j = 1, \ldots, s, \tag{9}$$

where C_0 is a small closed contour around the origin, containing no other singularity of the integrand.

Remark 1. The entries of the $s \times s$ matrix M may be expressed in terms of ASM's plain and refined enumerations, as follows:

$$\begin{aligned} M_{ij} = \frac{1}{A_{n-s+i} A_{n-s+j-1}} \sum_{k \geqslant 0} \sum_{l=0}^{k} \sum_{p=1}^{n-s+i} \sum_{q=1}^{n-s+j} (-1)^{i+j-k-p-q} \binom{k}{l} \\ \times \left[\binom{i-1}{i+l-k-p} - (-1)^i \binom{i-1}{i+l-k-p-1} \right] \\ \times \left[\binom{j-1}{j-l-q} + (-1)^j \binom{j-1}{j-l-q-1} \right] A_{n-s+i,p} A_{n-s+j,q} \end{aligned} \tag{10}$$

with $i, j = 1, \ldots, s$. This is easily worked out by Laurent expanding the integrand in (9) in the variables z and w, and evaluating the corresponding residues at the origin (that is, the constant term).

We propose the following formula for the frozen-corner enumeration.

Conjecture 2. The number $B_{n,s}$ may be evaluated in terms of a determinant:

$$B_{n,s} = A_n \det_{1 \leqslant i,j \leqslant s} (1 - M)$$

where the $s \times s$ matrix M is defined in (10), or equivalently, in terms of Eqs. (7), (8), and (9).

Conjecture 2 has already been stated in the physical context of the six-vertex model [CP4]. Here we reformulate the assertion in the ASM language, to bring it to the attention of the combinatorics community, where somebody might be able to build a rigorous proof. We also provide strong numerical evidence in support of it.

In Section 2 we explain the rationale behind Conjecture 2.

In Section 3 we show how the numbers $B_{n,s}$ may be evaluated, for moderate values of n ($n \leqslant 20$), by direct enumeration of corresponding monotone triangles satisfying some simple constraint. The obtained values coincides with those provided by evaluating the right hand side of Conjecture 2.

In Section 4 we provide further motivations and context behind the work that led us to state Conjecture 2. We also illustrate some nontrivial implications of it, in connection with the limit shape observed in large ASMs, its fluctuations, and the Tracy–Widom distribution.

2 Rationale

2.1 Why the frozen-square enumeration?

The motivation in studying frozen-corner enumeration of ASMs stems from asymptotic combinatorics. Large ASMs are known to develop a limit shape, with four 'frozen' regions, containing only zeroes, in the four corners, and a central 'disordered' region, containing all three kind of entries. The phenomenon is closely related to the Arctic Circle Theorem [JPS]. Indeed this occurs in the domino tilings of the Aztec Diamond, which can be mapped onto the 2-enumeration of ASMs [EKLP], i.e., when a weight 2^k is given to each ASM with k '-1' entries [MRR1].

In order to study the phenomenon, one could try to build a closed form expression for ASM enumeration under the refinement that, say, $a_{ij} = -1$ for some given i, j. Such an enumeration is currently beyond our capabilities. A simpler refinement consists in requiring the presence of a frozen square (or, more generally, rectangle) in one of the corners. This is the frozen-square enumeration described above. As we shall see, this may yield relevant information on the limit-shape phenomenon in large ASMs.

2.2 A closed formula for the frozen-square enumeration

Over the years, various closed form formulae have been provided for the frozen-square enumeration [CP2, CGP]. Here we present the most relevant one for our purposes.

