Cyclic Sieving of Multisets with Bounded Multiplicity and the Frobenius Coin Problem

Drew Armstrong

Submitted: Feb 24, 2026; Accepted: Jun 29, 2026; Published: Aug 28, 2026

Abstract

The two subjects in the title are related via the specialization of symmetric polynomials at roots of unity. Let $f(z_1,\dots,z_n) \in \mathbb{Z}[z_1,\dots,z_n]$ be a symmetric polynomial with integer coefficients and let ω be a primitive dth root of unity. If $d|n$ or $d|(n-1)$ then we have $f(1,\omega,\dots,\omega^{n-1}) \in \mathbb{Z}$. If $d|n$ then of course we have $f(\omega,\omega^2,\dots,\omega^n) = f(1,\omega,\dots,\omega^{n-1}) \in \mathbb{Z}$, but when $d|(n+1)$ we also have $f(\omega,\omega^2,\dots,\omega^n) \in \mathbb{Z}$. We investigate these three families of integers in the case $f = h_k^{(b)}$, where $h_k^{(b)}$ is the coefficient of t^k in the generating function $\prod_{i=1}^n (1+z_it+\cdots+(z_it)^{b-1})$. These polynomials were previously considered by several authors. They interpolate between the elementary symmetric polynomials ($b=2$) and the complete homogeneous symmetric polynomials ($b\to\infty$). When $\gcd(b,d)=1$ with $d|n$ or $d|(n-1)$ we find that the integers $h_k^{(b)}(1,\omega,\dots,\omega^{n-1})$ are related to cyclic sieving of multisets with multiplicities bounded above by b, generalizing the well-known cyclic sieving results for sets ($b=2$) and multisets ($b\to\infty$). When $\gcd(b,d)=1$ and $d|(n+1)$ we find that the integers $h_k^{(b)}(\omega,\omega^2,\dots,\omega^n)$ are related to the Frobenius coin problem with two coins. The case $\gcd(b,d)\neq 1$ is more complicated. At the end of the paper we combine these results with the expansion of $h_k^{(b)}$ in various bases of the ring of symmetric polynomials.

Mathematics Subject Classifications: 05A30,05E05,05E18,11D07

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University of Miami (d.armstrong@miami.edu).

1 Summary

We begin with a brief summary of our main results. Let the numbers $\binom{n}{k}^{(b)} \in \mathbb{N}$ be defined by

$$\sum_k \binom{n}{k}^{(b)} t^k = (1 + t + \cdots + t^{b-1})^n$$

and let the polynomials $\begin{bmatrix} n \\ k \end{bmatrix}_q^{(b)} \in \mathbb{N}[q]$ be defined by

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix}_q^{(b)} t^k = \prod_{i=0}^{n-1} (1 + q^i t + \cdots + (q^i t)^{b-1}).$$

If n or k is not an integer we define $\binom{n}{k}^{(b)} := 0$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q^{(b)} := 0$.

Let ω be a primitive dth root of unity. It is a basic fact (see Observation 3 and Remark 7) that $\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)}$ is an integer when $d|n$ or $d|(n-1)$, and that $\omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)}$ is an integer when $d|(n+1)$. We are interested in combinatorial interpretations of these integers.

If $d|n$ and $\gcd(b,d) = 1$ then we have (Theorem 5a):

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = (1 + t^d + \cdots + (t^d)^{b-1})^{n/d},$$

which implies that

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \binom{n/d}{k/d}^{(b)}.$$

Let X be the set of k-multisubsets from $\{1, \ldots, n\}$ whose multiplicities are less than b. It is easy to check (see Theorem 2a) that $\binom{n/d}{k/d}^{(b)}$ is also equal to the number of elements of X fixed by the permutation ρ of $\{1, \ldots, n\}$ defined by $i \mapsto i + n/d \bmod n$. The situation for $\gcd(b,d) \neq 1$ is more complicated.

If $d|(n-1)$ and $\gcd(b,d) = 1$ then we have (Theorem 5b):

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = (1 + t + \cdots + t^{b-1})(1 + t^d + \cdots + (t^d)^{b-1})^{(n-1)/d},$$

which implies that

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \sum_{\ell=0}^{b-1} \binom{(n-1)/d}{(k-\ell)/d}^{(b)}.$$

It is easy to check (see Theorem 2b) that this number also counts the elements of X fixed by the permutation $\tau : \{1, \ldots, n\}$ defined by $n \mapsto n$ and $i \mapsto i + (n-1)/d \bmod n-1$ for $i \in \{1, \ldots, n-1\}$. The situation for $\gcd(b,d) \neq 1$ is more complicated. These results generalize the two most basic examples of cyclic sieving [14, Theorem 1.1], which correspond to the cases $b = 2$ (subsets) and $b \to \infty$ (unbounded multisets).

If $d|(n+1)$ and $\gcd(b,d)=1$ then we have (Theorem 5c):

$$\sum_k \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = \frac{(1+t^d+\cdots+(t^d)^{b-1})^{(n+1)/d}}{1+t+\cdots+t^{b-1}},$$

which is a polynomial in $\mathbb{Z}[t]$. This time we do not have an interpretation in terms of cyclic actions on multisets. Instead, we find that the coefficients are related to the Frobenius coin problem. As a teaser, we mention one interesting result (Corollary 13). Let $(\delta_0,\ldots,\delta_{d-1})$ be the permutation of $(0,\ldots,d-1)$ defined by $\delta_r b \equiv r \bmod d$. Then for all $r \in \{0,\ldots,d-1\}$ we have

$$\sum_{k\geqslant 0} \omega^{r+kd} \begin{bmatrix} n \\ r+kd \end{bmatrix}_\omega^{(b)} t^{r+kd} = \varepsilon_r f_r(t^d),$$

where $f_r(t) \in \mathbb{N}[t]$ is a polynomial with positive, unimodal coefficients, and where

$$\varepsilon_r = \begin{cases} +1 & \delta_r < \delta_1, \\ -1 & \delta_r \geqslant \delta_1. \end{cases}$$

The situation for $\gcd(b,d) \neq 1$ is more complicated.

At the very end of this project we stumbled upon a surprising formula. Recall that a partition λ of length $\ell = l(\lambda)$ is a sequence $\lambda = (\lambda_1,\ldots,\lambda_\ell) \in \mathbb{N}^\ell$ where $\lambda_1 \geqslant \cdots \geqslant \lambda_\ell \geqslant 1$. We write $|\lambda| = k$ when $\sum_i \lambda_i = k$. Let $m_i = \#\{j : \lambda_j = i\}$ be the number of parts of λ equal to i and let $z_\lambda = \prod_i i^{m_i} \cdot m_i!$. For each $b \geqslant 2$ let $l_b(\lambda) = \#\{j : b|\lambda_j\}$ be the number of parts of λ divisible by b. Then for all $n,k,b \in \mathbb{N}$ we have (Proposition 14)

$$\binom{n}{k}^{(b)} = \sum_{|\lambda|=k} z_\lambda^{-1} (1-b)^{l_b(\lambda)} n^{l(\lambda)}.$$

We obtain this as the specialization of an identity involving symmetric polynomials. Corollary 16 gives the analogous specializations at roots of unity.

The paper is organized as follows. Section 2 discusses the cyclic sieving phenomenon. In this section we state and partially prove Theorems 2a and 2b on cyclic sieving of multisets with bounded multiplicity. In Section 3 we discuss the b-bounded symmetric polynomials and their specializations at roots of unity. The main result is Theorem 5. We use parts (a) and (b) of Theorem 5 to complete the proof of Theorem 2. Part (c) of Theorem 5 is not related to cyclic sieving. Instead, we show that it is related to the Frobenius coin problem, which we discuss in Section 4. Finally, in Section 5 we discuss the expansion of b-bounded symmetric polynomials in various bases for the ring of symmetric polynomials. In particular, the expansion in power sums yields the surprising formula.

2 Cyclic Sieving

This investigation began with the experimental observation (see Theorem 2a) that certain collections of multisets with bounded multiplicity obey a cyclic sieving property. We recall

the setup from Reiner, Stanton and White [14].[1] Suppose that a cyclic group $\langle\rho\rangle$ acts on a finite set X. Let $f(q) = \sum_{x\in X} q^{\mathrm{stat}(x)}$ be the generating polynomial for a certain function $\mathrm{stat} : X \to \mathbb{N}$, so $f(1) = \#X$. Let $N = \#\langle\rho\rangle$. We say that the triple $(X, f(q), \langle\rho\rangle)$ exhibits the *cyclic sieving phenomenon* (CSP) if for any primitive dth root of unity ω with $d|N$ we have

$$f(\omega) = \#X^{\rho^{N/d}} := \#\{x \in X : \rho^{N/d} \cdot x = x\}.$$

Reiner, Stanton and White observed that this concept unifies many disparate examples in which a combinatorially defined polynomial $f(q)$ yields integers when evaluated at certain roots of unity. The most basic examples of CSP occur when $f(q)$ is a q-binomial coefficient, which we define as usual:

$$[n]_q = 1 + q + \cdots + q^{n-1},\ [n]_q! = [n]_q[n-1]_q \cdots [1]_q \ \text{ and } \ \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

It is well-known that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial with positive integer coefficients. A *k-multisubset* of $\{1, \ldots, n\}$ has the form

$$\{1^{x_1}, 2^{x_2}, \ldots, n^{x_n}\} = \{\underbrace{1, \ldots, 1}_{x_1 \text{ times}}, \underbrace{2, \ldots, 2}_{x_2 \text{ times}} \ldots, \underbrace{n, \ldots, n}_{x_n \text{ times}}\},$$

where the symbol i is repeated x_i times and where $x_1 + \cdots + x_n = k$. We can identify each multiset with its vector of multiplicities $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{N}^n$. Subsets without repetition correspond to binary vectors $(x_1, \ldots, x_n) \in \{0, 1\}^n$. More generally, for any integers $b, n, k \in \mathbb{N}$ we consider k-multisubsets with multiplicities bounded above by b:

$$\begin{aligned} X_b^n &:= \{(x_1, \ldots, x_n) \in \mathbb{N}^n : x_i < b \text{ for all } i\}, \\ X_b^n[k] &:= \{(x_1, \ldots, x_n) \in \mathbb{N}^n : x_i < b \text{ for all } i \text{ and } \sum_i x_i = k\}. \end{aligned} \tag{1}$$

The notation is suggested by the fact that $\#X_b^n = b^n$. Note that $X_b^n[k] = X_{k+1}^n[k]$ for all $b \geqslant k$ hence we may define $X_\infty^n[k] := X_{k+1}^n[k]$. Thus $X_2^n[k]$ is the set of k-subsets and $X_\infty^n[k]$ is the set of k-multisubsets of $\{1, \ldots, n\}$. We have

$$\begin{aligned} \#X_2^n[k] &= \binom{n}{k}, \\ \#X_\infty^n[k] &= \binom{n-1+k}{k}. \end{aligned}$$

Consider the permutations $\rho, \tau : X_b^n[k] \to X_b^n[k]$ where ρ cycles all n coordinates and τ cycles the first $n-1$ coordinates, leaving the nth coordinate fixed:

$$\begin{aligned} \rho \cdot (x_1, x_2, \ldots, x_n) &:= (x_2, \ldots, x_n, x_1), \\ \tau \cdot (x_1, x_2, \ldots, x_n) &:= (x_2, \ldots, x_{n-1}, x_1, x_n). \end{aligned}$$

Here is the most basic result in the theory of cyclic sieving.

[1] See [15] for a more recent survey.

Proposition 1 ([14, Theorem 1.1])**.** *If $C = \langle\rho\rangle$ or $C = \langle\tau\rangle$ then the following exhibit CSP:*

$$\left(X_2^n[k], \begin{bmatrix} n \\ k \end{bmatrix}_q, C\right), \quad \left(X_\infty^n[k], \begin{bmatrix} n-1+k \\ k \end{bmatrix}_q, C\right).$$

In other words, if ω is a primitive dth root of unity with $d|n$ then we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega = \#X_2^n[k]^{\rho^{n/d}},$$
$$\begin{bmatrix} n-1+k \\ k \end{bmatrix}_\omega = \#X_\infty^n[k]^{\rho^{n/d}}.$$

and if ω is a primitive dth root of unity with $d|(n-1)$ then we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega = \#X_2^n[k]^{\tau^{(n-1)/d}},$$
$$\begin{bmatrix} n-1+k \\ k \end{bmatrix}_\omega = \#X_\infty^n[k]^{\tau^{(n-1)/d}}.$$

In general the set of vectors $X_b^n[k]$ corresponds to k-multisubsets of $\{1,\dots,n\}$ whose multiplicities are bounded above by b. We will use the notation

$$\binom{n}{k}^{(b)} = \#X_b^n[k],$$

so that

$$\binom{n}{k}^{(2)} = \binom{n}{k} \quad \text{and} \quad \binom{n}{k}^{(\infty)} = \binom{n-1+k}{k}.$$

The generating function is

$$\sum_{k=0}^{(b-1)n} \binom{n}{k}^{(b)} t^k = (1+t+\cdots+t^{b-1})^n.$$

These numbers have a long history and appear often in probability because $\binom{n}{k}^{(b)}/b^n$ is the chance of getting a sum of k in n rolls of a fair die with sides labeled $\{0,1,\dots,b-1\}$. Belbachir and Igueroufa [3] have collected an extensive list of references going back to Euler. It is surprising that there is no standard notation; we use a modified form of Euler's notation.

