A determinantal formula for cluster variables in cluster algebras from surfaces

Javier De Loera Chávez

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Abstract

For cluster algebras of surface type, Musiker, Schiffler and Williams gave an expansion formula for cluster variables in terms of perfect matchings of snake graphs. Building on this, we provide a simple determinantal formula for cluster variables via a weighting of the biadjacency matrix of the associated snake graphs, thus circumventing the enumeration of their perfect matchings.

Mathematics Subject Classifications: 13F60, 05C70.

1 Introduction

Cluster algebras are a broad class of commutative algebras introduced by Fomin and Zelevinsky in [6] as a combinatorial framework for the study of total positivity and canonical bases. They are recursively defined from an initial set of data by a combinatorial process called mutation. In [5], Fomin, Shapiro, and Thurston introduced and studied a class of cluster algebras whose combinatorics are governed by flips of tagged triangulations in surfaces with marked points. In [10], Musiker, Schiffler and Williams gave cluster expansion formulas for the cluster variables of these cluster algebras in terms of the perfect matchings of certain planar graphs called *snake graphs*. These graphs were studied further in [2], where a remarkable connection to continued fractions was developed.

In this note, we use a classical idea of matching theory to give a reformulation of Musiker's, Schiffler's, and Williams' cluster expansion formula into a determinantal formula for the cluster variables associated to plain arcs (Theorem 30). This gives a simple way to avoid the enumeration of the perfect matchings of the corresponding snake graph. The formula works by using an algebraic device to count all of the perfect matchings of a graph. The formulas for cluster variables associated to singly and doubly notched arcs given in [10] are expressed in terms of more restrictive matchings, and they were simplified by Wilson in [13], were they are instead obtained in terms of *good matchings*

Dipartimento SBAI, Sapienza Università di Roma, Via Scarpa 16, 00161, Roma, Italy.
(javieralejandro.deloerachavez@uniroma1.it)

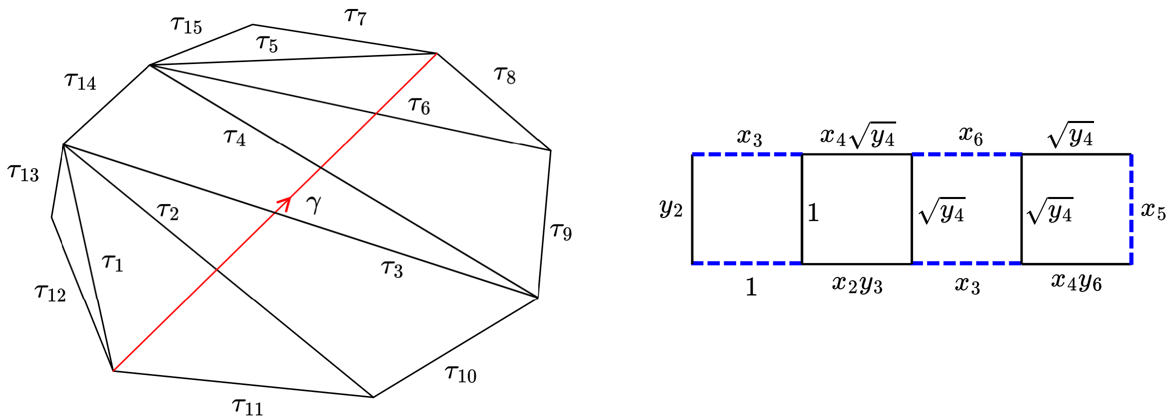

Figure 1: Example in type A. To the right, a principal weighting of $\mathcal{G}_\gamma$ with the minimal matching M_- depicted in blue.

of *loop graphs*. A related construction was used before in [11] to build an explicit linear basis for a cluster algebra coming from an unpunctured surface via *band graphs*. As an illustration, we provide a similar determinantal formula to compute the Laurent polynomials associated to band graphs, up to a pair of auxiliary variables (Theorem 41). Our main result is the following:

Theorem 1. *Let $\mathcal{A}(\Sigma, \mathbb{M})$ be a surface cluster algebra with principal coefficients at the initial seed $(T_0, \mathbf{x}_0, \mathbf{y}_0)$, where T_0 is an ideal triangulation of Σ. For every plain (generalized) arc $\gamma \notin T_0$, let $\mathcal{G}_\gamma$ be its associated snake graph. Let $B(w_\bullet)$ be a principal biadjacency matrix of $\mathcal{G}_\gamma$ and* $\operatorname{cross}(\gamma, T_0)$ *the crossing monomial of γ with respect to T_0. Then*

$$x_\gamma = \frac{\det B(w_\bullet)}{\operatorname{cross}(\gamma, T_0)}.$$

The definition of a principal biadjacency matrix of $\mathcal{G}_\gamma$ will be given in section 4. In a word, it is a matrix obtained from a weighted adjacency matrix of an orientation of the snake graph $\mathcal{G}_\gamma$ that is associated to an arc γ of the bordered surface $(\Sigma, \mathbb{M})$. The choice of weights allow to compute the cluster expansion of x_γ from the matchings of $\mathcal{G}_\gamma$ without having to compute the heights monomials that give the correct coefficients as in [10], while the orientation allows to compute the matchings algebraically by using a determinant. The main advantage of our formula is conceptual, as it expresses a cluster variable as a determinant of a matrix obtained from the base combinatorial object of the snake graph. Moreover, proposition 31 gives a way to express all cluster variables of a type A cluster algebra as minors of a square matrix, in a manner reminiscent but distinct to the usual Plucker coordiantes.

Example 2. Consider the cluster algebra with principal coefficients determined by the triangulated polygon of figure 1. The cluster variable associated to the red arc γ may be computed by using the perfect matchings of the snake graph $\mathcal{G}_\gamma$ to the right. A principal biadjacency matrix of $\mathcal{G}_\gamma$ is

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & y_2 \\ -x_2y_3 & x_3 & 0 & \sqrt{y_4} & 0 \\ 0 & -x_4y_6 & x_5 & 0 & 0 \\ 0 & -\sqrt{y_4} & -\sqrt{y_4} & x_6 & 0 \\ -1 & 0 & 0 & -x_4\sqrt{y_4} & x_3 \end{pmatrix},$$

and $\det(B) = x_3x_5x_6y_2 + x_2x_4^2y_2y_3y_4y_6 + x_2x_4x_5y_2y_3y_4 + x_4y_2y_4y_6 + x_5y_2y_4 + x_3^2x_5x_6 + x_3x_4y_4y_6 + x_3x_5y_4$. In this polynomial, every term corresponds to a matching M of the graph G_γ, and it is given by the product of the weights of each edge in M. Furthermore, the coefficients in the y-variables indicate the tiles enclosed by the symmetric difference $(M \cup M_-) \setminus (M \cap M_-)$, where M_- is the matching depicted in blue.

There is a multitude of methods for computing cluster expansions for surface type cluster variables using combinatorial information. In type A, several strategies are described in A. Claussen's thesis [4] and the references within. In [12], Pilaud, Reading and Schroll gave a general framework to compute cluster expansions in surface type by using the order ideals of certain posets that encode the crossing pattern of an arc with respect to an ideal triangulation. These posets are now usually called *fence posets*, as well as *loop* fence posets, and they are closely related to Wilson's loop graphs. In [8], Kantarcı Oğuz and Yıldırım gave a particularly efficient method of computation via *oriented posets*, using products of 2×2 matrices; we remark that since the biadjacency matrices of snake graphs are sparse and may be chosen to be 4-diagonal, the number of multiplications needed to compute our formula is also of linear order. It is often the case that cluster variables are given by minors or determinants of particular matrices, as in the case of the type A cluster algebra. However, our reformulation of the expansion formula in [10] appears to be new.

The paper is organized as follows. In section 2, we recollect some basic notions on cluster algebras from surfaces. In section 3, we recall the well known machinery that allows for the automatic enumeration of matchings of planar graphs, following [9]. In section 4 we recall the construction of snake graphs due to Musiker, Schiffler and Williams given in [10], and we introduce the notion of a principal weighting and of the principal biadjacency matrices of snake graphs and then apply it to give theorem 1. Finally, in section 5 we extend this approach to give a similar formula for band graphs.

