A combinatorial method for log-concavity of rows of triangular arrays

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Abstract

Recurrences of the form

$$T(n,k) = c(n,k)\,T(n-1,k) + d(n,k)\,T(n-1,k-1) + \delta_{n,0}\delta_{k,0}$$

arise in a wide range of classical combinatorial sequences, including the Stirling numbers of both kinds, Lah numbers, and Eulerian numbers. We provide a combinatorial interpretation for arrays $T(n,k)$ defined by such recurrences and use this framework to study the log-concavity of rows of triangular arrays. Our approach yields new and uniform combinatorial proofs of log-concavity for many well-known combinatorial sequences.

Motivated by recent generalizations of these classical arrays, we further investigate recurrences of the form

$$T(n,k) = (\alpha n + \beta k + \gamma)^l\,T(n-1,k) + (\alpha' n + \beta' k + \gamma')^l\,T(n-1,k-1) + \delta_{n,0}\delta_{k,0},$$

where l is a positive integer. We establish sufficient conditions under which the rows of triangular arrays defined by such recurrences are log-concave. This result implies the log-concavity of several previously unexplored combinatorial arrays and, in particular, confirms a conjecture of Žigon Tankosič (*J. Integer Seq. 26*, 2 (2023), article 23.2.6, 16) on the log-concavity of generalized Lah numbers.

Our main technique is to interpret the triangular array $(T(n,k))$ in terms of weighted lattice paths and to construct a weight-increasing injection that proves log-concavity. Finally, we introduce a two-parameter generalization of the Eulerian numbers, analogous to the generalized Stirling and Lah numbers. We show that this array is palindromic in k and conclude with remarks on its γ-nonnegativity and real-rootedness.

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1 Introduction

Let $\mathbb{N}$ denote the non-negative integers and for $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$. A sequence of real numbers $(a_k)_{k=0}^{n}$ is said to be log-concave if $a_k^2 \geqslant a_{k+1}a_{k-1}$ for all indices $k \in [n-1]$. Log-concavity is a well studied property of finite sequences and it appears in various areas of mathematics such as combinatorics, probability and algebra. See the surveys of Stanley [18], Bränden [5] for a wealth of results on log-concavity.

A *triangular array* $T(n, k)$ is a collection of real numbers that satisfies a recurrence of the form

$$T(n, k) = c(n, k)\ T(n-1, k) + d(n, k)\ T(n-1, k-1) + \delta_{n,0}\delta_{k,0}, \quad \text{for } n, k \geqslant 0 \tag{1}$$

and $T(n, k) = 0$ for $n < 0, k < 0$, or $k > n$ for some $c(n, k), d(n, k) \in \mathbb{R}$ and $\delta_{i,j}$ is the Kronecker delta function. Several interesting enumerative sequences obey a recurrence of this form, including the binomial coefficients, $\binom{n}{k}$, the Stirling number of first and second kinds, $s(n, k)$ and $S(n, k)$, the Lah numbers, $L(n, k)$, and the Eulerian numbers, $A(n, k)$. These sequences and their generating functions are rich in combinatorial structure and well studied in the literature, in particular with respect to properties such as log-concavity, unimodality, gamma positivity and real-rootedness. Moreover, in all these examples the expression for $c(n, k)$ and $d(n, k)$ are affine functions in the variables n and k.

To prove the log-concavity of the Stirling numbers of both kinds, Kurtz [8] gave the following sufficient condition for triangular arrays.

Theorem 1 ([8, Theorem 2]). *Let $T(n, k) = c(n, k)\ T(n-1, k) + d(n, k)\ T(n-1, k-1)$ and*

- $2c(n, k) \geqslant c(n, k-1) + c(n, k+1)$,
- $2d(n, k) \geqslant d(n, k-1) + d(n, k+1)$,

then, the array $T(n, k)$ is log-concave in k, for all n.

A strengthening of the Theorem was obtained by Sagan [15] wherein the hypothesis of concavity of the coefficients was weakened to log-concavity of the coefficients and concavity of the product. The theorem is as follows.

Theorem 2 ([15, Theorem 1]). *Let $T(n, k)$ be an extended triangular array satisfying $T(n, k) = c(n, k)\ T(n-1, k) + d(n, k)\ T(n-1, k-1)$ for all $n \geqslant 1$ and $T(n, k), c(n, k), d(n, k)$ are all non-negative integers. Suppose*

- *$c(n, k), d(n, k)$ are log-concave in k,*
- $d(n, k-1)c(n, k+1) + d(n, k+1)c(n, k-1) \leqslant 2d(n, k)c(n, k)$,

then, the $T(n, k)$ are log-concave in k.

The main contribution of this work is a combinatorial interpretation of triangular arrays in terms of weighted lattice paths. This allows us to produce a weight increasing injection that proves log-concavity. The advantage of this method is two-fold. It provides a combinatorial transparency to log-concavity of several classical enumerative sequences. Further, it is applicable, in more generality, to triangular arrays where $c(n,k), d(n,k)$ are not necessarily linear in n and k.

Using this interpretation, we prove a sufficient condition for log-concavity of triangular arrays that is applicable to a large number of arrays. The main result of the work is the following.

Theorem 3. *For $n \geqslant 1$ and $0 < k \leqslant n$, let $T(n,k)$ satisfy the following triangular recurrence.*

$$T(n,k) = c(n,k)\ T(n-1,k) + d(n,k)\ T(n-1,k-1).$$

The sequence $(T(n,k))_{k=0}^{n}$ is log-concave if (i), (ii) and (iii) hold.

(i) $c(n,k), d(n,k)$ *are log-concave in* k.

(ii) Suppose

- $n_2 > n_1$; $l_2 > k_2 > k_1$; $l_2 > l_1 > k_1$,
- $k_2 - k_1 = l_2 - l_1 + 1$,
- $n_2 - n_1 \geqslant l_2 - k_2$ *and* $n_2 - n_1 \geqslant l_1 - k_1$.

Then,

$$c(n_1,k_1)d(n_1,k_2)d(n_2,l_1)c(n_2,l_2) \leqslant c(n_1,k_1+1)d(n_1,k_2-1)d(n_2,l_1+1)c(n_2,l_2-1).$$

(iii) For all n, k, *we have* $d(n,k+1)c(n,k) \leqslant d(n,k)c(n,k+1)$.

