Quantum Chromatic Number of Subgraphs of Orthogonality Graphs and the Distance-2 Hamming Graph

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Abstract

The determination of the quantum chromatic number of graphs has attracted considerable attention recently. However, there are few families of graphs whose quantum chromatic numbers are determined. A notable exception is the family of orthogonality graphs, whose quantum chromatic numbers are fully determined. In this paper, we extend these results by determining the exact quantum chromatic number of several subgraphs of the orthogonality graphs. Using the technique of combinatorial designs, we also determine the quantum chromatic number of the distance-2 Hamming graph, whose edges consist of binary vectors of Hamming distance 2, for infinitely many lengths.

Mathematics Subject Classifications: 05C15, 05B30

1 Introduction

A *c-coloring* of a graph G with vertex set V is a map $V \to [c]$, where $[c] \coloneqq \{1, 2, \dots, c\}$, such that adjacent vertices receive distinct colors in $[c]$. The *chromatic number* $\chi(G)$ of G is the smallest c such that G admits a c-coloring, which has been thoroughly studied in the literature [11, Chapter 5]. In the past decades, motivated by the study in quantum information theory [1, 5], the c-coloring game of graphs was proposed, which can be thought of as a quantum generalization of the classical graph coloring if the players are accessible to quantum resources [6, 10].

In the c-coloring game of the graph G, two players, Alice and Bob, must convince a referee that they possess a c-coloring of G. Both of the players receive a vertex $v, w \in V$

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chosen randomly by the referee, respectively, and must then respond with a color $c_v, c_w \in [c]$ to the referee. During the game, Alice and Bob are not allowed to communicate with each other, but they are allowed to share classical or quantum resources and agree on some protocol a priori, which may depend on G. The referee is convinced that Alice and Bob possess a c-coloring of G, if $c_v = c_w$ when $v = w$ and $c_v \neq c_w$ when v and w are adjacent. While classical players can win the c-coloring game of G with probability 1 if and only if a classical c-coloring of G exists, for particular graphs, quantum players, who have access to a shared entangled quantum state and may perform local measurements on the shared entangled state, may still win with probability 1 even if classical c-coloring of G does not exist. The smallest number c for which a quantum winning protocol exists is called the *quantum chromatic number* of G, denoted as $\chi_q(G)$ [6, 20]. Notably, there exist families of graphs for which $\chi_q(G)$ is strictly smaller than $\chi(G)$, demonstrating a quantum advantage [8, 16, 21]. The quantum chromatic number is also related to other important parameters of graphs [20] or objects in quantum information theory [23].

In general, it is NP hard to determine the quantum chromatic number of a graph [14]. To our knowledge, there are only few classes of graphs whose quantum chromatic number was determined. For example, the complete graph on n vertices has quantum chromatic number n [6]; bipartite graphs have quantum chromatic number 2 [6]; $\chi(G) = 3$ if and only if $\chi_q(G) = 3$ [6], in particular, odd cycles have quantum chromatic number 3. Beyond these special graphs, the *orthogonality graph* $O_{n,p}$ is an important class of graphs studied in this topic, whose vertices are vectors of length n with entries the p-th roots of unity, and two vectors are adjacent if and only if they are orthogonal in $\mathbb{C}^n$. When p is a prime and $p \mid n$, $O_{n,p}$ is isomorphic to the Cayley graph over $\mathbb{Z}_p^n$ with generating set consisting of all balanced vectors in $\mathbb{Z}_p^n$. In particular, when $p = 2$, both the graph $O_{4t,2}$ (also known as the Hadamard graph) and one of its subgraphs have quantum chromatic number $4t$ [21]. For general p, Cao et al. [7] recently determined the quantum chromatic number of $O_{lp,p}$ when l is large enough with respect to p and $l(p-1)$ is even.

Motivated by the above results, we study the quantum chromatic number of subgraphs of orthogonality graphs. In particular, we determine the quantum chromatic numbers of five classes of subgraphs of the orthogonality graphs $O_{4t,2}$ and $O_{3l,3}$. Further, using the technique of combinatorial designs, we also determine the quantum chromatic number of the distance-2 Hamming graph, whose edges correspond to pairs of binary vectors of Hamming distance 2, that is, the Cayley graph over $\mathbb{Z}_2^n$ generated by all vectors of Hamming weight 2 for infinitely many n. The results mentioned above in the literature and in this paper are summarized in Table 1 for readers' convenience.

The paper is organized as follows. Section 2 introduces necessary preliminaries on quantum chromatic numbers and orthogonality graphs. In Section 3, we study the spectrum of the orthogonality graph $O_{3l,3}$, which will be used to determine the quantum chromatic number of its subgraphs. In Section 4, we determine the quantum chromatic number of five classes of subgraphs of $O_{4t,2}$ and $O_{3l,3}$. In Section 5, with the technique of combinatorial designs, we determine the quantum chromatic number of the distance-2 Hamming graph, that is, the Cayley graph over $\mathbb{Z}_2^n$ generated by vectors of Hamming weight 2 for certain n. Finally, we conclude this paper in Section 6 and discuss open

Table 1: Known Quantum Chromatic Numbers of Graphs

Graphs G	$\chi_q(G)$	References
K_n (the complete graph on n vertices)	n	[6]
bipartite graphs	2	[6]
graphs with $\chi(G) = 3$	3	[6]
$O_{4,4}$	4	[6]
$O_{4t,2} \cong \mathrm{Cay}(\mathbb{Z}_2^{4t}, (2t, 2t))$	$4t$	[8, 21]
$\mathrm{Cay}(\mathbb{Z}_2^{4t-1}, (2t-1, 2t))$	$4t$	[8]
$\mathrm{Cay}(\mathbb{Z}_p^{lp}, (l, l, \dots, l))$, p is an integer, l is large enough and $l(p-1)$ is even	lp	[7]
$\mathrm{Cay}(\mathbb{F}_q^{q^l}, (q^{l-1}, q^{l-1}, \dots, q^{l-1}))$, q is a prime power	q^l	[7]
$O_{3l,3} \cong \mathrm{Cay}(\mathbb{Z}_3^{3l}, (l, l, l))$	$3l$	Theorem 4
$\mathrm{Cay}(\mathbb{Z}_2^{4t-1}, (2t, 2t-1) \cup (2t-1, 2t))$	$4t$	Theorem 7
$\mathrm{Cay}(\mathbb{Z}_2^{4t-2}, (2t-1, 2t-1) \cup (2t-2, 2t))$	$4t$	Theorem 7
$\mathrm{Cay}(\mathbb{Z}_3^{3l-1}, (l-1, l, l) \cup (l, l-1, l) \cup (l, l, l-1))$	$3l$	Theorem 7
$\mathrm{Cay}(\mathbb{Z}_p^{lp-1}, (l-1, \dots, l) \cup \cdots \cup (l, \dots, l-1))$	lp	Theorem 7
$\mathrm{Cay}(\mathbb{Z}_3^{3l-1}, (l-1, l, l))$	$3l$	Theorem 8
$\mathrm{Cay}(\mathbb{Z}_3^{3l-2}, (l-2, l, l) \cup (l-1, l-1, l) \cup (l-1, l, l-1))$	$3l$	Theorem 10
$\mathrm{Cay}(\mathbb{Z}_2^{q}, (q-2, 2))$, $q \equiv 3 \pmod 4$ is a prime power	$q+1$	Table 2
$\mathrm{Cay}(\mathbb{Z}_2^{2^{t+2}-1}, (2^{t+2}-3, 2))$	2^{t+2}	Table 2
$\mathrm{Cay}(\mathbb{Z}_2^{q^2+2q}, (q^2+2q-2, 2))$, q and $q+2$ are both odd prime powers	q^2+2q+1	Table 2

[1] In the table, $\mathrm{Cay}(\mathbb{Z}_2^{4t}, (2t, 2t))$ is the Cayley graph over $\mathbb{Z}_2^{4t}$ with a generating set consisting of vectors of type $(2t, 2t)$ (see Section 2.1); Similar for other Cayley graphs.

problems.

2 Preliminaries

In this section, we introduce the notions and related results about the quantum chromatic number of orthogonality graphs.

Let $G = (V, E)$ be a simple graph. The quantum chromatic number of G was motivated by quantum information theory and was originally formulated in the framework of non-local games as described in the Introduction (also see [6] and the references therein). We adopt the mathematical definition of quantum chromatic numbers as in [12, 20, 23]. A *quantum c-coloring* of G is a collection of orthogonal projections $P_{v,i} : \mathbb{C}^d \to \mathbb{C}^d$ for $v \in V$ and $i \in [c]$ satisfying the following two conditions:

(i) for all $v \in V$, $\sum_{i \in [c]} P_{v,i} = I_d$, and

(ii) for all adjacent vertices $v \sim w$ and all $i \in [c]$, $P_{v,i}P_{w,i} = 0$.

