

Constructions for positional games and applications to domination games

Ali Deniz Bagdas^a Dennis Clemens^a
Fabian Hamann^a Yannick Mogge^b

Submitted: Sep 5, 2025; Accepted: Jul 3, 2026; Published: Aug 28, 2026

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Abstract

We present constructions regarding the general behaviour of biased positional games, and amongst others show that the outcome of such a game can differ in an arbitrary way depending on which player starts the game, and that fair biased games can behave highly non-monotonic. We construct a gadget that helps to transfer such results to Maker-Breaker domination games, and by this we extend a recent result by Gledel, Iršič, and Klavžar, regarding the length of such games. Additionally, we introduce Waiter-Client domination games, give tight results when they are played on trees or cycles, and using our transference gadget we show that in general the length of such games can differ arbitrarily from the length of their Maker-Breaker analogue.

Mathematics Subject Classifications: 05C57, 05C69

1 Introduction

Biased Maker-Breaker games. Maker-Breaker games, a special kind of positional games, have been studied extensively throughout the last decades. For a nice overview we recommend the monograph [26] as well as the survey [31]. In most generality, let a hypergraph $\mathcal{H} = (X, \mathcal{F})$ and two integers $m, b \geq 1$ be given. The $(m : b)$ *Maker-Breaker game* on $(X, \mathcal{F})$ is played as follows. Maker and Breaker alternate in moves, where in a move Maker claims up to m unclaimed elements of the *board* X , and Breaker claims up to b unclaimed elements of X . Maker wins if during the course of the game she manages to claim all elements of a *winning set*, i.e. a hyperedge from $\mathcal{F}$, while Breaker wins otherwise. Surely, this outcome can depend on who makes the first move. Therefore, whenever it

^aHamburg University of Technology, Germany (ali.bagdas@tuhh.de, dennis.clemens@tuhh.de, fabian.hamann@tuhh.de).

^bUniversité Claude Bernard Lyon 1, France (yannick.mogge@univ-lyon1.fr).

makes a difference in the following, we state clearly whom we assume to be the first player. If $m = b = 1$, the game is called *unbiased*; and otherwise it is called *biased*.

A nice property of $(m : b)$ Maker-Breaker games is that these are monotone with respect to each of the biases m and b . That is, roughly speaking, increasing the bias of one of the players can never be a disadvantage for this player; see e.g. [3, 26]. This observation leads to the natural definition of *threshold biases*, which have been studied extensively already and in many cases have proven to be related to properties of random graphs, see e.g. [4, 19, 22, 30, 34]. When both biases get increased simultaneously, we however cannot expect monotonicity in general, and interestingly not so much is known when both biases get increased at the same rate. For instance, a famous result of Beck [3], supporting a so-called random graph intuition, states that in the $(1 : 1)$ Maker-Breaker game on the edges of the complete graph K_n , Maker has a strategy to occupy a clique of size $(2 - o(1)) \log_2(n)$, but not larger. Still, it remains one of the bigger unsolved open problems stated in Beck's monograph (see Problem 31.1 in [3]), whether Maker can do the same in the $(2 : 2)$ Maker-Breaker game, see also [21]. On the other hand, Balogh et al. [2] considered the 2-diameter game in which Maker's goal is to occupy a spanning subgraph of K_n with diameter 2, and they proved that Breaker wins the $(1 : 1)$ variant of this game, while Maker has a winning strategy for the $(2 : b)$ variant even when $b \leq \frac{1}{9}n^{1/8}(\log n)^{-3/8}$, provided n is large enough. However, when we look only at fair $(b : b)$ games in this case, we can still observe monotonicity, as Maker wins for all $b \geq 2$ if n is large enough.

In contrast to that, for our first result we prove that fair biased Maker-Breaker games, i.e. when both players have the same bias, can behave very highly non-monotonic.