Proposition 3. *The frozen-square enumeration may be evaluated in terms of the multiple integral*

$$B_{n,s} = (-1)^s A_n \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{s} \frac{1}{z_j^{n-s}(z_j-1)^{s-j+1}} \times \prod_{1\leqslant j<k\leqslant s} \frac{1}{z_j z_k - z_j + 1} \det_{1\leqslant j,k\leqslant s} \left[(z_j-1)^{s-k} z_j^{k-1} h_{n-k+1}(z_j)\right] \frac{\mathrm{d}^s z}{(2\pi\mathrm{i})^s}, \quad (11)$$

where $h_n(z)$ is defined in (7)*, and C_0 is a small closed contour around the origin.*

Relation (11) is essentially a 'constant term identity' for $B_{n,s}$. Proposition 3 is a particular case of a more general result, valid when the frozen region is a rectangle, and originally derived in the context of the six-vertex model, for arbitrary values of its parameters, using integrability techniques [Ba], and the Quantum Inverse Scattering Method [TF, KBI]. Relation (11) may be obtained from, e.g., Eq. (5.14) in [CGP], simply by replacing therein $N \to n$, $r \to n-s$, specializing the values of the parameters to $\Delta = \frac{1}{2}$ and $t = 1$, and multiplying by A_n. Proposition 3 may be checked numerically against the data provided in Section 3.

2.3 Towards a Fredholm type determinant representation

Expression (11) is rather nice and compact. However, the evaluation of its asymptotic behaviour for large n, s is still out of reach. For such a purpose, a Fredholm determinant representation would be highly desiderable, in view of the various well-established techniques developed in the last decades.

The notion of Fredhom determinant generalizes that of determinant of a finite dimensional linear operator. It is defined for bounded operators on a Hilbert space which differ from the identity operator by a trace-class operator [GGK]. In order to build a Fredholm determinant representation, following [TW2], we have first tried to rewrite the above integral representation as a Fredholm-type determinant, that is as the determinant of some $s \times s$ matrix of the form $1 - M$.

We have managed to do so for the case $s = 1, \ldots, 4$. This has allowed us to guess a general structure for the matrix M, reported in (9). The general structure is constrained by two lemmas which can be easily proven, and by some natural recursive properties, see [CP4] for details. The resulting formula, Conjecture 2, has gone through all our checks, see below.

3 Numerical checks

3.1 Monotone triangles

To check Conjecture 2, we want here to evaluate the numbers $B_{n,s}$ by direct enumeration of ASMs with a frozen square. We use the well-known bijection between ASMs and monotone triangles [MRR2]. A monotone triangle of size n is a triangular array of numbers t_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, i$, such that:

1. for any given i, the sequence t_{ij} is strictly increasing with j: $t_{ij} < t_{i,j+1}$, $j = 1, \ldots, i-1$;
2. the sequences in two consecutive rows are weakly interlacing: $t_{i+1,j} \leqslant t_{ij} \leqslant t_{i+1,j+1}$;
3. the nth (bottom) row is: $t_{nj} = j$, $j = 1, \ldots, n$.

Recall the correspondence between ASMs and monotone triangles: given an $n \times n$ ASM with entries a_{ij}, compute a new $n \times n$ matrix with entries $b_{ij} = \sum_{l=1}^{i} a_{ij}$. It follows from the definition of ASMs that the ith row of this matrix has exactly i entries equal to 1, all others being 0. Denoting the position of the jth 1's in the ith row by t_{ij}, we indeed obtain a monotone triangle, see Fig. 2. The above described mapping is easily seen to be bijective.

3.2 Frozen-corner ASMs and constrained monotone triangles

Let us now consider an $n \times n$ ASM constrained to have an $s \times s$ square of zeroes in, say, the top-right corner (recall that the choice of the corner is irrelevant for the definition of $B_{n,s}$). It is clear from the above bijection that such a condition translates into the requirement that in the first s rows of the corresponding monotone triangle no entry is larger than $n - s$. In other words, $B_{n,s}$ is equal to the number of monotone triangle of size n, satisfying the additional condition

$$t_{ij} \leqslant n - s, \qquad i \leqslant s. \tag{12}$$

We have implemented on Wolfram Mathematica [W] the evaluation of the number of monotone triangle satisfying condition (12). The code is a minor adaptation of that for enumerating unconstrained monotone triangles, provided in [Br]. It is reported in Appendix B.