The *quantum chromatic number* $\chi_q(G)$ of G is the smallest c such that G admits a quantum c-coloring for some dimension $d \geqslant 1$.

A homomorphism from a graph $G = (V, E)$ to a graph $H = (V', E')$ is a map $f : V \to V'$ such that if $\{u, v\} \in E$, then $\{f(u), f(v)\} \in E'$. It is known that [6] if there is a homomorphism of graphs $G \to H$, then

$$\chi_q(G) \leqslant \chi_q(H). \tag{1}$$

In particular, if G is a subgraph of H, then $\chi_q(G) \leqslant \chi_q(H)$.

2.1 Orthogonality Graphs

Let $p \geqslant 2$ be an integer and let $\zeta_p := e^{2\pi\sqrt{-1}/p}$ be a primitive p-th root of unity. Let $U_p := \left\{\zeta_p^i : i \in \mathbb{Z}_p\right\}$ be the group generated by ζ_p, where $\mathbb{Z}_p$ denotes the ring of integers modulo p. The *orthogonality graph* $O_{n,p}$ has vertex set U_p^n, and two vertices $\mathbf{v}, \mathbf{w} \in U_p^n$ are adjacent if and only if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal with respect to the standard Hermitian inner product on $\mathbb{C}^n$.

Let G be a graph with vertex set V. An *orthogonal representation* of G is a map $\phi : V \to \mathbb{C}^d$ for some $d \geqslant 1$, such that $\phi(v)$ and $\phi(w)$ are orthogonal whenever v and w are adjacent in G. The orthogonal representation ϕ is *flat* if for all $v \in V$, each component of $\phi(v)$ is of the same modulus. It is known that if G admits a flat orthogonal representation $\phi : V \to \mathbb{C}^d$, then one can construct a quantum d-coloring of G from this representation [19], and thus $\chi_q(G) \leqslant d$.

The orthogonality graph $O_{n,p}$ is naturally equipped with a flat orthogonal representation over $\mathbb{C}^n$. Thus,

$$\chi_q(O_{n,p}) \leqslant n. \tag{2}$$

For the lower bound of $\chi_q(O_{n,p})$, there exists a spectral lower bound from [12],

$$\chi_q(G) \geqslant 1 - \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)}, \tag{3}$$

where $\lambda_{\max}(G)$ and $\lambda_{\min}(G)$ are the largest and smallest eigenvalues of G, respectively.

To compute $\lambda_{\max}$ and $\lambda_{\min}$ of $O_{n,p}$, we consider it as Cayley graphs over $\mathbb{Z}_p^n$, whose eigenvalues can be expressed conveniently in terms of characters of abelian groups. For $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{Z}_p^n$, the *type* of $\mathbf{v}$ is the sequence $T(\mathbf{v}) := (t_0, t_1, \ldots, t_{p-1})$ given by

$$\mathrm{wt}_i(\mathbf{v}) := t_i := \left|\{j \in [n] : v_j = i\}\right|, \quad i \in \mathbb{Z}_p. \tag{4}$$

It is clear that $T(\mathbf{v})$ is an ordered non-negative p-partition of n. For any such partition $T = (t_0, \ldots, t_{p-1})$ of n, we denote by $\mathbb{Z}_p^n(t_0, \ldots, t_{p-1})$ the set of vectors in $\mathbb{Z}_p^n$ of type T. When $p \mid n$, vectors of type $(n/p, n/p, \ldots, n/p)$ are called *balanced* vectors. Under these notations, when p is a prime and $p \mid n$,

$$O_{n,p} \cong \mathrm{Cay}(\mathbb{Z}_p^n, \mathbb{Z}_p^n(n/p, n/p, \ldots, n/p)). \tag{5}$$

Note that when p is prime and $p \nmid n$, $O_{n,p}$ is empty.

When $p = 2$, if $n \equiv 2 \pmod 4$, then $O_{n,2}$ is bipartite, and thus $\chi_q(O_{n,2}) = \chi(O_{n,2}) = 2$ [6]. If $n \equiv 0 \pmod 4$, then $\chi_q(O_{n,2}) = n$ [6,8] while $\chi(O_{n,2})$ is exponential in n, showing the advantage of quantum entanglement in the coloring game of $O_{n,2}$.

When $p = 3$, if $3 \mid n$, the quantum chromatic number of $O_{n,3}$ was shown to be n in [7] for large n. In fact, the authors of [7] showed $\chi_q(\mathrm{Cay}(\mathbb{Z}_p^n, \mathbb{Z}_p^n(n/p, \ldots, n/p))) = n$ for every integer p and large n satisfying $p \mid n$ and $(p-1)n/p$ even. Since $\mathrm{Cay}(\mathbb{Z}_p^n, \mathbb{Z}_p^n(n/p, \ldots, n/p))$ is a subgraph of $O_{n,p}$ for non-prime p, by Equation (1), we know that $\chi_q(O_{n,p}) = n$ for large n satisfying $p \mid n$ and $(p-1)n/p$ even.

In this paper, we focus on determining the quantum chromatic number of several classes of subgraphs of $O_{n,p}$ with $p = 2, 3$. The upper bound is trivially n, while the main work is to give the matching spectral lower bound by determining the smallest eigenvalue of $O_{n,p}$, which is the one with the second largest absolute value.

2.2 Eigenvalues of Cayley Graphs

We recall the eigenvalues of a Cayley graph. Suppose $G = \mathrm{Cay}(A, S)$ is a Cayley graph over a finite abelian group A generated by an inversion-closed subset $S \subset A \setminus \{0\}$. For $a \in A$, we denote by $\chi_a : A \to \mathbb{C}^*$ the character of A corresponding to a [18, Chapter 5]. Then all eigenvalues of $G = \mathrm{Cay}(A, S)$ are given by (see [4, Chapter 1])

$$\lambda(a) \coloneqq \sum_{s \in S} \chi_a(s), \ a \in A. \tag{6}$$

For Cayley graphs over $\mathbb{Z}_p^n$, it is known that all the characters are [18, Chapter 5]

$$\chi_{\mathbf{v}} : \mathbb{Z}_p^n \to \mathbb{C}^*, \quad \mathbf{x} \mapsto \zeta_p^{\mathbf{v}\cdot\mathbf{x}}, \tag{7}$$

where $\mathbf{v} = (v_1, \ldots, v_n), \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}_p^n$ and $\mathbf{v} \cdot \mathbf{x} = \sum_{i=1}^n v_i x_i$.

2.2.1 $p = 2$

For simplicity, denote $L_r \coloneqq \mathbb{Z}_2^n(n-r, r)$ for $r \in [0, n]$, i.e., the set of vectors in $\mathbb{Z}_2^n$ of Hamming weight r. Consider $\mathrm{Cay}(\mathbb{Z}_2^n, L_r)$. For $\mathbf{v} \in \mathbb{Z}_2^n$ with $\mathrm{wt}(\mathbf{v}) = w$, by Equation (6), the eigenvalue corresponding to $\mathbf{v}$ is given by

$$\begin{aligned} \lambda(\mathbf{v}) &= \sum_{\mathbf{b} \in L_r} (-1)^{\mathbf{v}\cdot\mathbf{b}} = \sum_j (-1)^j \binom{w}{j}\binom{n-w}{r-j} \\ &= \left[x^{n-r}y^r\right]\left((x+y)^{n-w}(x-y)^w\right) \coloneqq K_r(w). \end{aligned} \tag{8}$$

In Equation (8), $[x^{n-r}y^r]\, g(x,y)$ stands for the coefficient of the monomial $x^{n-r}y^r$ in $g(x,y)$, and $K_r(w)$ is known as the *Krawtchouk polynomial* [3]. So the eigenvalue $\lambda(\mathbf{v})$ depends only on the Hamming weight w of $\mathbf{v}$, and $K_r(w)$, $w \in [0, n]$ are all eigenvalues of $\mathrm{Cay}(\mathbb{Z}_2^n, L_r)$ without counting multiplicities. The largest eigenvalue of $\mathrm{Cay}(\mathbb{Z}_2^n, L_r)$ is given by

$$\lambda(\mathbf{0}) = K_r(0) = |L_r| = \binom{n}{r} = (-1)^r K_r(n).$$

In particular, $K_r(0)$ and $K_r(n)$ are eigenvalues of the largest absolute value. Moreover, when r is odd, $K_r(n) = -\binom{n}{r}$ is the smallest eigenvalue of $\mathrm{Cay}(\mathbb{Z}_2^n, L_r)$. For even $r \geqslant \frac{n}{2}$, the smallest eigenvalue of $\mathrm{Cay}(\mathbb{Z}_2^n, L_r)$ has been determined in [3]. When $n \equiv 0 \pmod 4$ and $r = \frac{n}{2}$, the spectral lower bound shows $\chi_q(O_{n,2}) \geqslant n$, and hence $\chi_q(O_{n,2}) = n$ [21].