2 Cluster algebras from surfaces

We begin by briefly recalling the definition of surface cluster algebras following [5]. Let Σ be a Riemann surface with boundary $\partial\Sigma$, and let $\mathbb{M} \subset \Sigma$ be a non-empty finite set of *marked points* such that $\mathbb{M} \cap C \neq \emptyset$ for each connected component C of $\partial\Sigma$. The pair $(\Sigma, \mathbb{M})$ is a *bordered surface*, the marked points are called $\partial\Sigma$ *ideal points*, and those in the interior of Σ *punctures*. A *boundary arc* of Σ is a curve with endpoints in M that is contained in $\delta\Sigma$. A *generalized arc* in Σ is a curve γ with endpoints in $\mathbb{M}$ such that its relative interior is disjoint from M and it is not homotopic to a boundary arc or to a point, while an *arc* is a generalized arc with no self-intersections. All such curves are considered up to isotopy. We call an arc with an endpoint at a puncture a *radius.* Finally, a *kink* in a generalized arc γ is any segment of γ from one of its interior point to itself. Note that kinks of γ cannot be resolved with an isotopy.

Two arcs are said to be *compatible* if they do not intersect in their relative interiors, up to homotopy. A maximal collection T of distinct pairwise compatible arcs in Σ along with all the boundary arcs of Σ is an *ideal triangulation* of $(\Sigma, \mathbb{M})$. All ideal triangulations of $(\Sigma, \mathbb{M})$ have $6g + 3b + 3p + c - 6$ arcs. It may happen that not all triangles of an ideal triangulation T have three sides: when a loop l encircles a puncture p, there is exactly one arc r joining the base of l to p that is compatible with l. The shape formed by l and r is called a *self-folded triangle*, where r is the self-folded side.

Pick an initial ideal triangulation T_0 with arcs $\tau_1, \ldots, \tau_n$. For each triangle Δ in T_0, let $B^\Delta = (b^\Delta_{i,j})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant n}$, where

$$b^\Delta_{i,j} = \begin{cases} 1 & \text{if } t_j \text{ follows } t_i \text{ in clockwise order,} \\ -1 & \text{if } t_j \text{ follows } t_i \text{ in counterclockwise order,} \\ 0 & \text{otherwise.} \end{cases}$$

The *signed adjacency matrix* $B = B(T_0)$ is given by $B = \sum_\Delta B^\Delta$, where the sum is taken over all triangles of T_0 that are not self-folded.

Consider the *tropical semifield* $\mathbb{P} := \mathrm{Trop}(y_1, \ldots, y_n)$ defined as the multiplicative abelian group freely generated by $y_1, \ldots, y_n$ with the *auxiliary sum*

$$\prod_{i=1}^n y_i^{a_i} \oplus \prod_{i=1}^n y_i^{b_i} := \prod_{i=1}^n y_i^{\min(a_i,b_i)} .$$

Take a tuple of $\mathbf{x}_0 = (x_1, \ldots, x_n)$ of algebraically independent generators of the field of rational functions $\mathbb{QP}(u_1, \ldots, u_n)$. Such a tuple is called a *cluster*. Let $\mathbf{y}_0 \in \mathbb{P}^n$. The triple $(\mathbf{x}_0, \mathbf{y}_0, B)$ is called the *initial seed*. For each $k \in \{1, \ldots, n\}$ define the *mutation* in direction k $\mu_k(\mathbf{x}_0, \mathbf{y}_0, B_0) := (\mathbf{x}', \mathbf{y}', B')$ by the following rule:

- The matrix $B' = (b'_{ij})$ is defined by

$$b'_{i,j} = \begin{cases} -b_{i,j} & \text{if } k \in \{i, j\}\,, \\ b_{i,j} + \frac{|b_{i,k}|b_{k,j} + b_{i,k}|b_{k,j}|}{2} & \text{otherwise}\,. \end{cases}$$

- The cluster $\mathbf{x}' = (x'_1, \ldots, x'_n)$ is given by $x'_i = x_i$ if $i \neq k$, and

$$x'_k = \frac{1}{(y_k \oplus 1)x_k} \left(y_k \prod_{B_{k,j}>0} x_j^{B_{k,j}} + \prod_{B_{k,j}<0} x_j^{-B_{k,j}} \right) .$$

- The coefficient tuple $\mathbf{y}'$ is given by

$$y'_j = \begin{cases} y_j^{-1} & i = k\,, \\ y_j y_k^{\max(0,b_{kj})}(y_k \oplus 1)^{-b_{kj}} & \text{otherwise}\,. \end{cases}$$

The matrix $B' = \mu_k(B(T_0))$ is in fact the adjacency matrix of the triangulation obtained by flipping the edge τ_k, and the set $\mu_k(\mathbf{x}_0)$ is again a cluster. Let $\mathcal{X}$ be the set of *cluster variables*: this is, all variables contained in all possible clusters obtained by sequences of mutations starting from the initial seed $\mathbf{x}_0$. The *cluster algebra* with principal coefficients associated to the surface with marked points $(\Sigma, \mathbb{M})$ is the algebra $\mathcal{A} := \mathbb{QP}[\mathcal{X}]$. It is independent of the chosen triangulation T_0, so it really is associated to the surface $(\Sigma, \mathbb{M})$. The cluster algebra $\mathcal{A}$ has principal coefficients (at t_0) when $\mathbf{y}_0 = (y_1, \ldots, y_n)$, and this is denoted by $\mathcal{A}_\bullet(T_0)$. There is a bijection between the cluster variables of $\mathcal{A}_\bullet(T_0)$ and the *tagged* arcs of Σ. With this in mind, we will also write y_{τ_i} for y_i. A tagged arc is obtained by drawing a notch at one or both ends of a regular or *plain* arc, up to certain restrictions, and a tagged triangulation is a maximal collection of tagged arcs subject to additional compatibility conditions.

As a consequence of the Laurent Phenomenon, for each arc of γ we may write the cluster variable x_γ as a Laurent polynomial in the cluster variables $x_1, \ldots, x_n$ that correspond to the diagonals of T_0. We call such an expression the *cluster expansion* of x_γ with respect to T_0.

The tagging of arcs, introduced and developed in [5], allows to fully model the mutation of all cluster variables associated with any initial tagged triangulation, as the self-folded side of a self-folded triangle cannot be flipped in the usual way. For simplicity, we have defined the cluster algebra $\mathcal{A}_\bullet(T_0)$ for an initial seed given by an ideal triangulation, but it can be done as well using a tagged triangulation. However, for the purpose of computing cluster expansions, it is enough to consider seeds given by ideal triangulations, as per Proposition 3.16 of [10].

3 Perfect matchings

In this section, we recall preliminary notions on perfect matchings of general planar graphs. In subsequent sections, we will use these results to provide our determinantal formula. The material of this section is well known and we mainly follow [9], though definition 3 and thus lemma 9 is given in slightly more generality here. We will consider only finite and loopless graphs. Let $\mathcal{G}$ be a graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set E. Denote by $E(i,j)$ the set of edges between v_i and v_j. Write $s(G)$ for the underlying simple graph of G, and $m(i,j) = |E(i,j)|$. Of course, if G is simple then $s(G) = G$ and $m(i,j) \in \{0,1\}$ for all $1 \leqslant i,j \leqslant n$. If $\overrightarrow{G}$ is an orientation of G, then $E(i,j)$ will also denote the set of arrows with tail v_i and head v_j.

A *perfect matching* of G is a subset $M \subseteq E$ such that each $v \in V$ is an endpoint of exactly one member of M. Thus, $|M| = |V|/2$, so a graph that admits a perfect matching has an even number of vertices. Graphs with an even number of vertices will be simply be called *even* graphs. Denote by $\mathrm{Match}(G)$ the set of perfect matchings of G, and let $\Phi(G) := |\,\mathrm{Match}(G)|$. We will use a simple but useful device to keep track of all perfect matchings of G, the so-called *method of variables*, as in [9].

Definition 3. (Weighted adjacency matrix) Let G be a graph with vertex set $V = \{v_1, \ldots, v_n\}$. For each edge e of $s(G)$, consider the formal variable z_e, and let $a_{i,j} := m(i,j)z_e$. The matrix $A_G(z) := (a_{i,j})$ is called the *weighted adjacency matrix* of G. Furthermore, if $\overrightarrow{G}$ is an orientation of G with no oriented 2-cycles, write

$$a'_{i,j} := \epsilon(i,j)m(i,j)z_e\,, \quad \text{where} \quad \epsilon(i,j) := \begin{cases} 1 & E(i,j) \neq \emptyset\,, \\ -1 & E(j,i) \neq \emptyset\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

$A_{\overrightarrow{G}}(z) := (a'_{i,j})$ is the *weighted skew-symmetric adjacency matrix* of $\overrightarrow{G}$.