3.3 Results and checks

We have run the code described above on a standard desktop and evaluated $B_{n,s}$ for all $n \leqslant 20$, with $s \leqslant n$. We report below the first few ones, till $n = 12$.

n	s	$B_{n,s}$
2	1	1
3	1	5
4	1	35
4	2	4
5	1	387
5	2	102
6	1	7007
6	2	2889
6	3	49
7	1	210 912
7	2	115 089
7	3	7007
8	1	10 631 868
8	2	6 994 572

8	3	901 849
8	4	1764
9	1	900 985 244
9	2	673 987 304
9	3	144 298 856
9	4	1 489 302
10	1	128 622 437 240
10	2	105 207 867 496
10	3	32 490 142 348
10	4	945 848 476
10	5	184 041
11	1	30 966 210 579 675
11	2	26 959 351 496 635
11	3	10 955 022 440 189
11	4	662 654 919 034
11	5	944 518 861
12	1	12 580 216 114 825 125
12	2	11 439 460 153 178 700
12	3	5 734 064 378 518 415
12	4	600 268 797 504 481
12	5	3 128 505 277 443
12	6	55 294 096

We have omitted vanishing values, occuring for $s > \lfloor n/2 \rfloor$. Further enumerations, for $n = 13, \ldots, 20$ are reported in appendix C.

Remark 4. The numbers $B_{n,s}$ do not seem to factorize into 'small' integers.

We have checked that the obtained values satisfy properties (3), (4), (5), (6) for all $n \leqslant 20$. We have also checked that the determinant formula proposed in Conjecture 2 reproduces the obtained numbers for all $n \leqslant 20$ and $s \leqslant n$.

4 Large ASMs, limit shape, and Tracy–Widom distribution

4.1 Arctic curve

As said, the motivation in studying frozen-corner enumeration of ASMs originates from asymptotic combinatorics, and in particular, from the observed limit-shape phenomenon in large ASMs. To be more precise, for any given ASM $(a_{ij})_{1 \leqslant i,j \leqslant n}$, we say that $(i,j) \in [1,n] \times [1,n]$ is in the top-left frozen region if $a_{kl} = 0$ for all $k \leqslant i$ an $l \leqslant j$. We may similarly define the three other frozen regions. If (i,j) is not in any frozen region, then we say that (i,j) is in the disordered region.

Consider now the square lattice graph underlying any ASM; let Λ be its dual graph. The polygon living on Λ and separating the frozen and disordered regions is uniquely defined. It is called the *frozen boundary.* Clearly each given ASM will have a given frozen boundary. A natural question concerns the distribution of the frozen boundaries, when considering uniformly sampled ASMs of given size n, and in particular if such distribution concentrates, and where, in the limit $n \to \infty$.

Let us introduce the rescaled variables x and y, with $(x,y) \in [0,1] \times [0,1]$, related to the matrix indices i,j as follows, $i = \lceil yn \rceil$, $j = \lceil xn \rceil$.

Theorem 5. *In the limit $n \to \infty$, the frozen boundary concentrate on a smooth curve (arctic curve) lying in the unit square. Restricting to the top left quadrant, $x,y \in [0,1/2]$ for simplicity, the expression of the arctic curve in parametric form reads*

$$x = 1 - \frac{2\omega - 1}{2\sqrt{\omega^2 - \omega + 1}}, \qquad y = 1 - \frac{\omega + 1}{2\sqrt{\omega^2 - \omega + 1}}, \qquad \omega \in [1, \infty). \tag{13}$$

The expression of the arctic curve in the other three quadrants may be easily obtained from symmetry considerations.