2.2.2 $p = 3$

Let $n = 3l$. For simplicity, let $B_3 \coloneqq \mathbb{Z}_3^n(l,l,l)$ be the set of balanced vectors in $\mathbb{Z}_3^n$. Then $O_{n,3} \cong \mathrm{Cay}(\mathbb{Z}_3^n, B_3)$. For each $\mathbf{v} \in \mathbb{Z}_3^n$ and $a \in \mathbb{Z}_3$, let

$$B_3(\mathbf{v}, a) \coloneqq \{\mathbf{b} \in B_3 \,:\, \mathbf{v} \cdot \mathbf{b} = a\}. \tag{9}$$

Clearly, $\mathbf{b} \mapsto -\mathbf{b}$ is a bijection between $B_3(\mathbf{v}, 1)$ and $B_3(\mathbf{v}, 2)$. Thus, $|B_3(\mathbf{v}, 1)| = |B_3(\mathbf{v}, 2)|$ and $|B_3(\mathbf{v}, 0)| + 2\,|B_3(\mathbf{v}, 1)| = |B_3| = \binom{n}{l,l,l}$. Then by Equation (6), we have the eigenvalue

$$\begin{aligned} \lambda(\mathbf{v}) &= \sum_{\mathbf{b} \in B_3(\mathbf{v},0)} \zeta_3^{\mathbf{v}\cdot\mathbf{b}} + \sum_{\mathbf{b} \in B_3(\mathbf{v},1)} \left(\zeta_3^{\mathbf{v}\cdot\mathbf{b}} + \zeta_3^{-\mathbf{v}\cdot\mathbf{b}}\right) && (10) \\ &= |B_3(\mathbf{v}, 0)| - |B_3(\mathbf{v}, 1)| && (11) \\ &= \frac{3}{2}\,|B_3(\mathbf{v}, 0)| - \frac{1}{2}\,|B_3|. && (12) \end{aligned}$$

Now, we introduce some notions from coding theory to express Equation (12). An $[n, k, d]$ ternary code C is a $\mathbb{Z}_3$-linear subspace of $\mathbb{Z}_3^n$ of dimension k and minimum Hamming weight d. The dual code of C is defined as

$$C^{\perp} \coloneqq \{\mathbf{v} \in \mathbb{Z}_3^n \,:\, \forall\ \mathbf{c} \in C, \mathbf{v} \cdot \mathbf{c} = 0\}. \tag{13}$$

The (complete) weight enumerator of C is defined as

$$A_C(x, y, z) \coloneqq \sum_{\mathbf{c} \in C} x^{\mathrm{wt}_0(\mathbf{c})} y^{\mathrm{wt}_1(\mathbf{c})} z^{\mathrm{wt}_2(\mathbf{c})} \coloneqq \sum_{r+s+t=n} A_C[r, s, t] x^r y^s z^t, \tag{14}$$

where $\mathrm{wt}_0, \mathrm{wt}_1, \mathrm{wt}_2$ are defined in Equation (4), and $A_C[r, s, t]$ is the coefficient of the monomial $x^r y^s z^t$ in $A_C(x, y, z)$, which is also the number of codewords in C of type (r, s, t). The weight enumerators of C and $C^{\perp}$ are related by the MacWilliams transform [22] as follows

$$A_{C^{\perp}}(x, y, z) = \frac{1}{|C|} A_C(x + y + z, x + \zeta_3 y + \zeta_3^2 z, x + \zeta_3^2 y + \zeta_3 z). \tag{15}$$

With these notions, we express Equation (12) in terms of coefficients of the weight enumerator of some code in the following. For $\mathbf{v} \in \mathbb{Z}_3^n$, let

$$C_{\mathbf{v}} \coloneqq \begin{cases} \{\mathbf{0}\}, & \mathbf{v} = \mathbf{0}, \\ \{\mathbf{0}, \mathbf{v}, -\mathbf{v}\}, & \mathbf{v} \neq \mathbf{0}, \end{cases} \tag{16}$$

be the code generated by $\mathbf{v}$. Note that $C_{\mathbf{v}}$ is of dimension 0 when $\mathbf{v} = 0$, and dimension 1 when $\mathbf{v} \neq 0$. Assume that $T(\mathbf{v}) = (t_0, t_1, t_2)$, then

$$A_{C_{\mathbf{v}}}(x,y,z) = \begin{cases} x^n & \mathbf{v} = \mathbf{0} \\ x^n + x^{t_0}y^{t_1}z^{t_2} + x^{t_0}y^{t_2}z^{t_1} & \mathbf{v} \neq \mathbf{0}. \end{cases} \tag{17}$$

Clearly, $A_{C_{\mathbf{v}}}(x,y,z)$ and $A_{C_{\mathbf{v}}^{\perp}}(x,y,z)$ depend only on the type $T(\mathbf{v}) = (t_0, t_1, t_2)$ of $\mathbf{v}$, so we define

$$A_{(t_0,t_1,t_2)}(x,y,z) \coloneqq A_{C_{\mathbf{v}}}(x,y,z), \tag{18}$$

$$A_{(t_0,t_1,t_2)^{\perp}}(x,y,z) \coloneqq A_{C_{\mathbf{v}}^{\perp}}(x,y,z). \tag{19}$$

By definition, $|B_3(\mathbf{v},0)| = A_{C_{\mathbf{v}}^{\perp}}[l,l,l]$. Next, we encode $|B_3(\mathbf{v},0)|$ for all $\mathbf{v} \in \mathbb{Z}_3^n$ into the weight enumerator of a certain code by the following lemma.

Lemma 1. *Let (s_0, s_1, s_2) and (t_0, t_1, t_2) be two types of vectors in $\mathbb{Z}_3^n$. Then*

$$\binom{n}{s_0, s_1, s_2} \cdot A_{(s_0,s_1,s_2)^{\perp}}[t_0,t_1,t_2] = \binom{n}{t_0, t_1, t_2} \cdot A_{(t_0,t_1,t_2)^{\perp}}[s_0,s_1,s_2]. \tag{20}$$

Proof. Consider the following bipartite graph on the vertices $\mathbb{Z}_3^n(s_0,s_1,s_2) \cup \mathbb{Z}_3^n(t_0,t_1,t_2)$, where $\mathbf{v} \in \mathbb{Z}_3^n(s_0,s_1,s_2)$ and $\mathbf{w} \in \mathbb{Z}_3^n(t_0,t_1,t_2)$ are adjacent if and only if $\mathbf{v} \cdot \mathbf{w} = 0$. The degree of $\mathbf{v}$ in this bipartite graph is exactly $A_{(s_0,s_1,s_2)^{\perp}}[t_0,t_1,t_2]$. Similarly, the degree of $\mathbf{w}$ is exactly $A_{(t_0,t_1,t_2)^{\perp}}[s_0,s_1,s_2]$. Therefore, we have that

$$|\mathbb{Z}_3^n(s_0,s_1,s_2)| \cdot A_{(s_0,s_1,s_2)^{\perp}}[t_0,t_1,t_2] = |\mathbb{Z}_3^n(t_0,t_1,t_2)| \cdot A_{(t_0,t_1,t_2)^{\perp}}[s_0,s_1,s_2].$$

The proof is completed by noting that $|\mathbb{Z}_3^n(s_0,s_1,s_2)| = \binom{n}{s_0,s_1,s_2}$ and $|\mathbb{Z}_3^n(t_0,t_1,t_2)| = \binom{n}{t_0,t_1,t_2}$. $\square$

Now, with Theorem 1,

$$|B_3(\mathbf{v},0)| = A_{T(\mathbf{v})^{\perp}}[l,l,l] = \frac{\binom{n}{l,l,l}}{\binom{n}{T(\mathbf{v})}} A_{(l,l,l)^{\perp}}[T(\mathbf{v})]. \tag{21}$$

Note that $A_{(l,l,l)}(x,y,z) = x^n + 2(xyz)^{n/3}$. Apply the MacWilliams transform (Equation (15)),

$$A_{(l,l,l)^{\perp}}(x,y,z) = \frac{1}{3}\big((x+y+z)^n + 2(x^3+y^3+z^3-3xyz)^{n/3}\big). \tag{22}$$

Finally, by Equation (12), Equation (21) and Equation (22), we conclude that

$$\lambda(\mathbf{v}) = \frac{1}{2}\frac{\binom{n}{l,l,l}}{\binom{n}{T(\mathbf{v})}}\big((x+y+z)^n + 2(x^3+y^3+z^3-3xyz)^{n/3}\big)[T(\mathbf{v})] - \frac{1}{2}|B_3| \tag{23}$$

$$= \frac{\binom{n}{l,l,l}}{\binom{n}{T(\mathbf{v})}}(x^3+y^3+z^3-3xyz)^{n/3}[T(\mathbf{v})]. \tag{24}$$

From Equation (24), $\lambda(\mathbf{v})$ depends only on the type of $\mathbf{v}$, so we write

$$\lambda(T(\mathbf{v})) := \lambda(\mathbf{v}). \tag{25}$$

Moreover, as the polynomial in Equation (24) is symmetric, for any type (t_0, t_1, t_2) of vectors in $\mathbb{Z}_3^n$, $\lambda(t_0, t_1, t_2)$ is independent of the order of t_0, t_1, t_2. So we can always assume that $t_0 \leqslant t_1 \leqslant t_2$.