Theorem 1. *Let $B \subseteq \mathbb{N}$ be any finite set. Then there exists a hypergraph $\mathcal{H} = (X, \mathcal{F})$, such that Maker wins the $(b : b)$ Maker-Breaker game on $\mathcal{H}$ if and only if $b \notin B$, when Maker is the first player.*

Next to studying threshold biases, another central topic in positional games is to find fast winning strategies, see e.g. [9, 10, 17, 18, 24, 33]. For many results in these papers, e.g. when Maker aims for a perfect matching, a Hamilton cycle or a fixed spanning tree in the $(1 : 1)$ game on K_n , the outcome turns out to be independent of who starts the game, for large enough n . Therefore, it becomes a natural question whether the time that Maker needs for winning can differ a lot when we compare the case when Maker starts with the case when Breaker starts. Our next result roughly states that almost anything can happen. Additionally, we are able to give an analogous statement when we consider the smallest winning set that Maker can occupy.

Theorem 2. *Let m, b, s, s', t, t' be positive integers with $s' \geq s \geq 2m + 1$, $m \leq b$, $t' \geq t \geq \lceil \frac{s}{m} \rceil$ and $t' \geq \lceil \frac{s'}{m'} \rceil$. Then there exists a hypergraph $\mathcal{H} = (X, \mathcal{F})$ which satisfies the following properties:*

- (1) *Let Maker be the first player in the $(m : b)$ Maker-Breaker game on $\mathcal{H}$. Then she has a strategy to occupy a winning set of size s within t rounds, but she can neither win within less than t rounds nor occupy a winning set of size smaller than s .*

- (2) Let Maker be the second player in the $(m : b)$ Maker-Breaker game on $\mathcal{H}$. Then she has a strategy to occupy a winning set of size s' within t' rounds, but she can neither win within less than t' rounds nor occupy a winning set of size smaller than s' .

Note that all conditions on the parameters m, b, s, s', t, t' in the theorem above are optimal except for maybe the inequalities $s \geq 2m + 1$ and $m \leq b$. We discuss this in more detail in the concluding remarks.

Maker-Breaker domination games. Many publications on positional games consider the case when X is the edge set of some given graph G , and $\mathcal{F}$ contains the edge sets of e.g. all spanning subgraphs of G that can be described by a monotone graph property, such as being connected [22, 32], containing a Hamilton cycle [25, 30], or similar. More recently, Duchêne et al. [15] introduced *Maker-Breaker domination games*, which in contrast to the above games are played on the vertex set of a given graph G by two players called Dominator and Staller, where Dominator (who is playing as Maker) wins if and only if she manages to occupy all vertices of a dominating set. Note that the renaming of the players Maker and Breaker as Dominator and Staller is done to be consistent with the usual domination games; for an overview on these games we recommend the book [7].

While Duchêne et al. [15] proved that deciding who wins an unbiased Maker-Breaker domination game is PSPACE-complete, Gledel et al. [23] put a focus on the number of rounds which Dominator needs to win. Following their notation, let $\gamma_{MB}(G, m : b)$ denote the smallest number of rounds in which Dominator can always win the $(m : b)$ domination game on G , provided that she starts the game, and where we set $\gamma_{MB}(G, m : b) = \infty$ if Dominator does not have a winning strategy. Similarly, let $\gamma'_{MB}(G, m : b)$ denote the smallest number of rounds for the case when Staller starts the game, and for short, let $\gamma_{MB}(G) := \gamma_{MB}(G, 1 : 1)$ and $\gamma'_{MB}(G) := \gamma'_{MB}(G, 1 : 1)$. Gledel et al. [23] determined $\gamma_{MB}(G)$ and $\gamma'_{MB}(G)$ precisely when G is a tree or a cycle, and partial results for Cartesian products [14, 20] and Corona products [12] of graphs were obtained afterwards as well. Recently, biased variants of these games were considered in [1, 6]. Additionally, Gledel et al. [23] provided examples which show that the domination number $\gamma(G)$ of a graph G and the Maker-Breaker domination numbers $\gamma_{MB}(G)$ and $\gamma'_{MB}(G)$ can take almost arbitrary values.