We use this theorem to study the log-concavity of arrays that satisfy "super-recurrence" relations of the form

$$T(n,k) = (\alpha n+\beta k+\gamma)^l\ T(n-1,k)+(\alpha' n+\beta' k+\gamma')^l\ T(n-1,k-1)+\delta_{n,0}\delta_{k,0}, \text{ for } n,k \geqslant 0, \tag{2}$$

and $T(n,k) = 0$ for $n < 0$, $k < 0$, or $k > n$, where $\alpha, \alpha', \beta, \beta', \gamma, \gamma' \in \mathbb{R}$, and $\delta_{i,j}$ is the Kronecker delta function.

The second main result of this work is the following.

Theorem 4. *Let α, β, γ and α', β', γ' be real numbers such that the following hold:*

(i) $\alpha, \alpha', \beta \geqslant 0$

(ii) $\beta' \in [-\alpha', 0]$

(iii) $\alpha + \beta + \gamma,\ \alpha' + \beta' + \gamma' \geqslant 0$

Then

$$T(n,k) = (\alpha n + \beta k + \gamma)^l \; T(n-1,k) + (\alpha' n + \beta' k + \gamma')^l \; T(n-1,k-1) \tag{3}$$

is log-concave in k *for all* n.

We give a list of combinatorial sequences that are $l = 1$ specialisations of the triangular arrays where Theorem 4 applies. Note that Theorem 1 and Theorem 2 are not applicable to these arrays when $l \geqslant 2$ but the rows of these arrays are, in fact, log-concave by Theorem 4 for all $l \in \mathbb{N}$, yielding the log-concavity of rows of a large family of triangular arrays.

$(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma};\ \boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\gamma}')$	**Name ($l = 1$) and Reference**	**OEIS Entry**
$(0,1,1;\nu,-1,1-\nu)$	ν-order Eulerian numbers [2]	A173018, A008517, A219512
$(0,j,1;\ j,-j,0)$	$1/j$-Eulerian numbers [16]	A185410
$(0,1,0;\ 0,0,1)$	Stirling subset numbers	A008277
$(1,1,-1;\ 0,0,1)$	Lah numbers	A015278
$(1,m,-1;\ 0,0,m)$	$m-$associated Lah numbers	
$(2,1,-2;\ 0,0,1)$	Generalisation of Lah, Stirling numbers [9]	A035342
$(0,1,r-1;\ 0,0,1)$	$S(r,n,k)$ [6]	A143494, A143495, A143496
$(0,0,1;\ 0,0,1)$	Binomial Coefficients	A007318
$(1,0,-1;\ 0,0,1)$	Stirling cycle numbers	A132393
$(1,2,-1;\ 3,-2,1)$	Scaled type A Narayana numbers [10]	
$(1,2,0;\ 3,-2,0)$	Scaled type B Narayana numbers [10]	
$(2,1,-1;\ 0,0,1)$	Holiday numbers of first kind [19]	
$(2,1,0;\ 0,0,1)$	Holiday numbers of second kind[19]	

Table 1: Some sequences of combinatorial interest satisfying Theorem 4. OEIS entries are presently unavailable for unfilled rows.

Recently, some of the enumerative sequences listed above have been generalized by introducing more parameters. Let $r \in \mathbb{N}$. The r-Stirling number of the first kind, $s_r(n,k)$ (with $n \geqslant k \geqslant r$), is defined to be the number of permutations in $\mathfrak{S}_n$ with k cycles such that $1, 2, \ldots, r$ are in different cycles. The r-Stirling number of the second kind, $S_r(n,k)$ (with $n \geqslant k \geqslant r$), is defined to be the number of partitions of $[n]$ into k blocks such that $1, 2, \ldots, r$ are in different blocks. These numbers have been studied in great detail, see [6, 12]. The closely related r-Lah number, $L_r(n,k)$ (with $n \geqslant k \geqslant r$), is defined to be the number of partitions of $[n]$ into k nonempty ordered subsets such that $1, 2, \ldots, r$ are in different ordered subsets. These were studied in [14, 17, 13]. The r-Stirling numbers and r-Lah numbers are also triangular arrays in the sense of (1), with coefficients that are affine functions in the variables.

These numbers have been further generalised with an additional parameter as follows (see [3, 20]). Let $l \in \mathbb{N}$. Define the *cycle leader* of a permutation $\pi \in \mathfrak{S}_n$, denoted $\mathrm{cl}(\pi)$, to be the set of minimum elements of each cycle in the cycle decomposition of π. The (l,r)-Stirling number of the first kind, $s_r^{(l)}(n,k)$, is defined to be the number of l-tuples of permutations $(\pi_1, \ldots, \pi_l)$ such that $[r] \subset \mathrm{cl}(\pi_1)$ and $\mathrm{cl}(\pi_1) = \cdots = \mathrm{cl}(\pi_l)$. Define the *block leader* of a partition Π of $[n]$, denoted $\mathrm{bl}(\Pi)$, to be the set of minimum elements of each block of the partition Π. The (l,r)-Stirling number of the second kind, $S_r^{(l)}(n,k)$, is defined similarly: it is the number of l-tuples of partitions $(\Pi_1,\dots,\Pi_l)$ such that $[r] \subset \mathrm{bl}(\Pi_1)$ and $\mathrm{bl}(\Pi_1) = \cdots = \mathrm{bl}(\Pi_l)$. The (l,r)-Lah number is defined in a similar manner to the (l,r)-Stirling number of the second kind. It counts the number of l-tuples of partitions $(\Pi_1,\dots,\Pi_l)$ into k linearly ordered blocks such that $[r] \subset \mathrm{bl}(\Pi_1)$ and $\mathrm{bl}(\Pi_1) = \cdots = \mathrm{bl}(\Pi_l)$.