Remark 4. The matrices of definition 3 are defined up to a relabeling of the vertex set of G, and thus a simultaneous permutation of rows and columns. We choose the notation $A_G(z)$ with the symbol z so as to underline the fact that the adjacency matrix is weighted. By setting $z_e = 1$, we recover the usual adjacency matrices of G and $\overrightarrow{G}$, so we denote them by $A_G(1)$ and $A_{\overrightarrow{G}}(1)$, respectively.

Definition 5. Let G be a graph and E' be the edge set of $s(G)$. The *matching generating function* of G is the polynomial $\phi_G \in \mathbb{Z}[z_e \mid e \in E']$ given by

$$\phi_G := \sum_{\{e_1,\ldots,e_k\} \in \mathrm{Match}(G)} z_{e_1} \cdots z_{e_k}\,.$$

Definition 6. Let $C = (c_{i,j})$ be a skew-symmetric matrix of size $2n \geqslant 2$ over a commutative unit ring A. For each partition into two element sets $P = \{\{i_1, j_1\}, \ldots, \{i_n, j_n\}\}$ of the set $\{1, 2, \ldots, 2n-1, 2n\}$, define

$$\sigma_P := \begin{pmatrix} 1 & 2 & \ldots & 2n-1 & 2n \\ i_1 & j_1 & \ldots & i_n & j_n \end{pmatrix}\,, \quad c_P := \mathrm{sgn}(\sigma_P) c_{i_1,j_1} \cdots c_{i_n,j_n}\,,$$

where $\mathrm{sgn}(\sigma_P)$ denotes the sign of the permutation σ. The *pfaffian* of C is $\mathrm{pff}(C) := \sum_P c_P\,.$

Remark 7. The term c_P is independent of the order of labeling of each part in P. Indeed, switching the roles of i_k and j_k changes the sign of σ_P by -1, but $c_{j_k,i_k} = -c_{i_k,j_k}$. Thus, c_P depends only on the partition P. Note that a partition of $1, \ldots, 2n$ into two element subsets corresponds to a perfect matching of K_{2n}, the complete graph in $2n$ vertices, and that $c_P \neq 0$ if and only if such a *potential* matching corresponds to an *actual* matching of G.

The pfaffian and determinant of a skew-symmetric matrix are closely related. The following lemma is due to Cayley [3]. A proof can be found in [1], where it appears as Theorem 5.6.

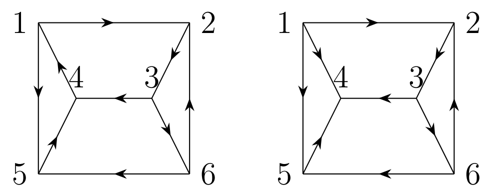

Figure 2: To the left, an example of an orientation that is not pfaffian. It can be made into a pfaffian orientation, pictured on the right, by reversing the orientation of the edge joining vertices 1 and 4.

Lemma 8. *If B is a skew-symmetric matrix of even size,* $\det B = \operatorname{pff}(B)^2$.

The next well known lemma gives a way to keep track of the perfect matchings of a graph G.

Lemma 9. *Let G be an even graph, and let $\overrightarrow{G}$ be an arbitrary orientation of G. The sum of the absolute value of the coefficients of* $\operatorname{pff}(A_{\overrightarrow{G}}(z))$ *is* $\Phi(G)$.

Proof. Take the vertex set V of G to be $\{1, \ldots, n\}$, and consider the simple graph $s(G)$ with edge set E'. Let $k = n/2$. Each partition $P = \{\{i_l, j_l\} \mid 1 \leqslant l \leqslant k\}$ as in definition 6 corresponds to a possible set of independent edges in the simple graph $s(G)$. For $C = A_{\overrightarrow{G}}(z)$, the associated term c_P is

$$c_P = \begin{cases} 0 & \text{if } \{i_s, j_s\} \notin E' \text{ for some } 1 \leqslant s \leqslant k\,, \\ \prod_{l=1}^{k} \epsilon(i_l, j_l) m(i_l, j_l) z_{\{i_l, j_l\}} & \text{otherwise.} \end{cases}$$

Thus, $c_P \neq 0$ if and only if P corresponds to a perfect matching M in $s(G)$, as P is a partition. Note that the coefficient $\prod_{l=1}^{k} m(i_l, j_l)$ is the number of ways of getting the matching M in $s(G)$ from a matching in G. Since different matchings of $s(G)$ differ by at least one edge, terms corresponding to distinct matchings cannot cancel each other. This completes the proof. □

As observed by Tutte in [14], the previous argument and Lemma 8 imply that a graph G admits a perfect matching if and only if there is some orientation $\overrightarrow{G}$ for which $\det(A_{\overrightarrow{G}}(z)) \neq 0$. Call the polynomial $\operatorname{pff}(A_{\overrightarrow{G}}(z))$ the *pfaffian* of $\overrightarrow{G}$, and denote it by $\operatorname{pff}(\overrightarrow{G})$. Note that this polynomial is defined only up to a sign that depends on the chosen labeling of the vertices of G.

Corollary 10. *Let $\overrightarrow{G}$ be an oriented graph. Then* $|\operatorname{pff}(A_{\overrightarrow{G}}(1))| \leqslant \Phi(G)$.

An orientation $\overrightarrow{G}$ of G for which the equality is attained in the previous corollary is a *pfaffian* orientation. Likewise, a graph is *pfaffian* if it admits a pfaffian orientation. For such an orientation, the pfaffian of $\overrightarrow{G}$ is a polynomial whose coefficients all have the same sign. Indeed, in such a case $\operatorname{pff}(\overrightarrow{G}) = \pm\phi_G$. We cite the following useful characterization of pfaffian orientations. Call a cycle C of a graph G *nice* if $G - V(C)$ admits a perfect matching. An oriented even cycle is *oddly oriented* if it has an odd number of arrows that agree with any of its two opposite cyclic orientations.

Theorem 11. *([9], Theorem 8.3.2) Let G be an even graph and $\overrightarrow{G}$ an orientation of G. Then $\overrightarrow{G}$ is pfaffian if and only if each even nice cycle of G is oddly oriented.*

Example 12. Consider the graph G on figure 2, with orientations $\overrightarrow{G}_1$ to the left and $\overrightarrow{G}_2$ to the right. The 4-cycles on $\{1,2,3,4\}$ and,$\{3,4,5,6\}$ are nice, while the 3-cycles on $\{1,4,5\}$ and $\{2,3,6\}$ are not. Observe that the cycle on $\{1,2,3,4\}$ is not oddly oriented. In $\overrightarrow{G}_1$, the partition $P = \{\{1,2\},\{3,4\},\{5,6\}\}$ has sign -1, while $P' = \{\{1,5\},\{3,4\},\{5,6\}\}$ has sign 1, so $\overrightarrow{G}_1$ is not Pfaffian. In the orientation on the right, all partitions corresponding to matchings have sign 1 because all nice cycles are oddly oriented.

Furthermore, the following theorem is classic. A proof can be found in [9], theorem 8.3.4.

Theorem 13. *(Kasteleyn) Every even planar graph is pfaffian.*

Let $\overrightarrow{G}$ be bipartite with bipartition $V = U \cup W$ such that $|U| = |W|$ (if $\overrightarrow{G}$ has a perfect matching, this is the case). The weighted skew symmetric adjacency matrix of $\overrightarrow{G}$ takes the form

$$A_{\overrightarrow{G}}(z) = \begin{pmatrix} 0 & B_{\overrightarrow{G}}(z) \\ -B_{\overrightarrow{G}}(z)^T & 0 \end{pmatrix}, \tag{3.1}$$

for any labeling $V = \{u_1,\dots,u_k,w_1,\dots,w_l\}$ such that $u_i \in U$ and $w_j \in W$ for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$. $B_{\overrightarrow{G}}(x)$ is called the *weighted biadjacency matrix* of $\overrightarrow{G}$. The following lemma is given as an exercise in [7] (though there seems to be a sign missing therein). We include a proof.

Lemma 14. *Let B be a square matrix of size n. If*

$$A = \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix},$$

then $\operatorname{pff}(A) = (-1)^{n(n-1)/2} \det B$.

