

New r -Euler–Mahonian statistics involving Denert’s statistic

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Abstract

Recently, we proved the equidistribution of the pairs of permutation statistics $(r\text{des}, r\text{maj})$ and $(r\text{exc}, r\text{den})$. Any pair of permutation statistics that is equidistributed with these pairs is said to be r -Euler–Mahonian. Several classes of r -Euler–Mahonian statistics were established by Huang–Lin–Yan and Huang–Yan. Inspired by their bijections, we provide a new bijective proof of the classical result that (exc, den) is Euler–Mahonian. Using this bijection, we further show that $(\text{exc}_r, \text{den})$ is r -Euler–Mahonian, where exc_r denotes the number of r -level excedances, that is, excedances at least r . Furthermore, by extending our bijection, we establish a more general result that encompasses all the aforementioned results.

Mathematics Subject Classifications: 05A05, 05A19

1 Introduction

1.1 Euler–Mahonian statistics

Let $\mathfrak{S}_n$ denote the symmetric group of permutations on $[n] := \{1, 2, \dots, n\}$. We treat permutations as words on distinct letters by writing them in one-line notation; that is, if $\sigma \in \mathfrak{S}_n$, then we write $\sigma = \sigma_1\sigma_2 \dots \sigma_n$ where $\sigma_i = \sigma(i)$ for all $i \in [n]$. Given a permutation $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in \mathfrak{S}_n$, a position i , $1 \leq i \leq n-1$, is called a *descent* of σ if $\sigma_i > \sigma_{i+1}$. Let $\text{Des}(\sigma)$ be the set of descents of σ , and let $\text{des}(\sigma) = |\text{Des}(\sigma)|$, where $|\cdot|$ indicates cardinality. A position i , $1 \leq i \leq n$, is called an *excedance* of σ if $\sigma_i > i$. Let $\text{Exc}(\sigma)$ be the set of excedances of σ , and let $\text{exc}(\sigma) = |\text{Exc}(\sigma)|$. It is well known that des and exc are equidistributed over $\mathfrak{S}_n$, and a permutation statistic equidistributed with them is said to be *Eulerian*.

Let $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in \mathfrak{S}_n$. A pair (i, j) of positions is called an *inversion* of σ if $i < j$ and $\sigma_i > \sigma_j$. Let $\text{Inv}(\sigma)$ be the set of inversions of σ , and let $\text{inv}(\sigma) = |\text{Inv}(\sigma)|$. Define the

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major index of σ , denoted by $\text{maj}(\sigma)$, to be the sum of the descents of σ , that is,

$$\text{maj}(\sigma) = \sum_{i \in \text{Des}(\sigma)} i.$$

In [9], MacMahon showed that inv and maj are equidistributed over $\mathfrak{S}_n$, and that

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} = [n]_q!,$$

where $[n]_q! = [n]_q[n-1]_q \dots [1]_q$ with $[n]_q = 1 + q + \dots + q^{n-1}$. In his honor, any permutation statistic with this distribution over $\mathfrak{S}_n$ is said to be *Mahonian*.

Denert [1] introduced an excedance-based Mahonian statistic; in this paper, we adopt an equivalent definition due to Foata and Zeilberger [3]. Let $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in \mathfrak{S}_n$. If i is an excedance, we call σ_i an *excedance-letter*. Let $\text{EXCL}(\sigma)$ be the subsequence of σ that consists of the excedance-letters, and let $\text{NEXCL}(\sigma)$ be the subsequence of σ that consists of the remaining letters (i.e., the non-excedance-letters). Define *Denert's statistic* of σ , denoted by $\text{den}(\sigma)$, by

$$\text{den}(\sigma) = \sum_{i \in \text{Exc}(\sigma)} i + \text{inv}(\text{EXCL}(\sigma)) + \text{inv}(\text{NEXCL}(\sigma)).$$

For example, if $\sigma = 715492638$, we have $\text{den}(\sigma) = 1 + 3 + 5 + \text{inv}(759) + \text{inv}(142638) = 13$.

A pair of permutation statistics that is equidistributed with (des, maj) is said to be *Euler–Mahonian*. Denert [1] conjectured that the pair (exc, den) is Euler–Mahonian. The first proof of Denert's conjecture was given by Foata and Zeilberger [3], and a direct bijective proof was later provided by Han [4]. In Section 2, we present another bijective proof.

1.2 r -Euler–Mahonian statistics

Throughout the paper, let $r \geq 1$ be an integer. Let $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in \mathfrak{S}_n$. A position i , $1 \leq i \leq n-1$, is called an r -descent (or r -gap descent) of σ if $\sigma_i \geq \sigma_{i+1} + r$. Let $r\text{Des}(\sigma)$ denote the set of r -descents of σ , and let $r\text{des}(\sigma) = |r\text{Des}(\sigma)|$. A position i , $1 \leq i \leq n$, is called an r -excedance (or r -gap excedance) of σ if $\sigma_i \geq i + r$. Let $r\text{Exc}(\sigma)$ denote the set of r -excedances of σ , and let $r\text{exc}(\sigma) = |r\text{Exc}(\sigma)|$. Clearly, when $r = 1$, $r\text{des}$ and $r\text{exc}$ reduce to des and exc , respectively. The statistics $r\text{des}$ and $r\text{exc}$ are equidistributed over $\mathfrak{S}_n$, and any permutation statistic equidistributed with them is said to be r -Eulerian.

We now review two interpolating Mahonian statistics: the r -major index, introduced by Rawlings [10], and the r -Denert's statistic, introduced by Han [5].

Let $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in \mathfrak{S}_n$. The r -major index of σ , denoted by $r\text{maj}(\sigma)$, is defined by

$$r\text{maj}(\sigma) = \sum_{i \in r\text{Des}(\sigma)} i + |r\text{Inv}(\sigma)|,$$

where $r\text{Inv}(\sigma) = \{(i, j) \in \text{Inv}(\sigma) : \sigma_i < \sigma_j + r\}$. Clearly, $r\text{maj}$ reduces to maj when $r = 1$ and to inv when $r \geq n$, thus the family of $r\text{maj}$ interpolates between maj and inv .

Let $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in \mathfrak{S}_n$. If i is an r -excedance, we call σ_i an r -excedance-letter. Let $r\text{EXCL}(\sigma)$ be the subsequence of σ that consists of the r -excedance-letters, and let $r\text{NEXCL}(\sigma)$ be the subsequence of σ that consists of the remaining letters (i.e., the non- r -excedance-letters). Define the r -Denert's statistic of σ , denoted by $r\text{den}(\sigma)$, by

$$r\text{den}(\sigma) = \sum_{i \in r\text{Exc}(\sigma)} (i + r - 1) + \text{inv}(r\text{EXCL}(\sigma)) + \text{inv}(r\text{NEXCL}(\sigma)).$$

Clearly, $r\text{den}$ reduces to den when $r = 1$ and to inv when $r \geq n$.

In [8], we proved that $(r\text{des}, r\text{maj})$ and $(r\text{exc}, r\text{den})$ are equidistributed over $\mathfrak{S}_n$. Any pair of permutation statistics equidistributed with these pairs is said to be r -Euler–Mahonian. In the remainder of this subsection, we recall other r -Euler–Mahonian statistics.

Let $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in \mathfrak{S}_n$. An excedance i of σ is called an r -level excedance of σ if $i \geq r$. Let $\text{Exc}_r(\sigma)$ and $\text{exc}_r(\sigma)$ denote the set and the number of r -level excedances of σ , respectively. That is,

$$\text{Exc}_r(\sigma) = \{i \in [n] : \sigma_i > i, i \geq r\} \quad \text{and} \quad \text{exc}_r(\sigma) = |\text{Exc}_r(\sigma)|.$$

For example, if $\sigma = 2715643$, we have $\text{exc}_1 = \text{exc} = |\{1, 2, 4, 5\}| = 4$, $\text{exc}_2(\sigma) = |\{2, 4, 5\}| = 3$, $\text{exc}_3(\sigma) = \text{exc}_4(\sigma) = |\{4, 5\}| = 2$, $\text{exc}_5(\sigma) = |\{5\}| = 1$, $\text{exc}_6(\sigma) = 0$.

Given a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in \mathfrak{S}_n$, an excedance-letter σ_i is called an r -level excedance-letter of σ if $\sigma_i \geq r$. The position i is then referred to as an r -level excedance-letter position. Let $\text{Exclp}_r(\sigma)$ be the set of r -level excedance-letter positions of σ , that is,

$$\text{Exclp}_r(\sigma) = \{i \in [n] : \sigma_i > i, \sigma_i \geq r\}.$$

Let $\text{EXCL}_r(\sigma)$ be the subsequence of σ that consists of the r -level excedance-letters, and let $\text{NEXCL}_r(\sigma)$ be the subsequence of σ that consists of the remaining letters (i.e., the non- r -level excedance-letters). Define the r -level Denert's statistic of σ , denoted by $\text{den}_r(\sigma)$, by

$$\text{den}_r(\sigma) = \sum_{i \in \text{Exclp}_r(\sigma)} i + \text{inv}(\text{EXCL}_r(\sigma)) + \text{inv}(\text{NEXCL}_r(\sigma)).$$

For example, let $\sigma = 2715643$, we have $\text{EXCL}(\sigma) = 2756$. If $r = 3$, we have $\text{EXCL}_3(\sigma) = 756$ and $\text{Exclp}_3(\sigma) = \{2, 4, 5\}$, then $\text{den}_3(\sigma) = 2 + 4 + 5 + \text{inv}(756) + \text{inv}(2143) = 15$. If $r = 6$, we have $\text{EXCL}_6(\sigma) = 76$ and $\text{Exclp}_6(\sigma) = \{2, 5\}$, then $\text{den}_6(\sigma) = 2 + 5 + \text{inv}(76) + \text{inv}(21543) = 12$.

Note that if i is an r -level excedance of σ , then i is an r -level excedance-letter position of σ , but not vice versa. Therefore, $\text{Exc}_r(\sigma) \subseteq \text{Exclp}_r(\sigma)$ and $\text{exc}_r(\sigma) \leq |\text{Exclp}_r(\sigma)|$.