3 Eigenvalues of $O_{3l,3}$ with Large Absolute Values

Let $n = 3l$. It is easy to check that $\lambda(0, 0, n) = \binom{n}{l,l,l}$ is the largest eigenvalue of $O_{n,3}$. Further, $\lambda(0, 0, n)$ is of the largest absolute value among all eigenvalues of $O_{n,3}$. In this section, we show that for each $n = 3l$, $\lambda(1, 1, n-2)$ is negative and with the second largest absolute value, and thus it is the smallest eigenvalue of $O_{n,3}$; $\lambda(2, 2, n-4)$ is positive and with the third largest absolute value, thus it is the second largest eigenvalue of $O_{n,3}$.

Note that during the preparation of this paper, an asymptotic version of Theorem 2 was established in the recent online paper [7]. In fact, in [7], the second largest absolute value of $\mathrm{Cay}(\mathbb{Z}_p^{lp}, \mathbb{Z}_p^{lp}(l, l, \dots, l))$ was determined for l large enough with respect to p and $(p-1)l$ even, where $p \geqslant 2$ is any positive integer. As our result is for n a multiple of 3 when $p = 3$, we decided to reserve a place for Theorem 2 in this paper.

Theorem 2. *Let $n = 3l$ and (t_0, t_1, t_2) be a type of vectors in $\mathbb{Z}_3^n$ such that $0 \leqslant t_0 \leqslant t_1 \leqslant t_2 \leqslant n$ and $(t_0, t_1, t_2) \neq (0, 0, n)$. Then,*

$$|\lambda(t_0, t_1, t_2)| \leqslant |\lambda(1, 1, n-2)|. \tag{26}$$

In particular, $\lambda(1, 1, n-2) = -\frac{\binom{n}{l,l,l}}{n-1}$ is the smallest eigenvalue of $O_{n,3}$.

Proof. By Equation (24) and Equation (25), we have

$$\lambda(t_0, t_1, t_2) = \frac{\binom{n}{l,l,l}}{\binom{n}{t_0,t_1,t_2}} \big(x^3 + y^3 + z^3 - 3xyz\big)^l [t_0, t_1, t_2]. \tag{27}$$

Especially,

$$\lambda(1, 1, n-2) = \frac{\binom{n}{l,l,l}}{\binom{n}{1,1,n-2}}(-3l) = -\frac{\binom{n}{l,l,l}}{n-1} < 0. \tag{28}$$

It is clear that

$$|\lambda(t_0, t_1, t_2)| \leqslant \frac{\binom{n}{l,l,l}}{\binom{n}{t_0,t_1,t_2}} \big(x^3 + y^3 + z^3 + 3xyz\big)^l [t_0, t_1, t_2], \tag{29}$$

with equality holds when $t_0 \leqslant 2$. Denote $h(t_0, t_1, t_2) := \big(x^3 + y^3 + z^3 + 3xyz\big)^l [t_0, t_1, t_2]$. If $t_0 \equiv t_1 \equiv t_2 \pmod 3$ does not hold, $h(t_0, t_1, t_2) = \lambda(t_0, t_1, t_2) = 0$. So it suffices to show that

$$h(t_0, t_1, t_2) \leqslant \frac{\binom{n}{t_0,t_1,t_2}}{n-1} \tag{30}$$

when $t_0 \equiv t_1 \equiv t_2 \pmod 3$. We will prove Equation (30) by induction on n. The base cases when $t_0 \leqslant 2$ can be obtained by Theorem 3 combining the fact that the equality in Equation (29) holds when $t_0 \leqslant 2$.

Claim 3. *When $t_0 \leqslant 2$, $t_0 \equiv t_1 \equiv t_2 \pmod 3$, $t_0 \leqslant t_1 \leqslant t_2$ and $(t_0, t_1, t_2) \neq (0, 0, n)$, $|\lambda(t_0, t_1, t_2)| \leqslant |\lambda(1, 1, n-2)|$.*

Proof. We only prove the case when $t_0 = 0$, while the proofs for $t_0 = 1, 2$ are similar and can be found in Section A.

Assume that $t_0 = 0$, $t_1 \equiv t_2 \equiv 0 \pmod 3$ and $0 < t_1 \leqslant t_2$. By Equation (27), $\lambda(0, t_1, t_2) = \frac{\binom{n}{l,l,l}}{\binom{n}{t_1}}\binom{l}{t_1/3}$. Then $|\lambda(0, t_1, t_2)| \leqslant |\lambda(1, 1, n-2)|$ is equivalent to that $n - 1 \leqslant \frac{\binom{n}{t_1}}{\binom{l}{t_1/3}}$. Let $f(t) \coloneqq \frac{\binom{n}{t}}{\binom{l}{t/3}}$, then for $t \equiv 0 \pmod 3$ such that $0 < t$ and $t + 3 \leqslant n/2$,

$$\frac{f(t+3)}{f(t)} = \frac{(n-t-1)(n-t-2)}{(t+2)(t+1)} \geqslant 1.$$

Therefore, $f(t)$ is increasing with respect to t, and $f(t) \geqslant f(3) = \frac{(n-1)(n-2)}{2} \geqslant n - 1$ for $n \geqslant 4$. ☐

Now assume that $t_0 \geqslant 3$, $t_0 \equiv t_1 \equiv t_2 \pmod 3$ and $t_0 \leqslant t_1 \leqslant t_2$. Then

$$\begin{aligned} h(t_0, t_1, t_2) &= 3h(t_0 - 1, t_1 - 1, t_2 - 1) + h(t_0 - 3, t_1, t_2) \\ &\quad + h(t_0, t_1 - 3, t_2) + h(t_0, t_1, t_2 - 3) \\ &\leqslant \frac{1}{n-4}\left(3\binom{n-3}{t_0 - 1, t_1 - 1, t_2 - 1} + \binom{n-3}{t_0 - 3, t_1, t_2}\right. \\ &\quad \left. + \binom{n-3}{t_0, t_1 - 3, t_2} + \binom{n-3}{t_0, t_1, t_2 - 3}\right), \end{aligned}$$

where the inequality follows from the induction hypothesis. To get Equation (30), it then suffices to have

$$3t_0t_1t_2 + t_0(t_0 - 1)(t_0 - 2) + t_1(t_1 - 1)(t_1 - 2) + t_2(t_2 - 1)(t_2 - 2) \leqslant (n-4)(n-2)n.$$

Let

$$g(t_0, t_1, t_2) \coloneqq (n-4)(n-2)n - t_0(t_0-1)(t_0-2) - t_1(t_1-1)(t_1-2) - t_2(t_2-1)(t_2-2) - 3t_0t_1t_2,$$

where $n = t_0 + t_1 + t_2$. It suffices to show that $g(t_0, t_1, t_2) \geqslant 0$ for all $t_0, t_1, t_2 \geqslant 3$. We think of t_1 and t_2 as being fixed, and show that $g(t_0, t_1, t_2)$ is increasing with respect to t_0. By calculation

$$g(t_0, t_1, t_2) - g(t_0 - 1, t_1, t_2) = 3t_1^2 - 15t_1 + 3t_2^2 - 15t_2 + 3t_1t_2 + 9 + t_0(6t_1 + 6t_2 - 6).$$

When $t_0, t_1, t_2 \geqslant 3$, we have $g(t_0, t_1, t_2) - g(t_0 - 1, t_1, t_2) \geqslant 0$. Thus, $g(t_0, t_1, t_2)$ is increasing with respect to $t_0 \geqslant 3$, for fixed $t_1, t_2 \geqslant 3$. By the symmetry of t_0, t_1, t_2, $g(t_0, t_1, t_2)$ is increasing with respect to $t_0, t_1, t_2 \geqslant 3$, respectly. In conclusion, for $t_0, t_1, t_2 \geqslant 3$, $g(t_0, t_1, t_2) \geqslant g(3, 3, 3) = 216 > 0$. The proof is completed. ☐

Theorem 2 immediately implies that $\lambda(1,1,3l-2)$ is the smallest eigenvalue of $O_{3l,3}$. Together with the spectral lower bound on quantum chromatic numbers, $\chi_q(O_{3l,3})$ can be determined for any l.