Proof. Let $V = \{1,\dots,2n\}$, $U = \{1,\dots,n\}$ and $W = \{n+1,\dots,2n\}$. Fix some partition $P = \{\{i_1,j_1\},\dots,\{i_n,j_n\}\}$. By remark 7, we can assume that $j_i = n+i$ for each $1 \leqslant i \leqslant n$, so

$$\sigma_P = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2n-1 & 2n \\ i_1 & n+1 & i_2 & n+2 & \dots & i_n & 2n \end{pmatrix}$$

Note that $i_1,\dots,i_n \leqslant n < n+1 = j_1$, so the indices i_l amount to $n-1$ inversions with respect to j_1. Similarly, $n-2$ of the i_l give $n-2$ inversions with j_2, and so on. This gives a total of $n(n-1)/2$ inversions. Thus,

$$\operatorname{sgn}(\sigma_P) = (-1)^{n(n-1)/2} \operatorname{sgn} \begin{pmatrix} 1 & 2 & \dots & n-1 & n & n+1 & \dots & 2n-1 & 2n \\ i_1 & i_2 & \dots & i_{n-1} & i_n & n+1 & \dots & 2n-1 & 2n \end{pmatrix}.$$

It follows that $\operatorname{sgn}(\sigma_P) = (-1)^{n(n-1)/2} \operatorname{sgn}(\sigma'_P)$, where

$$\sigma'_P := \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ i_1 & i_2 & \dots & i_{n-1} & i_n \end{pmatrix} .$$

Thus, $b_P = (-1)^{n(n-1)/2} \operatorname{sgn}(\sigma'_P) b_{i_1,n+1} \cdots b_{i_n,2n}$. Summing over all P we obtain the result. □

4 Snake graphs and cluster expansion formula

A *tile* is a graph isomorphic to a square drawn in the plane. A *snake graph* $\mathcal{G}$ is a plane graph obtained from a sequence of d tiles $G_1, \dots, G_d$ such that each successive tile is placed on top or to the right of the previous tile, as in figure 4. That is, for $1 \leqslant i \leqslant d-1$, tiles G_i and G_{i+1} share their right and left edge, respectively, or their top and bottom edge, respectively. The edges of $\mathcal{G}$ that lie on its unbounded face are called *boundary edges*, and the rest of the edges are *internal*. It is easily seen by induction that snake graphs are bipartite and always admit perfect matchings, and that a snake graph with d tiles has $2d+2$ vertices and $3d+1$ edges.

Snake graphs were used in [10] to provide manifestly positive expansions of the cluster variables in cluster algebras from surfaces. We briefly recall their construction. Fix a bordered surface $(\Sigma, \mathbb{M})$ and an initial seed $(\mathbf{x}_0, \mathbf{y}_0, B(T_0))$, where T_0 is an ideal triangulation and let $\mathcal{A}_\bullet(T_0)$ be the associated cluster algebra with principal coefficients at T_0. Let γ be a generalized arc of Σ that does not belong to T_0. Orient $\gamma \notin T_0$ so that it crosses the diagonals $\tau_{i_1}, \dots, \tau_{i_d}$ of T_0 in order. Each τ_{i_j} is contained in exactly two triangles Δ_j and Δ'_j. Let G_j be the quadrilateral formed by the triangles Δ_j and Δ'_j. Give each side s of G_j the weight $w(s) = x_k$ if it is the internal diagonal of T_0 associated to x_k, and give it the weight $w(s) = 1$ if it is a boundary edge of T_0. The graph $\mathcal{G}_\gamma$ is the weighted snake graph with tiles $G_1, \dots, G_d$ obtained by gluing G_i and G_{i+1} on the common side of $\Delta_{i'}$ and Δ_{i+1}. For complete details on how the tiles G_i are glued to obtain G_γ, see [10].

We now recall the definition of the specialized height monomial from [11].

Definition 15. Let $T_0 = \{\tau_1, \dots, \tau_n\}$ be an ideal triangulation of $(\Sigma, \mathbb{M})$ and $\gamma \notin T_0$ a plain arc. By lemma 4.7 of [10], the symmetric difference $M \ominus M_-$ encloses a union of tiles $\cup_{j \in J} G_{i_j}$. The *specialized height monomial* of M is given by

$$y(M) = \prod_{j \in J} y'_{\tau_{i_j}} ,$$

where $y'_\tau = y_\tau$ if τ is not a radius, and $y'_r = y_r/y_l$, when r is a radius inside a self folded triangle enclosed by the loop l.

Example 16. Observe figure 1. There is a unique matching M of $\mathcal{G}_\gamma$ that uses all of its vertical edges. The symmetric difference of M and M_- encloses tiles y_2 and y_4, so the specialized height monomial of M is $y_2 y_4$.

As discussed in the previous section, each tagged arc γ in Σ corresponds to a cluster variable x_γ, and if $\gamma \notin T_0$, then x_γ is not part of the initial cluster $\mathbf{x}_0$, so one must perform a series of mutations to find the expansion of x_γ with respect to $\mathbf{x}_0$. When γ is a plain arc, this can be avoided by virtue of the following.

Theorem 17. *([10], Theorem 4.9) Let (Σ, M) be a bordered surface and T_0 be a triangulation of Σ.. Let $\gamma \notin T_0$ be plain. Let γ cross the diagonals $\tau_{i_1}, \ldots, \tau_{i_d}$ of T_0 in order. Then*

$$x_\gamma = \frac{1}{\operatorname{cross}(\gamma, T_0)} \sum_{M \in \operatorname{Match}(\mathcal{G}_\gamma)} x(M) y(M)$$

where $x(M)$ is the product of all weights of the edges occurring in the matching M, $y(M)$ is the specialized height monomial of M, and $\operatorname{cross}(\gamma, T_0) := x_{i_1} x_{i_2} \cdots x_{i_d}$.

By theorem 5.3 of [11], the set of perfect matchings of $\mathcal{G}_\gamma$ is a distributive lattice $\operatorname{Match}(\mathcal{G}_\gamma)$. In this lattice, a matching M covers another matching M' when M can be obtained from M' via a local moved called a *face twist*: that is, there is a tile G_i of $\mathcal{G}_\gamma$ with edges e_1, e_2, e_3, e_4 such that $M \ominus M' = \{e_1, e_2, e_3, e_4\}$ and $M = (M' \setminus \{e_1, e_2\}) \cup \{e_3, e_4\}$. The perfect matching lattice of $\mathcal{G}_\gamma$ can be built up from sequences of face twists beginning at the minimal matching M_-. Thus, if the matching M is obtained by applying the sequence of face twists at tiles $G_{j_1}, \ldots, G_{j_k}$, $y(M) = y_{\tau_{j_1}} \cdots y_{\tau_{j_k}}$. There are exactly two matchings of $\mathcal{G}_\gamma$ consisting of boundary edges, M_- and M_+ whose union is the boundary cycle of $\mathcal{G}_\gamma$. If tile G_1 is drawn in the plane in such a way that its orientation agrees with its orientation on Σ, M_- is the matching that includes the south edge of G, and it is called the *minimal matching* of $\mathcal{G}_\gamma$. We will always assume that a snake graph coming from an arc $\gamma \notin T_0$ is drawn in such a way (in other words, in the language of [10], the tile G_1 has relative orientation 1).

Note that the height monomial $y(M)$ is the product of the variables y_i that correspond to the diagonals of the tiles enclosed by $M \ominus M_-$ (with a slight correction needed for the tiles corresponding to self-folded triangles). Observe that several tiles of $\mathcal{G}_\gamma$ may have the same diagonal.

Remark 18. A subset I of a poset P is an *order ideal* if for every $y \in I$, $x \leqslant_P y$ implies $x \in I$. It is well known that a finite distributive lattice L is isomorphic to the lattice of order ideals of the poset P_L of the join irreducible elements of L. In a finite distributive lattice, an element is join irreducible if and only if it covers exactly one other element. For a snake graph on d tiles, $\mathcal{G}$, the poset $P_\mathcal{G}$ of join irreducibles of $\operatorname{Match}(\mathcal{G})$ is always a *fence poset*, meaning its Hasse diagram is an orientation of a path on d vertices (theorem 5.4 of [11]). The order ideal of $P_\mathcal{G}$ that corresponds to a matching M is given by the set of tiles enclosed by the cycles of $M \ominus M_-$.