Huang, Lin and Yan [6] proved that the pair $(\text{exc}_r, \text{den}_r)$ is r -Euler–Mahonian, thereby confirming a conjecture proposed in [8].

Theorem 1 (Huang, Lin and Yan [6]). *The pair $(\text{exc}_r, \text{den}_r)$ is r -Euler–Mahonian.*

Let us generalize the above notions (see [8], where slightly different notations are used). Let $g, \ell \geq 1$, which we assume throughout the paper. Given a permutation $\sigma = \sigma_1 \sigma_2 \dots \sigma_n \in \mathfrak{S}_n$, a position i , $1 \leq i \leq n$, is called a g -gap ℓ -level excedance of σ if $\sigma_i \geq i + g$ and $i \geq \ell$. Let $g\text{Exc}_\ell(\sigma)$ and $g\text{exc}_\ell(\sigma)$ denote the set and the number of g -gap ℓ -level excedances of σ , respectively. That is,

$$g\text{Exc}_\ell(\sigma) = \{i \in [n] : \sigma_i \geq i + g, i \geq \ell\} \quad \text{and} \quad g\text{exc}_\ell(\sigma) = |g\text{Exc}_\ell(\sigma)|.$$

Let $\sigma = \sigma_1 \sigma_2 \dots \sigma_n \in \mathfrak{S}_n$. The letter σ_i is called a g -gap ℓ -level excedance-letter of σ if $\sigma_i \geq i + g$ and $\sigma_i \geq \ell$. The position i is then referred to as a g -gap ℓ -level excedance-letter position of σ . Let $g\text{Exclp}_\ell(\sigma)$ be the set of g -gap ℓ -level excedance-letter positions of σ ; i.e.,

$$g\text{Exclp}_\ell(\sigma) = \{i \in [n] : \sigma_i \geq i + g, \sigma_i \geq \ell\}.$$

Let $g\text{EXCL}_\ell(\sigma)$ be the subsequence of σ that consists of the g -gap ℓ -level excedance-letters, and let $g\text{NEXCL}_\ell(\sigma)$ be the subsequence of σ that consists of the remaining letters (i.e., the non- g -gap ℓ -level excedance-letters). Define the g -gap ℓ -level Denert's statistic of σ , denoted by $g\text{den}_\ell(\sigma)$, by

$$g\text{den}_\ell(\sigma) = \sum_{i \in g\text{Exclp}_\ell(\sigma)} (i + g - 1) + \text{inv}(g\text{EXCL}_\ell(\sigma)) + \text{inv}(g\text{NEXCL}_\ell(\sigma)).$$

Huang and Yan [7] proved that $(g\text{exc}_\ell, g\text{den}_\ell)$ is $(g + \ell - 1)$ -Euler–Mahonian, thereby confirming a conjecture proposed in [8]. (By setting $\ell = 1$ and $g = 1$ in this result, one obtains that $(\text{rexc}, \text{rden})$ and $(\text{exc}_r, \text{den}_r)$ are r -Euler–Mahonian, respectively.) Moreover, they obtained an instructive result showing that $(g\text{exc}_\ell, g\text{den}_{g+\ell})$ is also $(g + \ell - 1)$ -Euler–Mahonian.

Theorem 2 (Huang and Yan [7]). *Let $g, \ell \geq 1$. We have*

- (1) *The pair $(g\text{exc}_\ell, g\text{den}_\ell)$ is $(g + \ell - 1)$ -Euler–Mahonian.*
- (2) *The pair $(g\text{exc}_\ell, g\text{den}_{g+\ell})$ is $(g + \ell - 1)$ -Euler–Mahonian.*

1.3 The main results

In the spirit of the map $\gamma_{n,g,\ell}$ of Huang and Yan [7] and the map $\beta_{r,n}$ of Huang, Lin and Yan [6], we provide a new bijective proof of the classical result that (exc, den) is Euler–Mahonian. Using this bijection, we further prove that $(\text{exc}_r, \text{den})$ is r -Euler–Mahonian, which is somewhat surprising since the second statistic, den , is independent of r .

Theorem 3. *Let $r \geq 1$. The pair $(\text{exc}_r, \text{den})$ is r -Euler–Mahonian.*

Remark 4. We note that the result does *not* hold for the “gap” case—that is, $(\text{rexc}, \text{den})$ is *not* r -Euler–Mahonian.

By extending our bijection, we establish the following more general result.

Theorem 5. *Let $g, \ell \geq 1$. The pair $(g\text{exc}_\ell, g\text{den}_h)$ is $(g + \ell - 1)$ -Euler–Mahonian for all $1 \leq h \leq g + \ell$.*

Remark 6. Small examples show that $(g\text{exc}_\ell, g\text{den}_h)$ is *not* $(g + \ell - 1)$ -Euler–Mahonian when $h > g + \ell$. Thus, the bound $g + \ell$ indicated by Theorem 2 (2) is tight.

Setting $g = h = 1$ in Theorem 5 yields Theorem 3. Setting $h = \ell$ and $h = g + \ell$ recovers Theorem 2 (1) and (2), respectively. Finally, setting $g = 1$ yields the following result.

Corollary 7. *Let $r \geq 1$. The pair $(\text{exc}_r, \text{den}_h)$ is r -Euler–Mahonian for all $1 \leq h \leq r + 1$.*

This paper is organized as follows. Section 2 presents a new bijective proof of Denert’s conjecture. Section 3 proves Theorem 3 using this bijection, and Section 4 extends the bijection to prove Theorem 5.

2 A new proof of Denert’s conjecture

In this section, we present a bijective proof of Denert’s conjecture, which states that (exc, den) is Euler–Mahonian. To prove that a pair $(\text{stat}_1, \text{stat}_2)$ of permutation statistics is Euler–Mahonian, it suffices to show that there exists a bijection

$$\phi_n : \mathfrak{S}_{n-1} \times \{0, 1, \dots, n-1\} \rightarrow \mathfrak{S}_n$$

such that for $\sigma \in \mathfrak{S}_{n-1}$ and $0 \leq c \leq n-1$, we have

$$\begin{aligned} \text{stat}_1(\phi_n(\sigma, c)) &= \begin{cases} \text{stat}_1(\sigma), & \text{if } 0 \leq c \leq \text{stat}_1(\sigma), \\ \text{stat}_1(\sigma) + 1, & \text{otherwise,} \end{cases} \\ \text{stat}_2(\phi_n(\sigma, c)) &= \text{stat}_2(\sigma) + c, \end{aligned} \tag{2.1}$$

see [10, 4]. The main result of this section is the following theorem.

Theorem 8. *There is a bijection*

$$\phi_n^{\text{den}} : \mathfrak{S}_{n-1} \times \{0, 1, \dots, n-1\} \rightarrow \mathfrak{S}_n,$$

such that for $\sigma \in \mathfrak{S}_{n-1}$ and $0 \leq c \leq n-1$, we have

$$\begin{aligned} \text{Exc}(\phi_n^{\text{den}}(\sigma, c)) &= \begin{cases} \text{Exc}(\sigma), & \text{if } 0 \leq c \leq \text{exc}(\sigma), \\ \text{Exc}(\sigma) \cup \{k_d\}, & \text{otherwise} \end{cases} \\ \text{den}(\phi_n^{\text{den}}(\sigma, c)) &= \text{den}(\sigma) + c, \end{aligned} \tag{2.2}$$

where $d = c - \text{exc}(\sigma)$ and k_d is the d -th non-excedance of σ from smallest to largest.

It is clear that the first formula in (2.2) is a strengthening of the first formula in (2.1). Therefore, the bijection ϕ_n^{den} in Theorem 8 yields a bijective proof of Denert's conjecture. We now present this bijection.

The map $\phi_n^{\text{den}} : \mathfrak{S}_{n-1} \times \{0, 1, \dots, n-1\} \rightarrow \mathfrak{S}_n$

Let $\sigma = \sigma_1 \sigma_2 \dots \sigma_{n-1} \in \mathfrak{S}_{n-1}$ and let c be an integer with $0 \leq c \leq n-1$. Let $\text{exc}(\sigma) = s$, and let the excedance-letters of σ be $e_1, e_2, \dots, e_s$, where $e_1 < e_2 < \dots < e_s$. Let the non-excedance-letters of σ be $\sigma_{k_1}, \sigma_{k_2}, \dots, \sigma_{k_t}$ (equivalently, the non-excedances of σ are $k_1, k_2, \dots, k_t$), where $k_1 < k_2 < \dots < k_t$. We distinguish three cases.

- Case 1: $c = 0$. Define $\phi_n^{\text{den}}(\sigma, c) = \sigma_1 \sigma_2 \dots \sigma_{n-1} n$.
- Case 2: $1 \leq c \leq s$. Let $d = s + 1 - c$. Assume that $e_{j_1} < e_{j_2} < \dots < e_{j_x}$ are the excedance-letters of σ that lie strictly to the left of σ_{e_d} and are at least e_d . Clearly, $e_{j_1} = e_d$. Assume that $\sigma_{k_y}, \sigma_{k_{y+1}}, \dots, \sigma_{k_t}$ are the non-excedance-letters of σ that lie weakly to the right of σ_{e_d} . Define $\phi_n^{\text{den}}(\sigma, c)$ as the permutation obtained from σ by the following three steps:

Step 1. Adjusting Excedance-Letters

Replace e_{j_i} with $e_{j_{i+1}}$ for $i = 1, 2, \dots, x$, with the convention that $e_{j_{x+1}} = n$.

Step 2. Shifting Non-Excedance-Letters

Replace σ_{k_i} with $\sigma_{k_{i-1}}$ for $i = y + 1, y + 2, \dots, t, t + 1$, with the convention that $k_{t+1} = n$ (in other words, shift the letters $\sigma_{k_y}, \sigma_{k_{y+1}}, \dots, \sigma_{k_t}$ one non-excedance to the right).