Our main results in this paper have to do with the natural q-analogue defined by

$$\sum_{k=0}^{(b-1)n} \begin{bmatrix} n \\ k \end{bmatrix}_q^{(b)} t^k = \prod_{i=0}^{n-1}(1+q^it+(q^it)^2+\cdots+(q^it)^{b-1}) = \prod_{i=0}^{n-1}[b]_{q^it},$$

which is the generating function for the statistic $x_2 + 2x_3 + \cdots + (n-1)x_n$ on the set $X_b^n[k]$:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(b)} = \sum_{\mathbf{x} \in X_b^n[k]} q^{x_2+2x_3+\cdots+(n-1)x_n}.$$

This is a generalization of the q-binomial coefficients in the sense that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(2)} = q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q,$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(\infty)} = \begin{bmatrix} n-1+k \\ k \end{bmatrix}_q.$$

Here is the result that motivated the current paper.

Theorem 2 (Cyclic Sieving of Multisets with Bounded Multiplicity)**.** *Let $X_b^n[k]$ be the set of n-tuples $(x_1, \ldots, x_n) \in \{0, 1, \ldots, b-1\}^n$ with $x_1 + \cdots + x_n = k$ and consider the permutations ρ, τ of $X_b^n[k]$ defined by*

$$\rho \cdot (x_1, x_2, \ldots, x_n) := (x_2, \ldots, x_n, x_1),$$
$$\tau \cdot (x_1, x_2, \ldots, x_n) := (x_2, \ldots, x_{n-1}, x_1, x_n).$$

(a) If $\gcd(b, n) = 1$ then the triple $(X_b^n[k], \begin{bmatrix} n \\ k \end{bmatrix}_q^{(b)}, \langle \rho \rangle)$ exhibits the CSP. More precisely, if ω is a primitive dth root of unity with $d|n$ and $\gcd(b, d) = 1$ then we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \#X_b^n[k]^{\rho^{n/d}}.$$

(b) If $\gcd(b, n-1) = 1$ then the triple $(X_b^n[k], \begin{bmatrix} n \\ k \end{bmatrix}_q^{(b)}, \langle \tau \rangle)$ exhibits the CSP. More precisely, if ω is a primitive dth root of unity with $d|(n-1)$ and $\gcd(b, d) = 1$ then we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \#X_b^n[k]^{\tau^{(n-1)/d}}.$$

Here we will give the combinatorial portion of the proof. The algebraic portion of the proof follows from parts (a) and (b) of Theorem 5 in the next section.

Proof. Let ω be a primitive dth root of unity.

First suppose that $d|n$. If $d \nmid k$ then $X_b^n[k]^{\rho^{n/d}} = \emptyset$. If $d|k$ then we have a bijection $X_b^{n/d}[k/d] \to X_b^n[k]^{\rho^{n/d}}$ defined by

$$(x_1, \ldots, x_{n/d}) \mapsto (x_1, \ldots, x_{n/d}, \ldots, x_1, \ldots, x_{n/d}),$$

where the string $(x_1, \ldots, x_{n/d})$ is repeated d times on the right hand side. Hence we have

$$\#X_b^n[k]^{\rho^{n/d}} = \#X_b^{n/d}[k/d] = \binom{n/d}{k/d}^{(b)}.$$

In other words, we have the generating function

$$\sum_k \#X_b^n[k]^{\rho^{n/d}} t^k = (1 + t^d + \cdots + (t^d)^{b-1})^{n/d}.$$

Theorem 5a below shows that the numbers $\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)}$ satisfy the same generating function, but only when $\gcd(b, d) = 1$.

Next suppose that $d|(n-1)$. Any $\mathbf{x} \in X_b^n[k]^{\tau^{(n-1)/d}}$ has the form

$$(x_1, \ldots, x_n) = (x_1, \ldots, x_{(n-1)/d}, \ldots, x_1, \ldots, x_{(n-1)/d}, x_n),$$

where the string $(x_1, \ldots, x_{(n-1)/d})$ is repeated d times. Since the coordinates sum to k we have $x_n + d\sum_{i=1}^{(n-1)/d} x_i = k$. If $x_n = \ell$ then $x_1, \ldots, x_{(n-1)/d}$ sum to $(k-\ell)/d$, hence there are $\binom{(n-1)/d}{(k-\ell)/d}^{(b)}$ ways to choose $x_1, \ldots, x_{(n-1)/d}$. It follows that

$$\#X_b^n[k]^{\tau^{(n-1)/d}} = \sum_{\ell \geqslant 0} \binom{(n-1)/d}{(k-\ell)/d}^{(b)}.$$

In other words, we have the generating function

$$\sum_k \#X_b^n[k]^{\tau^{(n-1)/d}} t^k = (1 + t + \cdots + t^{b-1})(1 + t^d + \cdots + (t^d)^{b-1})^{(n-1)/d}.$$

Theorem 5b below shows that the numbers $\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)}$ satisfy the same generating function, but only when $\gcd(b, d) = 1$. □

Let us observe how Theorem 2 generalizes Proposition 1. Let ω be a primitive dth root of unity. If b is larger than k (and coprime to d) then we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(\infty)} = \begin{bmatrix} n-1+k \\ k \end{bmatrix}_\omega,$$

so in this case we recover the multiset portion of Proposition 1. If $b = 2$ then Theorem 2 only applies when d is odd, i.e., when $\gcd(2, d) = 1$. In this case one can show with some tedious checking of signs and vanishing conditions that

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(2)} = \omega^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_\omega = \begin{bmatrix} n \\ k \end{bmatrix}_\omega$$

whenever $d|n$ or $d|(n-1)$.[2] If d is even, so $\gcd(b, d) \neq 1$, it still turns out that $\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(2)} = \pm\begin{bmatrix} n \\ k \end{bmatrix}_\omega$, and the negative signs can be fixed by multiplying $\begin{bmatrix} n \\ k \end{bmatrix}_q^{(2)}$ with a suitable power of

[2] We regard this fact as a small coincidence. In general, it seems that one should just admit that $\begin{bmatrix} n \\ k \end{bmatrix}_q^{(2)} = q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q$ is the appropriate q-analogue of binomial coefficients when dealing with sets, while $\begin{bmatrix} n \\ k \end{bmatrix}_q^{(\infty)} = \begin{bmatrix} n-1+k \\ k \end{bmatrix}_q$ is the appropriate analogue when dealing with multisets.

q. A similar phenomenon happens whenever $\gcd(b,d)=2$, but for general $\gcd(b,d)\neq 1$ with $d|n$ or $d|(n-1)$ the integers $\begin{bmatrix}n\\k\end{bmatrix}_\omega^{(b)}$ seem unrelated to cyclic actions on $X_b^n[k]$. For example, consider $b=d=n=k=3$ with

$$\begin{bmatrix}3\\3\end{bmatrix}_q^{(b)} = q^4(1+2q+q^2+2q^3+q^4),$$

and consider the set

$$X_3^3[3] = \{(0,1,2),(0,2,1),(1,0,2),(1,2,0),(2,0,1),(2,1,0),(1,1,1)\}.$$

Since ρ acts on $X_3^3[3]$ by cycling all three coordinates we have $X_3^3[3]^\rho = \{(1,1,1)\}$, but if ω is a primitive 3rd root of unity then

$$\begin{aligned}\begin{bmatrix}3\\3\end{bmatrix}_\omega^{(3)} &= \omega^4(1+2\omega+\omega^2+2\omega^3+\omega^4)\\ &= \omega+2\omega^2+1+2\omega+\omega^2\\ &= 1+3\omega+3\omega^2\\ &= 1+3\omega+3(-1-\omega)\\ &= -2\\ &\neq \#X_3^3[3]^\rho.\end{aligned}$$

We first observed Theorem 2a in the case $b=n+1$ and then discovered the result for $\gcd(b,d)=1$ in the course of proving it. We discovered Theorem 2b much later, after a first draft of this paper was already written, and only then realized the connection to the "nearly free cyclic action" in Theorem 1.1 of Reiner, Stanton and White [14]. We believe this accidental rediscovery confirms the inevitability of the idea.

Reiner, Stanton and White gave two proofs of Proposition 1. Their first proof interprets the q-binomial coefficients evaluated at roots of unity as character values for certain GL_n representations, which can also be viewed as principal specializations of symmetric polynomials. Their second proof uses "brute force" arguments to show that enumeration and the q-substitutions have the same explicit formula.

Our point of view is that there is not a big distance between the two proofs. In Theorem 5 below we observe that Reiner, Stanton and White's ad hoc collection of q-binomial evaluations [14, Proposition 4.2] can be unified via specialization of certain symmetric polynomials at roots of unity. This is a case where generalization leads to simplification.

Unfortunately, it seems that a fully representation-theoretic proof of Theorem 2 along the lines of Reiner, Stanton and White might be difficult because the relevant symmetric polynomials do not have positive Schur coefficients (see Section 5). Perhaps the point of view in Doty and Walker [7] can be used to generalize the representation-theoretic proof in [14].

3 Bounded Symmetric Polynomials

In this section we develop the relationship between Theorem 2 and symmetric polynomials. Though this is not strictly necessary for the proof, we believe that it is the correct context for the problem and it leads to further interesting questions.

Recall that $f(z_1, \ldots, z_n) \in \mathbb{Z}[z_1, \ldots, z_n]$ is called *symmetric* if for any permutation π of $\{1, \ldots, n\}$ we have

$$f(z_{\pi(1)}, \ldots, z_{\pi(n)}) = f(z_1, \ldots, z_n).$$

The following observation guides the shape of this paper. It is surely well-known, but we have not seen it explicitly stated. We do not call this a lemma because we will not use these results in later proofs. Parts (b) and (c) of the observation use some basic Galois theory; however, they also follow from Theorem 5 below. See Remark 7.

Observation 3. *Let $f(z_1, \ldots, z_n) \in \mathbb{Z}[z_1, \ldots, z_n]$ be a symmetric polynomial with integer coefficients and let ω be a primitive dth root of unity.*

(a) If $d|n$ then we have $\omega^n = 1$, hence

$$f(\omega, \omega^2, \ldots, \omega^n) = f(\omega, \omega^2, \ldots, \omega^{n-1}, 1) = f(1, \omega, \ldots, \omega^{n-1}).$$

If, moreover, f is homogeneous of degree k with $d \nmid k$ then we have

$$f(\omega, \omega^2, \ldots, \omega^n) = f(1, \omega, \ldots, \omega^{n-1}) = 0.$$

(b) If $d|n$ or $d|(n-1)$ then $f(1, \omega, \ldots, \omega^{n-1}) \in \mathbb{Z}$.

(c) If $d|n$ or $d|(n+1)$ then $f(\omega, \omega^2, \ldots, \omega^n) \in \mathbb{Z}$.

(d) If $d|(n-1)$ then $f(\omega, \omega^2, \ldots, \omega^n)$ need not be an integer.

(e) If $d|(n+1)$ then $f(1, \omega, \ldots, \omega^{n-1})$ need not be an integer.

Proof. (a): If $d|n$ and if f is homogeneous of degree k then

$$f(\omega, \omega^2, \ldots, \omega^n) = \omega^k f(1, \omega, \ldots, \omega^{n-1}) = \omega^k f(\omega, \omega^2, \ldots, \omega^n).$$

If $f(\omega, \omega^2, \ldots, \omega^n) \neq 0$ then this implies that $\omega^k = 1$, hence $d|k$.

Parts (b) and (c) use the following setup from Galois theory. Let ζ_m denote a primitive mth root of unity. The Galois group of $\mathbb{Q}(\zeta_m)/\mathbb{Q}$ is $\{\varphi_r : \gcd(r, m) = 1\}$, where the automorphism $\varphi_r : \mathbb{Q}(\zeta_m) \to \mathbb{Q}(\zeta_m)$ is defined by $\varphi_r(\zeta_m) = \zeta_m^r$. If $\alpha \in \mathbb{Q}(\zeta_m)$ satisfies $\varphi_r(\alpha) = \alpha$ for all $\gcd(r, m) = 1$ then it follows that $\alpha \in \mathbb{Q}$. If we also have $\alpha \in \mathbb{Z}[\zeta_m]$ then α satisfies a monic polynomial over $\mathbb{Z}$, which implies that $\alpha \in \mathbb{Z}$.