We now describe an alternative weight scheme for the edges of $\mathcal{G}_\gamma$ that recovers the specialized height monomial of each matching.

Definition 19. (Principal weighting) Let $\mathcal{G}_\gamma$ be the snake graph on n tiles associated to the arc $\gamma \notin T_0$ as in theorem 17, and let w be its corresponding weighting. Consider the minimal matching M_-. We define the weighting $\tilde{w}$ of the edges of $\mathcal{G}_\gamma$ in three steps:

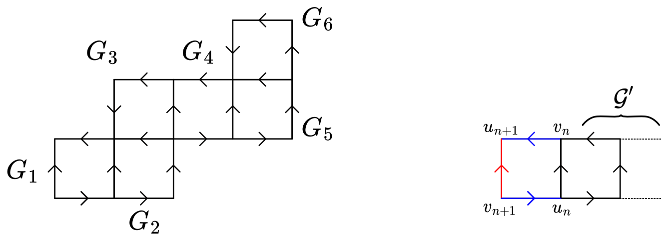

Figure 3: To the left, a snake graph $\mathcal{G}$ with the pfaffian orientation $\mathcal{P}(\mathcal{G})$. To the right, the inductive step in the proof of Proposition 23

1. Set $w_1(e) = 1$ for all edges of G_γ.

2. For each $1 \leqslant i \leqslant n$, pick a boundary edge of e of G_i such that $e \notin M_-$ and set $w_2(e) = y_{\tau_i}$, if such an edge exists. If tile G_i corresponds to the radius of a self-folded triangle whose loop is l, we set $w_2(e) = y_r/y_l$.

3. If all boundary edges of G_i are in M_-, then the consecutive tiles G_{i-1}, G_i, G_{i+1} form a straight snake subgraph of $\mathcal{G}_\gamma$. Pick a boundary edge e of G_{i-1} and a boundary edge e' of G_{i+1}, respectively, and let d, d' be the edges of G_i that are not in M_-. Set $w_3(x) = \sqrt{y_i}$ for each $x \in \{e, e', d, d'\}$.

Let $w_\bullet$ be the weighting of $\mathcal{G}_\gamma$ obtained by combining these weights together: write $w_\bullet(e) := w(e)w_1(e)w_2(e)w_3(e)$ for each edge e. We call $w_\bullet$ a *principal weighting* of $\mathcal{G}_\gamma$.

Example 20. Let us look at figure 1 once more. Notice that the third tile has no boundary edge e with $e \notin M_-$. Thus we must *split* the associated coefficient y_4 into the neighboring tiles, replacing it with $\sqrt{y_4}$.

Theorem 21. *Let $\mathcal{G}_\gamma$ be the snake graph of the arc $\gamma \notin T_0$. For each $M \in \mathrm{Match}(\mathcal{G}_\gamma)$, let $w_\bullet(M) := \prod_{e \in M} w_\bullet(e)$. Then*

$$x_\gamma = \frac{1}{\mathrm{cross}(\gamma, T_0)} \sum_{M \in \mathrm{Match}(\mathcal{G}_\gamma)} w_\bullet(M)\,.$$

Proof. By virtue of theorem 17, we just need to show that $w_\bullet(M) = x(M)y(M)$. By definition, $x(M)$ divides $w_\bullet(M)$ and there is no power of x_i that divides $w_\bullet(M)/x(M)$, for any i. Thus, it is enough to show that each variable y_j appears with equal multiplicity in both $w_\bullet(M)$ and $y(M)$. Now, y_j appears once in $y(M)$ for each tile G_i with diagonal τ_j that is enclosed by the edges $M \ominus M_-$. If there is a boundary edge e of G_i in M, then $e \notin M_-$, so M includes all boundary edges of G_i (one if G_i is a corner tile, two otherwise), and therefore $y_j \mid w_\bullet(M)$. If there is no boundary edge of G_i in M, then both are in M_-, and G_i is in such a position as in the third tile of figure 1. Let e, e' be the edges in $G_i \cap M_-$, with e adjacent to a_1, a_2 and e' adjacent to a_1', a_2'. Since G_i is enclosed by $M \ominus M_-$, M must include one element e_1 of $\{a_1, a_2\}$ and one element e_2 of $\{a_1', a_2'\}$. In all cases, $w_\bullet(e_1) = \sqrt{y_j} = w_\bullet(e_2)$, so $y_j \mid w_\bullet(M)$. Reciprocally, any matching M that includes the edge a_1 must also include either a_1' or a_2', so $M \ominus M_-$ encloses G_i and y_j appears in both $y(M)$ and $w_\bullet(M)$. $\square$

The main advantage of the previous theorem is that it allows one to ignore the lattice structure of the corresponding snake graph. To wit, the *height* of each matching, with respect to the minimal matching, is computed automatically from the weights. Thus we have encoded both the information of the initial cluster variables appearing in a monomial of x_γ as well as that of the its coefficients in the edges of the corresponding snake graph without having to appeal to a further structure.

Now, by Theorem 13, snake graphs are pfaffian, and Theorem 11 immediately suggests a pfaffian orientation:

Definition 22. Let $\mathcal{G}$ be a snake graph with tiles $G_1, \ldots, G_d$. Define the orientation $\mathcal{P}(\mathcal{G})$ as follows:

1. Orient the boundary edges of $\mathcal{G}$ cyclically and counterclockwise.
2. Orient every internal vertical edge from bottom to top, and every internal horizontal edge from right to left.
3. Reverse the orientation of the left edge of G_1.

The following is readily seen on account of Theorem 11, but we give a direct proof.

Theorem 23. *For every snake graph $\mathcal{G}$, $\mathcal{P}(\mathcal{G})$ is a pfaffian orientation.*

Proof. We proceed by induction in d, the number of tiles. When $d = 1$, say $\mathcal{G}$ has vertices u_1, w_1, u_2, w_2 listed in cyclic order. The weighted biadjacency matrix of $\mathcal{P}(\mathcal{G})$ is

$$\begin{pmatrix} z_{u_1w_1} & z_{u_2w_1} \\ -z_{u_1w_2} & z_{u_2w_2} \end{pmatrix}$$

so that $\operatorname{pff}(\mathcal{P}(\mathcal{G})) = z_{u_1w_1}z_{u_2w_2} + z_{u_2w_1}z_{u_1w_2}$. Take $\mathcal{G}$ with tiles $G_1, \ldots, G_{d+1}$, $d \geqslant 1$ and give it the orientation $\mathcal{P}(\mathcal{G})$. Let $\mathcal{G}'$ be the snake graph with tiles $G_2, \ldots, G_{d+1}$ obtained by deleting the two leftmost vertices of $\mathcal{G}$. Observe that the induced orientation on $\mathcal{G}'$ is $\mathcal{P}(\mathcal{G}')$, so by induction the coefficients of the pfaffian of $\mathcal{G}'$ all have the same sign $\epsilon \in \{1, -1\}$. By circling the boundary of $\mathcal{G}$ counterclockwise, we obtain a bipartition $\{u_1, \ldots, u_n\} \cup \{w_1, \ldots, w_n\}$ of its vertices. Relabel $V(\mathcal{G})$ so that G_1 has vertices $w_{n+1}, u_n, w_n, u_{n+1}$ as in figure 4, and write $a_{i,j}$ and $a'_{i,j}$ for the entries of $A_{\mathcal{P}(\mathcal{G})}(z)$ and $A_{\mathcal{P}(\mathcal{G}')}(z)$, respectively. Note that $A_{\mathcal{P}(\mathcal{G})}(x)$ is obtained from $A_{\mathcal{P}(\mathcal{G}')}(z)$ by adding a row and a column corresponding to vertices u_{n+1} and w_{n+1}.