Step 3. Placing e_d

Set the k_y -th letter to be e_d .

(See Example 9 for an instance in Case 2.)

- Case 3: $s + 1 \leq c \leq n - 1$. Let $d = c - s$. Define $\phi_n^{\text{den}}(\sigma, c)$ as the permutation obtained from σ by the following two steps:

Step i. Shifting Non-Excedance-Letters

Replace σ_{k_i} with $\sigma_{k_{i-1}}$ for $i = d + 1, d + 2, \dots, t, t + 1$, with the convention that $k_{t+1} = n$ (in other words, shift the letters $\sigma_{k_d}, \sigma_{k_{d+1}}, \dots, \sigma_{k_t}$ one non-excedance to the right).

Step ii. Placing n

Set the k_d -th letter to be n .

(See Example 10 for an instance in Case 3.)

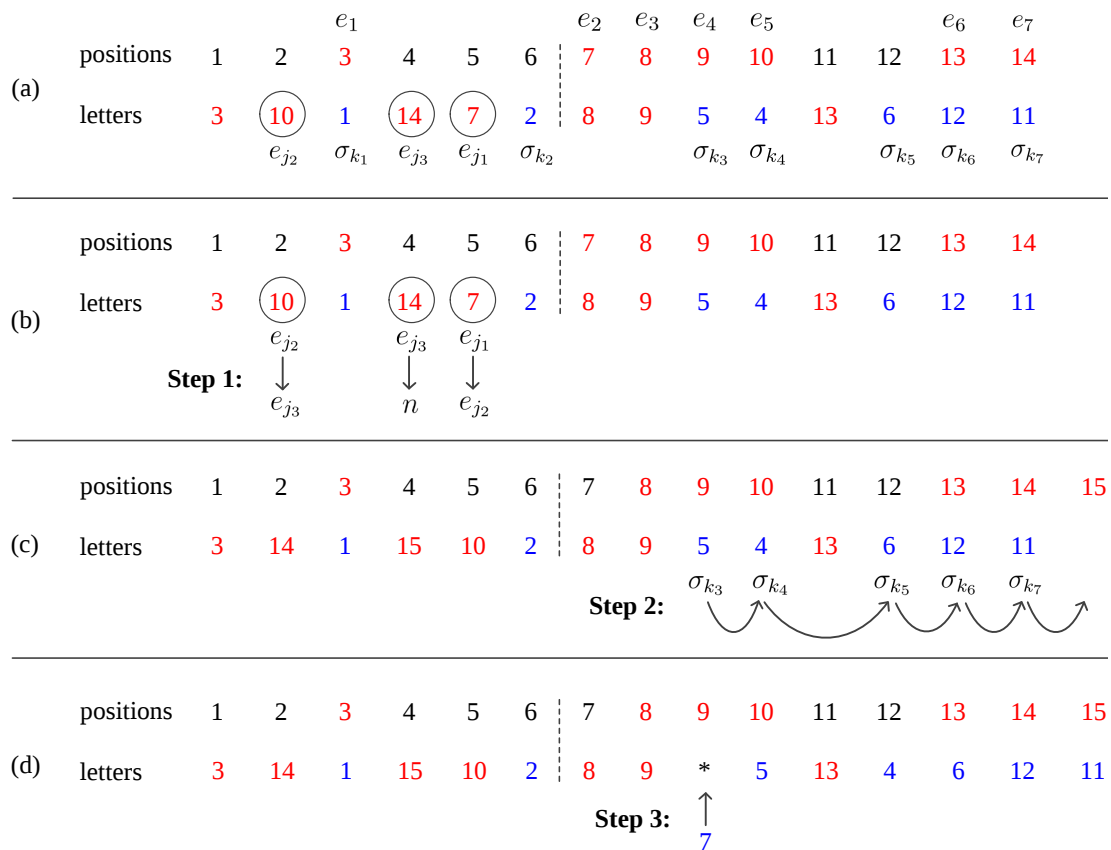


Figure 1: An illustration for Example 9.

Example 9. Let $\sigma = 3\ 10\ 1\ 14\ 7\ 2\ 8\ 9\ 5\ 4\ 13\ 6\ 12\ 11$, and let $c = 6$. Clearly, $s = \text{exc}(\sigma) = 7$, and the excedance-letters of σ are $e_1 = 3$, $e_2 = 7$, $e_3 = 8$, $e_4 = 9$, $e_5 = 10$, $e_6 = 13$, $e_7 = 14$. The non-excedance-letters are $\sigma_{k_1} = 1$, $\sigma_{k_2} = 2$, $\sigma_{k_3} = 5$, $\sigma_{k_4} = 4$, $\sigma_{k_5} = 6$, $\sigma_{k_6} = 12$, $\sigma_{k_7} = 11$. Note that $1 \leq c \leq s$. We have $d = s + 1 - c = 2$, so $e_d = e_2 = 7$. The excedance-letters of σ that lie strictly to the left of $\sigma_{e_d} = \sigma_7$ and that are at least $e_d = 7$ are $e_{j_1} = 7$, $e_{j_2} = 10$, $e_{j_3} = 14$. The non-excedance-letters of σ that lie weakly to the right of $\sigma_{e_d} = \sigma_7$ are $\sigma_{k_3} = 5$, $\sigma_{k_4} = 4$, $\sigma_{k_5} = 6$, $\sigma_{k_6} = 12$, $\sigma_{k_7} = 11$. See Fig. 1(a) for an illustration, where the positions and letters $e_1, e_2, \dots, e_7$ are in red, the non-excedance-letters are in blue, and the letters $e_{j_1}, e_{j_2}, e_{j_3}$ are circled.

Step 1. Adjusting Excedance-Letters

Replace $e_{j_1} = 7$ with $e_{j_2} = 10$, $e_{j_2} = 10$ with $e_{j_3} = 14$, and $e_{j_3} = 14$ with $e_{j_4} = n = 15$; see Fig. 1(b) for an illustration. This results in the sequence: $3\ 14\ 1\ 15\ 10\ 2\ 8\ 9\ 5\ 4\ 13\ 6\ 12\ 11$.

Step 2. Shifting Non-Excedance-Letters

Shift $\sigma_{k_3}, \sigma_{k_4}, \dots, \sigma_{k_7}$ one non-excedance to the right; see Fig. 1(c) for an illustration. The resulting sequence is $3\ 14\ 1\ 15\ 10\ 2\ 8\ 9\ * \ 5\ 13\ 4\ 6\ 12\ 11$, where $*$ is the k_3 -th letter.

Step 3. Placing e_d

Set the k_3 -th letter to be $e_d = 7$; see Fig. 1(d) for an illustration. Then we obtain $\phi_{15}^{\text{den}}(\sigma, 6) = 3\ 14\ 1\ 15\ 10\ 2\ 8\ 9\ 7\ 5\ 13\ 4\ 6\ 12\ 11$.

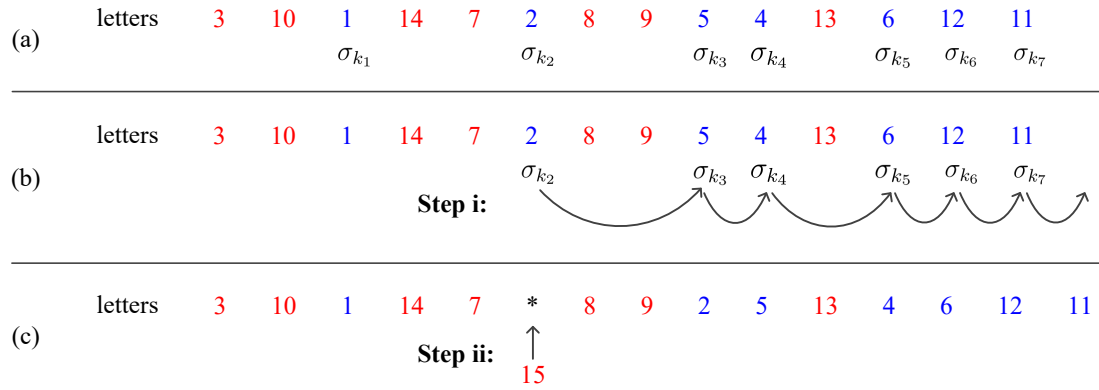


Figure 2: An illustration for Example 10.

Example 10. Consider the running example $\sigma = 3\ 10\ 1\ 14\ 7\ 2\ 8\ 9\ 5\ 4\ 13\ 6\ 12\ 11$, and let $c = 9$. Note that $c \geq s + 1 = 8$. In this case, $d = c - s = 9 - 7 = 2$. Recall that the non-excedance-letters are $\sigma_{k_1} = 1, \sigma_{k_2} = 2, \sigma_{k_3} = 5, \sigma_{k_4} = 4, \sigma_{k_5} = 6, \sigma_{k_6} = 12, \sigma_{k_7} = 11$. See Fig. 2(a) for an illustration, where the excedance-letters and non-excedance-letters of σ are in red and blue, respectively.

Step i. Shifting Non-Excedance-Letters

Shift $\sigma_{k_2}, \sigma_{k_3}, \dots, \sigma_{k_7}$ one non-excedance to the right; see Fig. 2(b) for an illustration. This results in the sequence: $3\ 10\ 1\ 14\ 7\ * \ 8\ 9\ 2\ 5\ 13\ 4\ 6\ 12\ 11$, where $*$ is the k_2 -th letter.

Step ii. Placing n

Set the k_2 -th letter to be $n = 15$; see Fig. 2(c) for an illustration. Then we obtain $\phi_{15}^{\text{den}}(\sigma, 9) = 3\ 10\ 1\ 14\ 7\ 15\ 8\ 9\ 2\ 5\ 13\ 4\ 6\ 12\ 11$.

To establish Theorem 8, we prove two lemmas.