Let ω be a primitive dth root of unity and let $\boldsymbol{\omega} = (\omega, \ldots, \omega^{d-1})$ and write powers to denote concatenation of sequences. If $\gcd(r, d) = 1$ then the automorphism $\varphi_r(\omega) = \omega^r$ permutes the sequence $\boldsymbol{\omega}$, hence it permutes sequences of the following four types:

$$(1, \boldsymbol{\omega})^m, \quad (\boldsymbol{\omega}, 1)^m, \quad ((1, \boldsymbol{\omega})^m, 1), \quad ((\boldsymbol{\omega}, 1)^m, \boldsymbol{\omega}).$$

(b): If $d|n$ and $\gcd(r,n) = 1$ then we also have $\gcd(r,d) = 1$, hence the sequence $(1,\omega,\ldots,\omega^{n-1}) = (1,\boldsymbol{\omega})^{n/d}$ is permuted by φ_r. Since f is symmetric it follows that

$$\varphi_r(f(1,\omega,\ldots,\omega^{n-1})) = f(\varphi_r(1,\boldsymbol{\omega})^{n/d}) = f((1,\boldsymbol{\omega})^{n/d}) = f(1,\omega,\ldots,\omega^{n-1}).$$

Then since $f(1,\omega,\ldots,\omega^{n-1}) \in \mathbb{Z}[\zeta_n]$ we have $f(1,\omega,\ldots,\omega^{n-1}) \in \mathbb{Z}$. If $d|(n-1)$ and $\gcd(r,n-1) = 1$ then we also have $\gcd(r,d) = 1$, hence the sequence $(1,\omega,\ldots,\omega^{n-1}) = ((1,\boldsymbol{\omega})^{(n-1)/d},1)$ is permuted by φ_r. Since f is symmetric, the automorphism φ_r of $\mathbb{Q}(\zeta_{n-1})$ fixes $f(1,\omega,\ldots,\omega^{n-1})$, and since $f(1,\omega,\ldots,\omega^{n-1}) \in \mathbb{Z}[\zeta_{n-1}]$ this implies that $f(1,\omega,\ldots,\omega^{n-1}) \in \mathbb{Z}$. (c): If $d|n$ then $(\omega,\omega^2,\ldots,\omega^n) = (\boldsymbol{\omega},1)^{n/d}$ and if $d|(n+1)$ then $(\omega,\omega^2,\ldots,\omega^n) = ((\boldsymbol{\omega},1)^{(n+1)/d-1},\boldsymbol{\omega})$. The rest of the proof is the same as (b).

(d): If $d|(n-1)$ and $f(z_1,\ldots,z_n) = z_1+\cdots+z_n$ then $f(\omega,\omega^2,\ldots,\omega^n) = \omega \notin \mathbb{Z}$. (e): If $d|(n+1)$ and $f(z_1,\ldots,z_n) = z_1+\cdots+z_n$ then $f(1,\omega,\ldots,\omega^{n-1}) = -\omega \notin \mathbb{Z}$. □

For an indeterminate q and a symmetric polynomial $f(z_1,\ldots,z_n) \in \mathbb{Z}[z_1,\ldots,z_n]$ we consider the *principal specialization*

$$f(1,q,q^2,\ldots,q^{n-1}) \in \mathbb{Z}[q].$$

The evaluation of q at roots of unity appears often in the literature for specific families of f, but we have not seen it spelled out that this is a general phenomenon with three different cases: $d|n$, $d|(n-1)$ and $d|(n+1)$. The incompatibility of the cases $d|(n-1)$ and $d|(n+1)$ leads to some ambiguity whether one should define the principal specialization as $f(1,\ldots,q^{n-1})$ or $f(q,\ldots,q^n)$. The former is the standard convention.

In this paper we are interested in the case when f is a certain generalization of the elementary symmetric polynomials e_k and the complete symmetric polynomials h_k. Recall the generating functions

$$E(t;z_1,\ldots,z_n) = \prod_{i=1}^{n}(1+z_it) = \sum_{k\geqslant 0} e_k(z_1,\ldots,z_n)t^k,$$

$$H(t;z_1,\ldots,z_n) = \prod_{i=1}^{n}(1-z_it)^{-1} = \sum_{k\geqslant 0} h_k(z_1,\ldots,z_n)t^k,$$

and the reciprocity relation $E(-t;z_1,\ldots,z_n)H(t;z_1,\ldots,z_n) = 1$. The principal specializations of e_k and h_k satisfy

$$e_k(1,q,\ldots,q^{n-1}) = q^{k(k-1)/2}\begin{bmatrix} n \\ k \end{bmatrix}_q, \tag{2}$$

$$h_k(1,q,\ldots,q^{n-1}) = \begin{bmatrix} n-1+k \\ n-1 \end{bmatrix}_q. \tag{3}$$

For any $b \geqslant 2$ we consider the *b-bounded symmetric polynomials* $h_k^{(b)}$ with generating function

$$H^{(b)}(t;z_1,\ldots,z_n) = \sum_{k\geqslant 0} h_k^{(b)}(z_1,\ldots,z_n)t^k = \prod_{i=1}^{n}(1+z_it+(z_it)^2+\cdots+(z_it)^{b-1}).$$

Note that $h_k^{(2)} = e_k$ and $h_k^{(b)} = h_k$ when $b > k$. We can express the generating function $H^{(b)}$ in terms of E and H. This formula appears in several places; see, e.g., [7, Theorem 3.12].

Lemma 4.

$$H^{(b)}(t; z_1, \ldots, z_n) = E(-t^b; z_1^b, \ldots, z_n^b) H(t; z_1, \ldots, z_n).$$

Proof.

$$\begin{aligned} H^{(b)}(t; z_1, \ldots, z_n) &= \prod_{i=1}^{n} (1 + z_i t + (z_i t)^2 + \cdots + (z_i t)^{b-1}) \\ &= \prod_{i=1}^{n} \frac{1 - (z_i t)^b}{1 - z_i t} \\ &= \prod_{i=1}^{n} (1 + z_i^b (-t^b)) \prod_{i=1}^{n} (1 - z_i t)^{-1} \\ &= E(-t^b; z_1^b, \ldots, z_n^b) H(t; z_1, \ldots, z_n). \end{aligned}$$

□

It follows that

$$\begin{aligned} \sum_{k \geqslant 0} h_k^{(b)}(z_1, \ldots, z_n) t^k &= \sum_{\ell \geqslant 0} e_\ell(z_1^b, \ldots, z_n^b)(-t^b)^\ell \sum_{m \geqslant 0} h_m(z_1, \ldots, z_n) t^m \\ &= \sum_{k \geqslant 0} \left(\sum_{b\ell + m = k} (-1)^\ell e_\ell(z_1^b, \ldots, z_n^b) h_m(z_1, \ldots, z_n) \right) t^k \\ &= \sum_{k \geqslant 0} \left(\sum_{\ell \geqslant 0} (-1)^\ell e_\ell(z_1^b, \ldots, z_n^b) h_{k - b\ell}(z_1, \ldots, z_n) \right) t^k \end{aligned}$$

and hence

$$h_k^{(b)}(z_1, \ldots, z_n) = \sum_{\ell \geqslant 0} (-1)^\ell e_\ell(z_1^b, \ldots, z_n^b) h_{k - b\ell}(z_1, \ldots, z_n), \tag{4}$$

where we define $h_k = 0$ for negative k. This result can also be proved by inclusion-exclusion. In terms of the set (1) we have

$$h_k^{(b)}(z_1, \ldots, z_n) = \sum_{(x_1, \ldots, x_n) \in X_b^n[k]} z_1^{x_1} \cdots z_n^{x_n},$$

so that $h_k^{(b)}$ is the generating function for k-multisets with multiplicities bounded above by b. Using notation from the previous section, we have

$$\binom{n}{k}^{(b)} = h_k^{(b)}(\underbrace{1, \ldots, 1}_{n \text{ times}}) = \#X_b^n[k],$$

where $\binom{n}{k}^{(b)}$ is the coefficient of t^k in $(1+t+\cdots+t^{b-1})^n$. The principal specialization gives a natural q-analogue

$$\begin{aligned}\begin{bmatrix} n \\ k \end{bmatrix}_q^{(b)} &:= h_k^{(b)}(1, q, \ldots, q^{n-1}) \\ &= \sum_{(x_1,\ldots,x_n)\in X_b^n[k]} q^{x_2+2x_3+\cdots+(n-1)x_n}.\end{aligned}$$

Since $e_\ell(1, q, \ldots, q^{n-1}) = q^{\ell(\ell-1)/2}\begin{bmatrix} n \\ \ell \end{bmatrix}_q$ and $h_m(1, q, \ldots, q^{n-1}) = \begin{bmatrix} n-1+m \\ m \end{bmatrix}_q$, equation (4) implies the explicit formula

$$\begin{aligned}\begin{bmatrix} n \\ k \end{bmatrix}_q^{(b)} &= \sum_{\ell\geqslant 0}(-1)^\ell e_\ell\left((q^0)^b, (q^1)^b, \ldots, (q^{n-1})^b\right) h_{k-b\ell}(1, q, \ldots, q^{n-1}) \\ &= \sum_{\ell\geqslant 0}(-1)^\ell e_\ell\left((q^b)^0, (q^b)^1, \ldots, (q^b)^{n-1}\right) h_{k-b\ell}(1, q, \ldots, q^{n-1}) \\ &= \sum_{\ell\geqslant 0}(-1)^\ell (q^b)^{\ell(\ell-1)/2}\begin{bmatrix} n \\ \ell \end{bmatrix}_{q^b}\begin{bmatrix} n-1+k-b\ell \\ k-b\ell \end{bmatrix}_q, \end{aligned} \tag{5}$$

though we will never use this formula because our proof of Theorem 5 below follows from more elegant generating function arguments.

The polynomials $h_k^{(b)}(z_1, \ldots, z_n)$ have been studied by several authors. Doty and Walker [7] called them *modular complete symmetric functions* and showed that when $b = p$ is prime they are related to mod p representation theory of GL_n. We take the notation $h_k^{(b)}$ from Fu and Mei [8] who called these the *truncated homogeneous symmetric functions*. More recently, Grinberg [9] computed the Schur expansion, which he showed has coefficients in $\{0, 1, -1\}$. He called $h_k^{(b)}$ a *Petrie symmetric function* because the Schur coefficients are determinants of certain *Petrie matrices* (see the end of Section 5). After publicizing his results, Grinberg learned about several previous occurrences of these functions and collected an extensive list of references in the introduction to [9]. In private communication, he indicated that he would not have chosen the name *Petrie* if he had known about the previous occurrences. It seems that such a basic concept should have a basic name; in consultation with Grinberg we arrived at the name *b-bounded*. Of course, one need not use the letter b.

Let ω be a primitive dth root of unity. Since $h_k^{(b)}(z_1, \ldots, z_n) \in \mathbb{Z}[z_1, \ldots, z_n]$ is symmetric we know from Observation 3 that

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = h_k^{(b)}(1, \omega, \ldots, \omega^{n-1}) = h_k^{(b)}(\omega, \omega^2, \ldots, \omega^n) \in \mathbb{Z} \text{ when } d|n,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = h_k^{(b)}(1, \omega, \ldots, \omega^{n-1}) \in \mathbb{Z} \text{ when } d|(n-1), \text{ and}$$

$$\omega^k\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = h_k^{(b)}(\omega, \omega^2, \ldots, \omega^n) \in \mathbb{Z} \text{ when } d|(n+1).$$

Our main theorem gives generating functions for these integers. The proof is not difficult but the result is quite natural and seems to have been overlooked until now. It can be viewed as a systematization and generalization of Proposition 4.2 of Reiner, Stanton and White [14]. See Remark 6 below.

Theorem 5 (Main Theorem)**.** *Let ω be a primitive dth root of unity. Recall that we define $\binom{n}{k}^{(b)} = 0$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q^{(b)} = 0$ when n or k is not an integer.*

(a) If $d|n$ and $\gcd(b,d) = g$ then

$$\sum_{k=0}^{(b-1)n} \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = \left(\frac{(1-t^{bd/g})^g}{1-t^d} \right)^{n/d}.$$

When $g = 1$ this becomes

$$\sum_{k=0}^{(b-1)n} \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = \left(\frac{1-(t^d)^b}{1-t^d} \right)^{n/d} = [b]_{t^d}^{n/d},$$

and hence

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \binom{n/d}{k/d}^{(b)}.$$

(b) If $d|(n-1)$ and $\gcd(b,d) = g$ then

$$\sum_{k=0}^{(b-1)n} \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = [b]_t \left(\frac{(1-t^{bd/g})^g}{1-t^d} \right)^{(n-1)/d}.$$

When $g = 1$ this becomes

$$\sum_{k=0}^{(b-1)n} \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = [b]_t [b]_{t^d}^{(n-1)/d},$$

and hence

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \sum_{\ell=0}^{b-1} \binom{(n-1)/d}{(k-\ell)/d}^{(b)}.$$

(c) If $d|(n+1)$ and $\gcd(b,d) = g$ then

$$\sum_{k=0}^{(b-1)n} \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = \frac{1}{[b]_t} \left(\frac{(1-t^{bd/g})^g}{1-t^d} \right)^{(n+1)/d}.$$

When $g = 1$ this becomes

$$\sum_{k=0}^{(b-1)n} \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = \frac{1}{[b]_t} [b]_{t^d}^{(n+1)/d}. \tag{6}$$

The explicit computation of the coefficients is related to the Frobenius coin problem. See Corollary 13 in the next section. For later use in Section 5 we simply mention the special case when $b = 2$ and $d = n + 1$. If $\gcd(2, n + 1) = g$ then one can check that

$$\frac{1}{[2]_t}\left(\frac{(1-t^{2(n+1)/g})^g}{1-t^{n+1}}\right) = 1 - t + t^2 - \cdots + (-1)^n t^n,$$

and hence

$$e_k(\omega, \omega^2, \ldots, \omega^n) = \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(2)} = \omega^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_\omega = \begin{cases} (-1)^k & k < n+1, \\ 0 & k \geqslant n+1. \end{cases}$$

Proof. Let ω be a primitive dth root of unity and let T be an indeterminate, so that $\prod_{j=0}^{d-1}(1 - \omega^j T) = 1 - T^d$. If $\gcd(b, d) = 1$, then the sequence $1, \omega^b, \ldots, \omega^{(d-1)b}$ is a permutation of $1, \omega, \ldots, \omega^{d-1}$, hence we also have $\prod_{j=0}^{d-1}(1-\omega^{jb}T) = 1-T^d$. More generally, let $g = \gcd(b, d)$, so that ω^b is a primitive (d/g)th root of unity. Since $\gcd(d/g, b/g) = 1$ this implies that $1, \omega^b, \ldots, \omega^{(d/g-1)b}$ is a permutation of $1, \omega^g, \ldots, \omega^{(d/g-1)g}$, hence

$$\prod_{j=0}^{d-1}(1 - \omega^{jb}T) = \left(\prod_{j=0}^{d/g-1}(1 - \omega^{jb}T)\right)^g = (1 - T^{d/g})^g.$$