Corollary 4. *$\chi_q(O_{3l,3}) = 3l$ for all $l \geqslant 1$.*

Before ending this section, we determine the eigenvalue with the third largest absolute value, which is the second largest eigenvalue of $O_{n,3}$.

Theorem 5. *Let $n = 3l \geqslant 9$ and (t_0,t_1,t_2) be a type of vectors in $\mathbb{Z}_3^n$ such that $0 \leqslant t_0 \leqslant t_1 \leqslant t_2 \leqslant n$ and $(t_0,t_1,t_2) \notin \{(0,0,n),(1,1,n-2)\}$. Then,*

$$|\lambda(t_0,t_1,t_2)| \leqslant |\lambda(2,2,n-4)|.$$

In particular, $\lambda(2,2,n-4) = \frac{2\binom{n}{l,l,l}}{(n-1)(n-2)}$ is the second largest eigenvalue of $O_{n,3}$.

The proof of Theorem 5 is similar to that of Theorem 2 and thus moved to Section B. Theorem 5 will not be used in the rest of this paper. But we hope that it will be useful for future study.

4 Quantum Chromatic Numbers of Certain Subgraphs

In this section, we determine the quantum chromatic numbers for several classes of Cayley graphs, which can be considered as subgraphs of $O_{n,p}$ with $p = 2,3$. We will frequently use Equation (1), that is, if there is a homomorphism from G to H, then $\chi_q(G) \leqslant \chi_q(H)$.

First, we give a general result on graphs and subgraphs with the same quantum chromatic number.

Theorem 6. *For any integer p and ordered non-negative p-partition $(t_0,t_1,\ldots,t_{p-1})$ of n with $t_i = t_{p-i}$ for $i \in [\lfloor p/2 \rfloor]$ let $G = \mathrm{Cay}(\mathbb{Z}_p^n, \mathbb{Z}_p^n(t_0,t_1,\ldots,t_{p-1}))$ and $H = \mathrm{Cay}(\mathbb{Z}_p^{n-1}, S)$ where $S = \mathbb{Z}_p^{n-1}(t_0-1,t_1,\ldots,t_{p-1}) \cup \mathbb{Z}_p^{n-1}(t_0,t_1-1,\ldots,t_{p-1}) \cup \cdots \cup \mathbb{Z}_p^{n-1}(t_0,t_1,\ldots,t_{p-1}-1)$. Suppose $\sum_{i=0}^{p-1} it_i \equiv 0 \pmod p$, then*

$$\chi_q(G) = \chi_q(H).$$

Proof. We prove by constructing homomorphisms between G and H. We claim that H can be embedded into G through the following map

$$\phi : \mathbb{Z}_p^{n-1} \to \mathbb{Z}_p^n, \quad \mathbf{x} \mapsto (\mathbf{x}, -\mathbf{x}\cdot\mathbf{1}),$$

where $\mathbf{1}$ is the all-one vector in $\mathbb{Z}_p^{n-1}$. In other words, $\phi(\mathbf{x})$ is derived by appending a check bit to $\mathbf{x}$. We prove that ϕ is indeed a homomorphism. For any two vectors $\mathbf{x}$ and $\mathbf{y}$ connected in H, $T(\mathbf{x}-\mathbf{y}) = (t_0,t_1,\ldots,t_j-1,\ldots,t_{p-1})$ for some $j \in [0,p-1]$. Then $(\mathbf{x}-\mathbf{y})\cdot\mathbf{1} \equiv \sum_{i=0}^{p-1} it_i - j \equiv -j \pmod p$. Since $\phi(\mathbf{x}) = (\mathbf{x}, -\mathbf{x}\cdot\mathbf{1})$ and $\phi(\mathbf{y}) = (\mathbf{y}, -\mathbf{y}\cdot\mathbf{1})$, $\phi(\mathbf{x})-\phi(\mathbf{y}) = (\mathbf{x}-\mathbf{y}, (\mathbf{y}-\mathbf{x})\cdot\mathbf{1}) = (\mathbf{x}-\mathbf{y}, j)$. So $T(\phi(\mathbf{x})-\phi(\mathbf{y})) = (t_0,t_1,\ldots,t_j,\ldots,t_{p-1})$, which means that $\phi(\mathbf{x})$ and $\phi(\mathbf{y})$ are connected in G. This shows that $\chi_q(H) \leqslant \chi_q(G)$.

Conversely, the projection onto the first $n-1$ coordinates is a homomorphism of graphs $G \to H$, i.e.,

$$\mathbb{Z}_p^n \to \mathbb{Z}_p^{n-1}, \quad (x_1, x_2, \ldots, x_{n-1}, x_n) \mapsto (x_1, x_2, \ldots, x_{n-1}).$$

This shows that $\chi_q(G) \leqslant \chi_q(H)$. $\square$

Based on previous results on quantum chromatic numbers in Table 1, we have the following corollary. Recall that L_r is the set of vectors of weight r.

Corollary 7. *(1) For any $t \geqslant 1$, $\chi_q(\mathcal{G}_1) = \chi_q(\mathcal{G}_2) = 4t$, where*

$$\mathcal{G}_1 \coloneqq \mathrm{Cay}(\mathbb{Z}_2^{4t-1}, L_{2t-1} \cup L_{2t}) \text{ and } \mathcal{G}_2 \coloneqq \mathrm{Cay}(\mathbb{Z}_2^{4t-2}, L_{2t-1} \cup L_{2t}).$$

(2) When $l \geqslant 1$, $\chi_q(\mathcal{G}_3) = 3l$, where

$$\mathcal{G}_3 \coloneqq \mathrm{Cay}(\mathbb{Z}_3^{3l-1}, \mathbb{Z}_3^{3l-1}(l-1, l, l) \cup \mathbb{Z}_3^{3l-1}(l, l-1, l) \cup \mathbb{Z}_3^{3l-1}(l, l, l-1)).$$

(3) When $p \geqslant 4$ is an integer, l is large enough and $l(p-1)$ is even, $\chi_q(\mathcal{G}_4) = lp$, where

$$\mathcal{G}_4 \coloneqq \mathrm{Cay}(\mathbb{Z}_p^{lp-1}, \mathbb{Z}_p^{lp-1}(l-1, l, \ldots, l) \cup \mathbb{Z}_p^{lp-1}(l, l-1, \ldots, l) \cup \cdots \cup \mathbb{Z}_p^{lp-1}(l, l, \ldots, l-1)).$$

4.1 Subgraphs of $O_{n,3}$

Consider the Cayley graph

$$\mathcal{G}_5 \coloneqq \mathrm{Cay}(\mathbb{Z}_3^{3l-1}, \mathbb{Z}_3^{3l-1}(l-1, l, l)),$$

that is the Cayley graph over $\mathbb{Z}_3^{3l-1}$ generated by vectors of type $(l-1, l, l)$. Then $\mathcal{G}_5$ can be embedded into $O_{3l,3} \cong \mathrm{Cay}(\mathbb{Z}_3^{3l}, \mathbb{Z}_3^{3l}(l, l, l))$ by the following homomorphism

$$\mathbb{Z}_3^{3l-1} \to \mathbb{Z}_3^{3l}, \quad \mathbf{x} \mapsto (\mathbf{x}, 0). \tag{31}$$

This shows that $\chi_q(\mathcal{G}_5) \leqslant \chi_q(O_{3l,3}) = 3l$. Next, we determine the smallest eigenvalue of $\mathcal{G}_5$.

Theorem 8. *When $l \geqslant 2$, the smallest eigenvalue of $\mathcal{G}_5 = \mathrm{Cay}(\mathbb{Z}_3^{3l-1}, \mathbb{Z}_3^{3l-1}(l-1, l, l))$ is given by*

$$\lambda_{\min}(\mathcal{G}_5) = \lambda(0, 1, 3l-2) = -\frac{3l-2}{l^2}\binom{3l-3}{l-1, l-1, l-1}.$$

Proof. Suppose $\mathbf{v} \in \mathbb{Z}_3^{3l-1}$ is of type $T(\mathbf{v}) = (t_0, t_1, t_2)$, where $l \geqslant 2$. Following the approach described in Section 2.2.2, the eigenvalue $\lambda(\mathbf{v})$ of $\mathcal{G}_5$ is given by

$$\lambda(\mathbf{v}) = \frac{\binom{3l-1}{l-1,l,l}}{\binom{3l-1}{t_0,t_1,t_2}}\big((x^3 + y^3 + z^3 - 3xyz)^{l-1}(x^2 + y^2 + z^2 - xy - yz - xz)\big)[t_0, t_1, t_2].$$

In the following, we illustrate how to determine the smallest eigenvalue. To distinguish between the eigenvalues of $\mathcal{G}_5$ and the eigenvalues of $O_{3l,3}$, we denote by $\lambda(t_0,t_1,t_2) \coloneqq \lambda(\mathbf{v})$ the eigenvalues of $\mathcal{G}_5$ as in the statement of the theorem, and $\lambda_{O_{3l-3,3}}(t_0,t_1,t_2)$ the eigenvalues of $O_{3l-3,3}$.

