On the Terwilliger algebras of quasi-thin Schurian association schemes

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Abstract

An association scheme is triply transitive if its automorphism group is transitive, and the centralizer algebra of a point stabilizer of its automorphism group coincides with the Terwilliger algebra and its subspace T_0. In this paper, we give necessary and sufficient conditions for a quasi-thin association scheme to be triply transitive. As a by-product, we give new infinite families of triply-transitive association schemes.

Mathematics Subject Classifications: 05E30, 05E18

1 Introduction

Associations schemes are highly-regular combinatorial objects that play a fundamental role in algebraic combinatorics. In 1992, Paul Terwilliger introduced Terwilliger algebras in [15, 16, 17] as a means of studying P- and Q- polynomial association schemes. The (algebraic) structure of Terwilliger algebras heavily depends on these underlying commutative association schemes, and is generally highly complex. Terwilliger algebras can also be defined, more generally, for non-P or non-Q polynomial commutative association schemes, as well as for non-commutative association schemes as in [1, 2, 13, 14]. Association schemes with small valencies are interesting from the viewpoint of Terwilliger algebras since they are often non-commutative, and their Terwilliger algebras are naturally restricted by the valencies. This paper is concerned with quasi-thin association schemes whose Terwilliger algebras are combinatorially and algebraically as small as possible.

Let Ω be a finite set and $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$, with $R_0 = \{(\omega,\omega) : \omega \in \Omega\}$, be a set of relations on Ω. The pair $\mathfrak{X} = (\Omega, \mathcal{R})$ is an ***association scheme*** if the following properties hold.

(AS1) $\mathcal{R}$ is a partition of $\Omega \times \Omega$.

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(AS2) For $1 \leqslant i \leqslant d$, the relation $R_i^* = \{(v,u) : (u,v) \in R_i\}$ belongs to $\mathcal{R}$. In particular, $*$ can be viewed as a permutation in $\mathrm{Sym}(d)$ of order dividing 2.

(AS3) For $0 \leqslant i,j,k \leqslant d$, there exists a non-negative integer p_{ij}^k such that for $(u,v) \in R_k$ we have

$$|\{w \in \Omega : (u,w) \in R_i \text{ and } (w,v) \in R_j\}| = p_{ij}^k.$$

In addition, the association scheme $\mathfrak{X}$ is called ***commutative*** if $p_{ij}^k = p_{ji}^k$, for all $0 \leqslant i,j,k \leqslant d$. The digraphs of the scheme $\mathfrak{X}$ are those of the form (Ω, R_i), where $1 \leqslant i \leqslant d$.

If $G \leqslant \mathrm{Sym}(\Omega)$ is a finite transitive group and $\mathcal{O}$ is the set of all orbitals of G, then the pair $(\Omega, \mathcal{O})$ satisfies (AS1)-(AS3), so it is an association scheme. The association scheme $(\Omega, \mathcal{O})$ is called the ***orbital scheme*** of $G \leqslant \mathrm{Sym}(\Omega)$, and its digraphs are called the orbital digraphs. It is well known that the orbital scheme of G is commutative if and only if (G, G_ω) is a Gelfand pair[1] for any $\omega \in \Omega$. Association schemes that can be obtained as orbitals of certain transitive groups are called ***Schurian***.

For any $1 \leqslant i \leqslant d$, the pair (Ω, R_i) is a simple[2] regular[3] digraph if $i^* \neq i$, and is a simple undirected regular graph if $i^* = i$. For any $1 \leqslant i \leqslant d$, the ***valency*** of the relation R_i is the degree of the graph (Ω, R_i) if $i^* = i$, and its in-degree if $i^* \neq i$. The valency of the diagonal relation R_0 is defined to be 1. The association scheme $\mathfrak{X}$ is called ***thin*** if all valencies are equal to 1, and is called ***quasi-thin*** if the valencies are equal to 1 or 2.

Recall that an automorphism of a graph X with vertex set Ω is a permutation of Ω preserving the edges and the non-edges of X. The set of all automorphisms of X is a permutation group of Ω. The automorphism group $\mathrm{Aut}(\mathfrak{X})$ of the association scheme $\mathfrak{X} = \left(\Omega, \{R_i\}_{i=0}^d\right)$ is the intersection of all automorphism groups of the digraphs (Ω, R_i) for $1 \leqslant i \leqslant d$. For any matrix A, let $A(u,v)$ be the (u,v)-entry of A. The adjacency matrix of (Ω, R_i), for $1 \leqslant i \leqslant d$, is the matrix A_i indexed by Ω in its rows and columns such that

$$A_i(u,v) = \begin{cases} 1 & \text{if } (u,v) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

We let A_0 be the identity matrix indexed by Ω in its rows and columns.

Let $\mathfrak{X} = (\Omega, \mathcal{R})$ be an association scheme, where $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ and $R_0 = \{(u,u) : u \in \Omega\}$. Fix $v \in \Omega$ and let $G = \mathrm{Aut}(\mathfrak{X})$. For any $0 \leqslant i \leqslant d$, we define the matrix $E_{i,v}^*$ to be the diagonal matrix indexed by Ω in its rows and columns, where

$$E_{i,v}^*(u,u) = \begin{cases} 1 & \text{if } (v,u) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

[1] i.e., if $\mathbf{1}_{G_\omega}$ is the trivial character of G_ω, then the induced character $\mathrm{Ind}^G \mathbf{1}_{G_\omega}$ is a sum of distinct irreducible characters of G

[2] No arcs of the form (u,u), for $u \in \Omega$. Moreover, there is no multiple arcs between two vertices.

[3] i.e., every vertex has equal in-degrees and out-degrees.

The Terwilliger algebra of $\mathfrak{X}$ with respect to v is the subalgebra of the complex matrix algebra $\mathrm{M}_\Omega(\mathbb{C})$ given by

$$T_v = \left\langle A_0, A_1, \ldots, A_d, E^*_{0,v}, E^*_{1,v}, \ldots, E^*_{d,v} \right\rangle .$$

Here, we emphasize on the fact that the Terwilliger algebra T_v depends on the choice of v, and two Terwilliger algebras with respect to two different vertices need not be isomorphic.

On the one hand, the Terwilliger algebra T_v is a subalgebra of the centralizer algebra $\mathrm{End}_{G_v}\left(\mathbb{C}^\Omega\right)$, which is the set of all matrices in $\mathrm{M}_\Omega(\mathbb{C})$ commuting with every permutation matrix of G_v. On the other hand, the Terwilliger algebra also contains the subspace

$$T_{0,v} = \mathrm{Span}\left\{ E^*_{i,v} A_j E^*_{k,v} : 0 \leqslant i, j, k \leqslant d \right\}. \tag{1}$$

Therefore, we have the inclusions

$$T_{0,v} \subseteq T_v \subseteq \mathrm{End}_{G_v}\left(\mathbb{C}^\Omega\right). \tag{2}$$

The gap in dimensions between the subspaces in the above inclusions can be arbitrarily large. One example where such cases occur is with the association scheme of Paley graphs [5]. Equality can also hold in (2) as in the case of the association schemes of the Higman-Sims graph and the McLaughlin graph (see [7]). When $T_{0,v} = T_v = \mathrm{End}_{G_v}\left(\mathbb{C}^\Omega\right)$, one may think of the Terwilliger algebra to be as small as possible. Terwilliger algebras with this property admit a highly-regular combinatorial structure given by $T_{0,v} = T_v$, and a highly-symmetric algebraic structure given by $T_v = \mathrm{End}_{G_v}\left(\mathbb{C}^\Omega\right)$.

The subspace $T_{0,v}$ is an algebra if and only if $T_{0,v} = T_v$. In this case, the association scheme $\mathfrak{X}$ is called ***triply regular.*** Examples of triply-regular association schemes are the Hamming scheme $H(n, 2)$, the association scheme of the strongly-regular graphs $H(2, d)$, and the affine polar graphs $\mathrm{VO}^{\varepsilon}_{2m}(2)$, where $\varepsilon = \pm$ and $m \geqslant 2$. We note that if there exists $v \in \Omega$ such that $T_{0,v} = T_v$, then $T_{0,u} = T_u$ for all $u \in \Omega$, whenever the underlying association scheme is P-polynomial [4, Lemma 4.1.2].

If $T_v = \mathrm{End}_{G_v}\left(\mathbb{C}^\Omega\right)$, then the structure of T_v is straightforward since $\mathrm{End}_{G_v}\left(\mathbb{C}^\Omega\right)$ is completely determined by the action of G_v on $\Omega \times \Omega$. Hence, one can study the Terwilliger algebra from a purely algebraic viewpoint in this case.