Finally, if $d|N$ then we note that

$$\prod_{j=0}^{N-1}(1 - \omega^{jb}T) = \left(\prod_{j=0}^{d-1}(1 - \omega^{jb}T)\right)^{N/d} = ((1 - T^{d/g})^g)^{N/d}.$$

Next we consider the generating function for the principal specializations of $h_k^{(b)}$:

$$\begin{aligned} H^{(b)}(t; 1, q, \ldots, q^{n-1}) &= \sum_{k \geqslant 0} h_k^{(b)}(1, q, \ldots, q^{n-1})t^k \\ &= \prod_{j=0}^{n-1}(1 + q^j t + \cdots + (q^j t)^{(b-1)}) \\ &= \prod_{j=0}^{n-1} \frac{1 - q^{jb}t^b}{1 - q^j t}. \end{aligned}$$

(a): If $d|n$ then taking $T = t^b$ gives

$$H^{(b)}(t; 1, \omega, \ldots, \omega^{n-1}) = \frac{\prod_{j=0}^{n-1}(1 - \omega^{jb}(t^b))}{\prod_{j=0}^{n-1}(1 - \omega^j t)} = \frac{(1 - (t^b)^{d/g})^{gn/d}}{(1 - t^d)^{n/d}} = \left(\frac{(1 - t^{bd/g})^g}{1 - t^d}\right)^{n/d}.$$

(b): For the case $d|(n-1)$ we first observe that

$$H^{(b)}(t; 1, q, \ldots, q^{n-1}) = \frac{1 - q^{(n-1)b}t^b}{1 - q^{n-1}t} \prod_{j=0}^{n-2} \frac{1 - q^{jb}t^b}{1 - q^j t} = [b]_{q^{n-1}t} \prod_{j=0}^{n-2} \frac{1 - q^{jb}t^b}{1 - q^j t}.$$

If $d|(n-1)$ then we have $[b]_{\omega^{n-1}t} = [b]_t$. So taking $N = n-1$ and $T = t^b$ gives

$$H^{(b)}(t; 1, \omega, \ldots, \omega^{n-1}) = [b]_t \prod_{j=0}^{n-2} \frac{1-\omega^{jb}t^b}{1-\omega^j t} = [b]_t \left(\frac{(1-t^{bd/g})^g}{1-t^d} \right)^{(n-1)/d}.$$

(c): For the case $d|(n+1)$ we first observe that

$$H^{(b)}(t; q, \ldots, q^n) = \prod_{j=1}^{n} \frac{1-q^{jb}t^b}{1-q^j t} = \frac{1}{[b]_{q^{n+1}t}} \prod_{j=0}^{n} \frac{1-q^{jb}t^b}{1-q^j t}.$$

If $d|(n+1)$ then we have $[b]_{\omega^{n+1}t} = [b]_t$. So taking $N = n+1$ and $T = t^b$ gives

$$H^{(b)}(t; \omega, \omega^2, \ldots, \omega^n) = \frac{1}{[b]_t} \prod_{j=0}^{n} \frac{1-\omega^{jb}t^b}{1-\omega^j t} = \frac{1}{[b]_t} \left(\frac{(1-t^{bd/g})^g}{1-t^d} \right)^{(n+1)/d}.$$

□

Remark 6. Parts (a) and (b) of Theorem 5 generalize and explain Proposition 4.2 in Reiner, Stanton and White [14]. Their identities 4.2(i) and 4.2(ii) correspond to part (a) with $b = \infty$ and $b = 2$, respectively. Their 4.2(iii) and 4.2(iv) correspond to part (b) with $b = \infty$ and $b = 2$, respectively. Part (c) has no analogue in [14].

Remark 7. Special cases of this theorem give a combinatorial proof of Observation 3 that avoids Galois theory. That is, in the case $b = 2$ this theorem shows that the appropriate specializations of $h_k^{(2)} = e_k$ are integers. (When d is odd this involves some sign bookkeeping, so perhaps it is more natural to use the case $b = \infty$ when $h_k^{(\infty)} = h_k$.) It is well known that any symmetric polynomial in $\mathbb{Z}[z_1, \ldots, z_n]$ can be expressed (uniquely) as a polynomial in $e_0, e_1, \ldots, e_n$ (also as a polynomial in $h_0, h_1, \ldots, h_n$) with integer coefficients.

Our original proof of Theorem 5a was based on the explicit formula (5) and involved lots of annoying case by case analysis, using the identities from [14, Prop 4.2]. A similar brute force proof of Theorem 5b could be carried out along the same lines. It seems more difficult to prove Theorem 5c without using generating functions.

In the next section we will study the coefficients in equation (6). To end this section we mention an interesting corollary of Theorem 5.

Corollary 8. *Let ω be a primitive dth root of unity.*

(a) If $d|n$ then we have

$$\sum_{k=0}^{(b-1)n} \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \begin{cases} b^{n/d} & \gcd(b,d) = 1, \\ 0 & \gcd(b,d) \neq 1. \end{cases}$$

(b) If $d|(n-1)$ then we have

$$\sum_{k=0}^{(b-1)n} \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \begin{cases} b^{(n-1)/d+1} & \gcd(b,d) = 1, \\ 0 & \gcd(b,d) \neq 1. \end{cases}$$

(c) If $d|(n+1)$ then we have

$$\sum_{k=0}^{(b-1)n} \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \begin{cases} b^{(n+1)/d-1} & \gcd(b,d)=1, \\ 0 & \gcd(b,d)\neq 1. \end{cases}$$

Proof. For $\gcd(b,d)=1$ this follows easily by setting $t=1$ in Theorem 5. For $\gcd(b,d)=g\neq 1$ we use L'Hospital's rule to observe that

$$\lim_{t\to 1}\frac{(1-t^{bd/g})^g}{1-t^d} = \lim_{t\to 1}\frac{\frac{d}{dt}(1-t^{bd/g})^g}{\frac{d}{dt}(1-t^d)} = \lim_{t\to 1} b(1-t^{bd/g})^{g-1}t^{d(b-g)/g} = 0.$$

□

The case $\gcd(b,d)=1$ of parts (a) and (b) also follows bijectively from Theorem 2. To see this for part (a), note from Theorem 2a that the sum in Corollary 8a is the size of the set $(X_b^n)^{\rho^{n/d}}$ of all words $(x_1,\ldots,x_n)\in\{0,1,\ldots,b-1\}^n$ fixed by the rotation $\rho^{n/d}$. But we have a bijection $X_b^{n/d}\to(X_b^n)^{\rho^{n/d}}$ sending

$$(x_1,\ldots,x_{n/d})\mapsto(x_1,\ldots,x_{n/d},\ldots,x_1,\ldots,x_{n/d}),$$

which shows that $\#(X_b^n)^{\rho^{n/d}}=b^{n/d}$. Part (c) is related to "q-analogues" of some number-theoretic congruences. Note that

$$\begin{aligned}\sum_{k=0}^{(b-1)n} q^k \begin{bmatrix} n \\ k \end{bmatrix}_q^{(b)} t^k &= \sum_{k=0}^{(b-1)n} \begin{bmatrix} n \\ k \end{bmatrix}_q^{(b)} (qt)^k \\ &= \prod_{i=0}^{n-1}(1+q^{i+1}t+\cdots+(q^{i+1}t)^{b-1}) \\ &= \prod_{i=1}^{n}[b]_{q^i t}.\end{aligned}$$

If ω is a primitive dth root of unity then substituting $q=\omega$ is the same as taking the equivalence class modulo the cyclotomic polynomial $\Phi_d(q)$. If $\gcd(b,d)=1$ and $d|(n+1)$ then Corollary 8c becomes

$$\prod_{i=1}^{n}[b]_{q^i} \equiv b^{(n+1)/d-1} \bmod \Phi_d(q).$$

If p is prime then the pth cyclotomic polynomial is $\Phi_p(q)=1+q+\cdots+q^{p-1}=[p]_q$. If $n+1=d=p$ and $p\nmid b$ then this becomes

$$\prod_{i=1}^{p-1}[b]_{q^i} \equiv 1 \bmod [p]_q,$$

which Pan and Sun [12, Formula 1.6] call a "q-analogue" of Fermat's little theorem. Other instances of Corollary 8c may be interesting in this context.

4 Sylvester Coin Polynomials

We have seen that parts (a) and (b) of Theorem 5 are related to cyclic sieving of multisets. In this section we investigate the combinatorial meaning of the integers in Theorem 5c. If $\gcd(b,d)=1$ then it follows indirectly from Theorem 5c that the expression $[b]_{t^d}/[b]_t$ is a polynomial with integer coefficients. While investigating this polynomial we discovered a relation to the Frobenius coin problem, which we now describe.

Given a list of positive integers $\mathbf{a}=(a_1,\ldots,a_m)\in\mathbb{N}\setminus\{0\}^m$ and a natural number $n\in\mathbb{N}$, let $\nu_{\mathbf{a}}(n)$ be the number of ways to express n as a linear combination of $a_1,\ldots,a_m$ with non-negative coefficients:

$$\nu_{\mathbf{a}}(n):=\#\{(k_1,\ldots,k_m)\in\mathbb{N}^m : k_1a_1+\cdots+k_ma_m=n\}.$$

In terms of generating functions we have

$$\sum_{n\geqslant 0}\nu_{\mathbf{a}}(n)t^n=\frac{1}{\prod_{i=1}^m(1-t^{a_i})}.$$

The numbers $\nu_{\mathbf{a}}(n)$ are associated with the names Frobenius and Sylvester. Sylvester [17] called these numbers "denumerants". Alfred Brauer [4] attributed the general problem to Frobenius, "who mentioned it occasionally in his lectures". Today it is called the Frobenius coin problem, or the Diophantine Frobenius problem. See [13] for a survey.

The first theorem in this subject says that if $\gcd(\mathbf{a})=1$ then there exists a largest n with $\nu_{\mathbf{a}}(n)=0$. (Alfonsín [13, Theorem 1.0.1] calls this a "folk result".) When $\gcd(\mathbf{a})=1$ we will denote the finite set of non-representable numbers by

$$S_{\mathbf{a}}:=\{n\in\mathbb{N}\setminus\{0\} : \nu_{\mathbf{a}}(n)=0\}.$$

The letter S is for Sylvester, who considered this set in the case of two parameters, and proved that $\#S_{a_1,a_2}=(a_1-1)(a_2-1)/2$. We call the generating polynomial for these numbers the *Sylvester coin polynomial*:

$$S_{\mathbf{a}}(t):=\sum_{s\in S_{\mathbf{a}}}t^s.$$

For example, we have $S_{3,5}(t)=t+t^2+t^4+t^7$. The general Frobenius problem is very difficult. In this paper we are interested in the easiest, but still quite interesting, case of two coprime parameters. We will call these parameters $(a_1,a_2)=(b,d)$ to agree with the notation in Theorem 5.

The main result connecting our Theorem 5c to the Frobenius coin problem is the following. Brown and Shiue [5] attribute this result to Özlük. Krattenthaler and Müller [10, Lemma 5] studied the polynomial $[b]_{t^d}/[b]_t$ in the context of cyclic sieving, but did not notice the connection to the Frobenius problem.

Lemma 9. *Let* $\gcd(b,d)=1$ *and consider the finite set* $S_{b,d}\subseteq\mathbb{N}$. *Then we have*

$$S_{b,d}(t)=\frac{t^{bd}-1}{(1-t^b)(1-t^d)}+\frac{1}{1-t},$$

and hence

$$1+(t-1)S_{b,d}(t) = 1+\left(\frac{(1-t^{bd})(t-1)}{(1-t^b)(1-t^d)}-1\right) = \frac{1-(t^d)^b}{1-t^d}\frac{1-t}{1-t^b} = \frac{[b]_{t^d}}{[b]_t}.$$

By symmetry we also have $[d]_{t^b}/[d]_t = 1+(t-1)S_{b,d}(t)$.

For example, when $(b,d)=(7,5)$ we have

$$\begin{aligned} S_{7,5} &= \{1,2,3,4,6,8,9,11,13,16,18,23\}, \\ S_{7,5}(t) &= t+t^2+t^3+t^4+t^6+t^8+t^9+t^{11}+t^{13}+t^{16}+t^{18}+t^{23}, \\ [7]_{t^5}\,/\,[7]_t &= 1+(t-1)S_{7,5}(t) \\ &= 1+t^5+t^7+t^{10}+t^{12}+t^{14}+t^{17}+t^{19}+t^{24} \\ &\quad -t-t^6-t^8-t^{11}-t^{13}-t^{16}-t^{18}-t^{23}. \end{aligned}$$

Note that the coefficients of $[7]_{t^5}/[7]_t$ are in $\{0,1,-1\}$. See Theorem 12a below for a nice interpretation of these coefficients. Though Lemma 9 is not new, we will give a complete proof because it will allow us to set up notation (the "double abacus") that we need for the proof of Theorem 12. We will obtain Lemma 9 as a corollary of Lemma 10.

