Four relations on the set of point-hyperplane antiflags

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Abstract

There are precisely four arrangements of two point-hyperplane antiflags. We consider the corresponding relations on the set of such antiflags and show that each of them can be recovered from any other except in one special case. If the field consists of two elements, then one of the relations cannot be used to recover each of the remaining three. This is related to a bijection between antiflags and exterior points of the hyperbolic polar space which exists in this case.

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1 Introduction

Let $\mathbb{F}$ be a field (not necessarily finite) and let n be a natural number not less than three. Consider the n-dimensional vector space $V = \mathbb{F}^n$ and the associated projective space $\mathrm{PG}(n-1, \mathbb{F})$. A *point-hyperplane antiflag* of $\mathrm{PG}(n-1, \mathbb{F})$ is a pair (p, H), where p is a point and H is a hyperplane such that $p \notin H$. The set of all such antiflags will be denoted by $\mathcal{A}$. We will work with the four relations $\sim_i$, $i \in \{1, 2, 3, 4\}$ on $\mathcal{A}$ defined as follows. For distinct antiflags $A_j = (p_j, H_j)$ with $j \in \{1, 2\}$ we write

$A_1 \sim_1 A_2$ if and only if either $p_1 = p_2$ or $H_1 = H_2$;

$A_1 \sim_2 A_2$ if and only if $p_1 \neq p_2$, $H_1 \neq H_2$, $p_1 \notin H_2$ and $p_2 \notin H_1$;

$A_1 \sim_3 A_2$ if and only if either $p_1 \in H_2$ and $p_2 \notin H_1$ or $p_1 \notin H_2$ and $p_2 \in H_1$;

$A_1 \sim_4 A_2$ if and only if $p_1 \in H_2$ and $p_2 \in H_1$.

It is clear that every pair of antiflags satisfies just one of the above four conditions. For every $i \in \{1, 2, 3, 4\}$ denote by Γ_i the simple graph whose vertex set is $\mathcal{A}$ and distinct

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antiflags A_1, A_2 are adjacent vertices of this graph if and only if $A_1 \sim_i A_2$. In what follows, we will say that antiflags A_1, A_2 are *i-adjacent* if $A_1 \sim_i A_2$.

Note that each of the relations $\sim_2, \sim_3$ and $\sim_4$ can be defined via $\sim_1$, or more precisely, via the distance d in the graph Γ_1. Let $A^{\sim_1}$ be the neighborhood in Γ_1 of an antiflag A. Then for distinct antiflags $A_j = \{p_j, H_j\}$ with $j \in \{1,2\}$ the following hold.

- $A_1 \sim_2 A_2$ if and only if $d(A_1, A_2) = 2$ and $|A_1^{\sim_1} \cap A_2^{\sim_1}| > 1$. In this case, $A_1^{\sim_1} \cap A_2^{\sim_1} = \{(p_1, H_2), (p_2, H_1)\}$.
- $A_1 \sim_3 A_2$ if and only if $d(A_1, A_2) = 2$ and $|A_1^{\sim_1} \cap A_2^{\sim_1}| = 1$. The unique element of $A_1^{\sim_1} \cap A_2^{\sim_1}$ is (p_1, H_2) or (p_2, H_1).
- $A_1 \sim_4 A_2$ if and only if $d(A_1, A_2) = 3$. In this case, $A_1^{\sim_1} \cap A_2^{\sim_1} = \emptyset$ and there are $A_1' \in A_1^{\sim_1}$ and $A_2' \in A_2^{\sim_1}$ such that $A_1' \sim_3 A_2$ and $A_1 \sim_3 A_2'$.

In particular, the graph Γ_1 has diameter 3.

It is well-known that $\sim_1$ can be recovered from $\sim_4$ (see, for example, [2, 10]). Hence, we can also recover $\sim_2$ and $\sim_3$ from $\sim_4$, since they are recoverable from $\sim_1$. In the present paper, we shall show that every $\sim_i$ with $i \neq 2$ can be recovered from $\sim_2$ and, when $|\mathbb{F}| \geqslant 3$, we can also recover every $\sim_i$ with $i \neq 3$ from $\sim_3$.

The projective space $\mathrm{PG}(n-1, \mathbb{F})$ can be recovered from Γ_1 up to duality (see [7] or [11, Section 3.7]). Consequently, the automorphism group of the graph Γ_1 is the semidirect product $\mathrm{P\Gamma L}(n, \mathbb{F}) \rtimes C_2$. In view of the main result of this paper, the same holds for $\mathrm{Aut}(\Gamma_i)$ for every $i \in \{2,3,4\}$, provided that $|\mathbb{F}| \geqslant 3$ when $i = 3$.

In the exceptional case, when $|\mathbb{F}| = 2$, the impossibility to recover any of the relations $\sim_1, \sim_2$ or $\sim_4$ from $\sim_3$ is due to the existence of a bijection between the antiflags of $\mathrm{PG}(n-1, 2)$ and the non-singular points of a hyperbolic form on $\mathbb{F}_2^{2n}$ or, equivalently, the exterior points of the quadric $\mathcal{O}^+(2n, 2)$ (see [12]; an alternative description of this bijection will be offered Section 4). This bijection is actually an isomorphism from Γ_3 to the collinearity graph of the point-line geometry with the non-singular points of $\mathrm{PG}(2n-1, 2)$ as points and the lines of $\mathrm{PG}(2n-1, 2)$ exterior to $\mathcal{O}^+(2n, 2)$ as lines. This point-line geometry admits the same automorphism group as $\mathcal{O}^+(2n, 2)$. Accordingly, when $|\mathbb{F}| = 2$ the automorphism group of Γ_3 is isomorphic to $\mathrm{O}^+(2n, 2)$. This group is larger than $\mathrm{SL}(n, 2) \rtimes C_2$ which, as explained above, when $|\mathbb{F}| = 2$ is isomorphic to $\mathrm{Aut}(\Gamma_i)$ for every $i \in \{1,2,4\}$.

Note that the relations $\sim_4$ and $\sim_1$ play an important role in the geometry of the classical groups [6, 7]. Suppose that $\mathrm{char}(\mathbb{F}) \neq 2$. For every involution $u \in \mathrm{GL}(n, \mathbb{F})$ there are subspaces $S_+(u), S_-(u) \subset V$ such that $V = S_+(u) \oplus S_-(u)$ and the restriction of u to $S_+(u)$ and $S_-(u)$ is id and $-id$, respectively. The involution u is called a $(k, n-k)$-*involution*, where $\dim(S_+(u)) = k$ (hence $\dim(S_-(u)) = n-k$). For every $k \in \{1, \ldots, n-1\}$ all $(k, n-k)$-involutions form a conjugacy class in $\mathrm{GL}(n, \mathbb{F})$. Every point-hyperplane antiflag of $\mathrm{PG}(n-1, \mathbb{F})$ determines some $(1, n-1)$-involution u as well as the $(n-1, 1)$-involution $-u$. For $k = 1, n-1$ two distinct $(k, n-k)$-involutions commute if and only if the corresponding antiflags are related by $\sim_4$. Recall that $t \in \mathrm{GL}(n, \mathbb{F})$ is a *transvection* if

$$t(x) = x + \alpha(x)x_0,$$

where $\alpha : V \to \mathbb{F}$ is a linear functional and x_0 is a non-zero vector in the kernel of α. For $k = 1, n-1$ the composition of two distinct $(k, n-k)$-involutions is a transvection if and only if the corresponding antiflags are related by $\sim_1$. The relations $\sim_4$ and $\sim_1$ applied to $(1, n-1)$-involutions are used in the description of automorphisms of the group $\mathrm{GL}(n, \mathbb{F})$ (see [6, 7] and [11, Section 3.7] for more information).