If the automorphism group $G = \mathrm{Aut}(\mathfrak{X})$ is transitive, then the Terwilliger algebras T_u and T_v are isomorphic for any $u, v \in \Omega$. Similarly, $\mathrm{End}_{G_u}\left(\mathbb{C}^\Omega\right)$ and $\mathrm{End}_{G_v}\left(\mathbb{C}^\Omega\right)$ are isomorphic for any $u, v \in \Omega$ whenever G is transitive. Therefore, for any fixed $v \in \Omega$, we will adopt the notations T for any Terwilliger algebra T_v, and $\tilde{T}$ for the centralizer algebra $\mathrm{End}_{G_v}\left(\mathbb{C}^\Omega\right)$, whenever G is transitive. We note that the dimension of $T_{0,v}$ only depends on the number of non-zero intersection numbers and not on $v \in \Omega$. As we will deal with association schemes with transitive automorphism groups, we fix $v \in \Omega$ and abbreviate $T_{0,v},\ T_v$, and $\mathrm{End}_{G_v}\left(\mathbb{C}^\Omega\right)$ as $T_0,\ T$, and $\tilde{T}$.

We say that an association scheme $\mathfrak{X}$ is ***triply transitive*** if $G = \mathrm{Aut}(\mathfrak{X})$ is transitive and $T_0 = T = \tilde{T}$. Therefore, a triply transitive association scheme is also triply regular. The converse of this statement does not hold in general: the association scheme of many orthogonal array graphs are triply regular but not triply transitive. Triply-transitive association schemes were first studied by Bannai and Munemasa in [1] for conjugacy class association schemes. In particular, they showed that the conjugacy class association scheme of abelian groups are triply transitive. In [11], the first author extended this result of Bannai and Munemasa to certain metacyclic groups that are “close” to being abelian. See [6, 19, 20] for results along the same line of research. In [7], triply-transitive association schemes arising from strongly-regular graphs were studied and almost classified except those coming from two families of graphs arising from finite geometry. Not much is known about triply-transitive association schemes except for these few examples (the conjugacy class association schemes and the association schemes of strongly-regular graphs).

This paper aims to provide more examples of triply-transitive association schemes. A by-product of this work is that we obtain a study of the Terwilliger algebras of quasi-thin association schemes over the complex numbers from a different perspective than those in the literature. We note that there are results on the Terwilliger algebras of such schemes in the literature, the majority of which are over fields of positive characteristic. In [10], Jiang studied Terwilliger $\mathbb{F}$-algebras of quasi-thin association schemes, where $\mathbb{F}$ is a field of positive characteristic. Recently, Chen and Xi [2] studied Terwilliger algebras of quasi-thin association schemes defined over a field of arbitrary characteristic from the viewpoint of representation theory of algebras. In particular, they showed that Terwilliger algebras of quasi-thin association schemes are always quasi-hereditary cellular algebras in the sense of Cline-Parshall-Scott and of Graham-Lehrer. The results in our paper are however more from a combinatorial and group-theoretical viewpoints.

1.1 Main results

If $\mathfrak{X}$ is a thin association scheme, then $\mathrm{Aut}(\mathfrak{X})$ admits a regular subgroup. It is an easy exercise to show that the subspace T_0 in fact coincides with the full matrix algebra $\mathrm{M}_\Omega(\mathbb{C})$. Therefore, any thin association scheme is triply transitive. For the rest of the paper, we will focus on quasi-thin, and non-thin, association schemes. We will consider the following notations for our purpose.

Hypothesis 1. Consider a quasi-thin association scheme $\mathfrak{X} = (\Omega, \mathcal{R})$ which is not thin, where $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$, and $R_0 = \{(v, v) : v \in \Omega\}$. For any $1 \leqslant i \leqslant d$, let Γ_i be the digraph (Ω, R_i). Assume that the automorphism group $G = \mathrm{Aut}(\mathfrak{X})$ is transitive[4]. In addition, we adopt the following notations.

(a) Let $A_0 = I_{|\Omega|}$ be the identity matrix, and let A_i be the adjacency matrix of the digraph Γ_i, for $1 \leqslant i \leqslant d$.

(b) Fix $v \in \Omega$, and denote the orbits of G_v by

$$\Delta_i = \Delta_i(v) := \{u \in \Omega : (v, u) \in R_i\}$$

for $0 \leqslant i \leqslant d$.

[4]Therefore, Theorem 7 implies that $\mathfrak{X}$ is Schurian

(c) Let $T_0, T,$ and $\tilde{T}$ be respectively the subspace $T_{0,v}$, the algebras T_v and $\mathrm{End}_{G_v}\left(\mathbb{C}^{\Omega}\right)$. We let $E_i^* = E_{i,v}^*$, for all $0 \leqslant i \leqslant d$.

In order to prove the main results, we introduce certain combinatorial structures in quasi-thin association schemes. We call these structures diamond pairs.

Definition 1. Let $\mathfrak{X}$ be an association scheme as in Hypothesis 1. We say that a pair (i, k), where $0 \leqslant i, k \leqslant d$, is a ***diamond pair*** of $\mathfrak{X}$ if there exists $0 \leqslant j \leqslant d$ such that $p_{ij}^k = |\Delta_i| = |\Delta_k| = 2$. In addition, if $H = G_v$ and $H_{|\Delta_i \cup \Delta_k}$ is the restriction of H onto $\Delta_i \cup \Delta_k$, then we say that (i, k) is an $\boldsymbol{H_{|\Delta_i \cup \Delta_k}}$***-diamond pair.***

As we will see later in Lemma 8, the group $H_{|\Delta_i \cup \Delta_k}$ is either isomorphic to the cyclic group $\mathbb{Z}_2$ or the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$. More details on diamond pairs are given later in Section 3.

The main result of this paper provides a complete characterization of Terwilliger algebras of quasi-thin association schemes with transitive automorphism group from a combinatorial and algebraic perspective. In particular, we show that triple transitivity is equivalent to the non-existence of $\mathbb{Z}_2$-diamond pairs.

We state the main result below.

Theorem 2. *If $\mathfrak{X}$ be an association scheme as in Hypothesis 1, then $\mathfrak{X}$ is triply transitive if and only if it does not admit any $\mathbb{Z}_2$-diamond pairs.*

We provide an infinite family of examples of triply transitive association schemes by using Theorem 2. We note that if G is a group and $h \in G$ is an involution, then the action of G on cosets of $\langle h \rangle$ by left multiplication gives rise to a quasi-thin association scheme.

Theorem 3. *If $\mathfrak{X}$ is the orbital scheme of a transitive group $G \leqslant \mathrm{Sym}(\Omega)$ with point stabilizer of order 2, and admits a regular subgroup R, then $\mathfrak{X}$ is triply transitive.*

We provide some examples illustrating the application of Theorem 3 in Examples 25 and 26.

1.2 Organization of the paper

This paper is organized as follows. In Section 2, we recall some results from permutation group theory and the general theory of Schurian association schemes. In Section 3, we give an in-depth analysis of $T_0, T,$ and $\tilde{T}$ whenever the association scheme admits diamond pairs. We prove Theorem 2 in Section 4, and give two examples of how to apply Theorem 2 in Section 5. The proof of Theorem 3 is given in Section 6.

2 Background results

2.1 Permutation group theory

Let Ω be a finite set and $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive group. For any $\omega \in \Omega$, we denote the point stabilizer of ω in G by G_ω. The permutation group $G \leqslant \mathrm{Sym}(\Omega)$ is permutation equivalent to the group action of G on right cosets of G_ω by right multiplication. Conversely, a group action is a permutation group whenever the action is faithful[5].

For any $v \in \Omega$, the ***suborbits*** of G with respect to v are the orbits of G_v on Ω. The suborbits of G with respect to v are in one-to-one correspondence with the orbitals of G, that is, the orbits of G on $\Omega \times \Omega$. In particular, a suborbit Δ of G with respect to v corresponds to the neighbours of v in the digraph (Ω, O), where $O = (v,u)^G$, and $u \in \Delta$. The rank of the transitive group $G \leqslant \mathrm{Sym}(\Omega)$ is the number of suborbits of G (or the number of orbitals).

2.2 Algebras and vector spaces related to the Terwilliger algebra

Let $\mathfrak{X} = (\Omega, \{R_i\}_{i=0}^d)$ be an association scheme with automorphism group $G = \mathrm{Aut}(\mathfrak{X})$. Let A_i be the adjacency matrix of the digraph (Ω, R_i), for $1 \leqslant i \leqslant d$. The Bose-Mesner algebra or the adjacency algebra of $\mathfrak{X}$ is the subalgebra $\mathcal{B}$ of $\mathrm{M}_\Omega(\mathbb{C})$ generated by the adjacency matrices $(A_i)_{i=0,1,\ldots,d}$. The Bose-Mesner algebra acts by left multiplication on the vector space $V = \mathbb{C}^\Omega$. Thus, the vector space V is a left $\mathcal{B}$-module. The centralizer algebra of $\mathfrak{X}$ is the algebra $\mathrm{End}_{G_v}(V)$ consisting of all matrices in $\mathrm{M}_\Omega(\mathbb{C})$ commuting with the elements of G_v, viewed as permutation matrices on Ω.