Theorem 3. [Theorem 3.1 in [23]] If G is a graph, then $\gamma(G) \leq \gamma_{MB}(G) \leq \gamma'_{MB}(G)$. Moreover, for any integers r, t, t' with $2 \leq r \leq t \leq t'$, there exists a graph G such that $\gamma(G) = r$, $\gamma_{MB}(G) = t$ and $\gamma'_{MB}(G) = t'$.

We note that similar constructive results were obtained very recently in [13] for *total domination games*, in which Maker's goal is to occupy a total dominating set.

Next to considering the number of rounds, it seems natural to ask for the smallest size of a dominating set that Dominator can always claim when playing the $(m : b)$ domination game on a graph G . Let $s_{MB}(G, m : b)$ and $s_{MB}(G) = s_{MB}(G, 1 : 1)$ denote this values when Dominator starts the game, and let $s'_{MB}(G, m : b)$ and $s'_{MB}(G) = s'_{MB}(G, 1 : 1)$ denote this value when Staller starts, where again we set such a value to ∞ if Dominator does not have a winning strategy. A priori it is not clear why the minimal number of

rounds and the minimal size of a dominating set that Dominator can achieve should be different. In fact, from the proofs in [23] it can be deduced easily that $\gamma_{MB}(G) = s_{MB}(G)$ and $\gamma'_{MB}(G) = s'_{MB}(G)$ hold when G is a tree or a cycle. Moreover, it is not hard to check that if G is the Erdős-Rényi random graph $G(n, p)$ with constant $p \in (0, 1)$ or a random geometric graph $\mathcal{G}(n, r)$ with constant radius $r \in (0, 1)$, then with high probability the parameters $\gamma_{MB}(G)$ and $s_{MB}(G)$ are asymptotically the same (and maybe equal), see also the concluding remarks. However, in contrast to this, we can construct graphs where the mentioned game parameters can take almost arbitrary values, hence extending Theorem 3.

Theorem 4. *Let m, b, r, s, s', t, t' be positive integers with $s' \geq s \geq \max\{r, 2m + 1\}$, $m \leq \min\{b, r - 1\}$, $t' \geq t \geq \lceil \frac{s}{m} \rceil$ and $t' \geq \lceil \frac{s'}{m'} \rceil$. Then there exists a graph G such that*

$$\begin{aligned}\gamma(G) &= r, \\ s_{MB}(G, m : b) &= s \quad \text{and} \quad \gamma_{MB}(G, m : b) = t \\ s'_{MB}(G, m : b) &= s' \quad \text{and} \quad \gamma'_{MB}(G, m : b) = t' .\end{aligned}$$

We again discuss the optimality of our conditions in Section 5. Also note that this theorem implies Theorem 2, but later we actually need Theorem 2 for the proof of Theorem 4. In fact, we construct a gadget (see Lemma 12) which allows us to transfer constructive results from general positional games to domination games.

With the help of the same transference argument it is possible to prove an analogue of Theorem 1 for Maker-Breaker domination games. Additionally, we are able to prove the following rather non-intuitive results which demonstrate that Dominator might need to employ very different strategies depending on whether she wants to win as fast as possible or claim a dominating set of smallest possible size. The first result roughly states that Dominator can be forced to claim an arbitrarily large dominating set before she is able to claim a smallest possible dominating set, i.e. she might need to play on after she already claimed a dominating set if she wants to minimize the size.

Theorem 5. *For any biases $m \leq b$ and any integers $t > s \geq 2m + 1$, there exists a graph G such that $s_{MB}(G, m : b) = s$, but Dominator cannot occupy a dominating set of size s before she has occupied another minimal dominating set of size t .*

The second result states that there are graphs for which Dominator can only achieve optimal time if she claims an arbitrarily large dominating set, while the smallest possible dominating set can take arbitrarily many rounds for her to claim, i.e. there are cases where Dominator needs to decide which goal she wants to pursue, since she cannot achieve both with the same strategy.