Fix $\gcd(b,d)=1$ and consider the labeling $\lambda : \mathbb{Z}^2 \to \mathbb{Z}$ defined by

$$\lambda(x,y) := xd+yb.$$

We note that λ increases by d when we take a step to the right and increases by b when we take a step up. We can also think of $\lambda(x,y)$ as the "height" of the point (x,y) relative to the line $xd+yb=0$. We are interested in the discrete half-plane of points with non-negative labels, which we call the *double abacus*:[3]

$$A := \{(x,y)\in\mathbb{Z}^2 : xd+yb \geqslant 0\}.$$

Figure 1 shows the double abacus when $(b,d)=(7,5)$. The proofs in this section are easiest to understand visually, so we advise the reader always to keep an eye on this figure.

Now we describe a certain decomposition of the double abacus, which is indicated in the figure by black lines. Note that the labeling λ is not injective. Indeed, for any $(x,y)\in\mathbb{Z}^2$ and $k\in\mathbb{Z}$ we have $\lambda(x-bk,y+dk)=\lambda(x,y)$. We can take as a fundamental domain any d consecutive rows (or any b consecutive columns). Let $R_k=\{(x,y)\in A : y=k\}$ be the kth row of A and define the *fundamental row abacus*:

$$R := R_0 \cup R_1 \cup \ldots \cup R_{d-1}.$$

This is the shaded region in Figure 1. Note that the leftmost label in row k is $\delta(k)$, where $\delta(k)\in\{0,1,\ldots,d-1\}$ satisfies $\delta(k)\equiv bk \bmod d$. Thus λ gives a bijection from the set of points R_k to the set of labels $\delta(k)+d\mathbb{N}$. Since $\gcd(b,d)=1$ we observe that δ defines

[3]The double abacus was introduced in the theory of simultaneous core partitions. See [1] and [16].

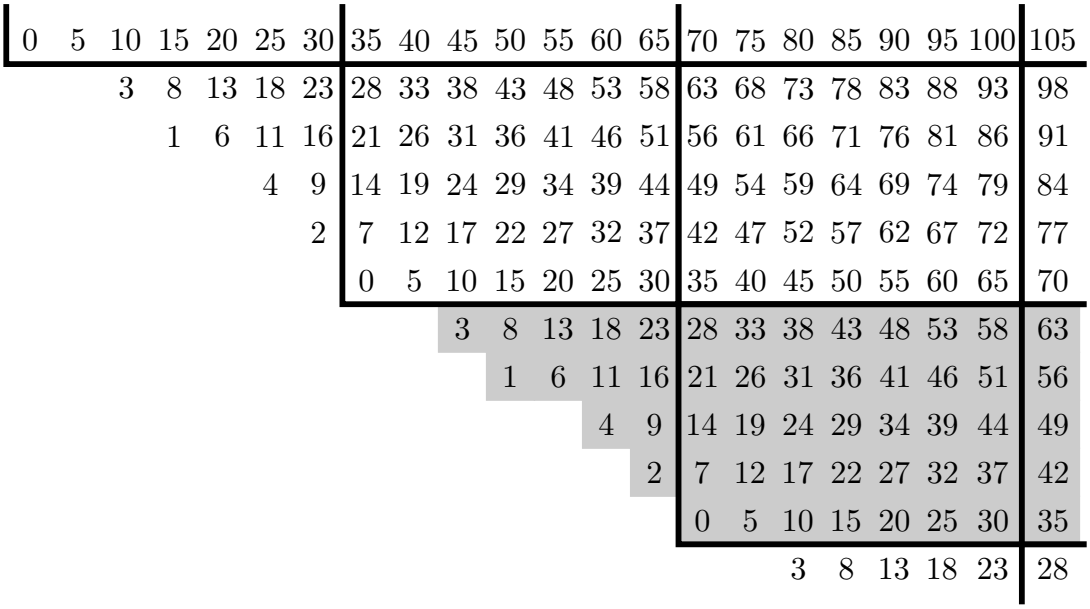

Figure 1: The double abacus for $(b, d) = (7, 5)$

a permutation of $\{0, 1, \ldots, d-1\}$, hence each label $n \in \mathbb{N}$ occurs exactly once in the row abacus R, and we obtain a bijection

$$\lambda : R \to \mathbb{N}.$$

The double abacus A is the disjoint union of the translated row abaci $R + k(-b, d)$ for all $k \in \mathbb{Z}$. We find it useful to consider a further decomposition by columns. For each $k \geqslant 0$ we consider the kth *block* of R:[4]

$$B_k := \{(x, y) \in R : (k-1)b \leqslant x < kb\}.$$

Thus R is the disjoint union of $B_0, B_1, B_2, \ldots$, where for each $k \geqslant 1$ the block B_k has the shape of a $b \times d$ rectangle and the zeroth block B_0 has the shape of a triangle. The translated sets $B_k + \ell(-b, d)$ with $k \geqslant 0$ and $\ell \in \mathbb{Z}$ give a decomposition of the whole double abacus into disjoint rectangles and triangles, which is shown in Figure 1. Observe that the labels inside the zeroth block B_0 are just the Sylvester set of non-representable numbers:

$$S_{7,5} = \{1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18, 23\}.$$

The following result generalizes this fact.

Lemma 10. *Fix* $\gcd(b, d) = 1$ *and recall that we have a bijection* $\lambda : R \to \mathbb{N}$ *from the fundamental row abacus to the set of all natural numbers. For each* $n \in \mathbb{N}$ *recall that*

[4]This is not related to the representation-theoretic meaning of "block" used in [16].

$\nu_{b,d}(n)$ is the number of distinct representations $n = kb + \ell d$ with $(k, \ell) \in \mathbb{N}^2$. Then $\nu_{b,d}(n) = k$ if and only if the label n occurs in the kth block of R:

$$\nu_{b,d}(n) = k \quad \Longleftrightarrow \quad \lambda^{-1}(n) \in B_k.$$

If $n \geqslant bd$ then it follows from this that $\nu_{b,d}(n - bd) = \nu_{b,d}(n) - 1$.

Proof. For the first statement, fix a point $(x, y) \in A$ in the double abacus and let $n = \lambda(x, y) = xd + yb$ be its label. Consider the following two sets:

$$X = \{(k, \ell) \in \mathbb{N}^2 : kd + \ell b = n\},$$
$$Y = \{(x', y') \in A : x' \leqslant x,\ y' \leqslant y \text{ and } \lambda(x', y') = 0\}.$$

The set X consists of all representations of n as an $\mathbb{N}$-linear combination of b and d, so that $\#X = \nu_{b,d}(n)$. The set Y consists of all points in the double abacus that are below and to the left of (x, y), and have label 0. It is easy to check that the maps $(k, \ell) \mapsto (x - k, y - \ell)$ and $(x', y') \mapsto (x - x', y - y')$ are inverse bijections between X and Y, hence $\#Y = \nu_{b,d}(n)$. Now the result follows from the observation that each point in the kth block B_k has exactly k points below and to its left with label 0.

For the second part of the lemma, suppose that $n \geqslant bd$. Since $\lambda : R \to \mathbb{N}$ is a bijection there exist unique points $(x, y), (x', y') \in R$ with $\lambda(x, y) = n$ and $\lambda(x', y') = n - bd$. But $\lambda(x - b, y) = (x - b)d + yb = (xd + yb) - bd = n - bd$ so we must have $(x', y') = (x - b, y)$. To complete the proof we observe that

$$\begin{aligned} \nu_{b,d}(n) = k \quad &\Longleftrightarrow \quad (x, y) \in B_k \\ &\Longleftrightarrow \quad (x - b, y) \in B_{k-1} \\ &\Longleftrightarrow \quad \nu_{b,d}(n - bd) = k - 1. \end{aligned}$$

□

For example, consider the double abacus in Figure 1 with $(b, d) = (7, 5)$. The label $n = 41$ occurs at position $(x, y) = (4, 3)$ in block B_1. The unique point (x', y') with $x' \leqslant x$, $y' \leqslant y$ and $\lambda(x', y') = 0$ is $(x', y') = (0, 0)$, which corresponds to the unique representation $41 = 5(x - x') + 7(y - y') = 5(4) + 7(3)$. Next consider the label $n = 88$ at position $(x, y) = (5, 9)$ which is in a translation of block B_2. (The bijection in the proof of Lemma 10 is invariant under translation by $(-b, d)$.) The two relevant points (x', y') with $x' \leqslant x$, $y' \leqslant y$ and $\lambda(x', y') = 0$ are $(0, 0)$ and $(-7, 5)$, which correspond to the representations $88 = 5(5 - 0) + 7(9 - 0) = 5(5) + 7(9)$ and $88 = 5(5 - (-7)) + 7(9 - 5) = 5(12) + 7(4)$.

Now we can prove Lemma 9.

Proof. Fix $\gcd(b, d) = 1$ and note that

$$\frac{1}{(1 - t^b)(1 - t^d)} = (1 + t^b + t^{2b} + \cdots)(1 + t^d + t^{2d} + \cdots) = \sum_{n \geqslant 0} \nu_{b,d}(n) t^n.$$

Applying Lemma 10 gives

$$\begin{aligned}\frac{t^{bd}-1}{(1-t^b)(1-t^d)} &= \sum_{n\geqslant 0}\nu_{b,d}(n)t^{n+bd} - \sum_{n\geqslant 0}\nu_{b,d}(n)t^n \\ &= \sum_{n\geqslant bd}\nu_{b,d}(n-bd)t^n - \sum_{n\geqslant 0}\nu_{b,d}(n)t^n \\ &= -\sum_{n=0}^{bd-1}\nu_{b,d}(n)t^n + \sum_{n\geqslant bd}[\nu_{b,d}(n-bd)-\nu_{b,d}(n)]t^n \\ &= -\sum_{n=0}^{bd-1}\nu_{b,d}(n)t^n - \sum_{n\geqslant bd}t^n,\end{aligned}$$

and hence

$$\frac{t^{bd}-1}{(1-t^b)(1-t^d)} + \frac{1}{1-t} = \sum_{n=0}^{bd-1}[1-\nu_{b,d}(n)]t^n.$$

But every number $0 \leqslant n \leqslant bd-1$ satisfies $\nu_{b,d}(n) = 0$ or $\nu_{b,d}(n) = 1$, and every number $n \in \mathbb{N}$ with $\nu_{b,d}(n) = 0$ satisfies $n \leqslant bd-1$. Hence

$$\frac{t^{bd}-1}{(1-t^b)(1-t^d)} + \frac{1}{1-t} = \sum_{\substack{n\in\mathbb{N}\\ \nu_{b,d}(n)=0}} t^n = S_{b,d}(t). \qquad \square$$

We will use this result to give two different explicit formulas (Theorem 11 and Corollary 13) for the coefficients in Theorem 5c. Recall that $\binom{n}{k}^{(b)}$ is the coefficient of t^k in $[b]_t^n$ and $q^k\begin{bmatrix}n\\k\end{bmatrix}_q^{(b)}$ is the coefficient of t^k in $[b]_q[b]_{q^2}\cdots[b]_{q^n}$. If ω is a primitive dth root of unity with $d|(n+1)$ and if $\gcd(b,d)=1$ then Theorem 5c says that

$$\sum_{k=0}^{(b-1)n}\omega^k\begin{bmatrix}n\\k\end{bmatrix}_\omega^{(b)}t^k = \frac{1}{[b]_t}[b]_{t^d}^{(n+1)/d}.$$

Combining this with Lemma 9 gives the following.

Theorem 11. *Let* $\gcd(b,d)=1$ *and recall that* $S_{b,d}$ *is the finite set of integers* $s\geqslant 1$ *that cannot be represented as* $s=bk+d\ell$ *with* $k,\ell\geqslant 0$. *If* ω *is a primitive* d*th root of unity with* $d|(n+1)$ *then we have*

$$\omega^k\begin{bmatrix}n\\k\end{bmatrix}_\omega^{(b)} = \binom{(n+1)/d-1}{k/d}^{(b)} + \sum_{s\in S_{b,d}}\binom{(n+1)/d-1}{(k-1-s)/d}^{(b)} - \sum_{s\in S_{b,d}}\binom{(n+1)/d-1}{(k-s)/d}^{(b)},$$

where we define $\binom{u}{v}^{(b)}=0$ *when* u *or* v *is not an integer.*

Proof. From Theorem 5c and Lemma 9 we have

$$\begin{aligned}\sum_{k=0}^{(b-1)n} \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k &= \frac{[b]_{t^d}}{[b]_t} [b]_{t^d}^{(n+1)/d-1} \\ &= (1 + (t-1)S_{b,d}(t))\, [b]_{t^d}^{(n+1)/d-1} \\ &= [b]_{t^d}^{(n+1)/d-1} + (t-1)S_{b,d}(t)[b]_{t^d}^{(n+1)/d-1} \\ &= \sum_\ell \binom{(n+1)/d-1}{\ell}^{(b)} t^{\ell d} + (t-1) \sum_{s \in S_{b,d}} t^s \sum_\ell \binom{(n+1)/d-1}{\ell}^{(b)} t^{\ell d}.\end{aligned}$$

Taking the coefficient of t^k gives the result. □

It is not clear from this alternating formula when the coefficients are zero, positive or negative. Corollary 13 will give an explicit criterion for this. It is based on the following theorem, which we believe has independent interest.