Lemma 11. For $\sigma \in \mathfrak{S}_{n-1}$ and $0 \leq c \leq n - 1$, we have

$$\begin{aligned} \text{Exc}(\phi_n^{\text{den}}(\sigma, c)) &= \begin{cases} \text{Exc}(\sigma), & \text{if } 0 \leq c \leq \text{exc}(\sigma), \\ \text{Exc}(\sigma) \cup \{k_d\}, & \text{otherwise} \end{cases} \\ \text{den}(\phi_n^{\text{den}}(\sigma, c)) &= \text{den}(\sigma) + c, \end{aligned}$$

where $d = c - \text{exc}(\sigma)$ and k_d is the d -th non-excedance of σ from smallest to largest.

Proof. We keep the notation of the construction of ϕ_n^{den} . Assume that $w = \phi_n^{\text{den}}(\sigma, c)$.

Case 1: $c = 0$. Since $w = \sigma_1 \sigma_2 \dots \sigma_{n-1} n$, it is clear that

$$\begin{aligned} \text{Exc}(w) &= \text{Exc}(\sigma), \\ \text{den}(w) &= \text{den}(\sigma) = \text{den}(\sigma) + c, \end{aligned}$$

and the result holds in this case.

Case 2: $1 \leq c \leq s$. Recall that $d = s + 1 - c$, and that $e_{j_1} < e_{j_2} < \dots < e_{j_x}$ are the excedance-letters of σ that lie strictly to the left of σ_{e_d} and are at least e_d . Let σ' be the

resulting sequence after Step 1; that is, σ' is obtained from σ by replacing e_{j_i} with $e_{j_{i+1}}$ for $i = 1, 2, \dots, x$, where $e_{j_{x+1}} = n$. We claim that

$$\text{inv}(\text{EXCL}(\sigma')) = \text{inv}(\text{EXCL}(\sigma)) + c - x. \quad (2.3)$$

- *Proof.* Consider the permutation σ . Let A be the set of positions of letters $e_1, e_2, \dots, e_{d-1}$. Let B (resp. C) be the set of positions of the letters in $\{e_d, \dots, e_s\}$ that lie strictly to the left (resp. weakly to the right) of σ_{e_d} in σ . Note that B is the set of positions of letters $e_{j_1}, e_{j_2}, \dots, e_{j_x}$. So $|B| = x$ and $|C| = s - d + 1 - |B| = c - x$. Let $\text{inv}_\sigma(X, Y)$ denote the number of inversions (i, j) in σ so that $i \in X, j \in Y$ or $i \in Y, j \in X$. Clearly, $\text{inv}(\text{EXCL}(\sigma)) = \text{inv}_\sigma(A, A) + \text{inv}_\sigma(A, B) + \text{inv}_\sigma(A, C) + \text{inv}_\sigma(B, B) + \text{inv}_\sigma(B, C) + \text{inv}_\sigma(C, C)$. It is easy to see that Step 1 does not affect $\text{inv}_\sigma(A, A)$, $\text{inv}_\sigma(A, C)$, and $\text{inv}_\sigma(C, C)$. Note that Step 1 preserves the relative order of the letters at the positions in B , so $\text{inv}_{\sigma'}(B, B) = \text{inv}_\sigma(B, B)$. It is not hard to see that $\text{inv}_{\sigma'}(A, B) = \text{inv}_\sigma(A, B)$, since $\sigma_i < \sigma_j$ if $i \in A$ and $j \in B$. It remains to examine the effect of Step 1 on $\text{inv}_\sigma(B, C)$. Note that the following facts hold: (1) $i < j$ if $i \in B$ and $j \in C$, (2) $e_{j_1} = e_d < \sigma_j$ for all $j \in C$, and (3) $e_{j_{x+1}} = n > \sigma_j$ for all $j \in C$. Then $\text{inv}_{\sigma'}(B, C) = \text{inv}_\sigma(B, C) + |C| = \text{inv}_\sigma(B, C) + c - x$. It follows that $\text{inv}(\text{EXCL}(\sigma')) = \text{inv}(\text{EXCL}(\sigma)) + c - x$, as claimed.

Clearly, Step 1 does not affect $\text{Exc}(\sigma)$ and $\text{inv}(\text{NEXCL}(\sigma))$. Before considering the effect of Steps 2 and 3, we make the following two observations.

1. There are x letters in $\{\sigma_{k_y}, \sigma_{k_{y+1}}, \dots, \sigma_{k_t}\}$ that are less than e_d .

- *Proof.* We need the following result of Dumont [2]: Let σ be a permutation and v a positive integer, then $|\{\sigma_i : \sigma_i < v \leq i\}| = |\{\sigma_i : \sigma_i \geq v > i\}|$. Setting $v = e_d$ and noting that $\{\sigma_i : \sigma_i \geq e_d > i\} = \{e_{j_1}, e_{j_2}, \dots, e_{j_x}\}$, we have $|\{\sigma_i : \sigma_i < e_d \leq i\}| = x$, which means that there are x non-excedance-letters that lie weakly to the right of σ_{e_d} and are less than e_d . This yields the desired result.

2. Any non-excedance-letter of σ that lies to the left of σ_{e_d} is less than e_d .

- *Proof.* Let σ_i be a non-excedance-letter of σ that lies to the left of σ_{e_d} . Then we have $\sigma_i \leq i < e_d$, as desired.

It follows from the two observations above that Steps 2 and 3 increase $\text{inv}(\text{NEXCL}(\sigma'))$ by x . Moreover, Steps 2 and 3 do not affect $\text{Exc}(\sigma')$ and $\text{inv}(\text{EXCL}(\sigma'))$. In summary, Step 1 only increases $\text{inv}(\text{EXCL}(\sigma))$ by $c - x$, while Steps 2 and 3 only increase $\text{inv}(\text{NEXCL}(\sigma'))$ by x . Then

$$\begin{aligned} \text{Exc}(w) &= \text{Exc}(\sigma), \\ \text{den}(w) &= \text{den}(\sigma) + (c - x) + x = \text{den}(\sigma) + c. \end{aligned}$$

Therefore, the result holds in this case.

Case 3: $s + 1 \leq c \leq n - 1$. Recall that $d = c - s$. Assume that there are u (resp. v) excedance-letters of σ to the left (resp. right) of σ_{k_d} . Thus, $u + v = s$. It is not hard to see that $u + d = k_d$. After Steps i and ii, we

1. create a new excedance k_d ,
2. increase $\text{inv}(\text{EXCL}(\sigma))$ by v , and
3. do not change $\text{inv}(\text{NEXCL}(\sigma))$.

Hence, we have

$$\begin{aligned}\text{Exc}(w) &= \text{Exc}(\sigma) \cup \{k_d\}, \\ \text{den}(w) &= \text{den}(\sigma) + k_d + v = \text{den}(\sigma) + u + d + v \\ &= \text{den}(\sigma) + s + d = \text{den}(\sigma) + c.\end{aligned}$$

Thus, the result holds in this case. $\square$

Lemma 12. *The map*

$$\phi_n^{\text{den}} : \mathfrak{S}_{n-1} \times \{0, 1, \dots, n-1\} \rightarrow \mathfrak{S}_n$$

is a bijection.

Proof. For any positive integers a and b with $a < b$, let (a, b) denote the set of all integers i with $a < i < b$, and let $[a, b) := (a, b) \cup \{a\}$. Let $w = w_1 w_2 \dots w_n \in \mathfrak{S}_n$, and let w_i be a non-excedance-letter of w (so $w_i \leq i$). We call w_i called a *critical non-excedance-letter* if $[w_i, i) \subseteq \text{Exc}(w)$. Note that if $w_i = i$, then w_i is a critical non-excedance-letter, since $[w_i, i) = \emptyset \subseteq \text{Exc}(w)$. Any permutation $w \in \mathfrak{S}_n$ has at least one non-excedance-letter—specifically, w_n —and therefore has at least one critical non-excedance-letter, as the leftmost non-excedance-letter must be critical.

We keep the notation of the construction of ϕ_n^{den} . Let $w = w_1 w_2 \dots w_n = \phi_n^{\text{den}}(\sigma, c)$. Assume that $w_z = n$, and that the rightmost critical non-excedance-letter of w is letter a . The image w satisfies the following properties.

- In Case 1, we have $z = n$.
- In Case 2, we have $z < a = e_d$.

Proof. Recall that the letter n in w arises from replacing e_{j_x} in σ , with e_{j_x} lying to the left of σ_{e_d} in σ . Hence $z < e_d$. In the remainder of the proof, we show that $a = e_d$. Note that the non-excedance-letters of w weakly to the right of w_{e_d} are $w_{k_y}, w_{k_{y+1}}, \dots, w_{k_{t+1}}$, where $w_{k_y} = e_d$ and $w_{k_i} = \sigma_{k_{i-1}}$ for $y + 1 \leq i \leq t + 1$. We prove that $a = e_d$ by showing that the non-excedance-letter $w_{k_y} = e_d$ is critical, while w_{k_i} is not critical for $i = y + 1, y + 2, \dots, t + 1$. Since σ_{k_y} is the first (i.e., leftmost) non-excedance-letter that lies weakly to the right of σ_{e_d} , it follows that $[w_{k_y}, k_y) = [e_d, k_y) \subseteq \text{Exc}(\sigma) = \text{Exc}(w)$. So $w_{k_y} = e_d$ is a critical non-excedance-letter of w . Consider w_{k_i} with $y + 1 \leq i \leq t + 1$. Note that $w_{k_i} = \sigma_{k_{i-1}}$. Since $\sigma_{k_{i-1}}$ is a non-excedance-letter of σ , we have $\sigma_{k_{i-1}} \leq k_{i-1}$. Then $w_{k_i} = \sigma_{k_{i-1}} \leq k_{i-1} < k_i$. Since $k_{i-1} \notin \text{Exc}(\sigma) = \text{Exc}(w)$ and $k_{i-1} \in [w_{k_i}, k_i)$, we see that w_{k_i} is not a critical non-excedance-letter of w , completing the proof.

- In Case 3, we have $a \leq z < n$.