The (l,r)-Stirling numbers and (l,r)-Lah numbers are also triangular arrays; in particular, equation (1) for these numbers take the following forms:

$$\begin{aligned} s_r^{(l)}(n,k) &= (n-1)^l s_r^{(l)}(n-1,k) + s_r^{(l)}(n-1,k-1) \\ S_r^{(l)}(n,k) &= k^l S_r^{(l)}(n-1,k) + S_r^{(l)}(n-1,k-1) \\ L_r^{(l)}(n,k) &= (n+k-1)^l L_r^{(l)}(n-1,k) + L_r^{(l)}(n-1,k-1) \end{aligned}$$

for $1 \leqslant r \leqslant k \leqslant n$. While it is easy to show that the (l,r)-Stirling numbers of both kinds are log-concave, it was conjectured in [20] that the (l,r)-Lah numbers are log-concave in k, which we affirm using Theorem 4.

Motivated by the definitions of the (l,r)-Stirling and (l,r)-Lah numbers, we generalise the Eulerian numbers to a two parameter (l,r)-Eulerian number. A subexceedant function is a function $f : [n] \to [n]$ such that $0 < f(i) \leqslant i$ for all $i \in [n]$. Dumont (see [11]) showed that the Eulerian number $\mathcal{A}(n,k)$ counts the number of subexceedant functions such that the cardinality of the image of f is $k+1$. Similar to the Stirling and Lah counterparts, we define the block leader of a subexceedant function. The block leader of a subexceedant function f is the set $\{i \in [n] : f(i) \notin f([i-1])\}$. Let us denote it by $\mathrm{bl}(f)$. Note that the $|\mathrm{bl}(f)|$ is the size of the image of f because for each element in the image of f the smallest element in the pre-image is in $\mathrm{bl}(f)$.

Let l be a positive integer. Define $\mathcal{A}_r^{(l)}(n,k)$ to count the number of l-tuples $(f_1,\dots,f_l)$ such that $[r] \subset \mathrm{bl}(f_1)$, $|\mathrm{bl}(f_1)| = k+1$, and $\mathrm{bl}(f_1) = \cdots = \mathrm{bl}(f_l)$. We show that there is a natural connection between the r-Stirling numbers and r-Eulerian numbers (setting $l = 1$), which is done by showing that the r-Eulerian numbers count the number of permutations in $\mathfrak{S}_n$ with k descents such that $[r]$ is subset of the set of first letters of the increasing runs in π. Finally, we show that the associated polynomials of the $(l,1)$-Eulerian numbers are palindromic and make a remark about their gamma-nonnegativity and real-rootedness.

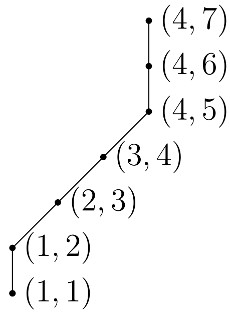
2 Definitions and Preliminaries

Suppose that $n \geqslant 1$ and $0 < k \leqslant n$, a triangular recurrence is a recurrence of the form

$$T(n,k) = c(n,k)\; T(n-1,k) + d(n,k)\; T(n-1,k-1)$$

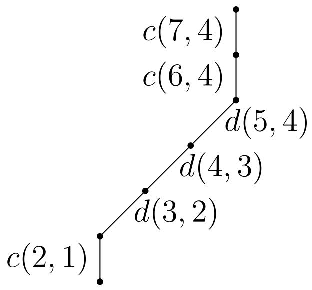
with boundary conditions $T(1,1) = 1$ and the coefficients $c(n,k), d(n,k)$ are non-negative real numbers.

For $A, B \in \mathbb{Z}^2$, we denote by $\mathcal{P}_{A\to B}$ the set of lattice paths starting with A and ending at B that use the step $N = (0,1)$ (north) and the step $C = (1,1)$ (cross). We can encode these paths as words in the alphabet $\{N, C\}$. For example, we write $p = NCCCNN$ for the lattice path below.



Since most of the paths in this paper will start from the point $(1, 1)$, we will write $\mathcal{P}_A$ instead of $\mathcal{P}_{(1,1)\to A}$.

We shall assign weights to the step of the lattice path. For a north step N that starts at $(k, n-1)$ and ends at (k, n), we assign the weight $c(n, k)$. For the cross step C that starts at $(k-1, n-1)$ and ends at (k, n), we assign the weight $d(n, k)$.



The weight of a path $p \in \mathcal{P}_{A\to B}$ is the product of the weights of the steps that make up the path. In our running example, it would be $c(2,1)d(3,2)d(4,3)d(5,4)c(6,4)c(7,4)$.

3 A log-concavity criterion for triangular arrays

3.1 Triangular arrays

We have the following interpretation of a triangular array in terms of weighted lattice paths:

Theorem 5. *For $n \geqslant 1$ and $0 < k \leqslant n$, let $T(n, k)$ satisfy the following triangular recurrence.*

$$T(n, k) = c(n, k)\ T(n-1, k) + d(n, k)\ T(n-1, k-1).$$

Then,

$$T(n, k) = \sum_{p \in \mathcal{P}_{(k,n)}} \mathrm{wt}(p),$$

where $\mathrm{wt}(p)$ *is defined as in Section 2.*

Proof. We prove by induction on n. Each path in $\mathcal{P}_{(k,n)}$ is either path in $\mathcal{P}_{(k,n-1)}$ followed by an N step or a path in $\mathcal{P}_{(k-1,n-1)}$ followed by a cross step C. Since the weight of a path is the product of weight of the constituting steps, the statement follows from the recurrence. □

Under this interpretation, we have the following interpretation of the log-concavity of rows.