Any perfect matching M of $\mathcal{G}$ either contains the edge $u_{n+1}w_{n+1}$ or the edges $u_{n+1}w_n$ and u_nw_{n+1}. In the first case, M restricts to a perfect matching of $\mathcal{G}'$, and its associated term in $\mathrm{pff}(\mathcal{P}(\mathcal{G}))$ is

$$\begin{aligned} a_M &= \operatorname{sgn}\begin{pmatrix} 1 & 2 & \dots & 2n-1 & 2n & 2n+1 & 2n+2 \\ i_1 & j_1 & \dots & i_n & j_n & 2n+1 & 2n+2 \end{pmatrix} a_{i_1,j_1}\dots a_{i_n,j_n}(-z_{u_{n+1}w_{n+1}}) \\ &= \operatorname{sgn}\begin{pmatrix} 1 & 2 & \dots & 2n-1 & 2n \\ i_1 & j_1 & \dots & i_n & j_n \end{pmatrix} a_{i_1,j_1}\dots a_{i_n,j_n}(-z_{u_{n+1}w_{n+1}}) \\ &= -\epsilon z_{u_{i_1}w_{j_1}}\cdots z_{u_{i_n}w_{j_n}} z_{u_{n+1}w_{n+1}} \end{aligned}$$

In the second case, M contains edges $u_{n+1}w_n$ and u_nw_{n+1}. Such matchings are in bijective correspondence to matchings M' of $\mathcal{G}'$ containing u_nw_n. The associated term a_M in $\mathrm{pff}(\mathcal{P}(\mathcal{G}))$ is

$$\operatorname{sgn}\begin{pmatrix} 1 & 2 & \dots & 2n-1 & 2n & 2n+1 & 2n+2 \\ i_1 & j_1 & \dots & 2n-1 & 2n+2 & 2n+1 & 2n \end{pmatrix} a_{i_1,j_1}\dots a_{i_{n-1}j_{n-1}} z_{u_nw_{n+1}} z_{u_{n+1}w_n}\,.$$

The corresponding term $a_{M'}$ is

$$a_M = \operatorname{sgn}\begin{pmatrix} 1 & 2 & \dots & 2n-1 & 2n \\ i_1 & j_1 & \dots & 2n-1 & 2n \end{pmatrix} a_{i_1,j_1}\cdots a_{i_{n-1}j_{n-1}} z_{u_n,w_n}\,.$$

Note a_M can be obtained from $a_{M'}$ by substituting z_{u_n,w_n} with $-z_{u_nw_{n+1}}z_{u_{n+1}w_n}$, for their associated permutations have opposite signs. Thus,

$$a_M = -\epsilon z_{u_{i_1},w_{j_1}}\cdots z_{u_{i_{n-1}},w_{j_{n-1}}} z_{u_nw_{n+1}} z_{u_{n+1}w_n}\,,$$

so every term of $\mathrm{pff}(\mathcal{P}(\mathcal{G}))$ has coefficient $-\epsilon$. □

Remark 24. As in the previous proof, one can label the vertices cyclically by circling the boundary of $\mathcal{G}$ in the clockwise direction, starting from the leftmost bottom vertex v_1. Call this the *cyclical labeling* of $\mathcal{G}$. With this labeling, the perfect matching of $\mathcal{P}(\mathcal{G})$ given by the boundary edges $v_1v_2, v_3v_4, \dots, v_{2n-1}v_{2n}$ has positive sign in $\mathrm{pff}(\mathcal{G})$, as it is associated to the identity permutation. Thus, every coefficient of $\mathrm{pff}(\mathcal{P}(\mathcal{G}))$ is positive with respect to this labeling.

Remark 25. The adjacency matrix of $\mathcal{G}$ with respect to the cyclic labeling is not of the form 3.1. To fix this, we relabel the vertices of $\mathcal{G}$ via the permutation σ given by $\sigma(i) = (i+1)/2$ for odd i and $\sigma(j) = i/2 + n$ for even j, which maps the odd numbered vertices in the cyclic labeling of $\mathcal{G}$ onto the set $\{1, \dots, n\}$, and the even numbered vertices onto the set $\{n+1, \dots, 2n\}$. Moreover, $\operatorname{sgn}(\sigma) = (-1)^{n(n-1)/2}$, as can be seen by counting the negative slope segments with endpoints in $\{(i, \sigma(i)) \in \mathbb{R}^2 \mid i \in \{1, \dots, 2n\}\}$.

Definition 26. Let $\mathcal{G}$ be a snake graph. Let $B_{\mathcal{P}(\mathcal{G})}(z)$ denote the weighted biadjacency matrix of $\mathcal{G}$ obtained from the relabeling in remark 25. That is, when walking the boundary cycle of $\mathcal{G}$ in the counterclockwise sense starting from the southwest vertex of the first tile of $\mathcal{G}$, we get the sequence of labels $1, n/2+1, 2, n/2+2, \dots, n/2, n$.

Note that all non zero entries of $B_{\mathcal{P}(\mathcal{G})}(z)$ below the have negative sign below the diagonal, and positive sign above it. In view of lemma 14 and remarks 24 and 25, we have the following:

Corollary 27. *Let $\mathcal{G}$ be a snake graph. Then* $\det B_{\mathcal{P}(\mathcal{G})}(z) = \phi_{\mathcal{G}}$ *.*

Remark 28. There is a shorter proof of proposition 23 that uses the lattice structure of $\mathrm{Match}(\mathcal{G})$. To wit, let the matching M be covered by M', and write σ_M, $\sigma_{M'}$ for their associated permutations, with respect to some labeling of $V(\mathcal{G})$. If M' is obtained by twisting the face f with the vertices u_i, w_j, u_k, w_l, then $\sigma' = (i\ l)\sigma$, but since f is oddly oriented by theorem 11, $b_{i,j}b_{l,k} = -b_{i,l}b_{j,k}$, so the terms corresponding to M and M' have the same sign in $\operatorname{pff}\mathcal{G}$. Thus, we may more generally take any pfaffian orientation of $\mathcal{G}$ and label its vertices in such a way that for some given matching M (say, the minimal matching M_-), the permutation σ_M is the identity. Similarly, one can see inductively that it is also possible to label the edges of $\mathcal{G}$ in such a way that if vertices i and j' are adjacent, then $i - j' \leqslant 2$, as all vertices have degree at most 4. Thus, one can take $B_{\mathcal{P}(\mathcal{G})}(z)$ to be 4-diagonal.

With remark 28 in mind, we give the following definition.

Definition 29. Let $\mathcal{G}_\gamma$ be the snake graph associated to the plain arc $\gamma \notin T_0$. Fix any pfaffian orientation of $\mathcal{G}_\gamma$ and label its vertex set so that $\sigma_{M_-} = \mathrm{Id}$. Let $B(z)$ be the corresponding weighted biadjacency matrix of $\mathcal{G}_\gamma$. Denote by $B(w_\bullet)$ the matrix obtained by substituting the weight $w_\bullet(e)$ in each entry of $B(z)$. We will say $B(w_\bullet)$ is a *principal* biadjacency matrix of $\mathcal{G}_\gamma$.

We can now compactly state our determinantal formula, which follows from corollary 27.

Theorem 30. *Let $\mathcal{A}_\bullet(\Sigma, \mathbb{M})$ be a surface cluster algebra with principal coefficients at the initial seed $(T_0, \mathbf{x}_0, \mathbf{y}_0)$, where T_0 is an ideal triangulation of Σ. For every plain (generalized) arc $\gamma \notin T_0$, let $\mathcal{G}_\gamma$ be its associated snake graph. Let $B(w_\bullet)$ be a principal biadjacency matrix of $\mathcal{G}_\gamma$. Then*

$$x_\gamma = \frac{\det B(w_\bullet)}{\operatorname{cross}(\gamma, T_0)} \,.$$

A nice consequence of the previous formula is that, in the case of the cluster algebra associated to a triangulated polygon Σ, it allows to compute all cluster variables from the matrix associated to the *longest* arc of Σ, in the following way.