As explained in the previous section, the Terwilliger algebra T_v is a subalgebra of the centralizer algebra $\mathrm{End}_{G_v}(V)$. The next lemma gives the dimension of these centralizer algebras; its proof is omitted as it is a simple exercise.

Lemma 4. *For any $v \in \Omega$, the dimension of $\mathrm{End}_{G_v}(V)$ is equal to the number of orbits of G_v on $\Omega \times \Omega$.*

Assume now that the automorphism group G of $\mathfrak{X}$ is transitive. Consequently, we may fix $v \in \Omega$ and use T_0, T, and $\tilde{T}$ instead of $T_{0,v}, T_v$, and $\mathrm{End}_{G_v}\left(\mathbb{C}^\Omega\right)$. Let $\Delta_0 = \{v\}, \Delta_1, \ldots, \Delta_d$ be the orbits of G_v, and for $0 \leqslant i \leqslant d$, choose a representative $u_i \in \Delta_i$. If $O = (u,w)^{G_v}$ is an orbital of G_v, then there exists $0 \leqslant i \leqslant d$ such that $u \in \Delta_i = u_i^{G_v}$. Therefore, there exists $w' \in \Omega$ such that $O = (u,w)^{G_v} = (u_i, w')^{G_v}$. We deduce that the number of orbits of G_v on $\Omega \times \Omega$ whose first component is in Δ_i is equal to the number of orbits of $G_v \cap G_{u_i}$ on Ω. Let M_i be the number of orbits of $G_v \cap G_{u_i}$ on Ω. Then, it is not hard to see that

$$\dim(\tilde{T}) = \sum_{i=0}^{d} M_i.$$

Another important subspace of the Terwilliger algebra T is the subspace T_0. Its dimension can be computed using the intersection numbers of $\mathfrak{X}$, which is given in the next lemma.

Lemma 5. *[1] For any $v \in \Omega$, $E_i^* A_j E_k^* \neq 0$ if and only if $p_{ij}^k \neq 0$. Moreover, we have*

$$\dim T_0 = \left|\left\{(i,j,k) : p_{ij}^k \neq 0\right\}\right|,$$

[5] i.e., the kernel of the action is trivial

and

$$\dim E_i^* T_0 E_k^* = \left|\left\{j : p_{ij}^k \neq 0\right\}\right|,$$

for $0 \leqslant i, k \leqslant d$.

2.3 Triply-transitive association schemes

Let $\mathfrak{X} = (\Omega, \{R_i\}_{i=0}^d)$ be an association scheme with transitive automorphism group $G = \mathrm{Aut}(\mathfrak{X})$. Fix $v \in \Omega$. For any $0 \leqslant i \leqslant d$, we define

$$N_i = N_i(v) := \{u \in \Omega : (v, u) \in R_i\}.$$

Let $\Delta_0 = \{v\}, \Delta_1, \ldots, \Delta_r$ be the orbits of G_v. Note that N_i is a union of orbits of G_v, and $\mathfrak{X}$ is Schurian if and only if $d = r$, and so up to reordering, we may assume that $N_i = \Delta_i$, for $0 \leqslant i \leqslant d$. As we saw before, $\mathfrak{X}$ is triply transitive if G is transitive and $T_0 = T = \tilde{T}$. We show that such association schemes are Schurian.

Lemma 6. *If $\mathfrak{X}$ is triply transitive, then it is Schurian.*

Proof. Since $\mathfrak{X}$ is triply transitive, we have

$$\dim E_i^* T_0 E_k^* = \dim E_i^* \tilde{T} E_k^*,$$

for all $0 \leqslant i, k \leqslant d$. On the one hand, for any $0 \leqslant k \leqslant d$, we have

$$\dim E_0^* T_0 E_k^* = \left|\left\{0 \leqslant j \leqslant d : p_{0j}^k \neq 0\right\}\right| = |\{k\}| = 1.$$

On the other hand, for any $0 \leqslant k \leqslant d$, we note that $\dim E_0^* \tilde{T} E_k^*$ is the number of orbits of G_v on N_k. We deduce that G_v is transitive on every N_k, for $0 \leqslant k \leqslant d$. Hence, $r = d$, and $\mathfrak{X}$ is Schurian. □

2.4 Thin and quasi-thin association schemes

Let $\mathfrak{X} = \left(\Omega, \{R_i\}_{i=0}^d\right)$ be an association scheme with automorphism group $G = \mathrm{Aut}(\mathfrak{X})$. If $\mathfrak{X}$ is thin, then it was shown in [3] that it is Schurian. Consequently, for any $\omega \in \Omega$, the orbits of G_ω are of size 1 and therefore G is a regular group. If $\mathfrak{X}$ is a quasi-thin association scheme however, then it can be non-Schurian. For instance, the quasi-thin association scheme `as16n173` in [12] is non-Schurian, and in addition its automorphism group is not transitive. It turns out that this is no coincidence, and holds more generally. Indeed, Hirasaka and Muzychuk [8, 9] showed that a quasi-thin association scheme has a transitive automorphism group if and only if it is Schurian. We state this result in the next theorem as we will refer to it later.

Theorem 7 (Hirasaka-Muzychuk)**.** *If $\mathfrak{X}$ is a quasi-thin association schemes, then* $\mathrm{Aut}(\mathfrak{X})$ *is transitive if and only if $\mathfrak{X}$ is Schurian.*

For the remainder of this section, we let $\mathfrak{X}$ be a quasi-thin association scheme which is not thin, and admits a transitive automorphism group. By Theorem 7, the association scheme $\mathfrak{X}$ is Schurian. Hence, the valencies of $\mathfrak{X}$ are exactly the numbers $|\Delta_0|, |\Delta_1|, \dots,$ and $|\Delta_d|$. Using the fact that the valencies of $\mathfrak{X}$ are 1 or 2, we can derive some results on the point stabilizer of $G = \mathrm{Aut}(\mathfrak{X})$ as follows.

Lemma 8. *Let $v \in \Omega$ and denote the subgroup G_v by H. The following statements hold.*

(i) H is an elementary abelian 2-group.

(ii) If $\Delta_i = \{u_1, u_2\}$ and $\Delta_k = \{v_1, v_2\}$ are two orbits of H, then the restriction $H_{|\Delta_i \cup \Delta_k}$ of H on $\Delta_i \cup \Delta_k$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. (i) Since $\mathfrak{X}$ is not thin, $H \neq \{1\}$. Given $h \in H$ such that $h \neq 1$, it is clear that $\langle h \rangle$ also has orbits of length 1 or 2, and thus h has order equal to 2. Consequently, every non-identity element of H has order 2, so H is an elementary abelian 2-group.

(ii) The restriction $H_{|\Delta_i \cup \Delta_k}$ is intransitive on $\Delta_i \cup \Delta_k$, but is transitive on Δ_i and Δ_k. The only subgroups of $\mathrm{Sym}(\Delta_i \cup \Delta_k)$ that have such properties are isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

3 Diamond pairs

Diamond pairs were already introduced in Definition 1. In this section, we give an in-depth analysis of diamond pairs and the combinatorial consequences of their existence. Henceforth, we let $\mathfrak{X} = (\Omega, \mathcal{R})$ be a quasi-thin association scheme as in Hypothesis 1.

Recall that the pair (i, k), where $0 \leqslant i, k \leqslant d$, is a diamond pair of $\mathfrak{X}$ if there exists $0 \leqslant j \leqslant d$ such that $p_{ij}^k = |\Delta_i| = |\Delta_k| = 2$. In other words, if (i, k) is a diamond pair, then $\Delta_i = \{u_1, u_2\}$, and $\Delta_k = \{w_1, w_2\}$, and we have the configuration in Figure 1. Note that if (i, k) is a diamond pair, then $|\Delta_j| = 2$.

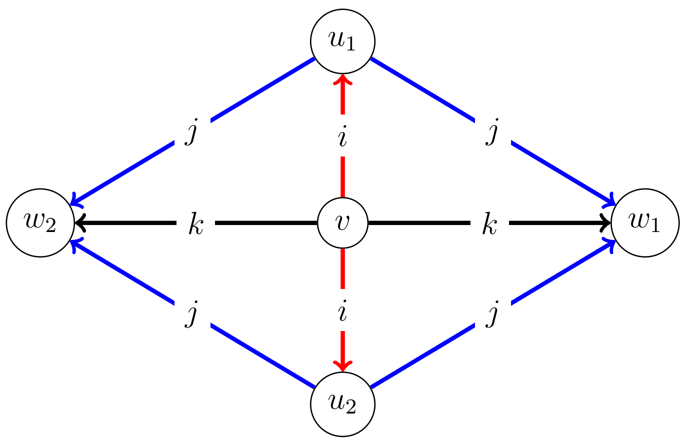

Figure 1: Configuration for diamond pairs.