Corollary 6. *The rows of the triangular array $T(n,k)$ are log-concave if and only if*

$$\sum_{(p,q)\in\mathcal{P}_{(k+1,n)}\times\mathcal{P}_{(k-1,n)}} \mathrm{wt}(p)\,\mathrm{wt}(q) \leqslant \sum_{(p',q')\in\mathcal{P}_{(k,n)}\times\mathcal{P}_{(k,n)}} \mathrm{wt}(p')\,\mathrm{wt}(q')$$

Proof. The inequality $T(n,k)^2 \geqslant T(n,k-1)T(n,k+1)$ is equivalent to

$$\left(\sum_{p\in\mathcal{P}_{(k,n)}} \mathrm{wt}(p)\right)^2 \geqslant \left(\sum_{p\in\mathcal{P}_{(k-1,n)}} \mathrm{wt}(p)\right)\left(\sum_{q\in\mathcal{P}_{(k+1,n)}} \mathrm{wt}(q)\right)$$

This can be written as

$$\sum_{(p',q')\in\mathcal{P}_{(k,n)}\times\mathcal{P}_{(k,n)}} \mathrm{wt}(p')\,\mathrm{wt}(q') \geqslant \sum_{(p,q)\in\mathcal{P}_{(k+1,n)}\times\mathcal{P}_{(k-1,n)}} \mathrm{wt}(p)\,\mathrm{wt}(q).$$

This completes our proof. □

3.2 An injection from $\mathcal{P}_{(k+1,n)} \times \mathcal{P}_{(k-1,n)}$ to $\mathcal{P}_{(k,n)} \times \mathcal{P}_{(k,n)}$

We shall describe an injective map that sends a pair of paths $(p,q) \in \mathcal{P}_{(k+1,n)} \times \mathcal{P}_{(k-1,n)}$ to a pair of paths $(p',q') \in \mathcal{P}_{(k,n)} \times \mathcal{P}_{(k,n)}$. The injection is based on an idea of Sagan [15] that is used to give injective proofs of log-concavity of the Stirling numbers of the first and second kinds.

Theorem 7 (Discrete Intermediate Value Theorem)**.** *Let f be an integer-valued function on the integers in the interval $[m,n]$. Suppose that $|f(i+1)-f(i)| \leqslant 1$ for $m \leqslant i \leqslant n$. If $f(m)f(n) < 0$, then $f(x) = 0$ for some integer $x \in [m,n]$.*

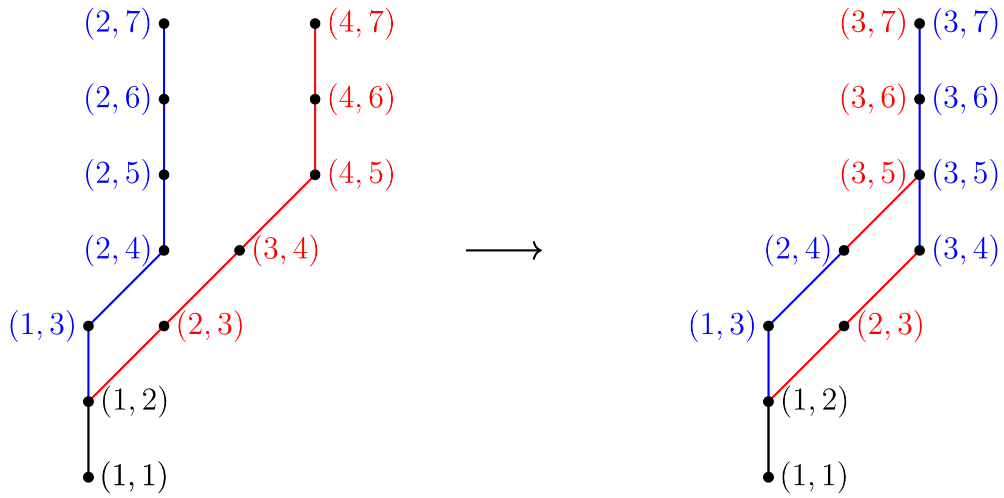
Proposition 8. *Let $(p,q) \in \mathcal{P}_{(k+1,n)} \times \mathcal{P}_{(k-1,n)}$ with $p = p_1 \dots p_{n-1}$ and $q = q_1 \dots q_{n-1}$. Then, there is an index $1 \leqslant i \leqslant n-1$ such that $p' = p_1 \dots p_i q_{i+1} \dots q_{n-1}$ and $q' = q_1 \dots q_i p_{i+1} \dots p_{n-1}$ are in $\mathcal{P}_{(k,n)} \times \mathcal{P}_{(k,n)}$.*

Proof. Let $N : [0, n-1] \to \mathbb{Z}$ be the function defined by $N(0) = -1$ and $N(i) = \#\{r \in [i] \mid p_r = C\} - \#\{r \in [i] \mid q_r = C\} - 1$ for $1 \leqslant i \leqslant n-1$. Here, $|N(i+1) - N(i)| = 1$ if either $p_{i+1} = C$ but $q_{i+1} \neq C$, or $p_{i+1} \neq C$ but $q_{i+1} = C$, and $|N(i+1) - N(i)| = 0$ otherwise. Also, $N(0) = -1$ and $N(n-1) = 1$ making $N(0)N(n-1) = -1 < 0$. Therefore, by Theorem 7, we have that $N(x) = 0$ for some $x \in [n-1]$. Since $N(n-1) - N(x) = 1$, we have that there is one more C step in the remaining subword of p than there is for q. Therefore, switching the subwords will have the intended effect. □

We are ready to define an injection I from $\mathcal{P}_{(k+1,n)} \times \mathcal{P}_{(k-1,n)}$ to $\mathcal{P}_{(k,n)} \times \mathcal{P}_{(k,n)}$. Given $(p,q) \in \mathcal{P}_{(k+1,n)} \times \mathcal{P}_{(k-1,n)}$ with $p = p_1 \dots p_{n-1}$ and $q = q_1 \dots q_{n-1}$, choose the largest index i such that $p' = p_1 \dots p_i q_{i+1} \dots q_{n-1}$ and $q' = q_1 \dots q_i p_{i+1} \dots p_{n-1}$ belongs to $\mathcal{P}_{(k,n)} \times \mathcal{P}_{(k,n)}$. Then $I(p,q) := (p', q')$. The map is well-defined because of the previous proposition.

Remark 9. A step in path $q \in \mathcal{P}_{k-1,n}$ after the index i is moved exactly 1 step to the right and a step that started at (k_1, n_1) would now start at (k_1+1, n_1) in the path $q' \in \mathcal{P}_{(k,n)}$. Similarly, any step in path $p \in \mathcal{P}_{k+1,n}$ would move exactly 1 step to the left after the index i. Any step that started at (k_2, n_1) would now start at $(k_2 - 1, n_1)$ in the path $p' \in \mathcal{P}_{(k,n)}$.