Remark 6. The portion of arctic curve (13) is the arc of ellipse of equation:

$$4x^2 + 4y^2 - 4xy - 4x - 4y + 1 = 0, \qquad x, y \in \left[0, \tfrac{1}{2}\right].$$

Note that the four points of tangency to the boundary of the unit square are singular (discontinuous in the second derivative). Expression (13) has been derived by the authors [CP3], however one crucial step in the derivation relied on some conjectural argument. Subsequent developments, in particular, the 'tangent method' [CS], have opened the way for a rigorous proof of Theorem 5, finally provided by Aggarwal [A].

4.2 Fluctuations around the arctic curve

Let us turn to the fluctuations of the frozen boundary. A full quantitative description of these is still beyond our capabilities. Let us therefore restrict to the *intersection*, (ξ, ξ), $\xi \in \{1, \ldots, \lfloor n/2 \rfloor\}$, of the frozen boundary with the ASM main diagonal. Under uniform sampling of the ASMs of size n, the quantity ξ is a discrete random variable. Its cumulative distribution function is simply

$$\mathbb{P}_n(\xi > s) = \frac{B_{n,s}}{A_n}.$$

A relevant question is the behaviour of $\mathbb{P}_n(\xi > s)$ in the limit of large n, s, with $s = \lfloor yn \rfloor$. However, to address it, one needs a suitable expression for $B_{n,s}$, amenable to asymptotic analysis. Remarkably, the formula proposed in Conjecture 2 complies with such requirement.

Indeed, the Fredholm-type determinant may be equivalently rewritten as the Fredholm determinant of some linear integral operator, with all dependence on parameters s and n encoded in the kernel of the integral operator. The asymptotic behaviour of the kernel as a function of the variable y may then be evaluated by using the saddle-point method.

In particular, in the vicinity of $y_{\mathrm{c}} := 1-\sqrt{3}/2$, that is the value of y for which the arctic curve (13) intersects the main diagonal $x = y$, one has the coalescence of two saddle-points. After suitable centering and rescaling of the variable y to $\sigma := 2^{4/3}3^{1/6}n^{2/3}(y_{\mathrm{c}}-y)$, a careful analysis shows that the kernel of the integral operator turns into the Airy kernel, and thus, the corresponding Fredholm determinant into the celebrated GUE Tracy–Widom distribution $\mathcal{F}_2(\sigma)$ [TW1]. More precisely, the following result holds.

Theorem 7. *In the large n limit, one has*

$$\lim_{n\to\infty}\left[\det_{1\leqslant i,j\leqslant s}(1-M)\Big|_{s=\left\lfloor ny_{\mathrm{c}}-\frac{n^{1/3}}{2^{4/3}3^{1/6}}\sigma\right\rfloor}\right]=\mathcal{F}_2(\sigma),$$

where M is the $s\times s$ matrix entering the right hand side of Conjecture 2.

For full details on the derivation of the above result, see [CP4].

4.3 Conclusion

If Conjecture 2 is true, then Theorem 7 implies that the fluctuations of the frozen boundary are governed by Tracy–Widom distribution. Thus, taken together, these two statements provide strong additional support to the occurrence of Tracy–Widom distribution in large ASMs, already conjectured in [ACJ], and checked against numerics in [LKV, PS].

Tracy–Widom distribution has already appeared in a variety of models of relevance in combinatorics (see, e.g., [D], and reference therein) or statistical mechanics (see, e.g., [JS], and references therein). However, most of these models are 'determinantal' (or free-fermionic, in physics), while ASMs are not. Our result thus extends the notion of 'universality' of Tracy–Widom distribution, in the sense of [D], to a wider class of models.

In conclusion, a rigorous derivation of Conjecture 2, besides being of interest by itself, would fill a gap in the proof that fluctuations of the frozen boundary of large ASMs are governed by Tracy–Widom distribution, which has been a long-standing open question.

Acknowledgements

The authors are indebted to L. Cantini, S. Chhita, H. Spohn, and J. Viti, for stimulating discussions. The work of A. G. Pronko was supported by the Ministry of Science and Higher Education of the Russian Federation (agreement 075-15-2025-344 dated 29/04/2025 for Saint Petersburg Leonhard Euler International Mathematical Institute at PDMI RAS). The authors are grateful to an anonymous referee for helpful remarks that improved the presentation.