It should be noted that various characterizations of distances in terms of other distances for Grassmann graphs and graphs related to polar spaces can be found in [1, 3, 4, 5, 9].

2 Main result

The following is the main result of this paper.

Theorem 1. *The following assertions are fulfilled:*

(1) *If $|\mathbb{F}| \geqslant 3$ or $i \neq 3$ then for any distinct $i, j \in \{1, 2, 3, 4\}$ the relation $\sim_j$ can be recovered from $\sim_i$.*

(2) *If $|\mathbb{F}| = 2$, then no $\sim_j$, $j \in \{1, 2, 4\}$ can be recovered from $\sim_3$.*

Every semilinear automorphism of V induces an automorphism of the graph Γ_i. Similarly, if u is a semilinear isomorphism of V to the dual vector space V^*, then the map

$$(p, H) \to (u(H)^0, u(p)^0)$$

(we write X^0 for the annihilator of a subspace X) is an automorphism of Γ_i.

We have previously noticed that $\mathrm{Aut}(\Gamma_1) \cong \mathrm{P\Gamma L}(n, \mathbb{F}) \rtimes C_2$. So, the first part of Theorem 1 yields the following.

Corollary 2. *Suppose that $|\mathbb{F}| \geqslant 3$ or $i \neq 3$. Then every automorphism of Γ_i is induced by a semilinear automorphism of V or a semilinear isomorphism of V to V^*. Hence $\mathrm{Aut}(\Gamma_i) \cong \mathrm{P\Gamma L}(n, \mathbb{F}) \rtimes C_2$.*

Remark 3. If $i \in \{3, 4\}$, then for any two pairs of i-adjacent antiflags there is a linear automorphism of V sending one of these pairs to the other. Similarly, any pair of 1-adjacent antiflags can be transferred to any other pair of 1-adjacent antiflags by a linear automorphism of V or a transformation of $\mathcal{A}$ induced by a linear isomorphism from V to V^*. So, Γ_i, $i \in \{1, 3, 4\}$ is edge-transitive. If (p_1, H_1) and (p_2, H_2) are 2-adjacent antiflags, then the line $\langle p_1, p_2 \rangle$ intersects $H_1 \cap H_2$ or it intersects H_1 and H_2 in distinct points. The second possibility occurs if and only if $|\mathbb{F}| \geqslant 3$. Therefore, Γ_2 is edge-transitive only when $|\mathbb{F}| = 2$.

Remark 4. If $|\mathbb{F}| \geqslant 3$, then $\sim_2$ can be split into two relations $\sim_2'$ and $\sim_2''$ as follows: $(p_1, H_1) \sim_2' (p_2, H_2)$ if the line $\langle p_1, p_2 \rangle$ intersects $H_1 \cap H_2$ and $(p_1, H_1) \sim_2'' (p_2, H_2)$ if this line intersects H_1 and H_2 in distinct points. By [8], the five relations $\sim_1, \sim_2', \sim_2'', \sim_3, \sim_4$ $, \sim_5$ define the structure of an association scheme on the set of point-line antiflags of a finite projective plane greater than the Fano plane. Verifying the same fact for a higher dimension requires complicated calculations.

The relation $\sim_3$ for the case when $|\mathbb{F}| = 2$ will be considered in Section 4. The second part of Theorem 1 will be obtained as a simple consequence of the description of the automorphism group of the graph Γ_3.

3 Proof of the first part of Theorem 1

3.1 Recovering from $\sim_1$ and $\sim_4$

If $\mathcal{X}$ is a subset of the set of antiflags $\mathcal{A}$, then for every $i \in \{1, 2, 3, 4\}$ denote by $\mathcal{X}^{\sim_i}$ the set of all antiflags i-adjacent to every antiflag from $\mathcal{X}$. By Mackey [10] (see also [11, Lemma 3.10] and Cohen et al. [2, Lemma 2.6]), the relation $\sim_1$ can be characterized in terms of the relation $\sim_4$ as follows.

Proposition 5. *Distinct antiflags A_1, A_2 are 1-adjacent if and only if for any distinct antiflags $A_1', A_2' \in (\{A_1, A_2\}^{\sim_4})^{\sim_4}$ we have*

$$\{A_1', A_2'\}^{\sim_4})^{\sim_4} = (\{A_1, A_2\}^{\sim_4})^{\sim_4}.$$

As noticed in the Introduction, the both relations $\sim_2$ and $\sim_3$ can be recovered from $\sim_1$. So, characterizations of $\sim_3$ and $\sim_2$ in terms of $\sim_4$ are implicit in Proposition 5. To sum up, for any distinct $i \in \{1, 4\}$ and $j \in \{1, 2, 3, 4\}$ the relation $\sim_j$ can be recovered from $\sim_i$.

3.2 Recovering from $\sim_2$

Let $\mathfrak{X}$ be the family of all subsets $\{A_1, A_2\}^{\sim_2}$, where A_1, A_2 are i-adjacent antiflags and $i \neq 2$. We characterize the relations $\sim_i$, $i \in \{1, 3, 4\}$ in terms of the poset $(\mathfrak{X}, \subset)$ for almost all cases.

Proposition 6. *Suppose that $|\mathbb{F}| \geqslant 3$ or $n \geqslant 4$. Then the following assertions are fulfilled:*

- *If $A_1 \sim_1 A_2$, then $\{A_1, A_2\}^{\sim_2}$ is a non-minimal and maximal element of $(\mathfrak{X}, \subset)$.*
- *If $A_1 \sim_3 A_2$, then $\{A_1, A_2\}^{\sim_2}$ is a minimal and non-maximal element of $(\mathfrak{X}, \subset)$.*
- *If $A_1 \sim_4 A_2$, then $\{A_1, A_2\}^{\sim_2}$ is a minimal and maximal element of $(\mathfrak{X}, \subset)$.*

The case when $|\mathbb{F}| = 2$ and $n = 3$ will be considered at the end of this subsection. To prove Proposition 6 we use the following.

Lemma 7. *For mutually distinct hyperplanes H_1, H_2, H_3 the following assertions are fulfilled:*

- *If $H_1 \cap H_2 \not\subset H_3$ and $|\mathbb{F}| \geqslant 3$ or $n \geqslant 4$, then there are at least two points outside of $H_1 \cup H_2 \cup H_3$.*
- *If $H_1 \cap H_2 \subset H_3$ and $|\mathbb{F}| \geqslant 3$, then there are more than two points outside of $H_1 \cup H_2 \cup H_3$.*

Proof. If $\mathbb{F}$ is infinite, then there are infinitely many points outside of $H_1 \cup H_2 \cup H_3$ in the both cases. Suppose that $|\mathbb{F}| = q$. If $H_1 \cap H_2 \not\subset H_3$, then

$$|H_1 \cup H_2 \cup H_3| = |H_1| + |H_2| + |H_3| - |H_1 \cap H_2| - |H_1 \cap H_3| - |H_2 \cap H_3| + |H_1 \cap H_2 \cap H_3| =$$

$$\frac{3(q^{n-1}-1) - 3(q^{n-2}-1) + q^{n-3} - 1}{q-1} = \frac{(3q^2 - 3q + 1)q^{n-3} - 1}{q-1}$$

which means that there are precisely

$$\frac{q^n - 1}{q-1} - |H_1 \cup H_2 \cup H_3| = \frac{(q^3 - 3q^2 + 3q - 1)q^{n-3}}{q-1} = (q-1)^2 q^{n-3}$$

points outside of $H_1 \cup H_2 \cup H_3$. This number is not less than 2 if $q \geqslant 3$ or $n \geqslant 4$. If $H_1 \cap H_2 \subset H_3$, then

$$|H_1 \cup H_2 \cup H_3| = \frac{3(q^{n-1}-1) - 2(q^{n-2}-1)}{q-1}$$

and we obtain that

$$\frac{q^n - 1}{q-1} - |H_1 \cup H_2 \cup H_3| = \frac{(q^2 - 3q + 2)q^{n-2}}{q-1} = (q-2)q^{n-2}.$$