Example 10. Take the two paths $NNCNNN \in \mathcal{P}_{(2,7)}$ and $NCCCNN \in \mathcal{P}_{(4,7)}$. Under the injection, it would go to $NNCCNN$ and $NCCNNN$ in $\mathcal{P}_{(3,7)}$. The index i where the shift occurs is 3.



Define $\mathrm{wt}(p,q) := \mathrm{wt}(p)\,\mathrm{wt}(q)$ to be the weight of a pair (p,q). In the next section, we give sufficient conditions for ensuring that $\mathrm{wt}(p,q) \leqslant \mathrm{wt}\, I(p,q)$ for $(p,q) \in \mathcal{P}_{(k+1,n)} \times \mathcal{P}_{(k-1,n)}$.

3.3 Proofs of Theorems 3 and 4

Usually, a 2-Motzkin path of length n is a lattice path starting from $(0,0)$ and $(n,0)$ that uses an up step $U := (1,1)$, a down step $D := (1,-1)$, a blue horizontal step $H_1 := (1,0)$ and a red horizontal $H_2 := (1,0)$. However, we will use the name 2-Motzkin path to mean a lattice path with the steps described above but starting from $(0,0)$ and ending in $(n,-1)$ such that the last step is a down step D and except for the last step, the remaining lattice path lies entirely above the x-axis.

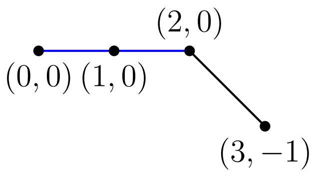
To obtain a sufficient condition for log-concavity, we will associate a weighted 2-Motzkin path to the pair of paths in $\mathcal{P}_{(k+1,n)} \times \mathcal{P}_{(k-1,n)}$ and use the weighted 2-Motzkin path to create a sufficient condition that ensures the weight of pair of paths increases under the injection defined in the previous subsection. Let $(p,q) \in \mathcal{P}_{(k+1,n)} \times \mathcal{P}_{(k-1,n)}$. Let $\mathrm{ch}(p) := p_{i+1} \dots p_{n-1}$ and $\mathrm{ch}(q) := q_{i+1} \dots q_{n-1}$ where i is largest index such that $p_1 \dots p_i\, \mathrm{ch}(q)$ and $q_1 \dots q_i\, \mathrm{ch}(p)$ are in $\mathcal{P}_{(k,n)}$.

The idea will be that the pair of paths p, q, when looked at from the end, start out with a spacing of 2 ($(k-1, n)$ and $(k+1, n)$), continue with a spacing greater than or equal to 2 and finally, getting down to a spacing of 1 at the index i prescribed by the injection. We will encode this portion of the paths as a 2-Motzkin path.

To this end, consider the reversal operator rev on words. We start with the two words $\mathrm{rev}(\mathrm{ch}(p))$ and $\mathrm{rev}(\mathrm{ch}(q))$. We build a weighted 2-Motzkin path out of the two words as follows. If the j^{th} steps of $\mathrm{rev}(\mathrm{ch}(p))$ and $\mathrm{rev}(\mathrm{ch}(q))$ are N and C respectively (resp. C and N or N and N or C and C), the j^{th} step of our weighted 2-Motzkin path is U (resp. D, H_1, H_2). The weight of the step in the 2-Motzkin path will be the product of the weight of the corresponding steps in p and q.

The fact that such a construction yields a weighted 2-Motzkin path follows the fact that i is the largest index such that $\mathrm{ch}(p)$ has one more C step than $\mathrm{ch}(q)$. The 2-Motzkin path will start at $(0,0)$ and end at $(n-i, -1)$. Except for the last D step, the entire path will be above the x-axis.

The 2-Motzkin path corresponding to our pair of paths is below. The blue steps correspond to H_1 steps.



Let us calculate the weight of the four different steps. The weight of U step is $c(n_1, k_1)d(n_1, k_2)$, and the D step is $d(n_1, k_1)c(n_1, k_2)$. The weight of an H_1 step is $c(n_1, k_1)c(n_1, k_2)$. The weight of an H_2 step is $d(n_1, k_1)d(n_1, k_2)$.

On a 2-Motzkin path, for each U step, there is a corresponding D step. That is if $(n_2, l_2) \in p$ and $(n_2, l_1) \in q$, there is a $(n_1, k_2) \in p$ and $(n_1, k_1) \in q$ such that the following are true.

- $n_2 > n_1$, $l_2 > k_2 > k_1$, $l_2 > l_1 > k_1$,
- $k_2 - k_1 = l_2 - l_1 + 1$,
- $n_2 - n_1 \geqslant l_2 - k_2$ and $n_2 - n_1 \geqslant l_1 - k_1$.

We are in a position to prove Theorem 3.

Proof of Theorem 3. The weight of a H_1 step is $c(n_1, k_1)c(n_1, k_2)$ which goes, under the injection, to a H_1 step whose weight is $c(n_1, k_1+1)c(n_1, k_2-1)$ (see Remark 9). Similarly, for H_2, $d(n_1, k_1)d(n_1, k_2)$ goes to $d(n_1, k_1+1)d(n_1, k_2-1)$. Therefore, Condition (i) ensures that $\mathrm{wt}(H_1) \leqslant \mathrm{wt}(I(H_1))$ and $\mathrm{wt}(H_2) \leqslant \mathrm{wt}(I(H_2))$.

Similarly, Condition (ii) ensures that the product of the weights of a U and its corresponding D step in the path is greater after applying the injection.

Finally, Condition (iii) ensures that the final D step is greater after applying the injection.

These three conditions ensure that

$$\sum_{(p,q)\in\mathcal{P}_{(k+1,n)}\times\mathcal{P}_{(k-1,n)}} \mathrm{wt}(p)\,\mathrm{wt}(q) \leqslant \sum_{(p',q')\in\mathcal{P}_{(k,n)}\times\mathcal{P}_{(k,n)}} \mathrm{wt}(p')\,\mathrm{wt}(q')$$

Therefore, the sequence $T(n,k)$ is log-concave in k. □

Remark 11. It should be possible to weaken Condition (*iii*) or completely remove it by sufficiently strengthening the other two conditions.