A

We sketch here the derivation of identity (6). This is much easier when resorting to the bijection between ASMs and path configurations, shortly described in Section 1.

$$\begin{pmatrix} * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 1 \\ * & * & * & * & 0 \end{pmatrix}$$

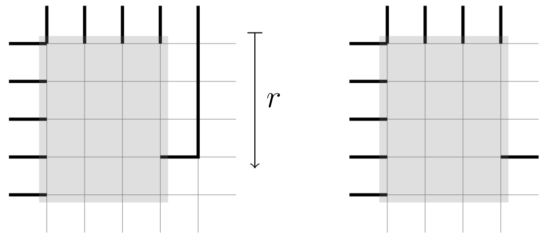


Figure 4: A $(2s+1) \times (2s+1)$ square ice grid, with an $s \times s$ frozen square in its top-left corner. The top-right and bottom-left rectangles correspond to the refined enumerations of $s \times s$ ASMs (see Fig. 3), while the bottom-right $(s+1) \times (s+1)$ square region contains only single path configurations.

Let us consider a $(2s+1) \times (2s+1)$ ASM, with an $s \times s$ frozen square in its top-left corner. In the osculating path picture, see Fig. 4, this becomes a set of $(2s+1)$ path on the $(2s+1) \times (2s+1)$ square grid, with an $s \times s$ square region in the top-left corner where all vertices are of type 2. Let us focus on the $s \times (s+1)$ top-right portion of the grid: there are $s+1$ paths flowing from the top; s of them have to exit from the left, while the $(s+1)$th may only exit downward, say in correspondence of the $(s+j)$th vertical edge, $j \in [1, \ldots, s+1]$. As explained above see Fig. 3, the number of such paths coincides with the refined enumeration $A_{s+1,j}$ of ASMs.

Similarly, the number of path configurations in the $(s+1) \times s$ bottom-left portion of the grid, with one path entering horizontally from the right at position $s+k$, coincides with the refined enumeration $A_{s+1,k}$. Finally, for given j and k, in the $(s+1) \times (s+1)$ bottom-right portion of the grid we only have one path, entering from the top, in correspondence of the $(s+j)$th vertical edge, and exiting from the left, in correspondence of the $(s+k)$th horizontal edge. The number of such single-path configurations is simply $\binom{j+k-2}{j-1}$. Multiplying these three enumerations, and summing over the allowed values of j and k, immediately yields (6).

B

We report here the Wolfram Mathematica code [W] we have used for the evaluation of the numbers $B_{n,s}$:

```
nextlist[n_, alist_] :=
  Select[Distribute[Apply[
    Range, Partition[Prepend[Append[alist, n], 1], 2, 1], 1
      ], List], # == Union[#] &];
nextlistnew[n_, s_, alist_] :=
  If[Length[alist] < s, nextlist[n - s, alist],
    nextlist[n, alist]]
MT[n_, s_][blist_] :=
  MT[n, s][blist] =
```

```
    If[Length[blist] >= n - 1, 1,
      Apply[Plus, Map[MT[n, s], nextlistnew[n, s, blist]]
        ], {1}];
ASMfrozen[n_, s_] := Sum[MT[n, s][{i}], {i, n - s}]
```

This is just a minor adaptation of the code for unconstrained monotone triangles provided in [Br], see page 63 therein.