Proposition 31. *Let Σ be a polygon with a fixed triangulation T_0. Let γ be the unique arc of Σ that crosses all diagonals of T_0, and write $G_1, \dots, G_d$ for the tiles of $\mathcal{G}_\gamma$. Let $B(w_\bullet)(G_\gamma)$ be a principal biadjacency matrix of G_γ. Let η be an arc of Σ that crosses the diagonals $i_1, \dots, i_m$ of T_0. Then $\mathcal{G}_\eta$ is isomorphic to the subgraph of G_γ on tiles $i_1, \dots, i_k$. Furthermore, write B' for the $|V(\mathcal{G}_\eta)| \times |V(\mathcal{G}_\eta)|$ submatrix of $B(w_\bullet)(\mathcal{G}_\gamma)$ indexed by $V(\mathcal{G}_\eta)$. If tile G_j does not appear in $\mathcal{G}_\eta$, set $y_j = 1$ in B'. Then*

$$x_\eta = \frac{\det B'}{\operatorname{cross}(\eta, T_0)} \,.$$

Proof. The tiles $G_1, \ldots, G_d$ are quadrilaterals on Σ that are numbered according to an orientation of γ. Notice that, since γ intersects all diagonals of T_0, all arcs of Σ have end-points at some G_i. Orient η so that its source is a vertex of tile G_{i_1} and its target a vertex of tile G_{i_2}, with $i_1 \leqslant i_2$. The arc η intersects a subset of the diagonals of T_0 in the same order and with the same intersection pattern as γ. Moreover, the weights for the edges of $\mathcal{G}_\eta$ that involve the cluster variables x_i are completely determined by T_0. Thus, ignoring y-variables, $\mathcal{G}_\eta$ coincides with the weighted snake graph given by tiles $G_{k_1}, \ldots, G_{k_2}$, up to a possible reflection when k_1 is even. The matrix B' is a biadjacency matrix for $\mathcal{G}_\eta$, and the restriction of the Pfaffian orientation $\mathcal{P}(\mathcal{G}_\gamma)$ to $\mathcal{G}_\eta$ is Pfaffian by theorem 11. Now, every matching M of $\mathcal{G}_\eta$ extends to a matching M' of $\mathcal{G}_\gamma$, and the specialized height monomial $y(M)$ if $\mathcal{G}_\eta$ is obtained from any corresponding monomial $y(M')$ by setting the variables not appearing in M to 1. The only case in which an irrelevant y-variable may occur in M is when there is some three-tile straight snake subgraph G' of G_γ such that its middle tile has no edges of M_- on its boundary and $\mathcal{G}_\eta$ does not contain the middle tile of G'. In that case, the corresponding weight $\sqrt{y_i}$ has been set it to 1 in B'. □

Observe again figure 1. The weighted snake subgraph given by the two rightmost tiles correctly calculates the cluster variable given by the arc η joining the vertex $\tau_{13} \cap \tau_{14}$ and $\tau_7 \cap \tau_8$, even if the induced weighting of its edges is not a principal weighting, as per definition 19. Notice, however, that this is not the case for the cluster variable associated to the flip of τ_6, since we need first to set $\sqrt{y_4} = 1$.

Remark 32. One can read the monomial $\operatorname{cross}(\eta)$ only by looking at the highest degree monomial in the y_i that occurs as a coefficient in one of the terms of $\det(B(w_\bullet)(\mathcal{G}_\gamma))$. Thus, when Σ is an n-gon, one can compute all cluster variables of $\mathcal{A}_\bullet(T_0)$ only by taking certain minors of a fixed square matrix of size $n-1$.

We finish this section with the following example involving a self-folded triangle.

Example 33. Consider figure 4. The snake graph associated to γ is depicted to the right, computed from left to right according to the rules in section 4.1 of [10]. When crossing a self folded triangle, one must attach a *triple tile* that is always a zig-zag snake graph on three tiles. Note that any As per definition 15, the unique boundary edge e of the fourth tile receives the weight $w_2(e) = y_4/y_3$, so that $w_\bullet(e) = x_3y_4/y_3$. The principal biadjacency matrix $B(w_\bullet)$ obtained from the labeling in remark 25 is

$$B(w_\bullet) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & y_1 \\ -x_1 & x_2 & 0 & 0 & 0 & x_5 & 0 \\ 0 & -y_3x_4 & x_4 & x_4 & 0 & x_4 & 0 \\ 0 & 0 & -y_3x_5 & 1 & x_2 & 0 & 0 \\ 0 & 0 & 0 & -y_5 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{y_4}{y_3}x_3 & x_3 & 0 \\ -1 & 0 & 0 & 0 & 0 & -y_2x_3 & x_2 \end{pmatrix}.$$

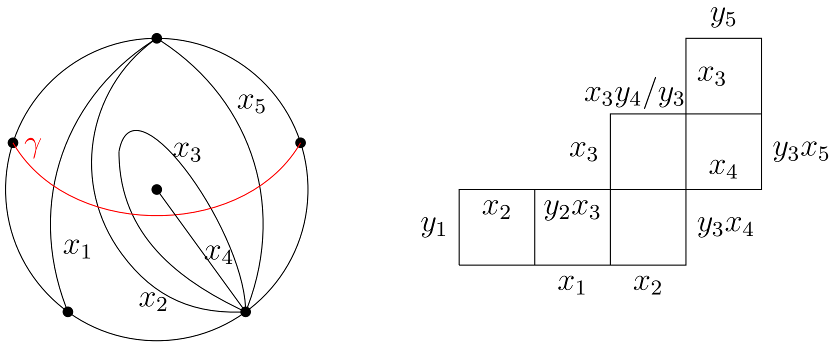

Figure 4: A triangulation of the punctured disc with a self-folded triangle. To the right, the snake graph $\mathcal{G}$ associated to the red arc γ, labeled with a principal weighting $w_\bullet$. Unlabeled edges have weight 1.

Computing $\det(B(w_\bullet))$, we see

$$\begin{aligned}x_\gamma = \frac{1}{x_1x_2x_3^2x_4x_5}&(x_1x_3^2x_4x_5y_1y_2y_3y_4y_5 + x_2x_3x_4x_5^2y_3y_4y_5 + x_3x_4x_5^2y_1y_3y_4y_5+\\ &x_2^2x_3^2x_4x_5y_3 + x_2x_3^2x_4x_5y_1y_3 + x_2^2x_3x_4x_5y_4y_5 + x_2x_3x_4x_5y_1y_4y_5+\\ &x_2^3x_3x_4y_5 + x_2^2x_3x_4y_1y_5 + x_2^2x_3^2x_4 + x_2x_3^2x_4y_1)\,.\end{aligned}$$

5 Band graphs

In this section we give a slight modification of principal weightings to express more general Laurent polynomials obtained from *band graphs*, introduced in [11]. We recall their definition.

Let $\mathcal{G}$ be a snake graph with tiles $G_1, \dots, G_d$. Let us write $N(G_i)$,$W(G_i)$, $E(G_i)$ and $S(G_i)$ for the north, west, east and south edge of G_i, respectively. We write $SW(G_i) := \{S(G_i), N(G_i)\}$ and $NE(G_i) = \{N(G_i), E(G_i)\}$. A *sign function* on $\mathcal{G}$ is an assignment of signs $-, +$ to its edges in such a way that, at each tile G_i, both elements of $SW(G_i)$ share the sign ϵ, while the elements of $NE(G_i)$ both have sign $-\epsilon$. Note that there are exactly two sign functions on any snake graph determined by the two possible choices of signs for the tile G_1.

Definition 34. (Band graph) Let $\mathcal{G}$ be a snake graph with tiles $G_1, \dots, G_d$. Let f be a sign function on G. Let $c \in SW(G_1)$ and $c' \in NE(G_d)$ such that $f(c) = f(c')$. If, as segments drawn in the plane, c and c' are parallel, we identify them with a twist: this is, we orient them in opposite directions and identify them accordingly. Otherwise, c and c' are orthogonal: If c is the west edge of G_1 and c' is the north edge of G_d, and we identify them by $x \sim x'$ and $y \sim y'$, where x is the north vertex of c, x' the west vertex of c', and y, y' the other vertices of c and c', respectively. On the other hand, if c is horizontal and c' vertical, we let x be the east vertex of c and x' the south vertex of c'. The resulting graph $\mathcal{G}^\circ$ is called a *band graph*, and the edge $c \sim c'$ is called the *cut edge*.

Observe that when c and c' are parallel, the band graph $\mathcal{G}^\circ$ sits inside a Möbius band, whereas when c and c' are orthogonal, $\mathcal{G}^\circ$ sits inside an annulus. If $c = \{x, y\}$ and $c' = \{x', y'\}$, are identified by $x \sim x'$ and $y \sim y'$, a twist can be seen if, by drawing a pair of arcs in the outside face of the snake graph $\mathcal{G}$, one connecting x and x' and the other $y \sim y'$, the connecting arcs intersect. The identification described above can be found in a more detailed form in [11], were a weighted band graph is associated to each closed loop ζ of a bordered surface with no punctures, in an analogous way as a snake graph is associated to a generalized arc. Let us denote such a band graph by $\mathcal{G}^\circ_\zeta$.

Definition 35. Let $\mathcal{G}^\circ$ be a band graph. A *good matching* of $\mathcal{G}^\circ$ is a perfect matching of $\mathcal{G}^\circ$ that can be extended to a perfect matching of $\mathcal{G}$.