Theorem 12. *Let* $\gcd(b,d) = 1$. *Let* $(\delta_0, \ldots, \delta_{d-1})$ *be the permutation of* $(0, \ldots, d-1)$ *defined by* $\delta_r b \equiv r \bmod d$ *and let* $(\beta_0, \ldots, \beta_{b-1})$ *be the permutation of* $(0, \ldots, b-1)$ *defined by* $\beta_r d \equiv r \bmod b$.

(a) We have the following formula that is symmetric in b *and* d*:*

$$\frac{[b]_{t^d}}{[b]_t} = \frac{[d]_{t^b}}{[d]_t} = [\beta_1]_{t^d}[\delta_1]_{t^b} - t[b-\beta_1]_{t^d}[d-\delta_1]_{t^b}.$$

The coefficients of $[\beta_1]_{t^d}[\delta_1]_{t^b}$ *are in* $\{0,1\}$*, the coefficients of* $-t[b-\beta_1]_{t^d}[d-\delta_1]_{t^b}$ *are in* $\{0,-1\}$*, and there is no cancellation. That is, there are* $\beta_1\delta_1$ *coefficients equal to* 1 *and* $(b-\beta_1)(d-\delta_1)$ *coefficients equal to* -1.

(b) We also have the following non-symmetric formula for the d*-multisections of* $[b]_{t^d}/[b]_t$. *For all* $r \in \{0,1,\ldots,d-1\}$ *define* $\gamma_r := (\delta_r b - r)/d \in \{0,\ldots,b-1\}$. *Then we have*

$$\frac{[b]_{t^d}}{[b]_t} = \sum_{r=0}^{d-1} t^r f_r(t^d),$$

where

$$f_r(t) = \begin{cases} t^{\gamma_r}[\beta_1]_t & \delta_r < \delta_1, \\ -t^{\gamma_r - b + \beta_1}[b-\beta_1]_t & \delta_r \geqslant \delta_1. \end{cases}$$

Before giving the proof, we would like to indicate that the expression in Theorem 12a has a simple geometric meaning. Namely, let $X_0 \subseteq R$ be the $\beta_1 \times \delta_1$ rectangle of points in the fundamental row abacus whose bottom left point has label 0, and let $X_1 \subseteq R$ be the $(b-\beta_1) \times (d-\delta_1)$ rectangle of points whose bottom left point has label 1. Because of the way that the labels λ are defined we have

$$\sum_{\mathbf{x} \in X_0} t^{\lambda(\mathbf{x})} = [\beta_1]_{t^d}[\delta_1]_{t^b} \quad \text{and} \quad \sum_{\mathbf{x} \in X_1} t^{\lambda(\mathbf{x})} = t[b-\beta_1]_{t^d}[d-\delta_1]_{t^b}.$$

Since the labeling $\lambda : R \to \mathbb{N}$ is injective and since the rectangles X_0, X_1 are disjoint, there can be no cancellation between these two polynomials.

For example, consider Figure 1 where $(b, d) = (7, 5)$. The relevant permutations are $(\beta_0, \ldots, \beta_6) = (0, 3, 6, 2, 5, 1, 4)$ and $(\delta_0, \ldots, \delta_4) = (0, 3, 1, 4, 2)$, hence $\beta_1 = \delta_1 = 3$. The $\beta_1 \times \delta_1 = 3 \times 3$ rectangle X_0 contains the labels

$$\lambda(X_0) = \{0, 5, 10, 7, 12, 17, 14, 19, 24\},$$

while the rectangle X_1 of shape $(b - \beta_1) \times (d - \delta_1) = 4 \times 2$ contains the labels

$$\lambda(X_1) = \{1, 6, 11, 16, 8, 13, 18, 23\}.$$

Hence

$$\begin{aligned}
[7]_{t^5}/[7]_t &= \sum_{\mathbf{x}\in X_0} t^{\lambda(\mathbf{x})} - \sum_{\mathbf{x}\in X_1} t^{\lambda(\mathbf{x})} \\
&= [\beta_1]_{t^d}[\delta_1]_{t^b} - t[b-\beta_1]_{t^d}[d-\delta_1]_{t^b} \\
&= [3]_{t^5}[3]_{t^7} - t[4]_{t^5}[2]_{t^7} \\
&= 1 + t^5 + t^7 + t^{10} + t^{12} + t^{14} + t^{17} + t^{19} + t^{24} \\
&\quad - t - t^6 - t^8 - t^{11} - t^{13} - t^{16} - t^{18} - t^{23}.
\end{aligned}$$

We will give an example of Theorem 12b at the end of this section.

Proof. We will apply Lemma 9 and the double abacus. Throughout the proof we advise the reader to refer to Figure 2. The key idea is that for any point $(x, y) \in A$ and $k, \ell \in \mathbb{N}$, the generating polynomial for labels in the $k \times \ell$ rectangle $\{(x + k', y + \ell') : 0 \leqslant k' \leqslant k - 1, 0 \leqslant \ell' \leqslant \ell - 1\}$ is

$$\begin{aligned}
\sum_{\substack{0\leqslant k'\leqslant k-1\\ 0\leqslant \ell'\leqslant \ell-1}} t^{\lambda(x+k',y+\ell')} &= \sum_{\substack{0\leqslant k'\leqslant k-1\\ 0\leqslant \ell'\leqslant \ell-1}} t^{(x+k')d+(y+\ell')b} \\
&= t^{xd+yb} \sum_{\substack{0\leqslant k'\leqslant k-1\\ 0\leqslant \ell'\leqslant \ell-1}} t^{k'd+\ell'b} \\
&= t^{xd+yb} \sum_{0\leqslant k'\leqslant k-1} t^{k'd} \sum_{0\leqslant \ell'\leqslant \ell-1} t^{\ell'b} \\
&= t^{xd+yb}[k]_{t^d}[\ell]_{t^b}.
\end{aligned}$$

If, furthermore, we have $k \leqslant b$ and $\ell \leqslant d$ then since $\gcd(b, d) = 1$ the points of the $k \times \ell$ rectangle have distinct labels, hence this polynomial has coefficients in $\{0, 1\}$.

We will prove part (a) by showing that $[b]_{t^d}/[b]_t$ has the form

$$[b]_{t^d}/[b]_t = \sum_{\mathbf{x}\in X_0} t^{\lambda(\mathbf{x})} - \sum_{\mathbf{x}\in X_1} t^{\lambda(\mathbf{x})},$$

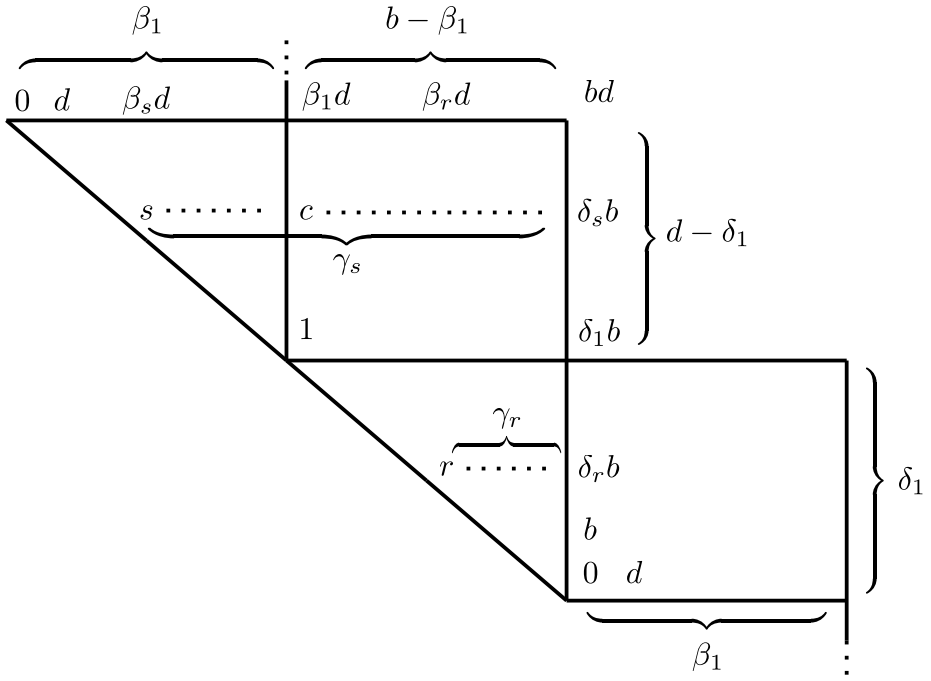

Figure 2: The configuration in the proof of Theorem 12

where X_0 is a $\beta_1 \times \delta_1$ rectangle whose bottom left point has label 0, and X_1 is a $(b-\beta_1) \times (d-\delta_1)$ rectangle disjoint from X_0 whose bottom left point has label 1. To show this, we recall that the Sylvester coin polynomial has the form

$$\sum_{\mathbf{x} \in B_0} t^{\lambda(\mathbf{x})},$$

where the "zeroth block" B_0 is the triangular set of points weakly to the left and below the point $(-1, d-1)$. From the definition of the numbers β_i and δ_i we note that the label 1 occurs at the lowest point of the column containing label $\beta_1 d$ and at the leftmost point of the row containing label $\delta_1 b$. That is, the label 1 occurs at point $(-b+\beta_1, \delta_1)$. Since $\lambda(b, -d) = 0$ we also have $\lambda(\beta_1, \delta_1 - d) = \lambda((-b+\beta_1)+b, (\delta_1)-d) = \lambda(-b+\beta_1, \delta_1) = 1$. Define the shifted triangle

$$B_0' := B_0 + (\beta_1, \delta_1 - d) = \{(x,y) + (\beta_1, \delta_1 - d) : (x,y) \in B_0\}.$$

We note that B_0' contains every point of the triangle weakly to the left and below the point $(-1, d-1) + (\beta_1, \delta_1 - d) = (\beta_1 - 1, \delta_1 - 1)$, **except for the point** $(0,0)$. Indeed, since

$$\lambda(x+\beta_1, y+\delta_1-d) = \lambda(x,y) + \lambda(\beta_1, \delta_1 - d) = \lambda(x,y) + 1,$$

we see that each label in B_0' is one larger than the corresponding label in B_0, so the label 0 does not occur. Since the labels in B_0 are the Sylvester set we know from Lemma 9 that

$$[b]_{t^d}/[b]_t = 1 + (t-1) \sum_{\mathbf{x} \in B_0} t^{\lambda(\mathbf{x})}$$

$$
\begin{aligned}
&= 1 + \sum_{\mathbf{x}\in B_0} t^{\lambda(\mathbf{x})+1} - \sum_{\mathbf{x}\in B_0} t^{\lambda(\mathbf{x})} \\
&= 1 + \sum_{\mathbf{x}\in B_0'} t^{\lambda(\mathbf{x})} - \sum_{\mathbf{x}\in B_0} t^{\lambda(\mathbf{x})} \\
&= \sum_{\mathbf{x}\in B_0'\cup\{(0,0)\}} t^{\lambda(\mathbf{x})} - \sum_{\mathbf{x}\in B_0} t^{\lambda(\mathbf{x})} \\
&= \sum_{\mathbf{x}\in B_0''} t^{\lambda(\mathbf{x})} - \sum_{\mathbf{x}\in B_0} t^{\lambda(\mathbf{x})},
\end{aligned}
$$

where $B_0'' := B_0' \cup \{(0,0)\}$. This can be simplified by observing that the triangular shapes B_0 and B_0'' are not disjoint. Their intersection is the smaller triangle T_1 below and to the left of the point $(-1, \delta_1 - 1)$. (This is the triangle containing the label r in Figure 2.) Hence

$$
[b]_{t^d}/[b]_t = \sum_{\mathbf{x}\in B_0''\setminus T_1} t^{\lambda(\mathbf{x})} - \sum_{\mathbf{x}\in B_0\setminus T_1} t^{\lambda(\mathbf{x})}.
$$

This can be simplified still further. Let T_2 be the triangle below and to the left of the point $(-b+\beta_1-1, d-1)$. (This is the triangle containing the label s in Figure 2.) Let X_1 be the rectangle of size $(b-\beta_1)\times(d-\delta_1)$ with bottom left corner $(-b+\beta_1, \delta_1)$ and let X_0 be the $\beta_1 \times \delta_1$ rectangle with bottom left corner $(0,0)$. Then we have $B_0 = T_1 \cup X_1 \cup T_2$ and $B_0'' = T_1 \cup X_0 \cup (T_2 + (b,-d))$. But the labels in the triangles T_2 and $T_2 + (b,-d)$ are the same. Hence

$$
\begin{aligned}
[b]_{t^d}/[b]_t &= \sum_{\mathbf{x}\in B_0''} t^{\lambda(\mathbf{x})} - \sum_{\mathbf{x}\in B_0} t^{\lambda(\mathbf{x})} \\
&= \sum_{\mathbf{x}\in X_0} t^{\lambda(\mathbf{x})} - \sum_{\mathbf{x}\in X_1} t^{\lambda(\mathbf{x})}.
\end{aligned}
$$

This completes the proof of part (a). To prove part (b), observe that the terms $t^{\lambda(\mathbf{x})}$ in $[b]_{t^d}/[b]_t$ with $\lambda(\mathbf{x}) \equiv r \bmod d$ come from the points $\mathbf{x}$ in the row containing label r, i.e., in the row containing label $\delta_r b$. Let $r \in \{0, \ldots, d-1\}$ and define $\gamma_r := (\delta_r b - r)/d \in \mathbb{N}$. This is the "width" of the triangle B_0 in the row containing label r, as in Figure 2. If $\delta_r < \delta_1$ then the row containing r intersects the rectangle X_0 at the sequence of points with labels $\delta_r b, \delta_r b + d, \ldots, \delta_r b + (\beta_1 - 1)d$. The sum of $t^{\lambda(\mathbf{x})}$ over these points is

$$
\sum_{k=0}^{\beta_1-1} t^{\delta_r b + kd} = t^{\delta_r b}[\beta_1]_{t^d} = t^r (t^d)_r^{\gamma} [\beta_1]_{t^d}.
$$

Hence

$$
\sum_{\mathbf{x}\in X_0} t^{\lambda(\mathbf{x})} = \sum_{\substack{0\leqslant r\leqslant d-1 \\ \delta_r < \delta_1}} t^r (t^d)^{\gamma_r} [\beta_1]_{t^d}.
$$

If $\delta_r \geqslant \delta_1$ then the row containing r intersects the rectangle X_1 at the sequence of points with labels $c, c+d, \ldots, c+(b-\beta_1-1)d$, where $c = r+(\gamma_r - b + \beta_1)d$. (In Figure 2 we have used the letter s for this case instead of r.) The sum of $t^{\lambda(\mathbf{x})}$ over these points is

$$\sum_{k=0}^{b-\beta_1-1} t^{c+kd} = t^c[b-\beta_1]_{t^d} = t^r(t^d)^{\gamma_r - b + \beta_1}[b-\beta_1]_{t^d}.$$

Hence

$$\sum_{\mathbf{x}\in X_1} t^{\lambda(\mathbf{x})} = \sum_{\substack{0\leqslant r\leqslant d-1 \\ \delta_r \geqslant \delta_1}} t^r(t^d)^{\gamma_r - b + \beta_1}[b-\beta_1]_{t^d}.$$

□

Finally, we apply Theorem 12 to obtain a more precise formula for the integers $\omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)}$ when $\omega^{n+1} = 1$. In particular, this formula determines the sign and vanishing of these integers.