C

We report here the number $B_{n,s}$, evaluated in terms of monotones triangles, for $n = 10, \dots, 20$, and $s = 1, \dots, \lfloor n/2 \rfloor$.

n	s	$B_{n,s}$
13	1	8 626 772 206 437 975 000
13	2	8 085 284 537 286 414 375
13	3	4 766 467 800 019 182 375
13	4	764 765 141 679 270 475
13	5	10 238 704 485 605 765
13	6	1 757 215 525 000
14	1	9 986 901 971 929 869 829 500
14	2	9 557 633 215 532 539 485 000
14	3	6 390 208 799 423 434 022 625
14	4	1 441 330 447 360 363 147 200
14	5	39 883 284 449 338 622 740
14	6	31 549 291 544 854 616
14	7	47 675 849 104
15	1	19 519 080 693 305 828 937 415 200
15	2	18 949 956 228 921 855 982 385 700
15	3	13 969 410 998 712 641 496 783 000
15	4	4 152 493 908 912 199 929 945 075
15	5	204 906 100 267 058 283 679 500
15	6	506 108 207 011 474 868 400
15	7	9 503 245 338 765 360
16	1	64 407 656 627 191 028 079 791 846 640
16	2	63 144 359 138 386 747 743 715 394 640
16	3	50 198 563 563 919 013 122 322 373 540
16	4	18 705 823 805 304 944 943 701 823 900
16	5	1 479 675 893 764 040 036 587 431 315
16	6	8 932 759 443 524 965 167 457 800
16	7	951 434 093 994 335 362 640
16	8	117 727 187 246 656
17	1	358 804 774 730 434 175 758 129 259 673 456

17	2	354 105 577 359 757 906 766 371 948 697 376
17	3	298 333 903 708 787 500 221 857 600 739 936
17	4	133 912 858 837 890 294 182 713 812 604 536
17	5	15 669 580 544 590 486 163 684 302 616 040
17	6	195 234 062 777 004 744 115 907 889 066
17	7	77 617 468 143 851 451 332 322 960
17	8	148 590 087 937 175 420 256
18	1	3 374 501 770 056 834 424 668 067 005 366 508 160
18	2	3 345 186 415 262 559 017 533 955 220 394 878 080
18	3	2 946 094 160 268 159 399 942 635 677 670 006 000
18	4	1 542 051 040 512 127 018 766 643 769 731 903 632
18	5	250 711 193 443 677 493 960 123 784 595 089 380
18	6	5 695 769 733 133 631 831 672 720 731 794 252
18	7	6 469 250 261 084 028 486 818 521 377 683
18	8	84 778 099 221 899 163 096 886 968
18	9	831 443 906 113 411 600
19	1	53 576 975 973 345 090 138 316 376 635 217 569 387 788
19	2	53 270 072 499 641 082 201 198 634 936 685 040 618 732
19	3	48 517 940 677 403 477 178 301 346 505 128 327 029 748
19	4	28 829 134 880 775 563 145 866 710 298 765 418 642 088
19	5	6 193 973 979 899 248 680 612 126 583 699 721 493 780
19	6	233 438 991 061 758 469 376 565 209 482 737 854 672
19	7	628 282 787 528 898 950 170 898 927 382 130 672
19	8	36 208 603 232 194 160 709 384 879 994 300
19	9	6 693 451 140 998 292 061 585 400
20	1	1 435 985 354 364 704 216 564 266 803 990 901 495 284 513 968
20	2	1 430 590 451 354 449 063 721 068 755 633 229 789 198 660 380
20	3	1 336 289 815 208 055 474 404 048 759 131 669 563 982 545 640
20	4	881 410 992 082 437 335 865 516 641 683 862 633 616 426 168
20	5	240 194 284 338 153 882 283 352 726 137 805 109 829 801 860
20	6	13 940 838 952 484 992 304 062 323 074 600 042 578 016 516
20	7	77 412 505 786 986 088 748 219 280 737 825 798 922 720
20	8	14 735 602 647 010 234 326 105 847 627 175 396 848
20	9	22 142 431 679 094 238 038 503 945 163 100
20	10	16 779 127 803 917 965 290 000

References

[A] A. Aggarwal. Arctic boundaries of the ice model on three-bundle domains. *Invent. Math.*, 220: 611–671, 2020. doi:10.1007/s00222-019-00938-6 (arXiv:1812.03847).