Remark 12. Note that Theorem 3 is applicable for many other triangular arrays and not just ones given by Equation 2. We give an example of such an application below.

The following is a quick application of this theorem. The Legendre-Stirling numbers were shown to be log-concave [1]. It is easy to verify that the coefficients satisfy the above theorem, therefore, giving us another proof of their log-concavity.

Corollary 13. *The Legendre-Stirling numbers defined by*

$$LS(n,k) = (k^2+k)LS(n-1,k) + LS(n-1,k-1) \tag{4}$$

are log-concave in k for all n.

The following is an observation that follows from the form of the conditions in Theorem 3.

Corollary 14. *For $c(n,k), d(n,k) \geqslant 0$, if the triangular array*

$$T(n,k) = c(n,k)T(n-1,k) + d(n,k)T(n-1,k-1)$$

satisfies the conditions of Theorem 3, then for $l \geqslant 1$, so does the array

$$\widetilde{T}(n,k) = c(n,k)^l\widetilde{T}(n-1,k) + d(n,k)^l\widetilde{T}(n-1,k-1).$$

Using the two aforementioned results, we can prove the main result of the paper.

Proof of Theorem 4. Condition (i) is easily verified. We insist that $\alpha, \alpha' \geqslant 0$. Otherwise, for all large n, we would have $\alpha n + \beta k + \gamma$ or $\alpha' n + \beta' k + \gamma'$ becoming negative. Similarly, we insist on $\alpha+\beta+\gamma, \alpha'+\beta'+\gamma' \geqslant 0$ for ensuring the non-negativity of $c(n,k), d(n,k)$. We want to show that conditions $(ii), (iii)$ of Theorem 4 are enough to apply Theorem 3.

Verifying Condition (ii):

$$\begin{aligned}
& c(n_1,k_1+1)c(n_2,l_2-1) - c(n_1,k_1)c(n_2,l_2) \\
= \; & (\alpha n_1 + \beta k_1 + \gamma + \beta)(\alpha n_2 + \beta l_2 + \gamma - \beta) - (\alpha n_1 + \beta k_1 + \gamma)(\alpha n_2 + \beta l_2 + \gamma) \\
= \; & \alpha\beta(n_2 - n_1) + \beta^2(l_2 - k_1 - 1) \geqslant 0
\end{aligned}$$

$$\begin{aligned}
& d(n_1,k_2-1)d(n_2,l_1+1) - d(n_1,k_2)d(n_2,l_1) \\
= \; & (\alpha' n_1 + \beta' k_2 + \gamma' - \beta')(\alpha' n_2 + \beta' l_1 + \gamma' + \beta') - (\alpha' n_1 + \beta' k_2 + \gamma')(\alpha' n_2 + \beta' l_1 + \gamma') \\
= \; & -\beta'(\alpha'(n_2 - n_1) + \beta'(l_1 - k_2 + 1)) \geqslant 0
\end{aligned}$$

The last line follows because $\alpha'(n_2 - n_1) \geqslant \alpha'(l_1 - k_1) \geqslant \alpha'(l_1 - k_2 + 1) \geqslant |\beta'|(l_1 - k_2 + 1)$ and so, $\alpha'(n_2 - n_1) + \beta'(l_1 - k_2 + 1) \geqslant 0$.

Verifying Condition (iii):

$$\frac{d(n_1, k+1)}{d(n_1, k)} \frac{c(n_1, k)}{c(n_1, k+1)} \leqslant 1$$

Since $\beta' \leqslant 0$, $d(n, k)$ increases as k decreases and since $\beta \geqslant 0$, $c(n, k)$ increases as k increases. □

Remark 15. The conditions $\alpha + \beta + \gamma \geqslant 0$ and $\alpha' + \beta' + \gamma' \geqslant 0$ exist to ensure that coefficients c, d are non-negative. One may increase γ, γ' by non-negative integers and retain log-concavity of the sequence.

Remark 16. Since the r-variants start at (r, r), we can make the substitution $c \to c + (a + b)(r - 1)$ and $c' \to c' + (a' + b')(r - 1)$ to make it a triangular array with the same values but starting at $(1, 1)$.

Corollary 17. *The (l, r)-Lah numbers are log-concave in k for all n.*

Proof. We can reparametrise to get the following recurrence for the (l, r)-Lah numbers:

$$T(n, k) = (n + k + 2r - 3)^l \; T(n - 1, k) + T(n - 1, k - 1) \tag{5}$$

with the initial condition $T(0, 0) = 1$. By Theorem 4, we have that this triangular array is log-concave in k for all n. □

4 Two parameter generalization of Eulerian numbers

Recall that we defined $\mathcal{A}_r^{(l)}(n, k)$ to count the number of l-tuples $(f_1, \ldots, f_l)$ such that $[r] \subset \mathrm{bl}(f_1)$, $|\mathrm{bl}(f_1)| = k + 1$, and $\mathrm{bl}(f_1) = \cdots = \mathrm{bl}(f_l)$. We first start out by proving a recurrence relation for the (l, r)-Eulerian numbers. They satisfy a triangular recurrence analogous to the ones satisfied by their Stirling and Lah counterparts.

Proposition 18. *For $n \geqslant k \geqslant r - 1 \geqslant 0$ and a positive integer l, we have*

$$\mathcal{A}_r^{(l)}(n, k) = (n - k)^l \mathcal{A}_r^{(l)}(n - 1, k - 1) + (k + 1)^l \mathcal{A}_r^{(l)}(n - 1, k) \tag{6}$$

and $\mathcal{A}_r^{(l)}(r, r) = 1$

Proof. Let f be a subexceedant function with $k + 1$ elements in $\mathrm{bl}(f)$ and $[r] \subset \mathrm{bl}(f)$. Then, f could have been obtained in one of the following two ways.

(i) if $n \in \mathrm{bl}(f_i)$, its image has one of $(n - k)^l$ choices,

(ii) if not, then its image has one of $(k + 1)^l$ choices.