Proof. Observe that in this case $z = k_d$. Clearly, $k_d < n$. We show below that $a \leq k_d$. Note that the non-excedance-letters of w are $w_{k_1}, w_{k_2}, \dots, w_{k_{d-1}}, w_{k_{d+1}}, \dots, w_{k_{t+1}}$, where $w_{k_i} = \sigma_{k_i}$ for $1 \leq i \leq d-1$, and $w_{k_i} = \sigma_{k_{i-1}}$ for $d+1 \leq i \leq t+1$. Assume that $a = w_{k_j}$. Our goal is to show that $w_{k_j} \leq k_d$. If $1 \leq j \leq d-1$, then $w_{k_j} \leq k_j < k_d$, and the result holds. If $j = d+1$, $w_{k_j} = w_{k_{d+1}} = \sigma_{k_d} \leq k_d$, and the result holds. If $d+2 \leq j \leq t+1$, then $w_{k_j} = \sigma_{k_{j-1}}$. Since $\sigma_{k_{j-1}} \leq k_{j-1} < k_j$, we have $k_{j-1} \in [\sigma_{k_{j-1}}, k_j] = [w_{k_j}, k_j]$. Combining this with that $k_{j-1} \notin \text{Exc}(w)$ (note that $d+1 \leq j-1 \leq t$), we see that w_{k_j} is not a critical non-excedance-letter of w , which is a contradiction.

Based on the above properties of w , we can uniquely recover (σ, c) from its image w as follows.

- If $z = n$, let σ be obtained from w by removing the letter n .
- If $z < a$, denote $e_d := a$. Assume that $e_{j_2} < \dots < e_{j_x} < e_{j_{x+1}} := n$ are the excedance-letters of w that lie strictly to the left of w_{e_d} and that are greater than e_d . Let σ be obtained from w by the following three steps:
 1. Delete the letter e_d and replace it with a star “*”;
 - 2. Shift the non-excedance-letters that lie to the right of the star one non-excedance to the left (we think of the position occupied by the star as a non-excedance);
 - 3. Replace e_{j_i} with $e_{j_{i-1}}$ for all $i = 2, 3, \dots, x+1$, where we set $e_{j_1} = e_d$.
- If $a \leq z < n$, let σ be obtained from w by the following two steps:
 1. Delete the letter n and replace it with a star “*”;
 - 2. Shift the non-excedance-letters that lie to the right of the star one non-excedance to the left (we think of the position occupied by the star as a non-excedance).

Now we have recovered σ from w . Finally, let $c = \text{den}(w) - \text{den}(\sigma)$, and we successfully recover (σ, c) from w . Therefore, the map $\phi_n^{\text{den}} : \mathfrak{S}_{n-1} \times \{0, 1, \dots, n-1\} \rightarrow \mathfrak{S}_n$ is an injection. Since $|\mathfrak{S}_{n-1} \times \{0, 1, \dots, n-1\}| = |\mathfrak{S}_n| = n!$, it follows that ϕ_n^{den} is a bijection. $\square$

By Lemmas 11 and 12, we obtain Theorem 8.

3 Proof of Theorem 3

As we will see in this section, the bijection ϕ_n^{den} introduced in the previous section can be used not only to prove that (exc, den) is Euler–Mahonian, but also to show that $(\text{exc}_r, \text{den})$ is r -Euler–Mahonian.

To prove that a pair $(\text{stat}_1, \text{stat}_2)$ of permutation statistics is r -Euler–Mahonian, it suffices to show that if $n \leq r$, $(\text{stat}_1, \text{stat}_2)$ is equidistributed with $(0, \text{inv})$ over $\mathfrak{S}_n$; if $n > r$, there exists a bijection

$$\phi_{r,n} : \mathfrak{S}_{n-1} \times \{0, 1, \dots, n-1\} \rightarrow \mathfrak{S}_n$$

such that for $\sigma \in \mathfrak{S}_{n-1}$ and $0 \leq c \leq n-1$, we have

$$\begin{aligned} \text{stat}_1(\phi_{r,n}(\sigma, c)) &= \begin{cases} \text{stat}_1(\sigma), & \text{if } 0 \leq c \leq \text{stat}_1(\sigma) + r - 1, \\ \text{stat}_1(\sigma) + 1, & \text{otherwise,} \end{cases} \\ \text{stat}_2(\phi_{r,n}(\sigma, c)) &= \text{stat}_2(\sigma) + c, \end{aligned} \quad (3.1)$$

see [10].

Clearly, when $n \leq r$, we have $\text{exc}_r(\sigma) = 0$ for any $\sigma \in \mathfrak{S}_n$. Note that den and inv are equidistributed. Thus, $(\text{exc}_r, \text{den})$ and $(0, \text{inv})$ are equidistributed over $\mathfrak{S}_n$ when $n \leq r$. In the remainder of this section, we assume that $n > r$.

Let $\sigma \in \mathfrak{S}_{n-1}$ and $0 \leq c \leq n-1$. Recall that

$$\text{Exc}_r(\sigma) = \{i : \sigma_i > i, i \geq r\} = \{i \in \text{Exc}(\sigma) : i \geq r\}.$$

Denote

$$A_r(\sigma) = \{i : \sigma_i > i, i \leq r-1\} \text{ and } B_r(\sigma) = \{i : \sigma_i \leq i, i \leq r-1\}.$$

Let $w = \phi_n^{\text{den}}(\sigma, c)$. By Theorem 8, we have

$$\begin{aligned} \text{Exc}(w) &= \begin{cases} \text{Exc}(\sigma), & \text{if } 0 \leq c \leq \text{exc}(\sigma), \\ \text{Exc}(\sigma) \cup \{k_d\}, & \text{otherwise} \end{cases} \\ \text{den}(w) &= \text{den}(\sigma) + c, \end{aligned}$$

where $d = c - \text{exc}(\sigma)$ and k_d is the d -th non-excedance of σ from smallest to largest. Then

- If $0 \leq c \leq \text{exc}(\sigma)$, we have $\text{Exc}(w) = \text{Exc}(\sigma)$, implying that $\text{Exc}_r(w) = \text{Exc}_r(\sigma)$.
- If $\text{exc}(\sigma) + 1 \leq c \leq \text{exc}(\sigma) + |B_r(\sigma)|$, we have $\text{Exc}(w) = \text{Exc}(\sigma) \cup \{k_d\}$. As $d = c - \text{exc}(\sigma)$, then $1 \leq d \leq |B_r(\sigma)|$. It follows that $k_d \in B_r(\sigma)$, and hence $k_d \leq r-1$. So $k_d \notin \text{Exc}_r(w)$. Combining this with $\text{Exc}(w) = \text{Exc}(\sigma) \cup \{k_d\}$, we have $\text{Exc}_r(w) = \text{Exc}_r(\sigma)$.
- If $c > \text{exc}(\sigma) + |B_r(\sigma)|$, we have $\text{Exc}(w) = \text{Exc}(\sigma) \cup \{k_d\}$. Note that $d = c - \text{exc}(\sigma) > |B_r(\sigma)|$. It follows that $k_d \notin B_r(\sigma)$, and hence $k_d \geq r$. So $k_d \in \text{Exc}_r(w)$. Combining this with $\text{Exc}(w) = \text{Exc}(\sigma) \cup \{k_d\}$, we have $\text{Exc}_r(w) = \text{Exc}_r(\sigma) \cup \{k_d\}$.

Thus, in summary,

$$\begin{aligned}\text{Exc}_r(w) &= \begin{cases} \text{Exc}_r(\sigma), & \text{if } 0 \leq c \leq \text{exc}(\sigma) + |B_r(\sigma)|, \\ \text{Exc}_r(\sigma) \cup \{k_d\}, & \text{otherwise,} \end{cases} \\ \text{den}(w) &= \text{den}(\sigma) + c.\end{aligned}\tag{3.2}$$

It is clear that $\text{Exc}(\sigma)$ is the disjoint union of $\text{Exc}_r(\sigma)$ and $A_r(\sigma)$. Thus,

$$\text{exc}(\sigma) = \text{exc}_r(\sigma) + |A_r(\sigma)|.$$

It follows that

$$\text{exc}(\sigma) + |B_r(\sigma)| = \text{exc}_r(\sigma) + |A_r(\sigma)| + |B_r(\sigma)| = \text{exc}_r(\sigma) + r - 1.$$

Then (3.2) becomes

$$\begin{aligned}\text{Exc}_r(w) &= \begin{cases} \text{Exc}_r(\sigma), & \text{if } 0 \leq c \leq \text{exc}_r(\sigma) + r - 1, \\ \text{Exc}_r(\sigma) \cup \{k_d\}, & \text{otherwise,} \end{cases} \\ \text{den}(w) &= \text{den}(\sigma) + c,\end{aligned}$$

implying that

$$\begin{aligned}\text{exc}_r(w) &= \begin{cases} \text{exc}_r(\sigma), & \text{if } 0 \leq c \leq \text{exc}_r(\sigma) + r - 1, \\ \text{exc}_r(\sigma) + 1, & \text{otherwise,} \end{cases} \\ \text{den}(w) &= \text{den}(\sigma) + c.\end{aligned}\tag{3.3}$$

Comparing with equation (3.1), we see that $(\text{exc}_r, \text{den})$ is r -Euler–Mahonian, completing the proof of Theorem 3.

Example 13. Let $\sigma = 621534$. We have $\text{Exc}(\sigma) = \{1, 4\}$ and $\text{den}(\sigma) = 7$. Clearly,

$$\text{exc}(\sigma) = 2, \text{exc}_2(\sigma) = 1, \text{exc}_3(\sigma) = 1, \text{exc}_4(\sigma) = 1, \text{exc}_5(\sigma) = 0, \text{exc}_6(\sigma) = 0.$$

In Table 1, we list the images of (σ, c) under the bijection ϕ_7^{den} for $c = 0, 1, \dots, 6$, together with the values of $(\text{exc}_r, \text{den})$ of $\phi_7^{\text{den}}(\sigma, c)$ for $r = 1, 2, \dots, 6$. From Table 1, one can verify equation (3.3).

4 Proof of Theorem 5

In this section, we present an extension of the bijection ϕ_n^{den} introduced in Section 2, followed by a proof of Theorem 5.