If (i,k) is not a diamond pair, then either $|\Delta_i| + |\Delta_k| \leqslant 3$ or $\Delta_i = \{u_1, u_2\}$ and $\Delta_k = \{w_1, w_2\}$, and $p_{ij}^k \neq 2$ for all $0 \leqslant j \leqslant d$. By definition of an association scheme, there exists $0 \leqslant j \leqslant d$ such that $p_{ij}^k \neq 0$. If $p_{ij}^k = 1$, for some $0 \leqslant j \leqslant d$, then $E_i^* A_j E_k^*$ is non-zero, and using the fact that $\mathfrak{X}$ is quasi-thin, there exist a unique $0 \leqslant j' \leqslant d$ such that $p_{ij'}^k = 1$ and $E_i^* A_{j'} E_k^*$ is non-zero. Therefore, if (i,k) is not a diamond pair and $|\Delta_i| = |\Delta_k| = 2$, then we have the configuration in Figure 2 for some $0 \leqslant j, j' \leqslant d$.

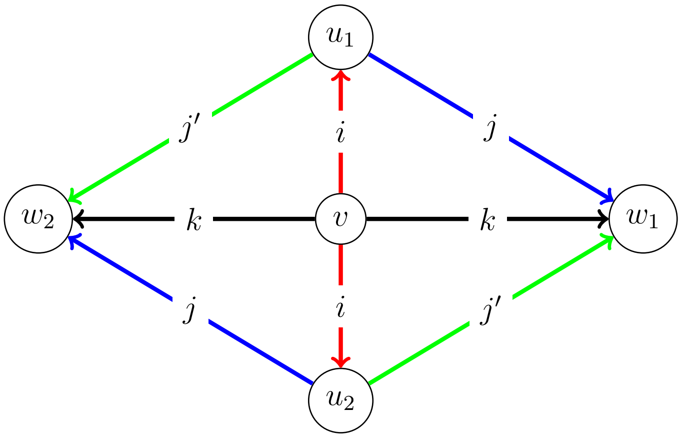

Figure 2: Configuration for non-diamond pairs.

By Lemma 8, we know that the restriction of a point stabilizer of a union of two orbits of size 2 must be isomorphic to the cyclic group $\mathbb{Z}_2$ or the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, there are two types of diamond pairs, depending on the orbits of point stabilizer G_v, which are introduced below.

Definition 9. Assume that (i,k) is a diamond pair of $\mathfrak{X}$. Fix $v \in \Omega$ and let $H = G_v$.

- If $H_{|\Delta_i \cup \Delta_k} \cong \mathbb{Z}_2$, then (i,k) is a ***$\mathbb{Z}_2$-diamond pair***.
- If $H_{|\Delta_i \cup \Delta_k} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then (i,k) is a ***$\mathbb{Z}_2 \times \mathbb{Z}_2$-diamond pair***.

Since the properties in the above definition depends on G_v, these two types of diamond pairs are indistinguishable from the viewpoint of the subspace T_0.

For the remainder of this section, we will exhibit the properties that diamond pairs (in the underlying association scheme) impose on the subspace T_0 and the algebra $\tilde{T}$.

3.1 The subspace T_0

Given a matrix $A \in \mathrm{M}_\Omega(\mathbb{C})$, the matrix $E_i^* A E_k^*$ is a matrix whose entries outside $\Delta_i \times \Delta_k$ are equal to 0, and those within $\Delta_i \times \Delta_k$ are equal to the entries of A. For any $A \in \mathrm{M}_\Omega(\mathbb{C})$, we let $\overline{E_i^* A E_k^*}$ be the submatrix of A determined by the rows Δ_i and columns Δ_k. For any $n \geqslant 2$, we denote the identity matrix and the all-ones matrix of size n by I_n and J_n, respectively.

We state the following propositions whose proofs are straightforward by counting the number of non-zero intersection numbers and using Figure 1 or Figure 2.

Proposition 10. *Let $0 \leqslant i, k \leqslant d$. If (i, k) is not a diamond pair and $|\Delta_i| = |\Delta_k| = 2$, then there exists a unique pair (j, j') of distinct integers with $0 \leqslant j, j' \leqslant d$ such that*

$$E_i^* T_0 E_k^* = \mathrm{Span}\{E_i^* A_j E_k^*, E_i^* A_{j'} E_k^*\}.$$

In particular, $\{\overline{E_i^* A_j E_k^*}, \overline{E_i^* A_{j'} E_k^*}\} = \{I_2, J_2 - I_2\}$, *and* $\dim E_i^* T_0 E_k^* = 2$.

Proposition 11. *If (i, k) is a diamond pair, then there exists a unique $0 \leqslant j \leqslant d$ such that* $E_i^* T_0 E_k^* = \mathrm{Span}\{E_i^* A_j E_k^*\}$ *where* $\overline{E_i^* A_j E_k^*} = J_2$. *In particular, we have* $\dim E_i^* T_0 E_k^* = 1$.

Propositions 10 and 11 provide an important tool to understanding the dimension of the subspace T_0. The case where $|\Delta_i| + |\Delta_k| \leqslant 3$ will be considered in the next subsection.

3.2 The centralizer algebra

We will determine precisely the pair of integers $0 \leqslant i, k \leqslant d$ such that the subspaces $E_i^* T_0 E_k^*$ and $E_i^* \tilde{T} E_k^*$ coincide. For $u, w \in \Omega$, we define the matrix $E_{u,w} \in \mathrm{M}_\Omega(\mathbb{C})$ indexed by Ω in its rows and columns, such that the (u, w)-entry of $E_{u,w}$ is 1, and every other entry is equal to 0. For any orbit O of a permutation group $H \leqslant G$ on $\Omega \times \Omega$, we define the matrix $A(\Omega, O)$ to be the adjacency matrix of the digraph (Ω, O). If $O_0, O_1, \ldots, O_l$ are the orbits of G_v on $\Omega \times \Omega$, then it is well known that

$$\tilde{T} = \mathrm{Span}\left\{A(\Omega, O_i) : 0 \leqslant i \leqslant l\right\}.$$

The following lemma will be useful in the proof of the main result of this section.

Lemma 12. *For any $0 \leqslant i \leqslant d$, the orbital R_i of G on Ω is a union of orbitals of G_v on Ω.*

Proof. The proof immediately follows from the fact that G_v is a subgroup of G. ☐

Lemma 13. *Let $0 \leqslant i, k \leqslant d$. If (i, k) is not a diamond pair of $\mathfrak{X}$, then*

$$E_i^* T_0 E_k^* = E_i^* \tilde{T} E_k^*.$$

Moreover, if $|\Delta_i| = |\Delta_k| = 2$, then $H_{|\Delta_i \cup \Delta_k} \cong \mathbb{Z}_2$, where $H = G_v$.

Proof. Assume that (i, k) is not a diamond pair of $\mathfrak{X}$. To show that $E_i^* T_0 E_k^* = E_i^* \tilde{T} E_k^*$, it is enough to show that $A(\Omega, O) \in T_0$ for all orbitals $O \subseteq \Delta_i \times \Delta_k$ of G_v on Ω.

Claim 14. *If $O \subseteq \Delta_i \times \Delta_k$ is an orbital of G_v on Ω and $|\Delta_i| = |\Delta_k| = 1$, then $A(\Omega, O) \in T_0$.*

Proof of Claim 14. Clearly, $O = \Delta_i \times \Delta_k$, and we may assume that $O = \Delta_i \times \Delta_k = \{(u, w)\}$. Note that $A(\Omega, O) = E_{u,w}$, since $|\Delta_i| = |\Delta_k| = 1$. If $0 \leqslant j \leqslant d$ such that $(u, w) \in R_j$, then

$$E_i^* A_j E_k^* = E_{u,w} = A(\Omega, O) \in T_0.$$

☐

Claim 15. *If $O \subseteq \Delta_i \times \Delta_k$ is an orbital of G_v on Ω and $|\Delta_i| = 1$ and $|\Delta_k| = 2$, then $A(\Omega, O) \in T_0$.*

Proof of Claim 15. Assume that $\Delta_i = \{u\}$ and $\Delta_k = \{w, w'\}$. Using the fact that $O \subseteq \Delta_i \times \Delta_k$, it is not hard to see that $O = \{(u, w), (u, w')\}$. Thus, $A(\Omega, O) = E_{u,w} + E_{u,w'}$. Let $0 \leqslant j \leqslant d$ such that $(u, w) \in R_j$. Since O is an orbital of G_v on Ω, by Lemma 12, the orbital R_j is a union of orbitals of G_v on Ω, thus including the orbital O. Hence, we deduce that $(u, w') \in R_j$, and consequently

$$A(\Omega, O) = E_{u,w} + E_{u,w'} = E_i^* A_j E_k^* \in T_0.$$

$\square$

Claim 16. *If $O \subseteq \Delta_i \times \Delta_k$ is an orbital of G_v on Ω with $|\Delta_i| = 2$ and $|\Delta_k| = 1$, then $A(\Omega, O) \in T_0$.*

Proof of Claim 16. Assume that $\Delta_i = \{u, u'\}$ and $\Delta_k = \{w\}$. Then, it is clear that the orbital O is equal to $\{(u, w), (u', w)\}$. Hence, $A(\Omega, O) = E_{u,w} + E_{u',w}$. Let $0 \leqslant j \leqslant d$ such that $(u, w) \in R_j$. Using Lemma 12, we deduce as in the previous claim that $(u', w) \in R_j$. Then, we have

$$A(\Omega, O) = E_{u,w} + E_{u',w} = E_i^* A_j E_k^* \in T_0.$$

$\square$

Claim 17. *If $O \subseteq \Delta_i \times \Delta_k$ is an orbital of G_v on Ω and $|\Delta_i| = |\Delta_k| = 2$, then $A(\Omega, O) \in T_0$.*

Proof of Claim 17. Let $\Delta_i = \{u_1, u_2\}$ and $\Delta_k = \{w_1, w_2\}$, and denote G_v by H. By Lemma 8, we know that $H_{|\Delta_i \cup \Delta_k}$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. We will distinguish these two cases.