This number is greater than 2 if $q \geqslant 3$. □

Lemma 8. *Suppose that $|\mathbb{F}| \geqslant 3$ or $n \geqslant 4$. If antiflags $\mathcal{A}_j = (P_j, H_j)$, $j \in \{1, 2\}$ are i-adjacent for $i \neq 2$, then the following assertions are fulfilled:*

(1) *For any point p distinct from p_1, p_2 there is an antiflag $(p', H') \in \{A_1, A_2\}^{\sim_2}$ such that $p \in H'$.*

(2) *For any hyperplane H distinct from H_1, H_2 there is an antiflag $(p'', H'') \in \{A_1, A_2\}^{\sim_2}$ such that $p'' \in H$.*

Remark 9. If $|\mathbb{F}| \geqslant 3$ or $n \geqslant 4$, then Lemma 8 shows that

$$(\{A_1, A_2\}^{\sim_2})^{\sim_2} = \{A_1, A_2\}$$

for any i-adjacent antiflags A_1, A_2 with $i \neq 2$. In the case when $|\mathbb{F}| = 2$ and $n = 3$, this fails, see Proposition 10.

Proof of Lemma 8. The statement (2) follows from the statement (1) by duality. Also, it is sufficient to prove the statement (1) for the case when A_1, A_2 are i-adjacent with $i \in \{3,4\}$.

Indeed, if $p_1 = p_2$, then we take any point $p'_2 \in H_1 \setminus (H_2 \cup \{p\})$. The antiflags $A_1 = (p_1, H_1)$ and $A'_2 = (p'_2, H_2)$ are 3-adjacent. If (p', H') belongs to $\{A_1, A'_2\}^{\sim_2}$, then $p' \notin H_2$ and $p_1 = p_2 \notin H'$ which means that (p', H') is 2-adjacent to A_2.

Similarly, if $H_1 = H_2$, then we consider $A'_2 = (p_2, H'_2)$, where H'_2 is a hyperplane which contains p_1 and does not contain p_2. As above, A_1 and A'_2 are 3-adjacent. If (p', H') belongs to $\{A_1, A'_2\}^{\sim_2}$, then $p' \notin H_1 = H_2$ and $p_2 \notin H'$, i.e. (p', H') is 2-adjacent to A_2.

From here on, we suppose that the antiflags A_1, A_2 are i-adjacent for $i \in \{3,4\}$. Then $p_1 \neq p_2$, $H_1 \neq H_2$ and at least one of p_j is contained in H_{3-j}. Without loss of generality we assume that $p_2 \in H_1$.

Consider the case when $|\mathbb{F}| \geqslant 3$. Suppose that p is not on the line $\langle p_1, p_2 \rangle$. By our assumption, H_1 intersects this line precisely in p_2. The intersection point of H_2 and $\langle p_1, p_2 \rangle$ may be distinct from p_1. Since there are at least four points on $\langle p_1, p_2 \rangle$, the line contains a point $\tilde{p} \notin H_1 \cup H_2$ distinct from p_1, p_2. We take any hyperplane H' passing through p and intersecting $\langle p_1, p_2 \rangle$ precisely in $\tilde{p}$.

Suppose now that $p \in \langle p_1, p_2 \rangle$. Then $p \notin H_1$ (since $p_2 \in H_1$ and $\langle p_1, p_2 \rangle \not\subset H_1$). If p is the intersection point of H_2 and $\langle p_1, p_2 \rangle$, then there is a hyperplane H' intersecting $\langle p_1, p_2 \rangle$ precisely in p and distinct from H_2. If $p \notin H_2$, then we take any hyperplane H' which intersects $\langle p_1, p_2 \rangle$ precisely in p.

In each of these cases, we have $H' \neq H_1, H_2$, $p \in H'$ and $p_1, p_2 \notin H'$. By Lemma 7, there are at least two points outside of $H_1 \cup H_2 \cup H'$. This fact implies the existence of a point $p' \notin H_1 \cup H_2 \cup H'$ distinct from p_1. The antiflag (p', H') is 2-adjacent to both A_1, A_2.

Consider the case when $|\mathbb{F}| = 2$ and $n \geqslant 4$.

If $p \in \langle p_1, p_2 \rangle$, then $p \notin H_1$. We choose a hyperplane S of H_1 distinct from $H_1 \cap H_2$ and such that $p_2 \notin S$. The hyperplane $H' = \langle p, S \rangle$ does not contain $H_1 \cap H_2$ and intersects $\langle p_1, p_2 \rangle$ precisely in p.

Suppose that $p \notin \langle p_1, p_2 \rangle$. Let ℓ be the line joining p with the point of $\langle p_1, p_2 \rangle$ distinct from p_1, p_2 and let $\tilde{p}$ be the intersection point of H_1 and ℓ. The points p and $\tilde{p}$ are not necessarily distinct. On the other hand, $\tilde{p} \neq p_2$ (otherwise, $\langle p_1, p_2 \rangle = \ell$ and $p \in \langle p_1, p_2 \rangle$ which contradicts our assumption). Since $n \geqslant 4$, hyperplanes are not lines. Therefore there is a hyperplane S of H_1 distinct from $H_1 \cap H_2$ and such that $\tilde{p} \in S$, $p_2 \notin S$. The hyperplane $H' = \langle \ell, S \rangle$ contains p, does not contain $H_1 \cap H_2$ and intersects $\langle p_1, p_2 \rangle$ in a point distinct from p_1 and p_2.

In each of these cases, we have $H_1 \cap H_2 \not\subset H'$, $p \in H'$ and $p_1, p_2 \notin H'$. Lemma 7 implies the existence of a point $p' \notin H_1 \cup H_2 \cup H'$ distinct from p_1. The antiflag (p', H') is as required. $\square$

Proof of Proposition 6. Let $A_j = (p_j, H_j)$, $j \in \{1,2\}$ be i-adjacent antiflags and let $A'_j = (p'_j, H'_j)$, $j \in \{1,2\}$ be i'-adjacent antiflags such that each of i, i' is distinct from 2. We shall show that the inclusions

$$\{A_1, A_2\}^{\sim_2} \subset \{A_1', A_2'\}^{\sim_2} \tag{1}$$

and

$$\{p_1', p_2'\} \subset \{p_1, p_2\} \text{ and } \{H_1', H_2'\} \subset \{H_1, H_2\} \tag{2}$$

are equivalent. The implication (2) $\Rightarrow$ (1) is obvious. Conversely, if p_j' is distinct from p_1, p_2, then, by Lemma 8, there is $(p, H) \in \{A_1, A_2\}^{\sim_2}$ such that $p_j' \in H$ which means that $(p, H) \notin \{A_1', A_2'\}^{\sim_2}$. Similarly, if H_j' is distinct from H_1, H_2, then there exists $(p, H) \in \{A_1, A_2\}^{\sim_2}$ such that $p \in H_j'$ and $(p, H) \notin \{A_1', A_2'\}^{\sim_2}$ again. So, (1) $\Rightarrow$ (2).

We are now going to prove the statement of the proposition. We will do it in several steps.

(I). Suppose that each of i, i' is distinct from 1. Then

$$p_1 \neq p_2, \ p_1' \neq p_2', \ H_1 \neq H_2, \ H_1' \neq H_2'.$$

The inclusion (2) implies that

$$\{p_1', p_2'\} = \{p_1, p_2\} \text{ and } \{H_1', H_2'\} = \{H_1, H_2\}.$$

Since $p_j \in H_{3-i}$ for at least one $j \in \{1, 2\}$, the inclusion (1) holds if and only if $\{A_1, A_2\} = \{A_1', A_2'\}$.