Table 1: Images of (σ, c) under ϕ_7^{den} and values of $(\text{exc}_r, \text{den})$							
c	$\phi_7^{\text{den}}(\sigma, c)$	(exc, den)	$(\text{exc}_2, \text{den})$	$(\text{exc}_3, \text{den})$	$(\text{exc}_4, \text{den})$	$(\text{exc}_5, \text{den})$	$(\text{exc}_6, \text{den})$
0	6215347	(2, 7)	(1, 7)	(1, 7)	(1, 7)	(0, 7)	(0, 7)
1	7215364	(2, 8)	(1, 8)	(1, 8)	(1, 8)	(0, 8)	(0, 8)
2	7216534	(2, 9)	(1, 9)	(1, 9)	(1, 9)	(0, 9)	(0, 9)
3	6725134	(3, 10)	(2, 10)	(1, 10)	(1, 10)	(0, 10)	(0, 10)
4	6275134	(3, 11)	(2, 11)	(2, 11)	(1, 11)	(0, 11)	(0, 11)
5	6215734	(3, 12)	(2, 12)	(2, 12)	(2, 12)	(1, 12)	(0, 12)
6	6215374	(3, 13)	(2, 13)	(2, 13)	(2, 13)	(1, 13)	(1, 13)

Recall that for $\sigma \in \mathfrak{S}_n$, if $\sigma_i \geq i + g$ and $\sigma_i \geq h$, the letter σ_i is called a g -gap h -level excedance-letter of σ , and the position i is called a g -gap h -level excedance-letter position. Recall also that

$$g\text{den}_h(\sigma) = \sum_{i \in g\text{Exclp}_h(\sigma)} (i + g - 1) + \text{inv}(g\text{EXCL}_h(\sigma)) + \text{inv}(g\text{NEXCL}_h(\sigma)),$$

where $g\text{EXCL}_h(\sigma)$ (resp. $g\text{NEXCL}_h(\sigma)$) is the subsequence of σ that consists of the g -gap h -level excedance-letters (resp. remaining letters), and $g\text{Exclp}_h(\sigma)$ is the set of g -gap h -level excedance-letter positions of σ (i.e., the positions corresponding to the letters of $g\text{EXCL}_h(\sigma)$). The main result of this section is the following theorem.

Theorem 14. *Let $g \geq 1$ and $1 \leq h \leq n$. There is a bijection*

$$\phi_{g,h,n}^{\text{den}} : \mathfrak{S}_{n-1} \times \{0, 1, \dots, n-1\} \rightarrow \mathfrak{S}_n,$$

such that for $\sigma \in \mathfrak{S}_{n-1}$ and $0 \leq c \leq n-1$, we have

$$\begin{aligned} g\text{Exclp}_h(\phi_{g,h,n}^{\text{den}}(\sigma, c)) &= \begin{cases} g\text{Exclp}_h(\sigma), & \text{if } 0 \leq c \leq |g\text{Exclp}_h(\sigma)| + g - 1, \\ g\text{Exclp}_h(\sigma) \cup \{k_d\}, & \text{otherwise} \end{cases} \\ g\text{den}_h(\phi_{g,h,n}^{\text{den}}(\sigma, c)) &= g\text{den}_h(\sigma) + c, \end{aligned} \quad (4.1)$$

where $d = c - |g\text{Exclp}_h(\sigma)| - g + 1$ and k_d is the d -th non- g -gap h -level excedance-letter position of σ from smallest to largest.

The map $\phi_{g,h,n}^{\text{den}} : \mathfrak{S}_{n-1} \times \{0, 1, \dots, n-1\} \rightarrow \mathfrak{S}_n$ for $1 \leq h \leq n$

Let $\sigma = \sigma_1\sigma_2\ldots\sigma_{n-1} \in \mathfrak{S}_{n-1}$ and let c be an integer with $0 \leq c \leq n-1$. Let $|g\text{Exclp}_h(\sigma)| = s$, and let the g -gap h -level excedance-letters of σ be $e_1, e_2, \dots, e_s$, where $e_1 < e_2 < \dots < e_s$. Let the non- g -gap h -level excedance-letters of σ be $\sigma_{k_1}, \sigma_{k_2}, \dots, \sigma_{k_t}$ (equivalently, the non- g -gap h -level excedance-letter positions of σ are $k_1, k_2, \dots, k_t$), where $k_1 < k_2 < \dots < k_t$. We distinguish three cases.

- Case 1: $0 \leq c \leq g - 1$. Define $\phi_{g,h,n}^{\text{den}}(\sigma, c)$ to be the permutation obtained from σ by inserting n immediately after the letter σ_{n-1-c} .
- Case 2: $g \leq c \leq s + g - 1$. Let $d = s + g - c$ and $p = e_d - g + 1$. Assume that $e_{j_1} < e_{j_2} < \dots < e_{j_x}$ are the g -gap h -level excedance-letters of σ that lie strictly to the left of σ_p and are at least e_d . Clearly, $e_{j_1} = e_d$. Assume that $\sigma_{k_y}, \sigma_{k_{y+1}}, \dots, \sigma_{k_t}$ are the non- g -gap h -level excedance-letters of σ that lie weakly to the right of σ_p . Define $\phi_{g,h,n}^{\text{den}}(\sigma, c)$ as the permutation obtained from σ by the following three steps:

Step 1: Replace e_{j_i} with $e_{j_{i+1}}$ for $i = 1, 2, \dots, x$, with the convention that $e_{j_{x+1}} = n$.

Step 2: Replace σ_{k_i} with $\sigma_{k_{i-1}}$ for $i = y+1, y+2, \dots, t, t+1$, with the convention that $k_{t+1} = n$ (in other words, shift the letters $\sigma_{k_y}, \sigma_{k_{y+1}}, \dots, \sigma_{k_t}$ one non- g -gap h -level excedance-letter position to the right).

Step 3: Set the k_y -th letter to be e_d .
- Case 3: $s + g \leq c \leq n - 1$. Let $d = c - s - g + 1$. Define $\phi_{g,h,n}^{\text{den}}(\sigma, c)$ to be the permutation obtained from σ by the following two steps:

Step i: Replace σ_{k_i} with $\sigma_{k_{i-1}}$ for $i = d+1, d+2, \dots, t, t+1$, with the convention that $k_{t+1} = n$ (in other words, shift the letters $\sigma_{k_d}, \sigma_{k_{d+1}}, \dots, \sigma_{k_t}$ one non- g -gap h -level excedance-letter position to the right).

Step ii: Set the k_d -th letter to be n .

Proof of Theorem 14. We keep the notation of the construction of $\phi_{g,h,n}^{\text{den}}$. Let

$$w = \phi_{g,h,n}^{\text{den}}(\sigma, c).$$

We first prove equation (4.1).

Case 1: $0 \leq c \leq g - 1$. Note that $w = \sigma_1 \sigma_2 \dots \sigma_{n-1-c} n \sigma_{n-c} \dots \sigma_{n-1}$. It is not hard to see that the letters $w_{n-c}, w_{n-c+1}, \dots, w_n$ (i.e., the letters $n, \sigma_{n-c}, \dots, \sigma_{n-1}$) are all non- g -gap h -level excedance-letters of w . Indeed, for $n - c \leq i \leq n$, we have $w_i \leq n \leq i + c < i + g$. Similarly, $\sigma_{n-c}, \sigma_{n-c+1}, \dots, \sigma_{n-1}$ are all non- g -gap h -level excedance-letters of σ . Therefore, $\text{inv}(g\text{NEXCL}_h(w)) - \text{inv}(g\text{NEXCL}_h(\sigma)) = c$. It is not hard to see that

$$\begin{aligned} g\text{Exclp}_h(w) &= g\text{Exclp}_h(\sigma), \\ g\text{den}_h(w) &= g\text{den}_h(\sigma) + c. \end{aligned}$$

Thus, (4.1) holds in this case.

Case 2: $g \leq c \leq s + g - 1$. Recall that $d = s + g - c$, $p = e_d - g + 1$, and that $e_{j_1} < e_{j_2} < \dots < e_{j_x}$ are the g -gap h -level excedance-letters of σ that lie strictly to the

left of σ_p and are at least e_d . Since there are x letters in $\{e_d, e_{d+1}, \dots, e_s\}$ that lie strictly to the left of σ_p (namely $e_{j_1}, e_{j_2}, \dots, e_{j_x}$), there are $s - d + 1 - x$ letters in $\{e_d, e_{d+1}, \dots, e_s\}$ that lie weakly to the right of σ_p . On the other hand, any g -gap h -level excedance-letters of σ that lie weakly to the right of σ_p must be greater than e_d . Indeed, if σ_i is such a letter, then $\sigma_i \geq i + g \geq p + g = e_d + 1 > e_d$. Then we see that there are $s - d + 1 - x$ g -gap h -level excedance-letters of σ that lie weakly to the right of σ_p , and each of them is greater than e_d . Let σ' be the resulting sequence after Step 1. That is, σ' is obtained from σ by replacing e_{j_i} with $e_{j_{i+1}}$ for $i = 1, 2, \dots, x$, where $e_{j_{x+1}} = n$. A similar argument as in the proof of (2.3) yields

$$\text{inv}(g\text{EXCL}_h(\sigma')) - \text{inv}(g\text{EXCL}_h(\sigma)) = s - d + 1 - x = c - g - x + 1.$$

Clearly, Step 1 does not affect $g\text{Exclp}_h(\sigma)$ and $\text{inv}(g\text{NEXCL}_h(\sigma))$. Before considering the effect of Steps 2 and 3, we make the following two observations.