If $H_{|\Delta_i \cup \Delta_k} \cong \mathbb{Z}_2$, then there are exactly two orbitals of H in $\Delta_i \times \Delta_k$, which are

$$O_1 = \{(u_1, w_1), (u_2, w_2)\} \text{ and } O_2 = \{(u_1, w_2), (u_2, w_1)\}.$$

In this case, we note that $E_i^* \tilde{T} E_k^* = \operatorname{Span}\{A(\Omega, O_1), A(\Omega, O_2)\}$. Since (i, k) is not a diamond pair, there exist two distinct integers $0 \leqslant j, j' \leqslant d$ such that $(u_1, w_1), (u_2, w_2) \in R_j$ and $(u_2, w_1), (u_1, w_2) \in R_{j'}$ (see Figure 2). Hence, we have

$$A(\Omega, O_1) = E_i^* A_j E_k^* \in T_0 \text{ and } A(\Omega, O_2) = E_i^* A_{j'} E_k^* \in T_0.$$

Since $O \in \{O_1, O_2\}$, we have $A(\Omega, O) \in T_0$.

If $H_{|\Delta_i \cup \Delta_k} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $i \neq k$ and there is an involution in H that fixes the two elements of Δ_i pointwise and swaps the elements of Δ_k. Similarly, there exists another involution in H which fixes the elements of Δ_k pointwise and swaps the elements of Δ_i. Consequently, the set $\Delta_i \times \Delta_k$ is itself an orbit of H in its action on $\Omega \times \Omega$. Therefore,

$$O = \{(u_1, w_1), (u_2, w_2), (u_1, w_2), (u_2, w_1)\}.$$

By Lemma 12, we note that there exists $0 \leqslant j \leqslant d$ such that

$$(u_1, w_1), (u_2, w_2), (u_1, w_2), (u_2, w_1) \in R_j.$$

Hence, $p_{ij}^k = 2$, and (i, k) must be a diamond pair, which is a contradiction. Therefore, $H_{|\Delta_i \cup \Delta_k}$ cannot be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ □

With Claims 14-17, we have shown that if (i, k) is not a diamond pair, then $E_i^* T_0 E_k^* = E_i^* \tilde{T} E_k^*$. In the proof of Claim 17, we also proved that if $|\Delta_i| = |\Delta_k| = 2$, then $H_{|\Delta_i \cup \Delta_k} \cong \mathbb{Z}_2$, where $H = G_v$. This completes the proof. □

Lemma 18. *If (i, k) is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-diamond pair of $\mathfrak{X}$, then $\dim E_i^* \tilde{T} E_k^* = 1$, and thus*

$$E_i^* \tilde{T} E_k^* = E_i^* T_0 E_k^*.$$

Proof. As (i, k) is a diamond pair, we have seen in Proposition 11 that $\dim E_i^* T_0 E_k^* = 1$. So it remains to show that $\dim E_i^* \tilde{T} E_k^* = 1$. Let $H = G_v$ and assume that $\Delta_i = \{u_1, u_2\}$ and $\Delta_k = \{w_1, w_2\}$. Using the fact that $H_{|\Delta_i \cup \Delta_k} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we know that there exists an element of $H_{|\Delta_i \cup \Delta_j}$ fixing Δ_i pointwise, and permuting the elements of Δ_k. Similarly, there exists an element of $H_{|\Delta_i \cup \Delta_k}$ fixing Δ_k pointwise and permuting Δ_i. Hence, the set $O = \Delta_i \times \Delta_k = \{(u_1, w_1), (u_1, w_2), (u_2, w_1), (u_2, w_2)\}$ is an orbit of H on $\Omega \times \Omega$. We deduce that

$$E_i^* \tilde{T} E_k^* = \operatorname{Span}\{A(\Omega, O)\}.$$

Consequently, we have that $\dim E_i^* \tilde{T} E_k^* = 1$, and $E_i^* \tilde{T} E_k^* = E_i^* T_0 E_k^*$. □

Lemma 19. *If (i, k) is a $\mathbb{Z}_2$-diamond pair of $\mathfrak{X}$, then $\dim E_i^* \tilde{T} E_k^* = 2$, and*

$$E_i^* \tilde{T} E_k^* = \operatorname{Span}\{E_i^* A E_k^*, E_i^* B E_k^*\},$$

for some matrices $A, B \in \tilde{T} \setminus T_0$ such that $\overline{E_i^* A E_k^*} = I_2$ and $\overline{E_i^* B E_k^*} = J_2 - I_2$.

Proof. Let $\Delta_i = \{u_1, u_2\}$ and $\Delta_k = \{w_1, w_2\}$. As (i, k) is a $\mathbb{Z}_2$-diamond pair, the subgroup $H_{|\Delta_i \cup \Delta_k}$ is isomorphic to $\mathbb{Z}_2$. That is, every element of $H = G_v$ that swaps the elements of Δ_i also swaps the element of Δ_k, and vice versa. Since $|\Delta_i| = |\Delta_k| = 2$, there are two orbitals of H in $\Delta_i \times \Delta_k$, which are

$$O_1 = \{(u_1, w_1), (u_2, w_2)\} \text{ and } O_2 = \{(u_1, w_2), (u_2, w_1)\}.$$

Hence,

$$E_i^* \tilde{T} E_k^* = \operatorname{Span}\{A(\Omega, O_1), A(\Omega, O_2)\}.$$

In particular, we have $\left\{\overline{E_i^* A(\Omega, O_1) E_k^*}, \overline{E_i^* A(\Omega, O_2) E_k^*}\right\} = \{I_2, J_2 - I_2\}$. As (i, k) is a diamond pair we know that there exists a unique $0 \leqslant j \leqslant d$ such that

$$(u_1, w_1), (u_1, w_2), (u_2, w_1), (u_2, w_2) \in R_j,$$

and

$$E_i^* T_0 E_k^* = \operatorname{Span}\{E_i^* A_j E_k^*\}$$

where $\overline{E_i^* A_j E_k^*} = J_2$. Therefore,

$$A(\Omega, O_1), A(\Omega, O_2) \notin T_0.$$

This completes the proof. □

4 Proof of Theorem 2

In this section, we prove Theorem 2. We first determine when such association schemes are triply regular, then we provide a necessary and sufficient condition for T to be equal to $\tilde{T}$. The proof of Theorem 2 follows immediately from these two results.

Throughout this section, we let $\mathfrak{X}$ be an association schemes as in Hypothesis 1. Recall that $\mathfrak{X}$ is triply regular if $T_0 = T$, or equivalently, T_0 is an algebra, so it is closed under multiplication. In the next lemma, we give a necessary and sufficient condition for $\mathfrak{X}$ to be triply regular.

Lemma 20. *Let $\mathfrak{X}$ be an association scheme as in Hypothesis 1. The following statements are equivalent.*

(i) $\mathfrak{X}$ is triply regular.

(ii) For any diamond pair (i, k) of $\mathfrak{X}$, there exists no $0 \leqslant \ell \leqslant d$ with $|\Delta_\ell| = 2$ such that (i, ℓ) and (ℓ, k) are not diamond pairs.

Proof. (i) $\Rightarrow$ (ii) Suppose that $T_0 = T$, or equivalently, T_0 is closed under multiplication. Assume by contradiction that there exists a diamond pair (i, k) of $\mathfrak{X}$ and an integer $0 \leqslant \ell \leqslant d$ with $|\Delta_\ell| = 2$ such that (i, ℓ) and (ℓ, k) are not diamond pairs. By Proposition 11, there exists a unique $0 \leqslant j \leqslant d$ such that $E_i^* T_0 E_k^* = \operatorname{Span}\{E_i^* A_j E_k^*\}$ and $\overline{E_i^* A_j E_k^*} = J_2$. By Proposition 10, we can find $0 \leqslant t, t' \leqslant d$ such that $\overline{E_i^* A_t E_\ell^*} = I_2$, and $\overline{E_\ell^* A_{t'} E_k^*} = I_2$. Consequently, we have $(E_i^* A_t E_\ell^*)(E_\ell^* A_{t'} E_k^*) \notin E_i^* T_0 E_k^* = \operatorname{Span}\{E_i^* A_j E_k^*\}$. This is impossible since T_0 is closed under multiplication. We conclude that (ii) holds.