(II). For $i = 1$ we have $p_1 = p_2$ or $H_1 = H_2$ which means that (2) holds if and only if $\{A_1, A_2\} = \{A_1', A_2'\}$. Since the inclusions (1) and (2) are equivalent, $\{A_1, A_2\}^{\sim_2}$ is a maximal element of the poset $(\mathfrak{X}, \subset)$.

(III). Consider the case when $i \in \{3, 4\}$ and $i' = 1$. Since $p_j' \notin H_{3-j}'$ for each $j \in \{1, 2\}$, the inclusion (2) guarantees that $p_j \notin H_{3-j}$ for a certain $j \in \{1, 2\}$ and, consequently, $i = 3$. So, the inclusion (1) is impossible for $i = 4$. This observation and the previous steps show that if $i = 4$ then $\{A_1, A_2\}^{\sim_2}$ is a minimal and maximal element of the poset $(\mathfrak{X}, \subset)$.

Suppose that $i = 3$ (and $i' = 1$ as above). Without loss of generality we assume that $p_1 \notin H_2$ and $p_2 \in H_1$. Then the inclusion (1) holds if and only if $\{A_1', A_2'\}$ is

$$\{A_1 = (p_1, H_1), (p_1, H_2)\} \text{ or } \{(p_1, H_2), A_2 = (p_2, H_2)\}.$$

By the step (II), the inclusion reverse to (1) fails for each of these cases. Therefore, $\{A_1, A_2\}^{\sim_2}$ is a minimal and non-maximal element of the poset $(\mathfrak{X}, \subset)$ if $i = 3$.

(IV). If A_1', A_2' are 1-adjacent, then $\{A_1', A_2'\}^{\sim_2}$ is a maximal element of the poset $(\mathfrak{X}, \subset)$. We need to show that this element is non-minimal, i.e. the inclusion (1) holds for some 3-adjacent A_1, A_2. If $p_1' \neq p_2'$ and $H_1' = H_2'$, then we take any hyperplane H_2 which contains p_1' and does not contain p_2' (such a hyperplane is distinct from $H_1' = H_2'$) and put

$$A_1 = A_1' = (p_1', H_1'), \ A_2 = (p_2', H_2).$$

In the case when $p_1' = p_2'$ and $H_1' \neq H_2'$, there is a point $p_2 \in H_1' \setminus H_2'$ (such a point is distinct from $p_1' = p_2'$) and

$$A_1 = A_1' = (p_1', H_1'),\ A_2 = (p_2, H_2')$$

are as required. □

Proposition 6 does not cover the case when $|\mathbb{F}| = 2$ and $n = 3$. In this case, we characterize $\sim_4$ in terms of $\sim_2$. Then $\sim_1$ can be characterized in terms of $\sim_2$ by Proposition 5. As we know, the relation $\sim_3$ can in turn be characterized in terms of $\sim_1$; hence, it can be characterized in terms of $\sim_2$ too.

Proposition 10. *Suppose that $|\mathbb{F}| = 2$ and $n = 3$. Then antiflags A_1, A_2 are 4-adjacent if and only if $\{A_1, A_2\}^{\sim_2}$ is non-empty.*

Proof. Since $n = 3$, hyperplanes are lines. Let $A_j = (p_j, \ell_j)$, $j \in \{1, 2\}$ be distinct antiflags.

Suppose that A_1, A_2 are 4-adjacent. Then $p_j \in \ell_{3-j}$ for each $j \in \{1, 2\}$. There is a unique line ℓ which intersects ℓ_1, ℓ_2 in two distinct points different from p_1, p_2 and there is a unique point $p \notin \ell \cup \ell_1 \cup \ell_2$. The antiflag (p, ℓ) is the unique element of $\{A_1, A_2\}^{\sim_2}$.

In the remaining cases (if A_1, A_2 are 1-adjacent, then we assume that $\ell_1 \neq \ell_2$ by duality), there is at most one point outside of $\ell_1 \cup \ell_2 \cup \{p_1, p_2\}$ which means that $\{A_1, A_2\}^{\sim_2}$ is empty. □

3.3 Recovering from $\sim_3$. The case when $|\mathbb{F}| \geqslant 4$

Let $\mathcal{C}$ be a coclique of the graph Γ_3, i.e. a subset of $\mathcal{A}$, where any two distinct antiflags are not 3-adjacent. We say that $\mathcal{C}$ is *linear* if all antiflags from $\mathcal{C}$ have a common hyperplane and there is a line containing the point of every antiflag from $\mathcal{C}$. The coclique is said to be *dually linear* if all antiflags from $\mathcal{C}$ have a common point and there is a line of the dual projective space containing the hyperplane of every antiflag from $\mathcal{C}$. Any two distinct antiflags in a linear or dually linear coclique are 1-adjacent. Every bijective transformation of $\mathcal{A}$ induced by a semilinear automorphism of V to V^* sends linear cocliques to dually linear cocliques and vice versa.

Lemma 11. *If $\mathcal{C}$ is a linear or dually linear coclique of Γ_3, then for every antiflag $A \notin \mathcal{C}$ one of the following possibilities is realized:*

(1) *$A^{\sim_3} \cap \mathcal{C}$ is empty,*

(2) *$A^{\sim_3} \cap \mathcal{C}$ is a singleton,*

(3) *$A^{\sim_3} \cap \mathcal{C}$ contains all antiflags of $\mathcal{C}$ except one,*

(4) *$\mathcal{C} \subset A^{\sim_3}$.*

Proof. We can assume that $\mathcal{C}$ is linear by duality. Let $A = (p, H) \in \mathcal{A} \setminus \mathcal{C}$ and let ℓ, H' be the line and hyperplane corresponding to the linear coclique $\mathcal{C}$.

Suppose first that $p \in H'$. If $\ell \subset H$, then A is 4-adjacent to every antiflag from $\mathcal{C}$ and, consequently, $A^{\sim_3} \cap \mathcal{C} = \emptyset$. Assume that $\ell \not\subset H$ and p' is the intersection point of ℓ and H. If the points of the antiflags of $\mathcal{C}$ are distinct from p', then A is 3-adjacent to all antiflags of $\mathcal{C}$, i.e. $\mathcal{C} \subset A^{\sim_3}$. Otherwise, the antiflag (p', H') belongs to $\mathcal{C}$ and $A^{\sim_3} \cap \mathcal{C}$ contains all antiflags of $\mathcal{C}$ except (p', H').

Suppose now that $p \notin H'$. If $\ell \subset H$, then A is 3-adjacent to all antiflags from $\mathcal{C}$ and we obtain that $\mathcal{C} \subset A^{\sim_3}$. Assume that $\ell \not\subset H$ and p' is the intersection point of ℓ and H. Suppose first that $p' \in H'$. If $p \notin \ell$, then all elements of $\mathcal{C}$ are 2-adjacent to A; while if $p \in \ell$, then $(p, H') \sim_1 A$ and all elements of $\mathcal{C} \setminus \{(p, H')\}$ are 2-adjacent to A. In both cases $A^{\sim_3} \cap \mathcal{C} = \emptyset$.

Now let $p' \notin H'$. Then $(p', H') \sim_3 A$. If $p \notin \ell$, then all antiflags in $\mathcal{C} \setminus \{(p', H')\}$ are 2-adjacent to A; while if $p \in \ell$, then $(p, H') \sim_1 A$ and all elements of $\mathcal{C} \setminus \{(p', H'), (p, H')\}$ are 2-adjacent to A. In both cases $A^{\sim_3} \cap \mathcal{C} = \{(p', H')\}$ or $A^{\sim_3} \cap \mathcal{C}$ is empty. $\square$

We say that a coclique $\mathcal{C}$ of Γ_3 satisfies the property (L) if for every antiflag $A \notin \mathcal{C}$ one of the possibilities described in Lemma 11 is realized. This property holds for every coclique of Γ_3 containing less than four elements. Note that linear and dually linear cocliques of Γ_3 containing four elements exist only when $|\mathbb{F}| \geqslant 4$.