By definition, the set of good matchings of $\mathcal{G}^\circ$ can be identified with a subset of the matchings of its underlying snake graph. Furthermore, as it is the case with the perfect matchings of snake graphs, the set of good matchings of $\mathcal{G}^\circ$ has the structure of a distributive lattice, as proved in [11]. We denote this lattice by $\mathrm{GMatch}(G^\circ)$.

Remark 36. Let $c = \{x, y\}$ and $c' = \{x', y'\}$ be a cut edge of $\mathcal{G}^*$. As explained in [11], a good matching M of $\mathcal{G}^\circ$ matches both x and y to vertices on the same side of the cut. In other words, either $c, c' \in M$ or x and y are both matched to vertices in the same tile of $\mathcal{G}$. Let G_c, G'_c and $G_{c,c'}$ denote the induced subgraphs of $\mathcal{G}$ on $V(\mathcal{G}) \setminus \{x, y\}$, $V(\mathcal{G}) \setminus \{x', y'\}$ and , respectively. For a band graph $\mathcal{G}^\circ$ with cut edge $c \sim c'$, notice that good matchings are obtained from extensions of the perfect matchings of $\mathcal{G}_c$, $\mathcal{G}_{c'}$ (these extensions coincide exactly when they can be restricted to matchings of $\mathcal{G}_{c,c'}$). Thus, the band graph $\mathcal{G}^\circ$ has exactly $|V(\mathcal{G}_c)| + |V(\mathcal{G}_{c'})| - |V(G_{c,c'})|$ good matchings.

Remark 37. For a band graph $\mathcal{G}^\circ$, one can obtain the poset $P_{\mathcal{G}^\circ}$ of its join irreducible good matchings by adding a single extra cover relation to the poset $P_{\mathcal{G}}$ of the join irreducible perfect matchings of $\mathcal{G}$, according to the following rule ([11], definition 5.6): if the cut edge of tile $\mathcal{G}_1$ is horizontal, then we add the relation $1 > d$. Otherwise, we add $d < 1$ (here we are under the convention that G_1 is drawn with relative orientation 1).

Definition 38. ([11], Definition 3.14) Let (Σ, M) be a marked and unpunctured surface. Let ζ be a non contractible closed loop with no contractible kinks. Suppose that, when looping around η once, the arcs of T_0 that are crossed are $\tau_{i_1}, \ldots, \tau_{i_d}$. Define

$$x_\zeta = \frac{1}{\mathrm{cross}(\zeta, T_0)} \sum_{M \in \mathrm{GMatch}(\mathcal{G}^\circ_\zeta)} x(M) y(M')$$

where $x(M)$ is the product of the weights of the edges in M, $y(M')$ is the specialized height monomial of the matching M' of $\mathcal{G}$ that extends M, and $\mathrm{cross}(\gamma, T_0) = x_{i_1} x_{i_2} \cdots x_{i_d}$.

We now give a slight modification of principal weightings that will be useful for band graphs.

Definition 39. Let $\mathcal{G}^\circ_\eta$ be the weighted band graph on tiles $G_1, \ldots, G_d$ obtained from the closed loop ζ, and write $\mathcal{G}$ for its underlying snake graph. Let $w_\bullet(\mathcal{G})$ be a principal weighting of $\mathcal{G}$. Write $P_{\mathcal{G}^\circ}$ for the poset from remark 37. Let c_1 and c_2 be the edges of $\mathcal{G}$ that are glued in $\mathcal{G}^\circ_\zeta$. Pick edges m_i, m_i' in G_i such that m_i is in the minimal matching M_- of $\mathcal{G}$ and m_i' is not, for $i = 1, d$. We choose m_1', m_d' in such a way that y_1 divides $w_\bullet(m_1')$ and y_d divides $w_\bullet(m_d')$. Consider a formal variable ε with $\varepsilon^2 = 0$. We define the *modified principal weighting* $w^\circ_\bullet(\mathcal{G})$ as follows: if $1 >_{P_{\mathcal{G}^\circ}} d$, set $w^\circ_\bullet(m_1') = \varepsilon y_d w_\bullet(m_1')$, and $w^\circ_\bullet(m_d) = \varepsilon w_\bullet(m_d)$. Similarly, if $1 <_{P_{\mathcal{G}^\circ}} d$, set $w^\circ_\bullet(m_1) = \varepsilon w_\bullet(m_1)$ and $w^\circ_\bullet(m_d') = \varepsilon y_1 w_\bullet(m_d')$. In both cases, set $w^\circ_\bullet(e) = w_\bullet(e)$ for all other edges of $\mathcal{G}$.

In analogy with definition 29, we consider the weighted biadjacency matrices obtained from modified principal weightings, as follows.

Definition 40. Let $\mathcal{G}^\circ$ denote a band graph obtained from the snake graph $\mathcal{G}$. Fix any pfaffian orientation of $\mathcal{G}$ and label its vertex set so that $\sigma_{M_-} = \mathrm{Id}$. Let $B(z)$ be the corresponding weighted biadjacency matrix of $\mathcal{G}$. Let $B(w^\circ_\bullet)$ be the matrix obtained by substituting the modified principal weighting $w^\circ_\bullet(e)$ in each entry of $B(z)$, We call $B(w^\circ_\bullet)$ a principal biadjacency matrix of $\mathcal{G}^\circ$.

Theorem 41. *Let (Σ, M) be an unpunctured bordered surface, and let $\mathcal{G}^\circ_\gamma$ be the weighted band graph $\mathcal{G} = \mathcal{G}_\gamma$ associated to the closed loop ζ. Write $w(c)$ for the weight of the cut edge $c \sim c'$. Let B' be the square submatrix of $B(w^\circ_\bullet)$ obtained from deleting the rows and columns corresponding to edges c and c'. Let $\tilde{\phi} = \det(B(w^\circ_\bullet)) - (w(c)^2 - w(c))\det(B')$. Write $\phi_{good}(\mathcal{G}^\circ_\zeta)$ for the expression obtained from $\tilde{\phi}$ after the substitution $\varepsilon = 1$. Then*

$$x_\gamma = \frac{\phi_{good}(\mathcal{G}^\circ_\zeta)}{\mathrm{cross}(\zeta, T_0)}\,.$$

Proof. Since $B(w^\circ_\bullet)$ is obtained from a pfaffian orientation of $\mathcal{G}$, each term of the polynomial $\det(B(w^\circ_\bullet))$ corresponds to some matching of $\mathcal{G}$. Write each such term as $b_M = \prod_{e\in M} w^\circ_\bullet(e)$. Note that $b_M = 0$ if and only if ε appears twice as a weight of an edge in M. By remark 18, this happens if and only if b_M corresponds to a matching M of $\mathcal{G}$ that involves the edges m_1' and m_d if $1 >_{P_{\mathcal{G}^\circ}} d$, or the edges m_1 and m_d' if $1 <_{P_{\mathcal{G}^\circ}} d$. Let M be such a matching, and observe that it is not a good matching: by definition of m_1, m_1' and m_d, m_d', M corresponds to a subset of $\{1, \ldots, d\}$ that either contains 1 but not d when $1 >_{P_{\mathcal{G}^\circ}} d$, or it contains d but not 1 when $1 <_{P_{\mathcal{G}^\circ}} d$. Thus, it is not an order ideal of $P_{\mathcal{G}^\circ}$. By remark 18, the good matchings of $\mathcal{G}^\circ$ correspond to the order ideals of $P_{\mathcal{G}^\circ}$. This implies that the non zero terms of $\det(B(w^\circ_\bullet))$ are in bijection with good matchings. The weights of a good matching not containing c or c' are computed by $w^\circ_\bullet$. However, in the expression $\det(B(w^\circ_\bullet))$, the weight of c appears with multiplicity 2 in each good matching that contains the cut edge, so we correct this by using the matrix B' whose determinant enumerates the matchings of M that include both c and c'. $\square$

Remark 42. Observe that by remark 36, we can avoid the use of the variable ε altogether and compute $\phi_{good} = \det B_1 + \det B_2 - (w(c)^2 - w(c))\det B_3$, where B_1, B_2 and B_3 are the submatrices of $B(w^\circ_\bullet)$ corresponding to $\mathcal{G}_c$, $\mathcal{G}_{c'}$ and $\mathcal{G}_{c,c'}$, respectively.

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