(i) $\Leftarrow$ (ii) Conversely, assume that for any diamond pair (i, k) of $\mathfrak{X}$, there exists no $0 \leqslant \ell \leqslant d$ with $|\Delta_\ell| = 2$ such that (i, ℓ) and (ℓ, k) are not diamond pairs. Assume by contradiction that T_0 is not closed under multiplication. Then, there exist three integers $0 \leqslant j, j', \ell \leqslant d$ such that $(E_i^* A_j E_\ell^*)(E_\ell^* A_{j'} E_k^*) \notin T_0$.

If (i, k) is not a diamond pair, then by Lemma 13, we know that $E_i^* T_0 E_k^* = E_i^* \tilde{T} E_k^*$. Using the fact that $(E_i^* A_j E_\ell^*)(E_\ell^* A_{j'} E_k^*) \in E_i^* \tilde{T} E_k^*$, we have $(E_i^* A_j E_\ell^*)(E_\ell^* A_{j'} E_k^*) \in E_i^* \tilde{T} E_k^* = E_i^* T_0 E_k^* \subseteq T_0$, which is impossible. Therefore, (i, k) is a diamond pair and $|\Delta_i| = |\Delta_k| = 2$. If $|\Delta_\ell| = 1$, then (i, ℓ) and (ℓ, k) are not diamond pairs, and we have

$$\overline{(E_i^* A_j E_\ell^*)(E_\ell^* A_{j'} E_k^*)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

implying that $(E_i^*A_jE_\ell^*)(E_\ell^*A_{j'}E_k^*) \in T_0$, a contradiction. Therefore, $|\Delta_\ell| = 2$. By (ii), we know that for $0 \leqslant \ell \leqslant d$ with $|\Delta_\ell| = 2$, one of (i, ℓ) or (ℓ, k) is a diamond pair. Without loss of generality, assume that (i, ℓ) is a diamond pair. By Proposition 11, there exists a unique integer $0 \leqslant j \leqslant d$ such that $E_i^*A_jE_\ell^* \neq 0$, and $\overline{E_i^*A_jE_\ell^*} = J_2$. As

$$\overline{E_\ell^*A_{j'}E_k^*} \in \{J_2, I_2, J_2 - I_2\},$$

we have $\overline{E_i^*A_jE_\ell^*E_\ell^*A_{j'}E_k^*}$ is a multiple of J_2. We conclude that $(E_i^*A_jE_\ell^*)(E_\ell^*A_{j'}E_k^*) \in E_i^*T_0E_k^* = \mathrm{Span}\{E_i^*A_jE_k^*\} \subseteq T_0$, which is a contradiction. Hence, T_0 must be closed under multiplication, and so $T_0 = T$. □

Lemma 21. *Let $\mathfrak{X}$ be an association scheme as in Hypothesis 1. The following statements are equivalent.*

(a) $T = \tilde{T}$.

(b) For any $\mathbb{Z}_2$-diamond pair (i, k) of $\mathfrak{X}$, there exists an integer $0 \leqslant \ell \leqslant d$ with $|\Delta_\ell| = 2$ such that (i, ℓ) and (ℓ, k) are not diamond pairs.

Proof. (a) ⇒ (b). Assume that $T = \tilde{T}$. If (i, k) is a $\mathbb{Z}_2$-diamond pair, then we know that $\dim E_i^*T_0E_k^* = 1$ and by (a), $\dim E_i^*TE_k^* = \dim E_i^*\tilde{T}E_k^* = 2$. We deduce that there exist $0 \leqslant \ell, j, j' \leqslant d$ such that $(E_i^*A_jE_\ell^*)(E_\ell^*A_{j'}E_k^*) \notin T_0$. If $|\Delta_\ell| = 1$, then (i, ℓ) and (ℓ, k) are not diamond pairs, and we have

$$\overline{(E_i^*A_jE_\ell^*)(E_\ell^*A_{j'}E_k^*)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

implying that $(E_i^*A_jE_\ell^*)(E_\ell^*A_{j'}E_k^*) \in T_0$, a contradiction. Hence, $|\Delta_\ell| = 2$. If one of (i, ℓ) or (ℓ, k) is a diamond pair, then we either have $\overline{E_i^*A_jE_\ell^*} = J_2$ or $\overline{E_\ell^*A_{j'}E_k^*} = J_2$. It is not hard to see that in either case, $(E_i^*A_jE_\ell^*)(E_\ell^*A_{j'}E_k^*) \in E_i^*T_0E_k^*$, which is a contradiction again. Therefore, none of (i, ℓ) and (ℓ, k) are diamond pairs.

(b) ⇒ (a) It is enough to show that $E_i^*TE_k^* = E_i^*\tilde{T}E_k^*$ for all $0 \leqslant i, k \leqslant d$. If (i, k) is not a diamond pair, then by Proposition 10 and Lemma 13, we have $E_i^*T_0E_k^* = E_i^*\tilde{T}E_k^*$, and in particular, $E_i^*TE_k^* = E_i^*\tilde{T}E_k^*$.

Assume that (i, k) is a diamond pair. Then, $\dim E_i^*T_0E_k^* = 1$. If (i, k) is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-diamond pair, then by Lemma 18, we know that $\dim E_i^*\tilde{T}E_k^* = 1$, so it is clear that $E_i^*T_0E_k^* = E_i^*TE_k^* = E_i^*\tilde{T}E_k^*$. If (i, k) is a $\mathbb{Z}_2$-diamond pair, then by (b) there exists an $0 \leqslant \ell \leqslant d$ such that (i, ℓ) and (ℓ, k) are not diamond pairs. So, there exist $0 \leqslant j, j' \leqslant d$ such that $(E_i^*A_jE_\ell)(E_\ell A_{j'}E_k^*) \in T \setminus T_0$. In particular, $1 = \dim E_i^*T_0E_k^* < \dim E_i^*TE_k^* \leqslant \dim E_i^*\tilde{T}E_k^* = 2$, so we must have $E_i^*TE_k^* = E_i^*\tilde{T}E_k^*$, for every $0 \leqslant i, k \leqslant d$ such that (i, k) is a $\mathbb{Z}_2$-diamond pair. Consequently, $T = \tilde{T}$.

□

Now, we are ready to prove Theorem 2.

Proof of Theorem 2. If $\mathfrak{X}$ is triply transitive, then $T_0 = T = \tilde{T}$. By Lemma 20 and Lemma 21, $\mathfrak{X}$ cannot admit $\mathbb{Z}_2$-diamond pairs. Conversely, if $\mathfrak{X}$ does not admit $\mathbb{Z}_2$-diamond pair, then Lemma 21(b) vacuously hold, so we have $T = \tilde{T}$. By assumption, $\mathfrak{X}$ admits only $\mathbb{Z}_2 \times \mathbb{Z}_2$-diamond pairs, so, it is enough to show that Lemma 20(ii) holds for $\mathbb{Z}_2 \times \mathbb{Z}_2$-diamond pairs. By contradiction, assume that (ii) does not hold. Then, there exists a $\mathbb{Z}_2 \times \mathbb{Z}_2$-diamond pair (i, k) of $\mathfrak{X}$ and an integer $0 \leqslant \ell \leqslant d$ with $|\Delta_\ell| = 2$ such that (i, ℓ) and (ℓ, k) are not diamond pairs. This implies $E_i^* T_0 E_k^*$ is not closed under multiplication, which is a contradiction to Lemma 18, so (ii) holds and $T_0 = T$. $\square$

5 Examples of triply-transitive quasi-thin association schemes

Theorem 2 gives a characterization of the triply-transitive quasi-thin association schemes by counting the number of diamond pairs. We give two examples of quasi-thin association schemes admitting transitive automorphism groups that are not triply transitive. These two examples are in fact only such examples with order up to 30.

We will see examples arising from transitive permutation group actions of a group by right multiplication on the subgroup generated by an involution. If G is a group and $H = \langle b \rangle \leqslant G$ has order 2, then the action of G on right cosets of H by right multiplication is faithful, and is therefore permutation equivalent to a transitive permutation group of degree $[G : H]$. By choice of H, the orbital scheme arising from this transitive group is quasi-thin, and if Δ_i and Δ_k are two orbits of H of size 2, then $H_{|\Delta_i \cup \Delta_k} = \mathbb{Z}_2$.

Example 22. Consider the group

$$G = \mathrm{Alt}(5) = \left\langle a, b | a^5 = b^2 = (ab)^3 = 1 \right\rangle$$

and its subgroup $H = \langle b \rangle$. There is a unique conjugacy class of subgroups of order 2 in G, so all actions of G on cosets of a subgroup of order 2 are permutation equivalent to G acting on cosets of H. This group is permutation equivalent to `TransitiveGroup(30,9)` in the library of transitive groups of `Sagemath` [18].