1. There are $x + g - 1$ letters in $\{\sigma_{k_y}, \sigma_{k_{y+1}}, \dots, \sigma_{k_t}\}$ that are less than e_d .

- *Proof.* We need the following result (see [8]): Let σ be a permutation, and v, g be positive integers with $v \geq g$, then $|\{\sigma_i : i \geq v - g + 1, \sigma_i < v\}| = |\{\sigma_i : i < v - g + 1, \sigma_i \geq v\}| + g - 1$. Set $v = e_d$ and $p = e_d - g + 1$. Then $|\{\sigma_i : i \geq p, \sigma_i < e_d\}| = |\{\sigma_i : i < p, \sigma_i \geq e_d\}| + g - 1$. Let $\sigma_i \in \{\sigma_i : i < p, \sigma_i \geq e_d\}$, we have $\sigma_i \geq e_d = p + g - 1 > i + g - 1$, then $\sigma_i \geq i + g$. Combining this with $\sigma_i \geq e_d \geq h$, we see that σ_i is a g -gap h -level excedance-letter of σ . Then we have $|\{\sigma_i : i < p, \sigma_i \geq e_d\}| = |\{e_{j_1}, e_{j_2}, \dots, e_{j_x}\}| = x$. It follows that $|\{\sigma_i : i \geq p, \sigma_i < e_d\}| = x + g - 1$. Now we let $\sigma_i \in \{\sigma_i : i \geq p, \sigma_i < e_d\}$. Since $\sigma_i < e_d = p + g - 1 \leq i + g - 1$, we see that σ_i is a non- g -gap h -level excedance-letter of σ . Then the equation $|\{\sigma_i : i \geq p, \sigma_i < e_d\}| = x + g - 1$ means that there are $x + g - 1$ non- g -gap h -level excedance-letters lying weakly to the right of σ_p and less than e_d . This yields the desired result.

2. Any non- g -gap h -level excedance-letter of σ that lies to the left of σ_p is less than e_d .

- *Proof.* Let σ_i be a non- g -gap h -level excedance-letter of σ that lies to the left of σ_p . Then we have $i < p$ and either $\sigma_i \leq i + g - 1$ or $\sigma_i < h$. If $\sigma_i \leq i + g - 1$, we have $\sigma_i \leq i + g - 1 < p + g - 1 = e_d$. If $\sigma_i < h$, we have $\sigma_i < h \leq e_d$. Thus, $\sigma_i < e_d$ holds in each case, as desired.

It follows from the two observations above that Steps 2 and 3 increase $\text{inv}(g\text{NEXCL}_h(\sigma'))$ by $x + g - 1$. Moreover, Steps 2 and 3 do not affect $g\text{Exclp}_h(\sigma')$ and $\text{inv}(g\text{EXCL}_h(\sigma'))$. In summary, Step 1 only increases $\text{inv}(g\text{EXCL}_h(\sigma))$ by $c - g - x + 1$, while Steps 2 and 3 only increase $\text{inv}(g\text{NEXCL}_h(\sigma'))$ by $x + g - 1$. Then we get

$$\begin{aligned} g\text{Exclp}_h(w) &= g\text{Exclp}_h(\sigma), \\ g\text{den}_h(w) &= g\text{den}_h(\sigma) + (c - g - x + 1) + (x + g - 1) = g\text{den}_h(\sigma) + c. \end{aligned}$$

Thus, (4.1) holds in this case.

Case 3: $s + g \leq c \leq n - 1$. Recall that, in this case, we have $d = c - s - g + 1$. Assume that there are u (resp. v) g -gap h -level excedance-letters of σ to the left (resp. right) of σ_{k_d} . Thus, $u + v = s$. It is easy to see that $u + d = k_d$. After Steps i and ii, we

1. create a new g -gap h -level excedance-letter position k_d ,

- *Proof.* Note that $k_d = u + d \leq s + d = c - g + 1 \leq (n - 1) - g + 1 = n - g$. Therefore, $w_{k_d} = n \geq k_d + g$. Combining this with the condition that $h \leq n$, we see that w_{k_d} is a g -gap h -level excedance-letter of w , and the proof follows.

2. increase $\text{inv}(g\text{EXCL}_h(\sigma))$ by v , and

3. do not change $\text{inv}(g\text{NEXCL}_h(\sigma))$.

Hence, we have

$$\begin{aligned} g\text{Exclp}_h(w) &= g\text{Exclp}_h(\sigma) \cup \{k_d\}, \\ g\text{den}_h(w) &= g\text{den}_h(\sigma) + k_d + g - 1 + v = g\text{den}_h(\sigma) + u + d + g - 1 + v \\ &= g\text{den}_h(\sigma) + s + d + g - 1 = g\text{den}_h(\sigma) + c. \end{aligned}$$

Thus, (4.1) holds in this case.

We have now completed the proof of (4.1). It remains to show that

$$\phi_{g,h,n}^{\text{den}} : \mathfrak{S}_{n-1} \times \{0, 1, \dots, n-1\} \rightarrow \mathfrak{S}_n$$

is a bijection. Let $w = w_1 w_2 \dots w_n \in \mathfrak{S}_n$, and let w_i be a non- g -gap excedance-letter of w (that is, $w_i < i + g$, and thus $w_i - g + 1 \leq i$), we say that w_i is a *critical non- g -gap excedance-letter* of w if $[w_i - g + 1, i] \subseteq g\text{Exc}(w) = \{j : w_j \geq j + g\}$. Any permutation $w \in \mathfrak{S}_n$ has at least one non- g -gap excedance-letter—specifically, w_n —and therefore has at least one critical non- g -gap excedance-letter, as the leftmost non- g -gap excedance-letter must be critical.

Let $w = w_1 w_2 \dots w_n = \phi_{g,h,n}^{\text{den}}(\sigma, c)$. Assume that $w_z = n$, and that the rightmost critical non- g -gap excedance-letter of w is letter a . The image w satisfies the following properties.

- In Case 1, we have $z + g > n$.

Proof. In this case, we have $0 \leq c \leq g - 1$ and $z = n - c$. Then $z + g = n - c + g > n$, as desired.

- In Case 2, we have $\max\{z + g, h\} \leq a = e_d$.

Proof. Recall that the letter n in w arises from replacing e_{j_x} in σ , with e_{j_x} lying to the left of σ_p in σ . Hence $z < p = e_d - g + 1$, so $z + g \leq e_d$. Since e_d is a g -gap h -level excedance-letter, we have $h \leq e_d$. Therefore, $\max\{z + g, h\} \leq e_d$. In the remainder of the proof, we show that $a = e_d$. The non- g -gap h -level excedance-letter of w weakly to the right of w_p are $w_{k_y}, w_{k_y+1}, \dots, w_{k_{t+1}}$, where $w_{k_y} = e_d$ and $w_{k_i} = \sigma_{k_{i-1}}$ for $y + 1 \leq i \leq t + 1$. We claim that the non- g -gap excedance-letters of w weakly to

the right of w_p are also $w_{k_y}, w_{k_{y+1}}, \dots, w_{k_{t+1}}$. Observe that a non- g -gap excedance-letter must be a non- g -gap h -level excedance-letter. It suffices to show that for any i with $y \leq i \leq t+1$, w_{k_i} is a non- g -gap excedance-letter of w . Since w_{k_i} is a non- g -gap h -level excedance-letter of w , we have $w_{k_i} < k_i + g$ or $w_{k_i} < h$. In either case, we have $w_{k_i} < k_i + g$, because $w_{k_i} < h \leq e_d = p + g - 1 \leq k_i + g - 1 < k_i + g$. Thus, w_{k_i} is a non- g -gap excedance-letter of w , which completes the proof of the claim. We prove that $a = e_d$ by showing that $w_{k_y} = e_d$ is a critical non- g -gap excedance-letter of w and w_{k_i} is not a critical non- g -gap excedance-letter of w for $i = y+1, y+2, \dots, t+1$. Since σ_{k_y} is the first (leftmost) non- g -gap h -level excedance-letter of σ lying weakly to the right of σ_p , we have that $[w_{k_y} - g + 1, k_y) = [e_d - g + 1, k_y) = [p, k_y) \subseteq g\text{Exclp}_h(\sigma) = g\text{Exclp}_h(w) \subseteq g\text{Exc}(w)$, hence w_{k_y} is a critical non- g -gap excedance-letter of w . Consider w_{k_i} with $y+1 \leq i \leq t+1$. Note that $w_{k_i} = \sigma_{k_{i-1}}$. Since $\sigma_{k_{i-1}}$ is a non- g -gap h -level excedance-letter of σ , we have $\sigma_{k_{i-1}} \leq k_{i-1} + g - 1$ or $\sigma_{k_{i-1}} < h$. In either case, we have $\sigma_{k_{i-1}} \leq k_{i-1} + g - 1$, since $\sigma_{k_{i-1}} < h \leq e_d = p + g - 1 \leq k_{i-1} + g - 1$. Thus, $w_{k_i} - g + 1 = \sigma_{k_{i-1}} - g + 1 \leq k_{i-1} < k_i$. So $k_{i-1} \in [w_{k_i} - g + 1, k_i)$. By our claim we see that $k_{i-1} \notin g\text{Exc}(w)$ (note that $y \leq i - 1 \leq t$). Thus, w_{k_i} is not a critical non- g -gap excedance-letter of w . We complete the proof.

- In Case 3, we have $a < \max\{z + g, h\} \leq n$.