For simplicity, denote $W(x,y,z) \coloneqq (x^3+y^3+z^3-3xyz)^{l-1}$, $Y(x,y,z) \coloneqq x^2+y^2+z^2-xy-yz-xz$ and write $T = T(\mathbf{v}) = (t_0,t_1,t_2)$. Assume that $(WY)[T] \neq 0$. Then by the formulas of W and Y, there is a unique pair of nonnegative ordered 3-partitions (T_0,T_1), where T_0 is a partition of $3l-3$ and T_1 is a partition of 2, such that $T = T_0+T_1$ (here $+$ denotes vector addition) and $(WY)[T] = W[T_0]\cdot Y[T_1]$. Further, the components of T_0 are congruent modulo 3 and

$$T_1 \in \{(2,0,0),(0,2,0),(0,0,2),(1,1,0),(1,0,1),(0,1,1)\}.$$

As $\lambda(t_0,t_1,t_2)$ is independent of the order of t_0,t_1,t_2, we only need to consider the following two cases that

$$T_1 \in \{(1,1,0),(2,0,0)\}.$$

If $T_1 = (1,1,0)$, we have

$$\lambda(t_0,t_1,t_2) = -\frac{\binom{3l-1}{l-1,l,l}}{\binom{3l-1}{t_0,t_1,t_2}}\frac{\binom{3l-3}{t_0-1,t_1-1,t_2}}{\binom{3l-3}{l-1,l-1,l-1}}\lambda_{O_{3l-3,3}}(t_0-1,t_1-1,t_2) \tag{32}$$

$$= -\frac{t_0t_1}{l^2}\lambda_{O_{3l-3,3}}(t_0-1,t_1-1,t_2). \tag{33}$$

We show that $\lambda(3l-2,1,0)$ is the smallest eigenvalue in this case. The proof is by contradiction. Assume that $\lambda(t_0,t_1,t_2) < \lambda(3l-2,1,0)$ for some (t_0,t_1,t_2) in this case. Then by Equation (32) and Equation (33),

$$-\frac{t_0t_1}{l^2}\lambda_{O_{3l-3,3}}(t_0-1,t_1-1,t_2) < -\frac{3l-2}{l^2}\lambda_{O_{3l-3,3}}(3l-3,0,0). \tag{34}$$

This implies that

$$\frac{t_0t_1}{3l-2} > \frac{\lambda_{O_{3l-3,3}}(3l-3,0,0)}{\lambda_{O_{3l-3,3}}(t_0-1,t_1-1,t_2)} \geqslant \frac{\lambda_{O_{3l-3,3}}(3l-3,0,0)}{|\lambda_{\min}(O_{3l-3,3})|} = 3l-4, \tag{35}$$

where the first inequality follows from the fact that $\lambda(t_0,t_1,t_2)$ is negative and thus $\lambda_{O_{3l-3,3}}(t_0-1,t_1-1,t_2)$ is positive, and the second inequality and the last equality follow from Theorem 2 and Equation (28). So we have $t_0t_1 > (3l-2)(3l-4)$, which contradicts to the assumption that $t_0+t_1+t_2 = 3l-1$ and $l \geqslant 2$.

If $T_1 = (2,0,0)$, we have

$$\lambda(t_0,t_1,t_2) = \frac{\binom{3l-1}{l-1,l,l}}{\binom{3l-1}{t_0,t_1,t_2}}\frac{\binom{3l-3}{t_0-2,t_1,t_2}}{\binom{3l-3}{l-1,l-1,l-1}}\lambda_{O_{3l-3,3}}(t_0-2,t_1,t_2) \tag{36}$$

$$= \frac{t_0(t_0-1)}{l^2}\lambda_{O_{3l-3,3}}(t_0-2,t_1,t_2). \tag{37}$$

Again, we show that $\lambda(t_0, t_1, t_2)$ cannot be smaller than $\lambda(3l-2, 1, 0)$. Otherwise, if $\lambda(t_0, t_1, t_2) < \lambda(3l-2, 1, 0)$, we have

$$\frac{t_0(t_0-1)}{l^2}\lambda_{O_{3l-3,3}}(t_0-2, t_1, t_2) < -\frac{3l-2}{l^2}\lambda_{O_{3l-3,3}}(3l-3, 0, 0). \tag{38}$$

Similar to Equation (35), we have

$$\frac{t_0(t_0-1)}{3l-2} > \frac{\lambda_{O_{3l-3,3}}(3l-3,0,0)}{\left|\lambda_{O_{3l-3,3}}(t_0-2, t_1, t_2)\right|} \geqslant \frac{\lambda_{O_{3l-3,3}}(3l-3,0,0)}{|\lambda_{\min}(O_{3l-3,3})|} = 3l-4, \tag{39}$$

which yields the contradiction $t_0(t_0-1) > (3l-2)(3l-4)$. □

Theorem 9. *When $l \geqslant 2$, $\chi_q(\mathcal{G}_5) = 3l$, where $\mathcal{G}_5 = \mathrm{Cay}(\mathbb{Z}_3^{3l-1}, \mathbb{Z}_3^{3l-1}(l-1, l, l))$.*

Proof. The result follows immediately from the spectral lower bound in Equation (3) and Theorem 8. □

Applying Theorem 6 to Theorem 9, we have the following corollary.

Corollary 10. *When $l \geqslant 2$, $\chi_q(\mathcal{G}_6) = 3l$, where*

$$\mathcal{G}_6 \coloneqq \mathrm{Cay}(\mathbb{Z}_3^{3l-2}, \mathbb{Z}_3^{3l-2}(l-2, l, l) \cup \mathbb{Z}_3^{3l-2}(l-1, l-1, l) \cup \mathbb{Z}_3^{3l-2}(l-1, l, l-1)).$$

5 Quantum Chromatic Number of $\mathrm{Cay}(\mathbb{Z}_2^n, L_2)$

Denote $H(n,r) \coloneqq \mathrm{Cay}(\mathbb{Z}_2^n, L_r)$. When r is odd, $H(n,r)$ is bipartite thus $\chi_q(H(n,r)) = \chi(H(n,r)) = 2$. For even $r \geqslant n/2$, it was shown in [8] that

$$r \leqslant \chi_q(H(n,r)) \leqslant 2r.$$

The open problem to find a good upper bound on $\chi_q(H(n,r))$ for even $r \leqslant n/2$ was raised in [8]. In this section, as a step toward this problem, we focus on the case $r = 2$ and determine $\chi_q(H(n,2))$ for infinitely many n by the theory of combinatorial designs.

We first compute the spectral lower bound of $\chi_q(H(n,2))$. As in Section 2.2.1, the eigenvalues of $H(n,2)$ are

$$\lambda(w) = K_2(w) = (n-2w)^2/2 - n/2, \quad 0 \leqslant w \leqslant n.$$

Thus,

$$\lambda_{\min}(H(n,2)) = \begin{cases} -n/2, & n \equiv 0 \pmod 2, \\ -(n-1)/2, & n \equiv 1 \pmod 2. \end{cases}$$

Clearly, $\lambda_{\max}(H(n,2)) = \binom{n}{2}$. So the spectral lower bound shows that

$$\chi_q(H(n,2)) \geqslant 1 - \frac{\lambda_{\max}(H(n,2))}{\lambda_{\min}(H(n,2))} = \begin{cases} n, & n \equiv 0 \pmod 2, \\ n+1, & n \equiv 1 \pmod 2. \end{cases} \tag{40}$$

To find an upper bound on $\chi_q(H(n,2))$, we use the theory of combinatorial designs to construct flat orthogonal representations of $H(n,2)$. To this end, we introduce the following combinatorial structure. Let $2^{[n]}$ denote the set of all subsets of $[n]$.

Definition 11 (balanced pair-separating family)**.** A family $\mathcal{F} \subset 2^{[n]}$ is called a *balanced pair-separating family with separation number θ* if for every pair of distinct elements $i, j \in [n]$, there are exactly θ members (called *blocks*) of $\mathcal{F}$ containing only one of i, j.

The following theorem shows that a balanced pair-separating family with a big separation number can help to construct flat orthogonal representations of $H(n, 2)$, and hence provide an upper bound of $\chi_q(H(n, 2))$. Note that there is a natural one-to-one correspondence between vectors in $\mathbb{Z}_2^n$ and subsets of $[n]$ by defining $\operatorname{supp}(\mathbf{v}) \coloneqq \{i \in [n] : v_i \neq 0\}$ for each $\mathbf{v} \in \mathbb{Z}_2^n$. Through this correspondence, we will not distinguish vectors in $\mathbb{Z}_2^n$ and subsets of $[n]$.