If $g \in G$ has a fixed point in this action, then g is conjugate in H. Therefore, the only permutations in G fixing a coset of H must be the identity or an involution. Moreover, the number of fixed cosets is either equal to $[G : H] = 30$ or the number of orbits of size 1 of H(equivalently, the number of cosets fixed by b). If $(Hx)b = Hx$ for some $x \in G$, then we have $xbx^{-1} \in H$. In other words, if Hx is fixed by b, then $x \in \mathrm{N}_G(\langle b \rangle)$. It is not hard to see that

$$\mathrm{N}_G(\langle b \rangle) = \mathrm{C}_G(\langle b \rangle) = \langle b \rangle \times \mathbb{Z}_2.$$

Therefore, b fixes two cosets of H. A direct consequence of this fact is that the rank of G acting by right multiplication on right cosets of H is $1 + 1 + \frac{28}{2} = 16$, and

$$\dim \tilde{T} = 2 \times 16 + 14 \times 30 = 452.$$

As the rank of G acting on cosets of H is $1 + 1 + 14 = 16$, we may assume that the orbits of H are $\Delta_1 = \{H\}$, $\Delta_2 = \{Hx_2\}$, $\Delta_3 = \{Hx_3, Hy_3\}$, $\ldots$, and $\Delta_{16} = \{Hx_{16}, Hy_{16}\}$.

Since the scheme is quasi-thin, we know that $p_{ij}^k \leqslant 2$, for all $0 \leqslant i, j, k \leqslant d$. There exists a unique pair (i, k) where $0 \leqslant i, k \leqslant d$ and $|\Delta_i| = |\Delta_k| = 2$ such that $p_{ij}^k = p_{kj}^i = 2$. Hence, these are $\mathbb{Z}_2$-diamond pairs. The association scheme can be represented by the matrix given below. The entries in blue and red correspond to the pairs of orbits forming $\mathbb{Z}_2$-diamond pairs.

```
[ 0  1  2  3  4  5  5  4  6  7  3  2  8  9 10 11 12 13  8  9 14 14 15 15 11 10 13 12  6  7]
[ 1  0 13 12  9  8  8  9 10 11 12 13  5  4  6  7  3  2  5  4 14 14 15 15  7  6  2  3 10 11]
[ 2  3  0  1  4  5  2  3 14 14  4  5 11 10  9  8  6  7 15 15  7  6  8  9 10 11 12 13 12 13]
[13 12  1  0  9  8 13 12 14 14  9  8  7  6  4  5 10 11 15 15 11 10  5  4  6  7  3  2  3  2]
[ 8  9  8  9  0  1 13 12  2  3 12 13  4  5  5  4 11 10  6  7  3  2  6  7 15 15 14 14 11 10]
[ 5  4  5  4  1  0  2  3 13 12  3  2  9  8  8  9  7  6 10 11 12 13 10 11 15 15 14 14  7  6]
[ 5  4  2  3  3  2  0  1  7  6  4  5 10 11 15 15 14 14  9  8 13 12 11 10  9  8  6  7 13 12]
[ 8  9 13 12 12 13  1  0 11 10  9  8  6  7 15 15 14 14  4  5  2  3  7  6  4  5 10 11  2  3]
[11 10 15 15  2  3  7  6  0  1 13 12  4  5  2  3  8  9 14 14  4  5 12 13  8  9  6  7 10 11]
[ 7  6 15 15 13 12 11 10  1  0  2  3  9  8 13 12  5  4 14 14  9  8  3  2  5  4 10 11  6  7]
[13 12  8  9 12 13  8  9  3  2  0  1 15 15  7  6  2  3  7  6 10 11  4  5  5  4 11 10 14 14]
[ 2  3  5  4  3  2  5  4 12 13  1  0 15 15 11 10 13 12 11 10  6  7  9  8  8  9  7  6 14 14]
[ 4  5  6  7  8  9 10 11  8  9 14 14  0  1 13 12  4  5  2  3 12 13 11 10  6  7  2  3 15 15]
[ 9  8 10 11  5  4  6  7  5  4 14 14  1  0  2  3  9  8 13 12  3  2  7  6 10 11 13 12 15 15]
[10 11  9  8  5  4 14 14  2  3  7  6  3  2  0  1 15 15  7  6  4  5 13 12 11 10 12 13  9  8]
[ 6  7  4  5  8  9 14 14 13 12 11 10 12 13  1  0 15 15 11 10  9  8  2  3  7  6  3  2  4  5]
[12 13 11 10  6  7 15 15  4  5  2  3  8  9 14 14  0  1 13 12 11 10  4  5  2  3  8  9  7  6]
[ 3  2  7  6 10 11 15 15  9  8 13 12  5  4 14 14  1  0  2  3  7  6  9  8 13 12  5  4 11 10]
[ 4  5 14 14 11 10  9  8 15 15  7  6  2  3  7  6  3  2  0  1 13 12 10 11 12 13  5  4  8  9]
[ 9  8 14 14  7  6  4  5 15 15 11 10 13 12 11 10 12 13  1  0  2  3  6  7  3  2  8  9  5  4]
[15 15  7  6 13 12  3  2  8  9 10 11 12 13  8  9  6  7  3  2  0  1 14 14  4  5 11 10  5  4]
[15 15 11 10  2  3 12 13  5  4  6  7  3  2  5  4 10 11 12 13  1  0 14 14  9  8  7  6  8  9]
[14 14  4  5 11 10  6  7 12 13  8  9  6  7  3  2  8  9 10 11 15 15  0  1 13 12  4  5  3  2]
[14 14  9  8  7  6 10 11  3  2  5  4 10 11 12 13  5  4  6  7 15 15  1  0  2  3  9  8 12 13]
[ 6  7 10 11 14 14  9  8  4  5  5  4 11 10  6  7  2  3 12 13  8  9  3  2  0  1 15 15 13 12]
[10 11  6  7 14 14  4  5  9  8  8  9  7  6 10 11 13 12  3  2  5  4 12 13  1  0 15 15  2  3]
[ 3  2 12 13 15 15 11 10 11 10  6  7  2  3 12 13  4  5  5  4  6  7  8  9 14 14  0  1  9  8]
[12 13  3  2 15 15  7  6  7  6 10 11 13 12  3  2  9  8  8  9 10 11  5  4 14 14  1  0  4  5]
[11 10 12 13  6  7  3  2 10 11 15 15 14 14  9  8  7  6  4  5  5  4 13 12  3  2  9  8  0  1]
[ 7  6  3  2 10 11 12 13  6  7 15 15 14 14  4  5 11 10  9  8  8  9  2  3 12 13  4  5  1  0]
```

The subspace T_0 coincides with $\tilde{T}$ everywhere except on the orbits corresponding to the blocks in blue and red. In each of these blocks, the difference in dimension between the block of T_0 and $\tilde{T}$ is 1. We deduce that $\dim T_0 = 450$, so $\mathfrak{X}$ is not triply transitive. Using Lemma 21, we also deduce that $T = \tilde{T}$.