[ACJ] A. Ayyer, S. Chhita, and K. Johansson. GOE fluctuations for the maximum of the top path in alternating sign matrices. *Duke Math. J.*, 172: 1961–2104, 2023. doi:10.1215/00127094-2022-0075 (arXiv:2109.02422).

[Ba] R. J. Baxter. *Exactly Solved Models in Statistical Mechanics*. Academic Press, San Diego, CA, 1982.

[BFK] R. Behrend, I. Fischer, and C. Koutschan. Diagonally symmetric alternating sign matrices. arXiv:2309.08446, 2023.

[Br] D. M. Bressoud. *Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture*. Cambridge University Press, Cambridge, 1999.

[CGP] F. Colomo, G. Di Giulio, and A. G. Pronko. Six-vertex model on a finite lattice: Integral representations for nonlocal correlation functions. *Nucl. Phys. B*, 972: 115535, 2021. doi:10.1016/j.nuclphysb.2021.115535 (arXiv:2107.13358).

[CP1] F. Colomo and A. G. Pronko. Square ice, alternating sign matrices, and classical orthogonal polynomials. *J. Stat. Mech. Theory Exp.*, 2005: P01005, 2005. doi:10.1088/1742-5468/2005/01/P01005 (arXiv:math-ph/0411076).

[CP2] F. Colomo and A. G. Pronko. Emptiness formation probability in the domain-wall six-vertex model. *Nucl. Phys. B*, 798: 340–362, 2008. doi:10.1016/j.nuclphysb.2007.12.016 (arXiv:0712.1524).

[CP3] F. Colomo and A. G. Pronko. The limit shape of large alternating-sign matrices. *SIAM J. Discrete Math.*, 24: 1558–1571, 2010. doi:10.1137/080730639 (arXiv:0803.2697).

[CP4] F. Colomo and A. G. Pronko. Evaluation of integrals for the emptiness formation probability in the square-ice model. *Nucl. Phys. B*, 1004: 116565, 2024. doi:10.1016/j.nuclphysb.2024.116565 (arXiv:2405.04358).

[CS] F. Colomo and A. Sportiello. Arctic curves of the six-vertex model on generic domains: The tangent method. *J. Stat. Phys.*, 164: 1488–1523, 2016. doi:10.1007/s10955-016-1590-0 (arXiv:1605.01388).

[D] P. Deift. Universality for mathematical and physical systems. In M. Sanz-Solé, J. Soria, J. L. Varona, and J. Verdera, editors, *Proceedings of the International Congress of Mathematicians, Madrid, 2006*, pages 125–152. European Mathematical Society, Zurich, 2007.

[EKLP] N. Elkies, G. Kuperberg, M. Larsen, and J. Propp. Alternating-sign matrices and domino tilings. *J. Algebraic Combin.*, 1: 111–132, 219–234, 1992. doi:10.1023/A:1022483817303 (arXiv:math/9201305).

[F1] I. Fischer. A new proof of the refined alternating sign matrix theorem. *J. Combin. Theory Ser. A*, 117: 1143–1157, 2010. doi:10.1016/j.jcta.2006.04.004 (arXiv:math/0507270)

[F2] I. Fischer. Short proof of the ASM theorem avoiding the six-vertex model. *J. Combin. Theory Ser. A*, 144: 139–156, 2016. doi:10.1016/j.jcta.2016.06.007.

[FK1] I. Fischer and M. Konvalinka. A bijective proof of the ASM theorem, Part I: the operator formula. *Elect. J. Combin.*, 27: P3.35, 2020. doi:10.37236/9082 (arXiv:1910.04198).

[FK2] I. Fischer and M. Konvalinka. A bijective proof of the ASM theorem, Part II: ASM enumeration and ASM-DPP relation. *Int. Math. Res. Not.*, 2022: 7203–7230, 2022. doi:10.1093/imrn/rnaa304 (arXiv:1912.01354).

[GGK] I. Gohberg, S. Goldberg, and N. Krupnik. *Traces and Determinants of Linear Operators*, volume 116 of *Operator Series, Advances and Applications*. Springer Basel AG, Basel, 2000.