Proposition 12. *Suppose that $|\mathbb{F}| \geqslant 4$. A 4-element coclique of Γ_3 is linear or dually linear if and only if it satisfies the property* (L).

If $|\mathbb{F}| \geqslant 4$, then any two 1-adjacent antiflags belong to a linear or dually linear 4-element coclique of Γ_3 and Proposition 12 yields the following characterization of the relation $\sim_1$ in terms of the relation $\sim_3$.

Corollary 13. *In the case when $|\mathbb{F}| \geqslant 4$, two distinct antiflags are 1-adjacent if and only if they belong to a 4-element coclique of Γ_3 satisfying the property* (L).

Therefore, since both $\sim_2$ and $\sim_4$ can be characterized in terms of $\sim_1$, if $|\mathbb{F}| \geqslant 4$, then $\sim_4$ and $\sim_2$ can be characterized in terms of $\sim_3$. In order to prove Proposition 12 we need the following.

Lemma 14. *If $|\mathbb{F}| \geqslant 4$, then for any set formed by five points of $\mathrm{PG}(n-1, \mathbb{F})$ and any point from this set there is a hyperplane of $\mathrm{PG}(n-1, \mathbb{F})$ intersecting this set precisely in this point.*

Proof. Let $\{p_1, \dots, p_5\}$ be a set of points. We shall show that there is a hyperplane intersecting this set precisely in p_5. Denote by S the subspace of $\mathrm{PG}(n-1, \mathbb{F})$ spanned by this set. If S is a line, then any hyperplane intersecting this line precisely in p_5 is as required. If S is a plane, then it contains at least five lines passing through p_5 and at least one of them does not intersect the set $\{p_1, p_2, p_3, p_4\}$; we take any hyperplane which intersects S precisely in this line.

Suppose that S is a solid. There is a plane $S' \subset S$ which does not contain p_5. For every $i \in \{1,2,3,4\}$ denote by p'_i the intersection point of S' and the line $\langle p_i, p_5 \rangle$. As above, there is a line $\ell \subset S'$ which does not intersects the set $\{p'_1, p'_2, p'_3, p'_4\}$. We take any hyperplane intersecting S precisely in the plane spanned by ℓ and p_5.

Similarly, if S is 4-dimensional, then we choose a solid $S' \subset S$ which does not contain p_5 and for every $i \in \{1,2,3,4\}$ denote by p'_i the intersection point of S' and the line $\langle p_i, p_5 \rangle$. As in the previous case, we establish the existence of a plane $S'' \subset S'$ which does not intersect the set $\{p'_1, p'_2, p'_3, p'_4\}$. Any hyperplane intersecting S precisely in the solid spanned by S'' and p_5 is as required. □

Proof of Proposition 12. By Lemma 11, the property (L) holds for every linear or dually linear coclique of Γ_3. Let $\mathcal{C} = \{(p_j, H_j)\}_{j=1}^4$ be a 4-element coclique of Γ_3 satisfying (L). We need to show that $\mathcal{C}$ is linear or dually linear.

Suppose that the points p_1, p_2, p_3, p_4 as well as the hyperplanes H_1, H_2, H_3, H_4 are mutually distinct. We consider $H_1 \cap H_2, H_1 \cap H_3, H_1 \cap H_4$ as hyperplanes of H_1 and choose a point

$$p \in H_1 \setminus (H_2 \cup H_3 \cup H_4).$$

It is possible that $p \in \{p_2, p_3, p_4\}$, but we can assume that $p \neq p_2$ without loss of generality. Lemma 14 implies the existence of a hyperplane H such that

$$H \cap \{p_1, p_2, p_3, p_4, p\} = \{p_2\}.$$

The antiflag (p, H) is 3-adjacent to (p_i, H_i) if and only if $i \in \{1, 2\}$. Then $(p, H) \notin \mathcal{C}$ (since $\mathcal{C}$ is a coclique of Γ_3) and (L) fails, a contradiction.

So, there are distinct $i, j \in \{1,2,3,4\}$ such that $p_i = p_j$ or $H_i = H_j$. We can assume that $|\{H_1, H_2, H_3, H_4\}| \leqslant 3$ by duality.

Consider the case when $|\{H_1, H_2, H_3, H_4\}| = 3$, say $H_1 = H_2 =: H'$ and H', H_3, H_4 are mutually distinct. By Lemma 14, there is a hyperplane H such that

$$H \cap \{p_1, p_2, p_3, p_4, p'\} = \{p'\},$$

where p' is a point outside of H'. Then $H' \neq H$. As above, we choose a point

$$p \in H' \setminus (H_3 \cup H_4 \cup H).$$

The antiflag (p, H) is 3-adjacent to (p_i, H_i) if and only if $i \in \{1, 2\}$ which contradicts (L). Therefore, $|\{H_1, H_2, H_3, H_4\}| \leqslant 2$.

Suppose now that $|\{H_1, H_2, H_3, H_4\}| = 2$, say $H_1 = H_2 = H_3 =: H'$ and $H_4 \neq H'$. Lemma 14 implies the existence of a hyperplane H such that

$$H \cap \{p_1, p_2, p_3, p_4, p'\} = \{p_3\},$$

where $p' \in H'$. Then $H' \neq H$ and there is a point

$$p \in H' \setminus (H_4 \cup H).$$

The antiflag (p, H) is 3-adjacent to (p_i, H_i) if and only if $i \in \{1,2\}$ which contradicts (L) again.

We obtain that $H_1 = H_2 = H_3 = H_4 =: H'$. If no line contains $\{p_1, p_2, p_3, p_4\}$, then for some distinct $i, j \in \{1,2,3,4\}$ the line $\langle p_i, p_j\rangle$ does not contain the remaining two p_t and it is easy to see that there is a hyperplane H intersecting $\{p_1, p_2, p_3, p_4\}$ precisely in two points. Let

$$H \cap \{p_1, p_2, p_3, p_4\} = \{p_3, p_4\}.$$

Since $H' \neq H$, there is $p \in H' \setminus H$. The antiflag (p, H) is 3-adjacent to (p_i, H') if and only if $i \in \{1,2\}$ in contrast to our assumption. Therefore, $\{p_1, p_2, p_3, p_4\}$ is contained in a line and the coclique $\mathcal{C}$ is linear. $\square$

3.4 Recovering from $\sim_3$. The case when $\mathbb{F}$ is finite

The case of $|\mathbb{F}| = 3$ remains to be considered. However, the arguments to be used to fix this case work as well for any finite field. So, throughout this subsection we assume that $\mathbb{F}$ is finite, $|\mathbb{F}| = 2$ being allowed. Thus, we shall be able to show why when $|\mathbb{F}| = 2$ the relations $\sim_i$ with $i \neq 3$ cannot be recovered from $\sim_3$.

Proposition 15. *Suppose that $|\mathbb{F}| = q$. Let $A_i = (p_i, H_i)$, $i \in \{1,2\}$ be distinct antiflags such that $A_1 \not\sim_3 A_2$. Then the following assertions are fulfilled:*

(1) *If $A_1 \sim_4 A_2$, then $|\{A_1, A_2\}^{\sim_3}| = 4(q^{n-2} - 1)(q-1)q^{n-3}$.*

(2) *If $A_1 \sim_1 A_2$, then $|\{A_1, A_2\}^{\sim_3}| = q^{2n-3} - 2q^{n-2}$.*

(3) *If $A_1 \sim_2 A_2$ and the line joining p_1 and p_2 intersects $H_1 \cap H_2$ in a point, then $|\{A_1, A_2\}^{\sim_3}| = 4q^{n-2}(q^{n-2} - q^{n-3}) - 2q^{n-2}$.*

(4) *If $A_1 \sim_2 A_2$ and the line joining p_1 and p_2 does not intersect $H_1 \cap H_2$, then $|\{A_1, A_2\}^{\sim_3}| = 4(q^{n-2} - 1)(q-1)q^{n-3} + 2q^{n-2}$.*

For $q = 2$ the fourth possibility of Proposition 15 cannot occur and the numbers (1)-(3) are mutually equal. If $q \geqslant 3$, then all above numbers are mutually different. Hence those numbers yields characterizations of each of the relations $\sim_i$, $i \in \{1,2,4\}$ in terms of $\sim_3$.