Corollary 13. *Let ω be a primitive dth root of unity with $d|(n+1)$ and let $\gcd(b,d) = 1$. Then for any $r \in \{0, \ldots, d-1\}$ we have*

$$\sum_{k\geqslant 0} \omega^{r+dk} \begin{bmatrix} n \\ r+kd \end{bmatrix}_\omega^{(b)} t^{r+dk} = t^r g_r(t^d),$$

where

$$g_r(t) = \begin{cases} t^{\gamma_r}[\beta_1]_t[b]_t^{(n+1)/d-1} & \delta_r < \delta_1, \\ -t^{\gamma_r - b + \beta_1}[b-\beta_1]_t[b]_t^{(n+1)/d-1} & \delta_r \geqslant \delta_1. \end{cases}$$

We observe that each $g_r(t)$ is (plus or minus) a unimodal polynomial since products of t-integers are unimodal; see [2, Theorem 3.9].

Proof. Recall from Theorem 5c that we have

$$\sum_{k=0}^{(b-1)n} \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = \frac{[b]_{t^d}}{[b]_t}[b]_{t^d}^{(n+1)/d-1}.$$

Now use Theorem 12b. □

To end this section, we complete our example with $(b,d) = (7,5)$. Recall that the relevant permutations are $(\beta_0, \ldots, \beta_6) = (0,3,6,2,5,1,4)$ and $(\delta_0, \ldots, \delta_4) = (0,3,1,4,2)$, hence $\beta_1 = 3$ and $b - \beta_1 = 4$. To compute the 5-multisections of $[7]_{t^5}/[7]_t$ we note that the numbers $\gamma_r = (\delta_r b - r)/d$ are $(\gamma_0, \ldots, \gamma_4) = (0,4,1,5,2)$, which are the lengths of the rows in the fundamental triangle B_0 (including the empty bottom row), permuted according to congruence class mod $d = 5$. Thus we have

$$[7]_{t^5}/[7]_t = \sum_{\substack{0\leqslant r\leqslant 4 \\ \delta_r < 3}} t^r(t^5)^{\gamma_r}[3]_{t^5} - \sum_{\substack{0\leqslant r\leqslant 4 \\ \delta_r \geqslant 3}} t^r(t^5)^{\gamma_r - 4}[4]_{t^5}$$

$$= t^0[3]_{t^5} - t^1[4]_{t^5} + t^2(t^5)^1[3]_{t^5} - t^3(t^5)^1[4]_{t^5} + t^4(t^5)^2[3]_{t^5}.$$

Finally, if ω is a $d = 5$th root of unity and $5|(n+1)$, then we have

$$\sum_{k=0}^{6n} \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(7)} t^k = \frac{[7]_{t^5}}{[7]_t} [7]_{t^5}^{(n+1)/5-1},$$

and hence

$$\sum_{k\geqslant 0} \omega^{r+5k} \begin{bmatrix} n \\ r+5k \end{bmatrix}_\omega^{(7)} t^{r+5k} = \begin{cases} [3]_{t^5}[7]_{t^5}^{(n+1)/5-1} & r=0, \\ -t^1[4]_{t^5}[7]_{t^5}^{(n+1)/5-1} & r=1, \\ t^2(t^5)^1[3]_{t^5}[7]_{t^5}^{(n+1)/5-1} & r=2, \\ -t^3(t^5)^1[4]_{t^5}[7]_{t^5}^{(n+1)/5-1} & r=3, \\ t^4(t^5)^2[3]_{t^5}[7]_{t^5}^{(n+1)/5-1} & r=4. \end{cases}$$

In particular, we have $\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(7)} \geqslant 0$ when $(k \bmod 5) \in \{0,2,4\}$ and $\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(7)} \leqslant 0$ when $(k \bmod 5) \in \{1,3\}$. The criterion for $\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(7)} = 0$ is a bit complicated to state.

5 Other Symmetric Polynomials

In this paper we have studied the principal specialization of the b-bounded symmetric polynomials $h_k^{(b)}$ at roots of unity. It may be interesting to compare these specializations to the expansion of $h_k^{(b)}$ in various bases for the ring of symmetric polynomials. Here we follow the standard notation from Macdonald [11].

An *integer partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \dots,) \in \mathbb{N}^\infty$ satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$, where only finitely many entries are non-zero. We define the *weight of* λ (also called *size*) by $|\lambda| = \sum_i \lambda_i$. If $|\lambda| = k$ then we say that λ is a *partition of* k (another notation is $\lambda \vdash k$). We let $l(\lambda)$ denote the maximum ℓ such that $\lambda_\ell \neq 0$, which is called the *length of* λ. We write $m_i(\lambda) = \#\{j : \lambda_j = i\}$ (or just m_i when λ is understood) for the number of parts of λ equal to i. We can also express this colloquially as $\lambda = (1^{m_1}2^{m_2}\cdots)$. Note that $|\lambda| = \sum_i i m_i$. The following quantity also shows up frequently:[5]

$$z_\lambda = \prod_{i\geqslant 1} i^{m_i} \cdot m_i!.$$

Its significance is the fact that for $|\lambda| = n$ the number of permutations of $\{1, 2, \dots, n\}$ with cycle type λ is $n! z_\lambda^{-1}$. For any integer partition λ we define the corresponding elementary and complete symmetric polynomials

$$e_\lambda(z_1, \dots, z_n) := \prod_i e_{\lambda_i}(z_1, \dots, z_n),$$

[5]The integer z_λ has nothing to do with the variables z_i.

$$h_\lambda(z_1,\dots,z_n) := \prod_i h_{\lambda_i}(z_1,\dots,z_n),$$

with the convention that $e_0 = h_0 = 1$. By analogy we define the b-bounded polynomials

$$h^{(b)}_\lambda(z_1,\dots,z_n) := \prod_i h^{(b)}_{\lambda_i}(z_1,\dots,z_n),$$

so that $h^{(2)}_\lambda = e_\lambda$ and $h^{(\infty)}_\lambda = h_\lambda$. Furthermore, we define the *power sum symmetric polynomials* $p_\lambda(z_1,\dots,z_n) := \prod_i p_{\lambda_i}(z_1,\dots,z_n)$, where

$$p_k(z_1,\dots,z_n) = z_1^k + \cdots + z_n^k,$$

and $p_0 = 1$. The Fundamental Theorem of Symmetric Polynomials says that the set $\{e_\lambda\}$ is a $\mathbb{Z}$-linear basis for the ring $\Lambda_n = \mathbb{Z}[z_1,\dots,z_n]^{S_n}$ of symmetric polynomials with integer coefficients. It is also true that $\{h_\lambda\}$ is a $\mathbb{Z}$-linear basis for Λ_n, while the set $\{p_\lambda\}$ is a $\mathbb{Q}$-linear basis for $\Lambda_n \otimes \mathbb{Q}$. The expansions of e_k and h_k in terms of p_λ are called "Newton's identities" [11, Equation 2.14′]:

$$h_k(z_1,\dots,z_n) = \sum_{|\lambda|=k} z_\lambda^{-1} p_\lambda(z_1,\dots,z_n),$$
$$e_k(z_1,\dots,z_n) = \sum_{|\lambda|=k} z_\lambda^{-1} (-1)^{|\lambda|-l(\lambda)} p_\lambda(z_1,\dots,z_n).$$

The generalization of these formulas to b-bounded symmetric polynomials follows from Proposition 3.15 and Remark 3.16(2) in Doty and Walker [7], which they attribute to Macdonald. We think it is interesting to reproduce the proof here. Note that Doty and Walker did not consider the specialization $z_i = 1$. In Corollary 16 we will consider the specializations $z_i = \omega^{i-1}$ and $z_i = \omega^i$ for roots of unity ω.

Proposition 14. *Let $b, n, k \in \mathbb{N}$ with $b \geqslant 2$. For an integer partition λ let $l_b(\lambda)$ denote the number of parts of λ that are divisible by b. Then we have*

$$h^{(b)}_k(z_1,\dots,z_n) = \sum_{|\lambda|=k} z_\lambda^{-1} (1-b)^{l_b(\lambda)} p_\lambda(z_1,\dots,z_n),$$

and substituting $z_i = 1$ for all i gives

$$\binom{n}{k}^{(b)} = \sum_{|\lambda|=k} z_\lambda^{-1} (1-b)^{l_b(\lambda)} n^{l(\lambda)}.$$

For example, we consider the case $(n,k) = (3,3)$ for various b. The partitions of $k = 3$ are $\lambda \in \{(3),(2,1),(1,1,1)\}$, with $z_{(3)} = 3$, $z_{(2,1)} = 2$ and $z_{(1,1,1)} = 6$. For any $b > 3$ we have $l_b(\lambda) = 0$ for all $|\lambda| = 3$, hence

$$\binom{3-1+3}{3} = \binom{3}{3}^{(b)} = \frac{(3)^1(1-b)^0}{3} + \frac{(3)^2(1-b)^0}{2} + \frac{(3)^3(1-b)^0}{6} = 10.$$

When $b = 3$ we have $l_3(3) = 1$, $l_3(2,1) = 0$ and $l_3(1,1,1) = 0$, hence

$$\binom{3}{3}^{(3)} = \frac{(3)^1(-2)^1}{3} + \frac{(3)^2(-2)^0}{2} + \frac{(3)^3(-2)^0}{6} = 7.$$

When $b = 2$ we have $l_2(3) = 0$, $l_2(2,1) = 1$ and $l_2(1,1,1) = 0$, hence

$$\binom{3}{3} = \binom{3}{3}^{(2)} = \frac{(3)^1(-1)^0}{3} + \frac{(3)^2(-1)^1}{2} + \frac{(3)^3(-1)^0}{6} = 1.$$

When $b > k$ we note that $l_b(\lambda) = 0$ for all $|\lambda| = k$ and when $b = 2$ one can check that $|\lambda| - l(\lambda)$ and $l_2(\lambda)$ have the same parity, so these cases reduce to the classical Newton identities.

The key idea in the proof is the introduction of the polynomials $p_k^{(b)}$. Doty and Walker called these *modular power sums* and they defined them in a slightly different way. Walker [18] says that the formula relating $p_k^{(b)}$ and p_k is a "surprising observation" of Macdonald.

Proof. Recall the generating function

$$H^{(b)}(t; z_1, \ldots, z_n) = \sum_{k \geqslant 0} h_k^{(b)}(z_1, \ldots, z_n) t^k = \prod_{i=1}^{n} \frac{1 - z_i^b t^b}{1 - z_i t}.$$

Taking logarithms gives

$$\begin{aligned}
\log H^{(b)}(t; z_1, \ldots, z_n) &= \sum_{i=1}^{n} \log(1 - z_i^b t^b) - \sum_{i=1}^{n} \log(1 - z_i t) \\
&= -\sum_{i=1}^{n} \sum_{k \geqslant 1} \frac{1}{k} z_i^{kb} t^{kb} + \sum_{i=1}^{n} \sum_{k \geqslant 1} \frac{1}{k} z_i^k t^k \\
&= -\sum_{k \geqslant 1} \frac{1}{k} p_{kb}(z_1, \ldots, z_n) t^{kb} + \sum_{k \geqslant 1} \frac{1}{k} p_k(z_1, \ldots, z_n) t^k.
\end{aligned}$$

It is convenient to write this last expression as

$$\sum_{k \geqslant 1} \frac{1}{k} p_k^{(b)}(z_1, \ldots, z_n) t^k,$$

where we define

$$p_k^{(b)}(z_1, \ldots, z_n) = \begin{cases} (1-b) p_k(z_1, \ldots, z_n) & b | k, \\ p_k(z_1, \ldots, z_n) & b \nmid k. \end{cases}$$

For any integer partition λ we also write

$$p_\lambda^{(b)}(z_1, \ldots, z_n) = \prod_i p_{\lambda_i}^{(b)}(z_1, \ldots, z_n) = (1-b)^{l_b(\lambda)} p_\lambda(z_1, \ldots, z_n).$$

Then taking exponentials gives[6]

$$\begin{aligned} H^{(b)}(t; z_1, \ldots, z_n) &= \prod_{i \geqslant 1} \exp\left(\frac{p_i^{(b)} t^i}{i}\right) \\ &= \prod_{i \geqslant 1} \sum_{m_i \geqslant 0} \frac{1}{m_i!} \left(\frac{p_i^{(b)} t^i}{i}\right)^{m_i} \\ &= \prod_{i \geqslant 1} \sum_{m_i \geqslant 0} \frac{1}{i^{m_i} \cdot m_i!} (p_i^{(b)})^{m_i} t^{i m_i} \\ &= \sum_{k \geqslant 0} \sum_{|\lambda| = k} z_\lambda^{-1} p_\lambda^{(b)} t^k \\ &= \sum_{k \geqslant 0} \sum_{|\lambda| = k} z_\lambda^{-1} (1-b)^{l_b(\lambda)} p_\lambda t^k. \end{aligned}$$

□

Let us see what happens to Proposition 14 under specialization at roots of unity. The specialization of the power sums is easy, so we omit the proof.