Proof. Note that in this case we have $z = k_d$. In the proof of Case 3 of (4.1), we showed that $k_d \leq n - g$, and hence $k_d + g \leq n$. Combining this with the condition that $h \leq n$, we obtain $\max\{k_d + g, h\} \leq n$. In the remainder of the proof, we show that $a < \max\{k_d + g, h\}$. Note that the non- g -gap h -level excedance-letters of w are $w_{k_1}, w_{k_2}, \dots, w_{k_{d-1}}, w_{k_{d+1}}, \dots, w_{k_{t+1}}$, where $w_{k_i} = \sigma_{k_i}$ for $1 \leq i \leq d - 1$, and $w_{k_i} = \sigma_{k_{i-1}}$ for $d + 1 \leq i \leq t + 1$. Observe that a non- g -gap excedance-letter of w must be a non- g -gap h -level excedance-letter of w . Assume that $a = w_{k_j}$, and our goal is to show that $w_{k_j} < \max\{k_d + g, h\}$. Since w_{k_j} is non- g -gap excedance-letter of w , we have $w_{k_j} < k_j + g$. If $1 \leq j \leq d - 1$, then $w_{k_j} < k_j + g < k_d + g$, and the result holds. If $j = d + 1$, we have $w_{k_j} = w_{k_{d+1}} = \sigma_{k_d}$. As σ_{k_d} is a non- g -gap h -level excedance-letter of σ , we have $\sigma_{k_d} < k_d + g$ or $\sigma_{k_d} < h$. Therefore, $w_{k_j} = \sigma_{k_d} < \max\{k_d + g, h\}$, and the result holds. If $d + 2 \leq j \leq t + 1$, then $w_{k_j} = \sigma_{k_{j-1}}$. Since $\sigma_{k_{j-1}}$ is a non- g -gap h -level excedance-letter of σ , we have $\sigma_{k_{j-1}} < k_{j-1} + g$ or $\sigma_{k_{j-1}} < h$. If $\sigma_{k_{j-1}} < h$, then $w_{k_j} = \sigma_{k_{j-1}} < h$, and the result holds. Below we assume that $\sigma_{k_{j-1}} \geq h$. Then we must have $\sigma_{k_{j-1}} < k_{j-1} + g$. It follows that $\sigma_{k_{j-1}} - g + 1 \leq k_{j-1} < k_j$. Thus, $k_{j-1} \in [\sigma_{k_{j-1}} - g + 1, k_j) = [w_{k_j} - g + 1, k_j)$. We claim that $w_{k_{j-1}} < k_{j-1} + g$. Note that $d + 1 \leq j - 1 \leq t$. Then $w_{k_{j-1}}$ is a non- g -gap h -level excedance-letter of σ . So $w_{k_{j-1}} < k_{j-1} + g$ or $w_{k_{j-1}} < h$. If $w_{k_{j-1}} < h$, since we assumed $\sigma_{k_{j-1}} \geq h$, it follows that $w_{k_{j-1}} < h \leq \sigma_{k_{j-1}} < k_{j-1} + g$. Thus, in either case we have $w_{k_{j-1}} < k_{j-1} + g$, as claimed. By our claim we see that $k_{j-1} \notin g\text{Exc}(w)$. Combining this with $k_{j-1} \in [w_{k_j} - g + 1, k_j)$, we see that w_{k_j} is not a critical non- g -gap excedance-letter of w , which is a contradiction.

Based on the above properties of w , we can uniquely recover (σ, c) from its image w as follows.

- If $z + g > n$, let σ be the permutation obtained from w by removing the letter n .
- If $\max\{z + g, h\} \leq a$, denote $e_d := a$. Let $p = e_d - g + 1$. Assume that $e_{j_2} < \dots < e_{j_x} < e_{j_{x+1}}$ are the g -gap h -level excedance-letters of w that lie strictly to the left of w_p and are greater than e_d . Let σ be obtained from w by the following three steps:
 1. Delete the letter e_d and replace it with a star “*”;
 - 2. Shift the non- g -gap h -level excedance-letters to the right of the star one non- g -gap h -level excedance-letter position to the left (we think of the position occupied by the star as a non- g -gap h -level excedance-letter position);
 - 3. Replace e_{j_i} with $e_{j_{i-1}}$ for all $i = 2, 3, \dots, x + 1$, where we set that $e_{j_1} = e_d$.
- If $a < \max\{z + g, h\} \leq n$, let σ be obtained from w by the following two steps:
 1. Delete the letter n and replace it with a star “*”;
 - 2. Shift the non- g -gap h -level excedance-letters to the right of the star one non- g -gap h -level excedance-letter position to the left (we think of the position occupied by the star as a non- g -gap h -level excedance-letter position).

Now we have recovered σ from w . Finally, let $c = g\text{den}_h(w) - g\text{den}_h(\sigma)$, and we successfully recover (σ, c) from w . Thus, the map $\phi_{g,h,n}^{\text{den}} : \mathfrak{S}_{n-1} \times \{0, 1, \dots, n-1\} \rightarrow \mathfrak{S}_n$ is an injection. Since $|\mathfrak{S}_{n-1} \times \{0, 1, \dots, n-1\}| = |\mathfrak{S}_n| = n!$, we see that $\phi_{g,h,n}^{\text{den}}$ is a bijection. $\square$

In the remainder of this section, we provide a proof of Theorem 5. To this end, we first state a lemma.

Lemma 15. *Let $g, \ell, h \geq 1$ with $h \leq g + \ell$, and let π be a permutation. Then we have*

$$g\text{Exc}_\ell(\pi) = \{i \in g\text{Exclp}_h(\pi) : i \geq \ell\}.$$

Proof. Recall that

$$\begin{aligned} g\text{Exc}_\ell(\pi) &= \{i : \pi_i \geq i + g, i \geq \ell\}, \text{ and} \\ g\text{Exclp}_h(\pi) &= \{i : \pi_i \geq i + g, \pi_i \geq h\}. \end{aligned}$$

If $i \in g\text{Exc}_\ell(\pi)$, we have $\pi_i \geq i + g \geq \ell + g \geq h$. Therefore,

$$g\text{Exc}_\ell(\pi) = \{i : \pi_i \geq i + g, \pi_i \geq h, i \geq \ell\} = \{i \in g\text{Exclp}_h(\pi) : i \geq \ell\},$$

completing the proof. $\square$

Proof of Theorem 5. If $n \leq g + \ell - 1$, then $g\text{exc}_\ell(\sigma) = |\{i : \sigma_i \geq i + g, i \geq \ell\}| = 0$ for any $\sigma \in \mathfrak{S}_n$. It is known that $g\text{den}_h$ is Mahonian; see [8, the paragraph preceding Section 3]. Therefore, $(g\text{exc}_\ell, g\text{den}_h)$ is equidistributed with $(0, \text{inv})$ over $\mathfrak{S}_n$. In what follows, we assume that $n > g + \ell - 1$. Clearly, $h \leq g + \ell \leq n$. Let $\sigma \in \mathfrak{S}_{n-1}$. Denote

$$A_{g,h,\ell}(\sigma) = \{i \in g\text{Exclp}_h(\sigma) : i \leq \ell - 1\} \text{ and } B_{g,h,\ell}(\sigma) = [\ell - 1] - A_{g,h,\ell}(\sigma).$$

Let $w = \phi_{g,h,n}^{\text{den}}(\sigma, c)$. Denote $s = |g\text{Exclp}_h(\sigma)|$. By Theorem 14, we have

$$g\text{Exclp}_h(w) = \begin{cases} g\text{Exclp}_h(\sigma), & \text{if } 0 \leq c \leq s + g - 1, \\ g\text{Exclp}_h(\sigma) \cup \{k_d\}, & \text{otherwise} \end{cases}$$

$$g\text{den}_h(w) = g\text{den}_h(\sigma) + c,$$

where $d = c - s - g + 1$ and k_d is the d -th non- g -gap h -level excedance-letter position of σ from smallest to largest. Then

- If $0 \leq c \leq s + g - 1$, we have $g\text{Exclp}_h(w) = g\text{Exclp}_h(\sigma)$. By Lemma 15, we see that $g\text{Exc}_\ell(w) = g\text{Exc}_\ell(\sigma)$.
- If $s + g \leq c \leq s + g + |B_{g,h,\ell}(\sigma)| - 1$, we have $g\text{Exclp}_h(w) = g\text{Exclp}_h(\sigma) \cup \{k_d\}$. Since $d = c - s - g + 1$, then $1 \leq d \leq |B_{g,h,\ell}(\sigma)|$. It follows that $k_d \in B_{g,h,\ell}(\sigma)$, and so $k_d \leq \ell - 1$. By Lemma 15, we have $g\text{Exc}_\ell(w) = g\text{Exc}_\ell(\sigma)$.
- If $c \geq s + g + |B_{g,h,\ell}(\sigma)|$, we have $g\text{Exclp}_h(w) = g\text{Exclp}_h(\sigma) \cup \{k_d\}$. Since $d = c - s - g + 1$, then $d \geq |B_{g,h,\ell}(\sigma)| + 1$. It follows that $k_d \notin B_{g,h,\ell}(\sigma)$, and hence $k_d \geq \ell$. By Lemma 15, we see that $g\text{Exc}_\ell(w) = g\text{Exc}_\ell(\sigma) \cup \{k_d\}$.

Therefore, in summary,

$$g\text{Exc}_\ell(w) = \begin{cases} g\text{Exc}_\ell(\sigma), & \text{if } 0 \leq c \leq s + g + |B_{g,h,\ell}(\sigma)| - 1 \\ g\text{Exc}_\ell(\sigma) \cup \{k_d\}, & \text{otherwise,} \end{cases} \quad (4.2)$$

$$g\text{den}_h(w) = g\text{den}_h(\sigma) + c.$$

By Lemma 15, we see that $g\text{Exclp}_h(\sigma)$ is the disjoint union of $g\text{Exc}_\ell(\sigma)$ and $A_{g,h,\ell}(\sigma)$. Thus, $s = g\text{exc}_\ell(\sigma) + |A_{g,h,\ell}(\sigma)|$. It follows that

$$\begin{aligned} s + g + |B_{g,h,\ell}(\sigma)| - 1 &= g\text{exc}_\ell(\sigma) + |A_{g,h,\ell}(\sigma)| + g + |B_{g,h,\ell}(\sigma)| - 1 \\ &= g\text{exc}_\ell(\sigma) + g + \ell - 2. \end{aligned}$$

Combining this with (4.2), we get

$$g\text{exc}_\ell(w) = \begin{cases} g\text{exc}_\ell(\sigma), & \text{if } 0 \leq c \leq g\text{exc}_\ell(\sigma) + g + \ell - 2, \\ g\text{exc}_\ell(\sigma) + 1, & \text{otherwise,} \end{cases}$$

$$g\text{den}_h(w) = g\text{den}_h(\sigma) + c.$$

Comparing with equation (3.1), we see that $(g\text{exc}_\ell, g\text{den}_h)$ is $(g + \ell - 1)$ -Euler–Mahonian. $\square$

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