Proof of Proposition 15. The complement of a hyperplane of $\mathrm{PG}(n-1,q)$ consists of q^{n-1} points and there are precisely $(q-1)q^{n-2}$ points outside of the union of two distinct hyperplanes of $\mathrm{PG}(n-1,q)$. Let H'_1, H'_2, H'_3 be mutually distinct hyperplanes of $\mathrm{PG}(n-1,q)$. If $H'_1 \cap H'_2 \not\subset H'_3$, then there are precisely $(q-1)^2 q^{n-3}$ points outside of $H'_1 \cup H'_2 \cup H'_3$. In the case when $H'_1 \cap H'_2 \subset H'_3$, the number of such points is $(q-2)q^{n-2}$.

(1). Suppose that $A_1 \sim_4 A_2$. Then $p_i \in H_{3-i}$ for $i \in \{1,2\}$. If an antiflag (p, H) belongs to $\{A_1, A_2\}^{\sim_3}$, then one of the following possibilities is realized:

- $p \in H_1 \cap H_2$ and H intersects $\langle p_1, p_2\rangle$ in a point distinct from p_1 and p_2;
- $p \notin H_1 \cup H_2 \cup \langle p_1, p_2\rangle$ and $\langle p_1, p_2\rangle \subset H$;

- for one of $i \in \{1,2\}$ we have $p \in H_i \setminus (H_{3-i} \cup \{p_{3-i}\})$, $p_i \notin H$ and $p_{3-i} \in H$.

There are precisely

$$\frac{q^{n-2}-1}{q-1} \cdot (q-1)^2 q^{n-3} = (q^{n-2}-1)(q-1)q^{n-3}$$

antiflags satisfying the first condition (we consider H as a point of the dual projective space which is not in the union of the hyperplanes corresponding to p, p_1, p_2). The second condition holds for precisely

$$((q-1)q^{n-2} - (q-1)) \cdot q^{n-3} = (q^{n-2}-1)(q-1)q^{n-3}$$

antiflags and for each $i \in \{1,2\}$ there are precisely

$$(q^{n-2}-1) \cdot (q-1)q^{n-3}$$

antiflags satisfying the third condition. We get the claim.

(2). Suppose that $A_1 \sim_1 A_2$. By duality, we assume that $p_1 = p_2$. Then for every antiflag (p, H) belonging to $\{A_1, A_2\}^{\sim_3}$ one of the following possibilities is realized:

- $p \in H_1 \cap H_2$ and $p_1 = p_2 \notin H$;
- $p \notin H_1 \cup H_2 \cup \{p_1 = p_2\}$ and $p_1 = p_2 \in H$.

There are precisely

$$\frac{q^{n-2}-1}{q-1} \cdot (q-1)q^{n-2} = (q^{n-2}-1)q^{n-2} \text{ and } ((q-1)q^{n-2}-1) \cdot q^{n-2}$$

antiflags satisfying the first and second condition, respectively. The sum of these two numbers is $q^{2n-3} - 2q^{n-2}$.

(3) and (4). If $A_1 \sim_2 A_2$, then $\{p_1, p_2\} \cap (H_1 \cup H_2) = \emptyset$ and for every antiflag (p, H) belonging to $\{A_1, A_2\}^{\sim_3}$ one of the following possibilities is realized:

- $p \in H_1 \cap H_2$ and H intersects $\langle p_1, p_2 \rangle$ in a point distinct from p_1 and p_2;
- $p \notin H_1 \cup H_2 \cup \langle p_1, p_2 \rangle$ and $\langle p_1, p_2 \rangle \subset H$;
- for one of $i \in \{1,2\}$ we have $p \in H_i \setminus H_{3-i}$, $p_i \notin H$ and $p_{3-i} \in H$.

Suppose first that $\langle p_1, p_2 \rangle$ intersects $H_1 \cap H_2$ in a point p'. There are precisely

$$\left(\frac{q^{n-2}-1}{q-1} - 1\right) \cdot (q-1)^2 q^{n-3}$$

antiflags (p, H) with $p \neq p'$ and $(q-2)q^{n-2}$ antiflags (p', H) which satisfy the first condition; the sum of these numbers is

$$(q^{n-2}-1)(q-1)q^{n-3} - q^{n-3}.$$

The second condition holds for precisely

$$((q-1)q^{n-2}-q)\cdot q^{n-3}$$

antiflags and for each $i \in \{1,2\}$ there are precisely

$$q^{n-3}\cdot (q-1)q^{n-2}$$

antiflags satisfying the third condition. A direct calculation shows that the sum

$$(q^{n-2}-1)(q-1)q^{n-3}-q^{n-3}+((q-1)q^{n-2}-q)q^{n-3}+2(q-1)q^{2n-5}$$

is as required.

Suppose now that $\langle p_1, p_2\rangle$ intersects H_1 and H_2 in distinct points p_1' and p_2', respectively. As in the case when $A_1 \sim_4 A_2$, the number of antiflags satisfying the first or second condition is

$$(q^{n-2}-1)(q-1)q^{n-3}.$$

Also, for each $i \in \{1,2\}$ we have the same number of antiflags (p,H) with $p \neq p_i'$ satisfying the third condition. Since the assumption that $p_{3-i} \in H$ and $p_i \notin H$ implies that $p_i' \notin H$, the number of antiflags (p_i', H) satisfying the third condition is q^{n-2}. We obtain the required equality. □

4 Non-singular points of the hyperbolic form over the field of two elements as antiflags

As above, we suppose that $V = \mathbb{F}^n$ and $n \geqslant 3$. If $x \in V$ and $x^* \in V^*$, then $x^* \cdot x$ denotes the value of the functional x^* on the vector x. Recall that for non-zero $x^* \in V^*$ the annihilator $(x^*)^0$ is the $(n-1)$-dimensional subspace of V formed by all $x \in V$ satisfying $x^* \cdot x = 0$.

Consider the non-degenerate quadratic form Q on the $2n$-dimensional vector space $V \times V^*$ defined as follows:

$$Q(\mathbf{x}) = x^* \cdot x \ \text{ for } \ \mathbf{x} = (x, x^*) \in V \times V^*.$$

It is clear that

$$\Pi = \{(x,0) : x \in V\} \ \text{ and } \ \Sigma = \{(0,x^*) : x^* \in V^*\}$$

are maximal singular subspaces of Q. Also, $(x,x^*) \notin \Pi \cup \Sigma$ is a non-singular vector of Q if and only if $\langle x\rangle$ and $(x^*)^0$ form an antiflag.

Singular points of $\mathrm{PG}(2n-1,\mathbb{F})$, i.e. corresponding to singular 1-dimensional subspaces of Q, form the hyperbolic polar space $\mathcal{O}^+(2n,\mathbb{F})$ (we write $\mathcal{O}^+(2n,q)$ in the case when $|\mathbb{F}| = q$). The remaining points of $\mathrm{PG}(2n-1,\mathbb{F})$ are said to be *non-singular*.