Lemma 15. *Let ω be a primitive dth root of unity. For an integer partition λ recall that $l_d(\lambda)$ is the number of parts of λ divisible by d.*

(a) If $d|n$ then we have

$$p_k(\omega, \omega^2, \ldots, \omega^n) = p_k(1, \omega, \ldots, \omega^{n-1}) = \begin{cases} n & d|k, \\ 0 & d \nmid k, \end{cases}$$

and hence

$$p_\lambda(\omega, \omega^2, \ldots, \omega^n) = p_\lambda(1, \omega, \ldots, \omega^{n-1}) = \begin{cases} n^{l(\lambda)} & d|\lambda_i \text{ for all } i, \\ 0 & \text{else.} \end{cases}$$

(b) If $d|(n-1)$ then we have

$$p_k(1, \omega, \ldots, \omega^{n-1}) = \begin{cases} n & d|k, \\ 1 & d \nmid k, \end{cases}$$

and hence $p_\lambda(1, \omega, \ldots, \omega^{n-1}) = n^{l_d(\lambda)}$.

(c) If $d|(n+1)$ then we have

$$p_k(\omega, \omega^2, \ldots, \omega^n) = \begin{cases} n & d|k, \\ -1 & d \nmid k, \end{cases}$$

and hence $p_\lambda(\omega, \omega^2, \ldots, \omega^n) = n^{l_d(\lambda)}(-1)^{l(\lambda) - l_d(\lambda)} = (-n)^{l_d(\lambda)}(-1)^{l(\lambda)}$.

[6] We omit variables to save space.

Corollary 16. *Let ω be a primitive dth root of unity.*

(a) If $d|n$ then combining Proposition 14 with Lemma 15a gives

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \sum_{\substack{|\lambda|=k \\ d|\lambda_i \text{ for all } i}} z_\lambda^{-1}(1-b)^{l_b(\lambda)} n^{l(\lambda)}.$$

For each term in the sum we can write $\lambda = d\mu$ for some partition $\mu \vdash k/d$. Then using the facts $l(\lambda) = l(\mu)$ and $z_\lambda = d^{l(\mu)} z_\mu$ gives

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \sum_{|\mu|=k/d} z_\mu^{-1}(1-b)^{l_b(d\mu)} (n/d)^{l(\mu)}.$$

In the special case $\gcd(b,d) = 1$ we also have $l_b(d\mu) = l_b(\mu)$, and hence

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \sum_{|\mu|=k/d} z_\mu^{-1}(1-b)^{l_b(\mu)} (n/d)^{l(\mu)} = \binom{n/d}{k/d}^{(b)},$$

which recovers the identity from Theorem 5a.

(b) If $d|(n-1)$ then we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \sum_{|\lambda|=k} z_\lambda^{-1}(1-b)^{l_b(\lambda)} n^{l_d(\lambda)}.$$

(c) If $d|(n+1)$ then we have

$$\omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \sum_{|\lambda|=k} z_\lambda^{-1}(1-b)^{l_b(\lambda)} (-1)^{l(\lambda)-l_d(\lambda)} n^{l_d(\lambda)}.$$

To end this paper we also consider the expansion of $h_k^{(b)}$ in the bases e_λ, m_λ and s_λ. The *monomial symmetric polynomials* $m_\lambda(z_1, \ldots, z_n)$ are defined as follows. Let λ be a partition of length $l(\lambda) = \ell \leqslant n$. We will write $\lambda = (\lambda_1, \ldots, \lambda_\ell, \lambda_{\ell+1}, \ldots, \lambda_n)$ with $\lambda_{\ell+1} = \lambda_{\ell+2} = \cdots = \lambda_n = 0$. Suppose that $\lambda = (0^{m_0} 1^{m_1} 2^{m_2} \cdots)$, i.e., that λ has m_i parts equal to i. In particular, we have $m_0 = n - \ell$ and $\sum_i m_i = n$. Then we define[7]

$$m_\lambda(z_1, \ldots, z_n) := \frac{1}{\prod_{i=0}^n m_i!} \sum_{\pi \in S_n} z_1^{\lambda_{\pi(1)}} z_2^{\lambda_{\pi(2)}} \cdots z_n^{\lambda_{\pi(n)}} \in \mathbb{Z}[z_1, \ldots, z_n].$$

This can be more simply described as the sum over all distinct monomials in the variables $z_1, \ldots, z_n$ whose coefficients are a rearrangement of the list $\lambda_1, \ldots, \lambda_n$. For example, we have

$$m_{(2,2)}(z_1, z_2, z_3) = m_{(2,2,0)}(z_1, z_2, z_3) = z_1^2 z_2^2 + z_1^2 z_3^2 + z_2^2 z_3^2,$$

[7]The use of the same letter m for the unrelated m_λ and m_i is unfortunate.

$$m_{(2,1)}(z_1,z_2,z_3) = m_{(2,1,0)}(z_1,z_2,z_3) = z_1^2z_2 + z_2^2z_1 + z_1^2z_3 + z_3^2z_1 + z_2^2z_3 + z_3^2z_2.$$

The *Schur polynomials* $s_\lambda(z_1,\dots,z_n)$ are most easily defined as a ratio of determinants of $n \times n$ matrices:

$$s_\lambda(z_1,\dots,z_n) = \det(z_i^{n-j+\lambda_j})/\det(z_i^{n-j}) \in \mathbb{Z}[z_1,\dots,z_n].$$

These polynomials have a rich theory that we will not discuss here. Given an integer partition $\lambda = (\lambda_1,\lambda_2,\dots)$, the *conjugate partition* $\lambda' = (\lambda'_1,\lambda'_2,\dots)$ is defined by $\lambda'_i := \#\{j : \lambda_j \geqslant i\}$, so that $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$. Note that $|\lambda| = |\lambda'|$. We have the following expansions for $h_k^{(b)}(z_1,\dots,z_n)$ in terms of various bases for Λ_n, which follow easily from the theory of duality for symmetric functions (see Macdonald [11, Chapter I.4]). None of these formulas is new, but we believe it is interesting to show that they all have the same proof. After the proof we will give references and discuss possible connections to Theorem 5.

Proposition 17. *Let $n,k,b \in \mathbb{N}$ with $b \geqslant 2$ and let ζ be a primitive bth root of unity. Then the polynomial $h_k^{(b)}(z_1,\dots,z_n)$ has the following expressions:*

(a) $\sum_{|\lambda|=k}(-1)^{-l(\lambda)}z_\lambda^{-1}p_\lambda(\zeta,\dots,\zeta^{b-1})p_\lambda(z_1,\dots,z_n)$,

(b) $(-1)^k\sum_{|\lambda|=k} e_\lambda(\zeta,\dots,\zeta^{b-1})m_\lambda(z_1,\dots,z_n)$,

(c) $(-1)^k\sum_{|\lambda|=k} m_\lambda(\zeta,\dots,\zeta^{b-1})e_\lambda(z_1,\dots,z_n)$,

(d) $(-1)^k\sum_{|\lambda|=k} s_{\lambda'}(\zeta,\dots,\zeta^{b-1})s_\lambda(z_1,\dots,z_n)$.

Proof. For any variables $x_1,x_2,\dots,y_1,y_2,\dots$, the *dual Cauchy kernel* has the following expansions, where each sum is over all integer partitions λ [11, Equations 4.1,′, 4.2′, 4.3′]:

$$\begin{aligned}\prod_{i=1}^{\infty}\prod_{j=1}^{\infty}(1+x_iy_j) &= \sum_\lambda (-1)^{|\lambda|-l(\lambda)}z_\lambda^{-1}p_\lambda(x_1,x_2,\dots)p_\lambda(y_1,y_2,\dots)\\ &= \sum_\lambda e_\lambda(x_1,x_2,\dots)m_\lambda(y_1,y_2,\dots)\\ &= \sum_\lambda m_\lambda(x_1,x_2,\dots)e_\lambda(y_1,y_2,\dots)\\ &= \sum_\lambda s_{\lambda'}(x_1,x_2,\dots)s_\lambda(y_1,y_2,\dots),\end{aligned}$$

Now let ζ be a primitive bth root of unity. We can express the generating function for $h_k^{(b)}$ as

$$H^{(b)}(t;z_1,\dots,z_n) = \sum_k h_k^{(b)}(z_1,\dots,z_n)t^k$$

$$= \prod_{j=1}^{n} (1 + z_j t + \cdots + (z_j t)^{b-1})$$
$$= \prod_{i=1}^{b-1} \prod_{j=1}^{n} (1 - \zeta^i z_j t).$$

We can obtain this generating function by substituting

$$(x_1, x_2, \ldots) = (\zeta, \ldots, \zeta^{b-1}, 0, 0, \ldots),$$
$$(y_1, y_2, \ldots) = (-z_1 t, -z_2 t, \ldots, -z_n t, 0, 0, \ldots),$$

into the dual Cauchy kernel. Then to compute the coefficient of t^k we use the fact that

$$f_\lambda(-z_1 t, -z_2 t, \ldots, -z_n t) = (-t)^{|\lambda|} f_\lambda(z_1, \ldots, z_n)$$

for any homogeneous polynomial f_λ of degree $|\lambda|$. □

Proposition 17a is equivalent to Proposition 14 because Lemma 15c says

$$p_\lambda(\zeta, \ldots, \zeta^{b-1}) = (b-1)^{l_b(\lambda)} (-1)^{l(\lambda) - l_b(\lambda)} = (1-b)^{l_b(\lambda)} (-1)^{l(\lambda)}.$$

Recall from Theorem 5c that

$$e_k(\omega, \omega^2, \ldots, \omega^{b-1}) = \begin{cases} (-1)^k & k < b, \\ 0 & k \geqslant b. \end{cases}$$

It follows from this that

$$e_\lambda(\zeta, \ldots, \zeta^{b-1}) = \begin{cases} (-1)^{|\lambda|} & \lambda_i < b \text{ for all } i, \\ 0 & \text{else.} \end{cases}$$

Thus Proposition 17b just says that $h_k^{(b)}(z_1, \ldots, z_n) = \sum_\lambda m_\lambda(z_1, \ldots, z_n)$, where the sum is over partitions λ with $|\lambda| = k$ in which each part is less than b, which is obviously true.

The last two parts are more interesting. Proposition 17c appears in Doty and Walker [7, Remark 3.10(3)]. We do not know a general formula for the integers $m_\lambda(\zeta, \ldots, \zeta^{b-1})$. Yamaguchi et al. [19] give formulas for some special cases. If ω is a primitive dth root of unity then the specializations of Proposition 17c at $z_i \mapsto \omega^i$ and $z_i \mapsto \omega^{i-1}$ should involve an interesting mix of bth and dth roots of unity.

The coefficients $s_{\lambda'}(\zeta, \ldots, \zeta^{b-1})$ in Proposition 17d were studied by Grinberg. His main result [9, Corollary 2.9] says that

$$h_k^{(b)}(z_1, \ldots, z_n) = \sum_{|\lambda|=k} \mathrm{pet}_b(\lambda) s_\lambda(z_1, \ldots, z_n),$$

where $\mathrm{pet}_b(\lambda) \in \{0, -1, 1\}$ is the determinant of a certain $b \times b$ *Petrie matrix* that has entries in $\{0, 1\}$ and is determined by λ. Cheng et al. [6, Theorem 1.3] gave an expression for $\mathrm{pet}_b(\lambda)$ in terms of the b-core and b-quotient of the partition λ. When ω is a primitive dth root of unity with $d|n$, Reiner, Stanton and White [14, Proposition 4.3] gave an expression for $s_\lambda(1, \omega, \ldots, \omega^{n-1})$ in terms of the d-core and d-quotient of λ. It would be interesting to see how these two formulas interact, particularly in the case $\gcd(b, d) = 1$. Perhaps there is a nice connection to the theory of simultaneous core partitions.

Acknowledgements

The author thanks Seamus Albion, Darij Grinberg, Christian Krattenthaler, Daniel Provder, Vic Reiner, Brendon Rhoades, Bruce Sagan and Nathan Williams for helpful conversations, and Heather Armstrong for help with LaTeX.

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