Theorem 12. *Suppose there is a balanced pair-separating family $\mathcal{F} \subset 2^{[n]}$ with separation number $\theta \geqslant |\mathcal{F}|/2$. Then $\chi_q(H(n, 2)) \leqslant 2\theta$.*

Proof. Let $\mathcal{F} = \{B_1, B_2, \ldots, B_b\}$. Then $2\theta \geqslant b$. Consider the following map

$$\phi : \mathbb{Z}_2^n \to \{\pm 1\}^{2\theta}, \quad \mathbf{v} \mapsto (\chi_1(\mathbf{v}), \chi_2(\mathbf{v}), \ldots, \chi_b(\mathbf{v}), \mathbf{1}_{2\theta - b}),$$

where $\chi_i \coloneqq \chi_{B_i}$ is the character of $\mathbb{Z}_2^n$ corresponding to B_i, and $\mathbf{1}_{2\theta-b}$ is the all-one vector of length $2\theta - b$. It suffices to prove that ϕ is a flat orthogonal representation of $H(n, 2)$ of dimension 2θ.

Given adjacent $\mathbf{v}$ and $\mathbf{w}$ in $H(n, 2)$, let $\mathbf{z} = \mathbf{v} + \mathbf{w}$. Then $\operatorname{wt}(\mathbf{z}) = 2$. We need to show that $\phi(\mathbf{v})$ and $\phi(\mathbf{w})$ are orthogonal. Consider the inner product

$$\langle \phi(\mathbf{v}), \phi(\mathbf{w}) \rangle = \sum_{i=1}^{b} \chi_i(\mathbf{v})\chi_i(\mathbf{w}) + (2\theta - b) = \sum_{i=1}^{b} \chi_i(\mathbf{z}) + (2\theta - b).$$

Note that $\chi_i(\mathbf{z}) = -1$ if and only if $|\operatorname{supp}(\mathbf{z}) \cap B_i| = 1$. Therefore,

$$\langle \mathbf{v}, \mathbf{w} \rangle = b - 2\theta + 2\theta - b = 0.$$

As $\phi(\mathbf{v})$ consists of only ± 1, ϕ is flat. $\square$

Next, we show the existence of balanced pair-separating families from balanced incomplete block designs (BIBD). An (n, k, λ)-BIBD is a family $\mathcal{B}$ of k-subsets of $[n]$, called blocks, such that any pair of distinct elements in $[n]$ is contained in exactly λ blocks [24, Chapter 1]. In such a BIBD, each element of $[n]$ is contained in exactly $r = \frac{\lambda(n-1)}{k-1}$ blocks and the number of blocks is

$$b \coloneqq |\mathcal{B}| = \frac{nr}{k} = \frac{\lambda n(n-1)}{k(k-1)}.$$

By definition, it is easy to see that if $\mathcal{B}$ is an (n, k, λ)-BIBD, then $\mathcal{B}$ is a balanced pair-separating family with separation number $\theta = 2(r - \lambda)$. By the requirement $\theta \geqslant |\mathcal{F}|/2$ in Theorem 12, and combining formulas on r and b, we have the following observation.

Table 2: Exact Quantum Chromatic Number of $H(n,2)$ from Symmetric BIBD

$(n=b, k=r, \lambda)$	$\chi_q(H(n,2))$	Comments and References
$(q,(q-1)/2,(q-3)/4)$	$q+1$	$q \equiv 3 \pmod 4$ is a prime power [24, Chapter 3]
$(2^{t+2}-1, 2^{t+1}-1, 2^t-1)$	2^{t+2}	t is a positive integer [9, II.6.8]
$(q^2+2q, \frac{q^2+2q-1}{2}, \frac{q^2+2q-3}{4})$	q^2+2q+1	q and $q+2$ are both odd prime powers [9, VI.18.5]
$(4s^2, 2s^2-s, s^2-s)$	$4s^2$	Existence of regular Hadamard matrix of order $4s^2$ [9, VI.18.6]

Observation 13. *If there is an (n,k,λ)-BIBD with $4k(n-k) \geqslant n(n-1)$, then*

$$\chi_q(H(n,2)) \leqslant \frac{4\lambda(n-k)}{k-1}.$$

In order to get upper bounds matching the spectral lower bounds in Equation (40), BIBDs with the following parameters are useful.

Corollary 14. *(1) If there exists an $(n,(n-1)/2,(n-3)/4)$-BIBD with $n \equiv 3 \pmod 4$, then $\chi_q(H(n,2)) = n+1$.*

(2) If there is a $(4s^2, 2s^2-s, s^2-s)$-BIBD for some integer s, then $\chi_q(H(4s^2,2)) = 4s^2$.

Proof. The upper bounds are from Theorem 13 and the lower bounds are from Equation (40). ☐

BIBDs with parameters in Theorem 14 are in fact symmetric BIBDs, i.e., satisfying $b = n$. In Table 2, we collect some known existence of symmetric (n,k,λ)-BIBDs satisfying the requirements of Theorem 14 and determine the corresponding quantum chromatic of $H(n,2)$.

We finish this section with a remark concerning the classical chromatic number of $H(n,2)$. It is clear that $\chi_q(G) \leqslant \chi(G)$ for any graph G. So, it is natural to consider $\chi(H(n,2))$ as an upper bound of $\chi_q(H(n,2))$. The values of $\chi(H(n,2))$ have been studied in [13, 15]. For $n \leqslant 16$, we collect known results on $\chi(H(n,2))$ in the literature [2, 13, 15, 17, 25] in Table 3. We can see from Table 3 that $\chi(H(n,2))$ does not match the spectral lower bound of $\chi_q(H(n,2))$ in general, leaving $\chi_q(H(n,2))$ undetermined. Also from Table 3, for $n \in \{9,10\}$, $\chi_q(H(n,2)) \leqslant 12$ while $13 \leqslant \chi(H(n,2))$ and for $n = 11$, $\chi_q(H(11,2)) = 12$ while $15 \leqslant \chi(H(11,2))$, illustrating a separation between the quantum and classical chromatic numbers of $H(9,2), H(10,2)$ and $H(11,2)$.

6 Conclusion

In this paper, we determined the quantum chromatic number of several Cayley graphs over $\mathbb{Z}_p^n$, which can be regarded as subgraphs of the orthogonality graphs. Further, we

Table 3: Chromatic Number of $H(n,2)$ [2,13,15,17,25]

n	2	3	4	5	6	7	8	9
$\chi(H(n,2))$	2	4	4	8	8	8	8	13
$\chi_q(H(n,2))$	2	4	4	$6-8$	$6-8$	8	8	$10-12$
n	10	11	12	13	14	15	16	
$\chi(H(n,2))$	$13-14$	$15-16$	$15-16$	16	16	16	16	
$\chi_q(H(n,2))$	$10-12$	12	$12-16$	$14-16$	$14-16$	16	16	

[1] References on $\chi(H(n,2))$: $n \leqslant 8$ in [25], $n \in [9,10]$ in [15,17], $n \in [11,16]$ in [2].

[2] The lower bounds on $\chi_q(H(n,2))$ is the spectral lower bound in Equation (40). For $n = 11$, $\chi_q(H(11,2)) = 12$ by Table 2. Note that $H(n,2)$ can be embedded into $H(n+1,2)$ for all n. Thus, $\chi_q(H(9,2)) \leqslant \chi_q(H(10,2)) \leqslant \chi_q(H(11,2)) = 12$. For the rest n, $\chi_q(H(n,2)) \leqslant \chi(H(n,2))$.

proposed an approach to construct flat orthogonal representations for the graph $H(n,2)$ by using tools in design theory and determined the quantum chromatic number of $H(n,2)$ for infinitely n. All the main results of this paper have been summarized in Table 1.

There are still many interesting questions that might be considered in the future. We list some of them below.

(1) Can one extend Theorem 9 into general integer p?

(2) Determine the quantum chromatic number of $H(n,2)$ for more n.

(3) Can one construct balanced pair-separating families with big separation number not from BIBDs?

(4) The definition of balanced pair-separating families can be easily extended to construct the flat orthogonal representations for $H(n,2t)$. However, even for $t = 2$, we failed to find any known 4-designs in the literature yielding an upper bound of $\chi_q(H(n,4))$ matching the spectral lower bound. Some new ideas are needed to determine the quantum chromatic number of $H(n,4)$.