Example 23. Let $G = \mathrm{AGL}_1(8)$. Then, G acts doubly transitively on the field $\mathbb{F}_8$ as affine transformations. Therefore, G also acts transitively on the 2-subsets of $\mathbb{F}_8$. This transitive permutation group is `TransitiveGroup(28,11)` in the library of transitive groups in `Sagemath`. The stabilizer of a 2-subsets of $\mathbb{F}_8$ is therefore a cyclic group of order 2, and the orbital scheme is quasi-thin. The resulting orbital scheme is given below, where the entries highlighted in colours correspond to $\mathbb{Z}_2$-diamond pairs.

```
[ 0  1  2  2  3  4  5  5  6  7  8  9 10 11 12 13 14 14  3  4 15 15  7  6  9  8 10 11]
[ 1  0 14 14  4  3 15 15  6  7  9  8 10 11 13 12  2  2  4  3  5  5  7  6  8  9 10 11]
[10 11  0  1  2  2  4  3 15 15  7  6  8  9 10 11 12 13 14 14  4  3  5  5  6  7  9  8]
[10 11  1  0 14 14  3  4  5  5  7  6  9  8 10 11 13 12  2  2  3  4 15 15  6  7  8  9]
[ 8  9 10 11  0  1  2  2  3  4  5  5  7  6  9  8 10 11 12 13 14 14  3  4 15 15  6  7]
[ 9  8 10 11  1  0 14 14  4  3 15 15  7  6  8  9 10 11 13 12  2  2  4  3  5  5  6  7]
[ 7  6  9  8 10 11  0  1  2  2  4  3 15 15  6  7  8  9 10 11 12 13 14 14  4  3  5  5]
[ 7  6  8  9 10 11  1  0 14 14  3  4  5  5  6  7  9  8 10 11 13 12  2  2  3  4 15 15]
[15 15  6  7  8  9 10 11  0  1  2  2  4  3  5  5  7  6  9  8 10 11 12 13 14 14  4  3]
[ 5  5  6  7  9  8 10 11  1  0 14 14  3  4 15 15  7  6  8  9 10 11 13 12  2  2  3  4]
[ 3  4  5  5  7  6  9  8 10 11  0  1  2  2  3  4 15 15  6  7  8  9 10 11 12 13 14 14]
[ 4  3 15 15  7  6  8  9 10 11  1  0 14 14  4  3  5  5  6  7  9  8 10 11 13 12  2  2]
[ 2  2  3  4  5  5  6  7  9  8 10 11  0  1 14 14  3  4 15 15  7  6  8  9 10 11 12 13]
[14 14  4  3 15 15  6  7  8  9 10 11  1  0  2  2  4  3  5  5  7  6  9  8 10 11 13 12]
[12 13  2  2  4  3 15 15  7  6  8  9 11 10  0  1 14 14  4  3  5  5  6  7  9  8 11 10]
[13 12 14 14  3  4  5  5  7  6  9  8 11 10  1  0  2  2  3  4 15 15  6  7  8  9 11 10]
[11 10 12 13  2  2  3  4  5  5  6  7  8  9 11 10  0  1 14 14  3  4 15 15  7  6  9  8]
[11 10 13 12 14 14  4  3 15 15  6  7  9  8 11 10  1  0  2  2  4  3  5  5  7  6  8  9]
[ 8  9 11 10 12 13  2  2  4  3 15 15  6  7  9  8 11 10  0  1 14 14  4  3  5  5  7  6]
[ 9  8 11 10 13 12 14 14  3  4  5  5  6  7  8  9 11 10  1  0  2  2  3  4 15 15  7  6]
[ 6  7  9  8 11 10 12 13  2  2  3  4  5  5  7  6  8  9 11 10  0  1 14 14  3  4 15 15]
[ 6  7  8  9 11 10 13 12 14 14  4  3 15 15  7  6  9  8 11 10  1  0  2  2  4  3  5  5]
[ 5  5  7  6  8  9 11 10 12 13  2  2  3  4 15 15  6  7  9  8 11 10  0  1 14 14  3  4]
[15 15  7  6  9  8 11 10 13 12 14 14  4  3  5  5  6  7  8  9 11 10  1  0  2  2  4  3]
[ 4  3 15 15  6  7  9  8 11 10 12 13  2  2  4  3  5  5  7  6  8  9 11 10  0  1 14 14]
[ 3  4  5  5  6  7  8  9 11 10 13 12 14 14  3  4 15 15  7  6  9  8 11 10  1  0  2  2]
[ 2  2  4  3 15 15  7  6  9  8 11 10 12 13 14 14  4  3  5  5  6  7  8  9 11 10  0  1]
[14 14  3  4  5  5  7  6  8  9 11 10 13 12  2  2  3  4 15 15  6  7  9  8 11 10  1  0]
```

There are exactly 376 non-zero intersection numbers, so $\dim T_0 = 376$. In total, there are 24 $\mathbb{Z}_2$-diamond pairs in the scheme, so $\dim \tilde{T} = 376 + 24 = 400$. Moreover, the scheme also satisfies the conditions of Lemma 21, so we have $T = \tilde{T}$.

6 Proof of Theorem 3

For convenience, we consider the following hypothesis.

Hypothesis 2. Let $\mathfrak{X}$ be an association scheme as in Hypothesis 1. We assume additionally that G has the following properties.

(a) The group G admits a subgroup R acting regularly.

(b) The stabilizer G_v is a subgroup generated by an involution φ.

Theorem 3 is restated as follows for convenience.

Theorem 24. *If $\mathfrak{X}$ is an association scheme as in Hypothesis 2, then it is triply transitive.*

Let $\mathfrak{X}$ be an association scheme as defined in Hypothesis 2. We will first derive a few important results form Hypothesis 2. For any $1 \leqslant i \leqslant d$, we let Γ_i be the digraph (Ω, R_i). By Hypothesis 2(a), we know that $R \leqslant \mathrm{Aut}(\Gamma_i)$ acts regularly for all $1 \leqslant i \leqslant d$. Hence, Γ_i is a Cayley digraph for all $1 \leqslant i \leqslant d$, and we may identify Ω and R. In particular, there is a one-to-one correspondence identifying v to the identity element $1 \in R$, which is a graph isomorphism.

We will assume from hereon that $\Delta_0 = \{1\}$ and $\Delta_i \subset R$ for all $1 \leqslant i \leqslant d$. Consequently, we may assume without loss of generality that for $1 \leqslant i \leqslant d$

$$\Gamma_i = \mathrm{Cay}(R, \Delta_i).$$

Since $\mathfrak{X}$ is Schurian, Γ_i is an orbital digraph of G for any $1 \leqslant i \leqslant d$. The orbits of the stabilizer G_1 of $1 \in R$ in G are therefore the sets $\Delta_0 = \{1\}, \Delta_1, \ldots, \Delta_d$. Assume that $G_1 = \langle \varphi \rangle$, where φ is the involution in Hypothesis 2.

We claim that the group G is a split extension of R. Indeed, since $R \leqslant G$ is a regular subgroup, we have $|R| = |\Omega| = \frac{|G|}{|G_1|} = \frac{|G|}{2}$ and $[G : R] = 2$. As $R \trianglelefteq G$ is a maximal subgroup, and $\varphi \notin R$, it is clear that $G = R\langle \varphi \rangle = RG_1$. Consequently, G is a split extension of R, and $G = R \rtimes_\Psi G_1$ for some homomorphism $\Psi : G_1 \to \mathrm{Aut}(R)$.

Proof of Theorem 3. By Theorem 2, it is enough to show that $\mathfrak{X}$ does not admit $\mathbb{Z}_2$-diamond pairs. If (i, k) is a $\mathbb{Z}_2$-diamond pair of $\mathfrak{X}$, and $\Delta_i = \{u_1, u_2\}$ and $\Delta_k = \{w_1, w_2\}$, then we have $(u_1, w_1), (u_2, w_1) \in R_j$ as illustrated below.

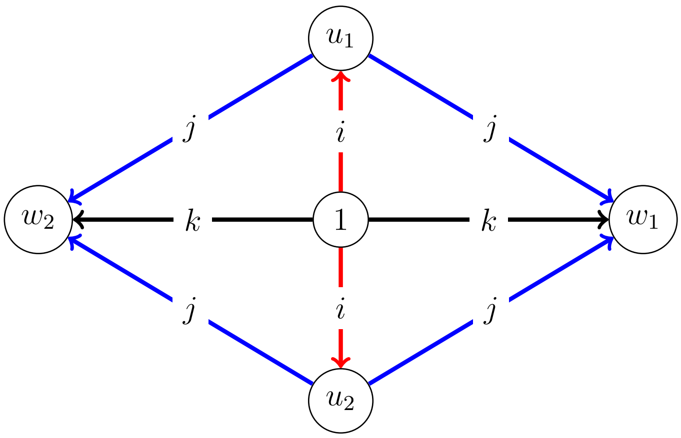

Figure 3: A diamond pair in $\mathfrak{X}$.

By definition of the adjacency in the digraphs (Γ_i), we have $\Delta_j = \{w_1u_1^{-1}, w_1u_2^{-1}\}$. On the one hand, since Δ_j is an orbit of $G_1 = \langle \varphi \rangle$, we have $\left(w_1u_1^{-1}\right)^\varphi = w_1u_2^{-1}$. On the other hand, using the fact that G_1 acts as an automorphism of R, we have

$$w_1u_2^{-1} = \left(w_1u_1^{-1}\right)^\varphi = w_1^\varphi \left(u_1^{-1}\right)^\varphi = w_2(u_1^\varphi)^{-1} = w_2u_2^{-1}.$$

Consequently, we have $w_1 = w_2$ which is impossible since $|\Delta_k| = 2$. Hence, $\mathfrak{X}$ does not admit any diamond pair, and is therefore triply transitive by Theorem 2. This completes the proof. $\square$

As an application of Theorem 3, we provide the next two examples.

Example 25. Consider the group $G = \mathrm{D}_{2n} = \langle a, b | a^n = b^2 = baba = 1\rangle$, $n \geqslant 3$, acting on cosets of the subgroup $\langle b \rangle$ by right multiplication. This group admits a regular subgroup, which is the cyclic group of order n, and the point stabilizer is a cyclic group of order 2. By Theorem 3, the corresponding association scheme[6] is triply transitive.

Example 26. Consider the group $G = \mathrm{Sym}(n) = \mathrm{Alt}(n) \rtimes \langle (1\ 2) \rangle$, $n \geqslant 3$, acting on cosets of the subgroup $\langle (1\ 2) \rangle$ by right multiplication. No conjugates of $\langle (1\ 2) \rangle$ can be contained in $\mathrm{Alt}(n)$, so the subgroup $\mathrm{Alt}(n)$ acts regularly. Hence, again by Theorem 3, the association scheme corresponding to this action is triply transitive.

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