[I] A. G. Izergin. Partition function of the six-vertex model in the finite volume. *Sov. Phys. Dokl.*, 32: 878–879, 1987.

[JPS] W. Jockush, J. Propp, and P. Shor. Random domino tilings and the arctic circle theorem. arXiv:math/9801068, 1998.

[JS] K. Jujimoto and T. Sasamoto. Quantum transport in interacting spin chains: Exact derivation of the Tracy–Widom distribution. *Phys. Rev. Lett.*, 136: 087102, 2026. doi:10.1103/lbl9-n9ws (arXiv:2412.20147).

[Ko] V. E. Korepin. Calculations of norms of Bethe wave functions. *Commun. Math. Phys.*, 86: 391–418, 1982. doi:10.1007/BF01212176.

[KBI] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin. *Quantum Inverse Scattering Method and Correlation Functions*. Cambridge University Press, Cambridge, 1993.

[Ku] G. Kuperberg. Another proof of the alternative-sign matrix conjecture. *Int. Math. Res. Not.*, 1996: 139–159, 1996. doi:10.1155/S1073792896000128 (arXiv:math/9712207).

[L] E. H. Lieb. Exact solution of the F model of an antiferroelectric. *Phys. Rev. Lett.*, 18: 1046–1048, 1967. doi:10.1103/PhysRevLett.18.1046

[LKV] I. Lyberg, V. Korepin, and J. Viti. Fluctuation of the phase boundary in the six-vertex model with domain wall boundary conditions: a Monte Carlo study. *J. Phys. A: Math. Theor.*, 56: 495002, 2023. doi:10.1088/1751-8121/ad0a43 (arXiv:2303.14669).

[MRR1] W. H. Mills, D. P. Robbins, and H. Rumsey. Proof of the Macdonald conjecture. *Invent. Math.*, 66:73–87, 1982. doi:10.1007/BF01404757.

[MRR2] W. H. Mills, D. P. Robbins, and H. Rumsey. Alternating-sign matrices and descending plane partitions. *J. Combin. Theory Ser. A*, 34: 340–359, 1983. doi:10.1016/0097-3165(83)90068-7.

[PS] M. Prähofer and H. Spohn. Domain wall fluctuations of the six-vertex model at the ice point. *J. Phys. A: Math. Theor.*, 57: 025001, 2023. doi:10.1088/1751-8121/ad13b4 (arXiv:2305.09502).

[RR] D. P. Robbins and H. Rumsey. Determinants and alternating-sign matrices. *Advances in Math.*, 62: 169–184, 1986. doi:10.1016/0001-8708(86)90099-X.

[TF] L. A. Takhtadjan and L. D. Faddeev. The quantum method of the inverse problem and the Heisenberg XYZ model. *Russ. Math. Surveys*, 34: 11–68, 1979. doi:10.1070/RM1979v034n05ABEH003909.

[TW1] C. A. Tracy and H. Widom. Level spacing distributions and the Airy kernel. *Comm. Math. Phys.*, 159: 51–174, 1994. doi:10.1007/BF02100489 (arXiv:hep-th/9211141).

[TW2] C. A. Tracy and H. Widom. A Fredholm determinant representation in ASEP. *J. Stat. Phys*, 132: 91–300, 2008. doi:10.1007/s10955-008-9562-7 (arXiv:0804.1379).

[W] Wolfram Mathematica, Ver. 14.3. Wolfram Research Inc., Champaign, IL, 2025. https://www.wolfram.com/mathematica.

[Z1] D. Zeilberger. Proof of the alternating sign matrix conjecture. *Elec. J. Comb.*, 3: R13, 1996. doi:10.37236/1271 (arXiv:math/9407211).

[Z2] D. Zeilberger. Proof of the refined alternating sign matrix conjecture. *New York J. Math.*, 2: 59–68, 1996. (arXiv:math/9606224).