□

Remark 19. By Theorem 4, the (l, r)-Eulerian numbers are log-concave and therefore, unimodal.

4.1 A bijection

Let $\pi \in \mathfrak{S}_n$. Define the block leaders of π, denoted by $\mathrm{bl}(\pi)$, as the set of first letters of the increasing runs of π. For example, let $\pi = 28|7|4|136|5$ with bars separating increasing runs. Then, $\mathrm{bl}(\pi) = \{1, 2, 4, 5, 7\}$. We show that $\mathcal{A}_r(n, k) := \mathcal{A}_r^{(1)}(n, k)$ also counts the number of permutations $\pi \in \mathfrak{S}_n$ with k descents such that $[r] \subset \mathrm{bl}(\pi)$.

Starting from a subexceedant function f that satisfies $[r] \subset \mathrm{bl}(f)$ and $|\,\mathrm{bl}(f)| = k+1$, we give a map Λ to a permutation π with k descents such that $[r] \subset \mathrm{bl}(\pi)$ with the additional property that $f(\mathrm{bl}(f)) = \mathrm{bl}(\pi)$.

We give the bijection Λ inductively. Start with the empty word and $i = 1$. If $f(i) = i$, add i to the left of all existing letters. If $f(i) \in f([i-1])$, then add i at the end of the increasing run starting at $f(i)$. If $f(i) \notin f([i-1])$, then add i immediately to the left of element $f(i)$. Increase i to $i+1$ and repeat.

Example 20. Starting with the subexceedant function $f = 12121547$, we show the steps.

$$\begin{aligned} \phi &\to 1 \to 21 \to 213 \to 2413 \to 24135 \\ &\to 241365 \to 2741365 \to 28741365 \end{aligned}$$

Therefore, $\Lambda(12121547) = 28741365$. Note that $f(\mathrm{bl}(f)) = \{1, 2, 4, 5, 7\} = \mathrm{bl}(28741365)$.

We can reconstruct the subexceedant function given a permutation π as follows. For $1 \leqslant i \leqslant n$, let $\pi|_i$ be the permutation with all letters greater than i removed. In our running example, $\pi|_5 = 24135$. If i appears to the left of all elements in $\pi|_i$, then $f(i) = i$. If i appears in front of some element $j \notin f([i-1])$, then $f(i) = j$. Else, $f(i)$ is the closest element from $f([i-1])$ to the left of i.

Example 21. Start with $\pi = 28741365$. Here, $\pi|_1 = 1$, $\pi|_2 = 21$, $\pi|_3 = 213$, $\pi|_4 = 2413$, $\pi|_5 = 24135$, $\pi|_6 = 241365$, $\pi|_7 = 2741365$ and $\pi|_8 = 28741365$. It is easy to check that $f(1) = 1, f(2) = 2$. $f(3) = 1$ because 1 the closest element in $f([2])$ to the left of 3. $f(4) = 2$, $f(5) = 1$, in the same way. $f(6) = 5$ because $5 \notin f([5])$. Similarly, $f(7) = 4$ and $f(8) = 7$.

Proposition 22. *The map Λ from the set of subexceedant functions from $[n]$ to $[n]$ such that $[r] \subset \mathrm{bl}(f)$, $|\,\mathrm{bl}(f)| = k+1$ to the set of permutations on $[n]$ such that $[r] \subset \mathrm{bl}(\pi)$, $|\,\mathrm{bl}(\pi)| = k+1$ is a bijection such that $f(\mathrm{bl}(f)) = \mathrm{bl}(\Lambda(f))$.*

Proof. It is clear that the function Λ is a bijection as we are able to describe its inverse. It remains to show that $f(\mathrm{bl}(f)) = \mathrm{bl}(\Lambda(f))$. Suppose $f(i) \notin f([i-1])$, then i goes behind $f(i)$, making $f(i)$ the first element of an increasing run. Similarly, the elements i such that $f(i) = i$ are added to the leftmost position and therefore, are first elements of an increasing run. Therefore, the block leaders of $\Lambda(f)$ are exactly the elements of $f(\mathrm{bl}(f))$. $\square$

We have the following connection between the r-Stirling number of the second kind and $\mathcal{A}_r(n, k)$. When $r = 1$, this identity specialises to Bóna [4, Theorem 1.18].

Theorem 23. *For $n \geqslant k \geqslant r-1 \geqslant 0$, we have*

$$k!S_r(n,k) = \sum_{i=r}^{k} \mathcal{A}_r(n,i-1)\binom{n-i}{k-i}. \tag{7}$$

Proof. The LHS is the number of ordered partitions Π of $[n]$ into k blocks such that $[r] \subset \mathrm{bl}(\Pi)$. We need to show that RHS counts the same objects. Let $\pi \in \mathfrak{S}_n$ with $i-1$ descents and $[r] \subset \mathrm{bl}(\pi)$. The i increasing runs of π give us an ordered partition of $[n]$. Now, the number of blocks is i, which can be smaller than k. However, we can make more blocks by making splits of the increasing runs. We need to make $k-i$ splits. This can be done by choosing $k-i$ of the $n-i$ different places where we can split.

We just need to now show that the ordered partitions thus formed are all distinct. However, given an ordered partition Π of $[n]$, we can arrange the blocks in ascending order internally and read from left to right a unique permutation and the choice of splitting from which it can be obtained. $\square$

Corollary 24. *For $n \geqslant k \geqslant r-1 \geqslant 0$, we have*

$$\mathcal{A}_r(n,k-1) = \sum_{i=1}^{k}(-1)^{k-i}S_r(n,i)\binom{n-i}{k-i}i! \tag{8}$$

Proof. Equation (7) can be rewritten as $\sum_{k=r}^{n} \mathcal{A}_r(n,k-1)(t+1)^{n-k} = \sum_{k=r}^{n} k!S_r(n,k)t^{n-k}$. The corollary follows from the substitution $t \to t-1$. $\square$

4.2 Palindromicity of the associated generalised Eulerian polynomials

In this section, let us restrict our attention to the case when $r=1$. We define associated $(l,1)$-Eulerian polynomials to be

$$\mathcal{A}_n^{(l)}(t) := \sum_{k=0}^{n-1} \mathcal{A}^{(l)}(n,k)t^k. \tag{9}$$

It is not immediately clear that this polynomial is palindromic. To make the symmetry manifest, we define

$$\mathcal{A}_n^{(l)}(s,t) := \sum_{k=0}^{n-1} \mathcal{A}^{(l)}(n,k)s^k t^{n-1-k}. \tag{10}$$

We show that the polynomial $\mathcal{A}_n^{(l)}(s,t)$ is symmetric in s,t, i.e., $\mathcal{A}_n^{(l)}(s,t) = \mathcal{A}_n^{(l)}(t,s)$. Let $\mathcal{D}_s = s\frac{\partial}{\partial s}$ and $\mathcal{D}_t = t\frac{\partial}{\partial t}$ in $\mathbb{Q}[s,t]$. The symmetry becomes apparent from the following proposition.