Let f be the map of the set of non-singular points of $\mathrm{PG}(2n-1,\mathbb{F})$ to the set of antiflags $\mathcal{A}$ sending every non-singular point $\langle (x,x^*)\rangle$ to the antiflag formed by $\langle x\rangle$ and $(x^*)^0$. This map is surjective, but it is not injective if $|\mathbb{F}| \geqslant 3$. Indeed, if $|\mathbb{F}| \geqslant 3$, $\mathbf{p} = \langle (x,x^*)\rangle$ is a non-singular point and $a \in \mathbb{F} \setminus \{0,1\}$, then $\mathbf{p'} = \langle (x, ax^*) \rangle$ is a non-singular point distinct from $\mathbf{p}$ and $f(\mathbf{p}) = f(\mathbf{p'})$.

So, f is bijective if and only if $|\mathbb{F}| = 2$.

From now on we assume that $|\mathbb{F}| = 2$. The next proposition is the main result of [12]. The setting we have introduced above allows us to give a very short proof of that result.

Proposition 16. *Suppose that $|\mathbb{F}| = 2$. For distinct non-singular points $\mathbf{p}_1, \mathbf{p}_2$ the third point on the line joining them is non-singular if and only if the antiflags $f(\mathbf{p}_1), f(\mathbf{p}_2)$ are 3-adjacent.*

Proof. Let $\mathbf{p}_i = \langle (x_i, x_i^*) \rangle$, $i \in \{1,2\}$ be distinct non-singular points. Then $x_i^* \cdot x_i = 1$ for every i. The third point on the line $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle$ is $\langle (x_1 + x_2, x_1^* + x_2^*) \rangle$ and

$$(x_1^* + x_2^*) \cdot (x_1 + x_2) = x_1^* \cdot x_1 + x_1^* \cdot x_2 + x_2^* \cdot x_1 + x_2^* \cdot x_2 = x_1^* \cdot x_2 + x_2^* \cdot x_1$$

is non-zero if and only if precisely one of $x_1^* \cdot x_2, x_2^* \cdot x_1$ is non-zero. This gives the claim. $\square$

Using the map f, we identify non-singular points of the form Q with antiflags of $\mathrm{PG}(n-1,2)$. By Proposition 16, Γ_3 is the collinearity graph of the point-line geometry of non-singular points and totally non-singular lines (lines formed by non-singular points).

Theorem 17. *The polar space $\mathcal{O}^+(2n,2)$ can be recovered from the graph Γ_3.*

The automorphism group of the polar space $\mathcal{O}^+(2n,2)$ is the orthogonal group $\mathrm{O}^+(2n,2)$. Every element of this group induces an automorphism of the graph Γ_3. In particular, elements of $\mathrm{O}^+(2n,2)$ preserving each of the singular subspaces Π and Σ correspond to linear automorphisms of V, elements transposing these subspaces are related to linear isomorphisms of V to V^*.

Conversely, Theorem 17 shows that every automorphism of Γ_3 induces an automorphism of the polar space $\mathcal{O}^+(2n,2)$. This fact implies the following.

Corollary 18. *If $|\mathbb{F}| = 2$, then* $\mathrm{Aut}(\Gamma_3) \cong \mathrm{O}^+(2n,2)$.

Let $i \in \{1,2,4\}$. By the first part of Theorem 1, the relation $\sim_3$ can be recovered from $\sim_i$. Therefore, if $\sim_i$ can be recovered from $\sim_3$, then the graphs Γ_3 and Γ_i have the same automorphism group which contradicts Corollaries 2 and 18. We obtain the second part of Theorem 1.

5 Proof of Theorem 17

Consider the non-degenerate bilinear form F on $V \times V^*$ defined as follows:

$$F(\mathbf{x}, \mathbf{y}) = y^* \cdot x + x^* \cdot y \quad \text{for} \quad \mathbf{x} = (x, x^*), \mathbf{y} = (y, y^*) \in V \times V^*.$$

The associated orthogonal relation on $\mathrm{PG}(2n-1,2)$ will be denoted by $\perp$. We use the following simple observations:

- Two distinct singular points are orthogonal if and only if the line joining them is singular.

- Two distinct non-singular points are non-orthogonal if and only if the line joining them is totally non-singular.

- A singular point $\mathbf{p}$ and a non-singular point $\mathbf{q}$ are orthogonal if and only if the third point on the line $\langle \mathbf{p}, \mathbf{q} \rangle$ is non-singular.

Recall that non-singular points of the form Q are identified with antiflags of $\mathrm{PG}(n-1, 2)$ and Γ_3 is the collinearity graph of the point-line geometry of non-singular points and totally non-singular lines. So, totally non-singular lines are cliques of the graph Γ_3. Our first step is the following characterization of such lines in therms of the relation $\sim_3$.

Proposition 19. *Let $\mathcal{C}$ be a 3-element clique of Γ_3. Then $\mathcal{C}$ is a line if and only if the following condition is satisfied:*

(NS) *every non-singular point $\mathbf{p} \notin \mathcal{C}$ is 3-adjacent to precisely two points of $\mathcal{C}$ or it is not 3-adjacent to any point of $\mathcal{C}$.*

This shows that every totally non-singular line is a maximal clique of Γ_3.

Proof of Proposition 19. Suppose that $\mathcal{C} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a totally non-singular line. Then $\mathbf{p}_1^\perp \cap \mathbf{p}_2^\perp \subset \mathbf{p}_3^\perp$ and the projective space $\mathrm{PG}(2n-1, 2)$ is the union of the hyperplanes $\mathbf{p}_1^\perp, \mathbf{p}_2^\perp, \mathbf{p}_3^\perp$. So, a non-singular point $\mathbf{p} \notin \mathcal{C}$ is contained in precisely one of these hyperplanes or in their intersection. This implies (NS); in particular, every totally non-singular line is a maximal clique of Γ_3.

We shall now show that (NS) fails if $\mathcal{C} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ spans a plane A. Let this be the case. There is a solid S containing A and such that the restriction of F to the 4-dimensional subspace of $V \times V^*$ corresponding to S is non-degenerate. We denote this restriction by F'. For every plane $X \subset S$ there is a unique point $\mathbf{p}_X \in X$ such that $S \cap \mathbf{p}_X^\perp = X$.

Since the line $\langle \mathbf{p}_i, \mathbf{p}_j \rangle$ is totally non-singular for any distinct $i, j \in \{1, 2, 3\}$, all points of A are non-singular except $\mathbf{p}_A$. Let B and C be the remaining two planes of S passing through the line $\langle \mathbf{p}_2, \mathbf{p}_3 \rangle$. If one of the points $\mathbf{p}_B, \mathbf{p}_C$ is non-singular, say $\mathbf{p}_B$, then $\mathbf{p}_B$ and $\mathbf{p}_1$ are 3-adjacent (these points are non-orthogonal, since F' is non-degenerate). On the other hand, $\mathbf{p}_B$ is not 3-adjacent to $\mathbf{p}_2, \mathbf{p}_3$, which contradicts (NS).

Suppose now that $\mathbf{p}_B, \mathbf{p}_C$ are singular. Since $\mathbf{p}_B$ is orthogonal to $\mathbf{p}_2, \mathbf{p}_3$ and $\langle \mathbf{p}_2, \mathbf{p}_3 \rangle$ is totally non-singular, every point of B distinct from $\mathbf{p}_B$ is non-singular. The same holds for the plane C. Therefore, all points of S are non-singular except $\mathbf{p}_A, \mathbf{p}_B, \mathbf{p}_C$. These points are mutually non-orthogonal (since F' is non-degenerate) and, consequently, they span a plane D. Then there is a plane of S intersecting D in a totally non-singular line. This plane is formed by non-singular points. However, this is impossible (every totally non-singular line is a maximal clique of Γ_3). So, at least one of the points $\mathbf{p}_B, \mathbf{p}_C$ is non-singular. It follows that (NS) fails. □

For every 2-element coclique $\{\mathbf{p}, \mathbf{q}\}$ of Γ_3 the third point on the line $\langle \mathbf{p}, \mathbf{q}\rangle$ is singular. Two distinct 2-element cocliques of Γ_3 are said to be *parallel* if the corresponding lines intersect in a singular point. Every singular point is determined by a class of parallel 2-element cocliques of Γ_3. So, we need to characterize such classes in terms of the relation $\sim_3$.