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A Remaining Proof of Theorem 3

Proof. When $t_0 = 1$, $t_1 \equiv t_2 \equiv 1 \pmod 3$ and $1 \leqslant t_1 \leqslant t_2$, by Equation (27)

$$\lambda(1, t_1, t_2) = -n\frac{\binom{n}{l,l,l}}{\binom{n}{1,t_1,t_2}}\binom{l-1}{(t_1-1)/3}.$$

Then $|\lambda(1, t_1, t_2)| \leqslant |\lambda(1, 1, n-2)|$ is equivalent to that $n - 1 \leqslant \frac{\binom{n-1}{t_1}}{\binom{l-1}{(t_1-1)/3}}$. Let $f(t) \coloneqq \frac{\binom{n-1}{t}}{\binom{l-1}{(t-1)/3}}$, then for $t \equiv 1 \pmod 3$ such that $1 \leqslant t$ and $t + 3 \leqslant (n - 1)/2$,

$$\frac{f(t+3)}{f(t)} = \frac{(n-t-1)(n-t-3)}{(t+2)(t+1)} \geqslant 1.$$

Therefore, $f(t)$ is increasing with respect to t, and $f(t) \geqslant f(1) = n - 1$.

When $t_0 = 2$, $t_1 \equiv t_2 \equiv 2 \pmod 3$ and $2 \leqslant t_1 \leqslant t_2$, $\lambda(2, t_1, t_2) = 9\frac{\binom{n}{l,l,l}}{\binom{n}{2,t_1,t_2}}\binom{l}{2}\binom{l-2}{(t_1-2)/3}$ by Equation (27). Then $|\lambda(2, t_1, t_2)| \leqslant |\lambda(1, 1, n-2)|$ is equivalent to that $n - 3 \leqslant \frac{\binom{n-2}{t_1}}{\binom{l-2}{(t_1-2)/3}}$. Write $k \coloneqq (t_1 - 2)/3$, then $k \in [0, \lfloor n/6 \rfloor - 1]$. Let $f(k) \coloneqq \frac{\binom{n-2}{3k+2}}{\binom{l-2}{k}}$, then for $0 \leqslant k$ and $k + 1 \leqslant \lfloor n/6 \rfloor - 1$,

$$\frac{f(k+1)}{f(k)} = \frac{(n-3k-4)(n-3k-5)}{(3k+4)(3k+5)} \geqslant 1.$$

Therefore, $f(k)$ is increasing with respect to k, and $f(k) \geqslant f(0) = \frac{(n-2)(n-3)}{2} \geqslant n - 3$ for $n \geqslant 6$. □

B Proof of Theorem 5

Proof. By Equation (24) and Equation (25),

$$\lambda(2, 2, n-4) = \frac{2\binom{n}{l,l,l}}{(n-1)(n-2)}$$

is a positive eigenvalue. We show that $\lambda(2, 2, n-4)$ is of the third largest absolute value. The proof follows from a similar approach as in the proof of Theorem 2. Recall in the proof of Theorem 2 that

$$|\lambda(t_0, t_1, t_2)| \leqslant \frac{\binom{n}{l,l,l}}{\binom{n}{t_0,t_1,t_2}}(x^3 + y^3 + z^3 + 3xyz)^l[t_0, t_1, t_2], \tag{41}$$

with equality holds when $t_0 \leqslant 2$, and denote $h(t_0, t_1, t_2) \coloneqq (x^3 + y^3 + z^3 + 3xyz)^l[t_0, t_1, t_2]$. It suffices to show that

$$h(t_0, t_1, t_2) \leqslant \frac{2\binom{n}{t_0,t_1,t_2}}{(n-1)(n-2)}. \tag{42}$$

Equation (42) will be shown by induction on n. The base case when $t_0 \leqslant 2$ can be obtained by Claims 15-17 with the fact that the equality in Equation (41) holds when $t_0 \leqslant 2$.

Claim 15. *When $t_0 = 0$, $t_1 \equiv t_2 \equiv 0 \pmod 3$ and $0 < t_1 \leqslant t_2$, $|\lambda(t_0, t_1, t_2)| \leqslant |\lambda(2, 2, n-4)|$.*

Proof. By Equation (27), $\lambda(0, t_1, t_2) = \frac{\binom{n}{l,l,l}}{\binom{n}{t_1}}\binom{l}{t_1/3}$. Then $|\lambda(0, t_1, t_2)| \leqslant |\lambda(2, 2, n-4)|$ is equivalent to that $\frac{(n-1)(n-2)}{2} \leqslant \frac{\binom{n}{t_1}}{\binom{l}{t_1/3}}$, which has been verified in the proof of Theorem 3. This essentially shows that $\lambda(2, 2, n-4) = \lambda(0, 3, n-3)$ share the same value. $\square$

Claim 16. *When $t_0 = 1$ and $t_1 \equiv t_2 \equiv 1 \pmod 3$ and $1 < t_1 \leqslant t_2$, $|\lambda(t_0, t_1, t_2)| \leqslant |\lambda(2, 2, n-4)|$.*

Proof. By Equation (27), $\lambda(1, t_1, t_2) = -n\frac{\binom{n}{l,l,l}\binom{l-1}{(t_1-1)/3}}{\binom{n}{1,t_1,t_2}}$ and $|\lambda(1, t_1, t_2)| \leqslant |\lambda(2, 2, n-4)|$ is equivalent to that $\frac{(n-1)(n-2)}{2} \leqslant \frac{\binom{n-1}{t_1}}{\binom{l-1}{(t_1-1)/3}}$. Let $f(t) \coloneqq \frac{\binom{n-1}{t}}{\binom{l-1}{(t-1)/3}}$, for $t \in [4, (n-1)/2]$ with $t \equiv 1 \pmod 3$. It has been verified in Section A that $f(t)$ is increasing with respect to t. Thus, $f(t) \geqslant f(4) = \frac{(n-1)(n-2)(n-4)}{8} \geqslant \frac{(n-1)(n-2)}{2}$ when $n \geqslant 8$. $\square$

Claim 17. *When $t_0 = 2$ and $t_1 \equiv t_2 \equiv 2 \pmod 3$ and $2 \leqslant t_1 \leqslant t_2$, $|\lambda(t_0, t_1, t_2)| \leqslant |\lambda(2, 2, n-4)|$.*

Proof. By Equation (27), $\lambda(2, t_1, t_2) = 9\frac{\binom{n}{l,l,l}\binom{l}{2}}{\binom{n}{2,t_1,t_2}}\binom{l-2}{(t_1-2)/3}$ and $|\lambda(2, t_1, t_2)| \leqslant |\lambda(2, 2, n-4)|$ is equivalent to that $\frac{(n-2)(n-3)}{2} \leqslant \frac{\binom{n-2}{t_1}}{\binom{l-2}{(t_1-2)/3}}$, which has been verfied in the proof of Section A. $\square$

Now assume that $t_0 \geqslant 3$, $t_0 \equiv t_1 \equiv t_2 \pmod 3$. Then

$$\begin{aligned}h(t_0, t_1, t_2) &= h(t_0-3, t_1, t_2) + h(t_0, t_1-3, t_2) + h(t_0, t_1, t_2-3) + 3h(t_0-1, t_1-1, t_2-1)\\ &\leqslant \frac{2}{(n-4)(n-5)}\left(\binom{n-3}{t_0-3, t_1, t_2} + \binom{n-3}{t_0, t_1-3, t_2} + \right.\\ &\qquad\left.\binom{n-3}{t_0, t_1, t_2-3} + 3\binom{n-3}{t_0-1, t_1-1, t_2-1}\right),\end{aligned}$$

where the inequality follow from the induction hypothesis. To get Equation (42), it suffices to have

$$t_0(t_0-1)(t_0-2) + t_1(t_1-1)(t_1-2) + t_2(t_2-1)(t_2-2) + 3t_0t_1t_2 \leqslant n(n-4)(n-5). \quad (43)$$

Let $f(t_0, t_1, t_2) \coloneqq n(n-4)(n-5) - t_0(t_0-1)(t_0-2) - t_1(t_1-1)(t_1-2) - t_2(t_2-1)(t_2-2) - 3t_0t_1t_2$ where $n = t_0 + t_1 + t_2$. It suffices to show that $f(t_0, t_1, t_2) \geqslant 0$ for all $t_0, t_1, t_2 \geqslant 3$.

We think of t_1, t_2 as fixed and show that $f(t_0, t_1, t_2)$ is increasing with respect to t_0. By calculation

$$f(t_0, t_1, t_2) - f(t_0 - 1, t_1, t_2) = 3(t_1^2 - 7t_1 + t_2^2 - 7t_2 + t_0t_1 + t_0t_2 + 8) + 6t_0(t_1 + t_2 - 2).$$

For $t_0, t_1, t_2 \geqslant 3$, $f(t_0, t_1, t_2) - f(t_0 - 1, t_1, t_2) \geqslant 0$ and $f(t_0, t_1, t_2)$ is increasing with respect to $t_0 \geqslant 3$. By the symmetry of t_0, t_1, t_2, $f(t_0, t_1, t_2)$ is increasing with respect to $t_0, t_1, t_2 \geqslant 3$, respectively. In conclusion, for $t_0, t_1, t_2 \geqslant 3$, $f(t_0, t_1, t_2) \geqslant f(3, 3, 3) = 81 > 0$. The proof is completed. $\square$

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