Proposition 25. *The polynomials $\mathcal{A}_n^{(l)}(s,t)$ satisfy the following partial differential equation.*

$$\mathcal{A}_n^{(l)}(s,t) = \frac{1}{s}\mathcal{D}_s^l(st\mathcal{A}_{n-1}^{(l)}(s,t)) + \frac{1}{t}\mathcal{D}_t^l(st\mathcal{A}_{n-1}^{(l)}(s,t)) \tag{11}$$

with $\mathcal{A}_0^{(l)}(s,t) = 1$ initially.

Proof.

$$\begin{aligned}\mathcal{D}_s^l(st\mathcal{A}_{n-1}^{(l)}(s,t)) &= t\mathcal{D}_s^l(s\mathcal{A}_{n-1}^{(l)}(s,t)) \\ &= t\Big(\mathcal{D}_s^l(\sum_{k=0}^{n-1}\mathcal{A}^{(l)}(n-1,k)s^{k+1}t^{n-1-k})\Big) \\ &= \sum_{k=0}^{n-1}(k+1)^l\mathcal{A}^{(l)}(n-1,k)s^{k+1}t^{n-k}\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{D}_t^l(st\mathcal{A}_{n-1}^{(l)}(s,t)) &= s\mathcal{D}_t^l(t\mathcal{A}_{n-1}^{(l)}(s,t)) \\ &= s\Big(\mathcal{D}_t^l(\sum_{k=0}^{n-1}\mathcal{A}^{(l)}(n-1,k)s^{k}t^{n-k})\Big) \\ &= \sum_{k=0}^{n-1}(n-k)^l\mathcal{A}^{(l)}(n-1,k)s^{k+1}t^{n-k}\end{aligned}$$

Therefore, the RHS of the proposition evaluates the polynomial

$$\sum_{k=0}^{n-1}(k+1)^l\mathcal{A}^{(l)}(n-1,k)s^{k}t^{n-k}+\sum_{k=0}^{n-1}(n-k)^l\mathcal{A}^{(l)}(n-1,k)s^{k+1}t^{n-1-k}$$

Comparing coefficients with (10) gives us the result. $\square$

A palindromic polynomial $p(x)=\sum_{k=0}^{n}a_kx^k$ is said to be gamma-positive of center of symmetry $\lfloor\frac{n}{2}\rfloor$ if it can be written in the form $p(x)=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\gamma_{n,i}t^i(1+t)^{n-2i}$. In light of the palindromicity of these polynomials, it would be the next natural question to ask if they are gamma-positive. It is not difficult to show that these polynomials are gamma-positive when $l=2$. The following is recurrence from Proposition (25) for the polynomials written out.

Proposition 26. *The polynomials $\mathcal{A}_n^{(2)}(s,t)$ satisfy the recurrence relation*

$$\mathcal{A}_n^{(2)}(s,t)=(s+t)\mathcal{A}_{n-1}^{(2)}(s,t)+3st(\frac{\partial}{\partial s}+\frac{\partial}{\partial t})\mathcal{A}_{n-1}^{(2)}(s,t)+st(s\frac{\partial^2}{\partial s^2}+t\frac{\partial^2}{\partial t^2})\mathcal{A}_{n-1}^{(2)}(s,t) \quad (12)$$

with $\mathcal{A}_0^{(2)}(s,t)=1$ initially.

It is straightforward to show that

$$s\frac{\partial^2}{\partial s^2}+t\frac{\partial^2}{\partial t^2}$$

sends an element of the gamma-basis $(st)^i(s+t)^{n-1-i}$ to a positive linear combination of gamma basis elements as do the other two terms. Since these are linear operators, we have that the polynomials are gamma-positive by induction. If we set $s=1$, we get

Proposition 27. *The polynomials* $\mathcal{A}_n^{(2)}(t) := \sum_{k=0}^{n-1} \mathcal{A}^{(2)}(n,k)t^k$ *are gamma-positive for all* $n \geqslant 2$.

Here are the polynomials $\mathcal{A}_n^{(3)}(t)$ and their expansion in the gamma-basis.

$$\begin{aligned}
\mathcal{A}_3^{(3)}(t) &= 1+16t+t^2 = (1+t)^2+14t\\
\mathcal{A}_4^{(3)}(t) &= 1+155t+155t^2+t^3 = (1+t)^3+152t(1+t)\\
\mathcal{A}_5^{(3)}(t) &= 1+1304t+8370t^2+1304t^3+t^4 = (1+t)^4+1300t(1+t)^2+5764t^2\\
\mathcal{A}_6^{(3)}(t) &= 1+10557t+309446t^2+309446t^3+10557t^4+t^5\\
&= (1+t)^5+10552t(1+t)^3+277780t^2(1+t)
\end{aligned}$$

Remark 28. We know that the Eulerian polynomials are gamma-positive and we have proved that the $(2,1)$-Eulerian polynomials are gamma-positive. Based on the limited data that we were able to compute, the $(l,1)$-Eulerian polynomials seem to be gamma-positive for all $l \geqslant 3$.

Remark 29. A similar remark can be made about real-rootedness. The Eulerian polynomials are known to be real rooted due to Frobenius [7]. From our limited data, it appears that the $(l,1)$-Eulerian polynomials are real rooted for $l \geqslant 2$.

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