For any two parallel 2-element cocliques of Γ_3 the corresponding lines are intersecting in a singular point $\mathbf{p}$ and span a plane contained in $\mathbf{p}^\perp$. There are precisely two possibilities:

(1) all points on the plane except $\mathbf{p}$ are non-singular,

(2) singular points of the plane form a line.

We say that these are parallel cocliques of *the first* or *the second type* if the corresponding possibility is realized.

Proposition 19 provides a characterization of totally non-singular lines in terms of $\sim_3$. Therefore, parallel cocliques of the first type can be characterized in terms of $\sim_3$ as follows.

Lemma 20. *Let $\mathcal{C}_1, \mathcal{C}_2$ be distinct 2-element cocliques of Γ_3. These are parallel cocliques of the first type if and only if there are two distinct totally non-singular lines which intersect each of these cocliques and are intersecting outside of $\mathcal{C}_1 \cup \mathcal{C}_2$.*

Proof. Easy verification. □

Lemma 21. *Let $\mathcal{C}_1, \mathcal{C}_2$ be distinct 2-element cocliques of Γ_3 which are not parallel cocliques of the first type. These are parallel cocliques of the second type if and only if there is a 2-element coclique $\mathcal{C}'$ such that $\mathcal{C}_i, \mathcal{C}'$ are parallel cocliques of the first type for every $i \in \{1, 2\}$.*

Proof. Let ℓ_1 and ℓ_2 be the lines containing $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. If there is $\mathcal{C}'$ such that $\mathcal{C}_i, \mathcal{C}'$ are parallel cocliques of the first type for every $i \in \{1, 2\}$, then ℓ_1 and ℓ_2 are intersecting in a singular point which means that $\mathcal{C}_1, \mathcal{C}_2$ are parallel.

Suppose now that $\mathcal{C}_1, \mathcal{C}_2$ are parallel cocliques of the second type and the lines ℓ_1, ℓ_2 intersect in a singular point $\mathbf{p}$. We take non-singular points $\mathbf{p}_1 \in \ell_1$ and $\mathbf{p}_2 \in \ell_2$. These points are orthogonal (since $\mathcal{C}_1, \mathcal{C}_2$ are parallel cocliques of the second type). The line $\langle \mathbf{p}_1, \mathbf{p}_2\rangle$ does not pass through $\mathbf{p}$ which guarantees that the intersections of $\mathbf{p}^\perp$ with $\mathbf{p}_1^\perp$ and $\mathbf{p}_2^\perp$ are distinct. It is sufficient to show that there is a non-singular point

$$\mathbf{q} \in \mathbf{p}^\perp \setminus (\mathbf{p}_1^\perp \cup \mathbf{p}_2^\perp).$$

Indeed, if $\mathbf{q}' \in \langle \mathbf{p}, \mathbf{q}\rangle$ is distinct from $\mathbf{p}$ and $\mathbf{q}$, then $\mathbf{q}'$ is non-singular and the coclique $\mathcal{C}' = \{\mathbf{q}, \mathbf{q}'\}$ is as required.

If $n > 3$, then there is a 6-dimensional subspace of $V \times V^*$ containing $\ell_1 \cup \ell_2$ and such that the restriction of F to this subspace is non-degenerate of Witt index equal to 3. For this reason we can assume with no loss that $n = 3$. Then $\mathbf{p}^\perp \cap \mathbf{p}_1^\perp \cap \mathbf{p}_2^\perp$ is the plane spanned by $\ell_1 \cup \ell_2$; we denote this plane by S. For every $i \in \{1, 2\}$ there are precisely four non-singular points in $(\mathbf{p}^\perp \cap \mathbf{p}_i^\perp) \setminus \mathbf{p}_{3-i}^\perp$ (the line joining $\mathbf{p}_i$ with any singular point from $(\mathbf{p}^\perp \cap \mathbf{p}_i^\perp) \setminus \mathbf{p}_{3-i}^\perp$ contains a non-singular point). These four non-singular points are lying on two lines passing through $\mathbf{p}$. Let S_i be the plane spanned by these lines. Then $S \cap S_i$ is the singular line of S passing through $\mathbf{p}$. Therefore, $\mathbf{p}^\perp \cap \mathbf{p}_i^\perp$ is spanned by S_i and ℓ_{3-i}. We take any non-singular point

$$\mathbf{q}_1 \in (\mathbf{p}^\perp \cap \mathbf{p}_1^\perp) \setminus \mathbf{p}_2^\perp.$$

Then $\ell_1 \subset \mathbf{q}_1^\perp$ and, consequently, $\mathbf{q}_1^\perp$ does not contain S_2 (otherwise $\mathbf{p}^\perp \cap \mathbf{q}_1^\perp$ coincides with $\mathbf{p}^\perp \cap \mathbf{p}_2^\perp$ which is impossible, since $\mathbf{p} \notin \langle \mathbf{q}_1, \mathbf{p}_2 \rangle$). So, $\mathbf{q}_1^\perp$ intersects S_2 in a line. This implies the existence of a non-singular point

$$\mathbf{q}_2 \in (\mathbf{p}^\perp \cap \mathbf{p}_2^\perp) \setminus \mathbf{p}_1^\perp$$

non-orthogonal to $\mathbf{q}_1$. The non-singular point $\mathbf{q} \in \langle \mathbf{q}_1, \mathbf{q}_2 \rangle$ distinct from $\mathbf{q}_1, \mathbf{q}_2$ is as required. □

Combining Lemmas 20 and 21 we obtain a characterization of parallel classes of 2-element cocliques of Γ_3 in terms of the relation $\sim_3$. So, the set of points of the polar space $\mathcal{O}^+(2n, 2)$ can be recovered from the graph Γ_3. Singular lines are recovered from Γ_3 as follows.

Proposition 22. *Let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ be mutually distinct singular points and let $\mathfrak{C}_i$, $i \in \{1,2,3\}$ be the parallel class of 2-element cocliques corresponding to $\mathbf{p}_i$. These points form a line if and only if there are cocliques $\mathcal{C}_1 \in \mathfrak{C}_1, \mathcal{C}_2 \in \mathfrak{C}_2, \mathcal{C}_3 \in \mathfrak{C}_3$ such that $C_i \cap C_j$ is a point for every choice of distinct indices $i, j \in \{1,2,3\}$ and the points $C_1 \cap C_2$, $C_2 \cap C_3$ and $C_3 \cap C_1$ are mutually distinct.*

Proof. If $\ell = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a line, then $\ell^\perp$ is a $(2n-3)$-dimensional subspace of $\mathrm{PG}(2n-1, 2)$ and, consequently, it contains a non-singular point $\mathbf{p}$ (since $n \geqslant 3$). There are precisely four non-singular points on the plane spanned by ℓ and $\mathbf{p}$. This plane contains three cocliques satisfying the required condition.

Conversely, suppose that $\mathcal{C}_1 \in \mathfrak{C}_1, \mathcal{C}_2 \in \mathfrak{C}_2, \mathcal{C}_3 \in \mathfrak{C}_3$ are mutually intersecting in three distinct points. Then $\ell_i = \mathcal{C}_i \cup \{\mathbf{p}_i\}$ is a line for every $i \in \{1,2,3\}$ and these lines span a plane. Therefore, the third point on the line $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle$ belongs to $\ell_3 \setminus (\ell_1 \cup \ell_2)$ and, consequently, this point is $\mathbf{p}_3$. □

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