Theorem 6. *For any biases $m \leq b$ and any integers $s' \geq s \geq 2m + 1$, and $t' \geq t \geq \lceil \frac{s'}{m} \rceil$ there exists a graph G such that $\gamma_{MB}(G, m : b) = t$ and $s_{MB}(G, m : b) = s$, but in the $(m : b)$ Maker-Breaker domination game on G Breaker has a strategy such that all of the following hold:*

- *Maker cannot claim a dominating set of size smaller than s .*

- *Maker cannot win in less than t rounds.*
- *Maker cannot claim a dominating set of size smaller than s' if she wins in less than t' moves.*

Waiter-Client domination games. In the last years, *Waiter-Client games* (earlier called Picker-Chooser games, see e.g. [3]) have received increasing attention, ranging from results on fast winning strategies [10, 16] over biased games [5, 27, 34] to games played on random graphs [11, 28]. In the following we stick to the case of unbiased Waiter-Client games only. On a given hypergraph $\mathcal{H} = (X, \mathcal{F})$, these games are played almost the same way as Maker-Breaker games with the following difference: In every round, Waiter chooses two unclaimed elements of the board X and then Client decides which of these elements goes to Waiter while the other one goes to Client. (If in the last round there is only one unclaimed element left, then it is claimed by Client.) Waiter wins if and only if she manages to claim all elements of a winning set from $\mathcal{F}$.¹

So far, domination games have not been studied in this setting. So, we aim to give first results for Waiter-Client domination games and also study how Waiter-Client and Maker-Breaker domination games are related. Given a graph G , we define the *Waiter-Client domination game* on G in the obvious way: Dominator (playing as Waiter) offers two unclaimed vertices of G and then Staller (playing as Client) picks one of these vertices for himself and the other goes to Dominator. In accordance with previous notation, we denote with $\gamma_{WC}(G)$ the smallest number of rounds in which Dominator can always occupy a dominating set in the Waiter-Client domination game on G , and we let $s_{WC}(G)$ denote the size of the smallest dominating set that Waiter can always claim. For our first results in this setup, we see that for cycles and trees the game behaves the same way as in the Maker-Breaker setting [23] (when Breaker starts).

Theorem 7. *Let T be a tree on n vertices. If T has a perfect matching, then*

$$\gamma_{WC}(T) = s_{WC}(T) = \frac{n}{2}.$$

In all other cases, Dominator does not win the $(1 : 1)$ Waiter-Client domination game on T .

Theorem 8. *For every $n \geq 3$,*

$$\gamma_{WC}(C_n) = s_{WC}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor.$$

Due to these results and due to the fact that Waiter-Client games often tend to be easier wins for Waiter than for Maker in the corresponding Maker-Breaker games, one could ask whether a relation such as $\gamma_{WC}(G) \leq \gamma'_{MB}(G)$ can be proven for any graph G .

¹In some papers, Waiter-Client games are defined so that Waiter's goal is to force Client to claim a winning set; and then accordingly if there is a unique element in the last round, it goes to Waiter. Both versions are essentially the same.

By using our transference lemma together with suitable constructions of hypergraphs $\mathcal{H} = (X, \mathcal{F})$ for Maker-Breaker and Waiter-Client games, we answer this question negatively as follows.

Theorem 9. *For all integers $s, t \geq 7$ there is a graph G such that $\gamma'_{MB}(G) = s$ and $\gamma_{WC}(G) = t$.*

Organization of the paper. In Section 2 we prove our central lemma which helps to transfer general results on positional games into results on domination games. Afterwards, in Section 3, we prove Theorem 1 and Theorem 2, and building on the transference lemma, we prove Theorem 4, Theorem 5 and Theorem 6. In Section 4 we then prove all our results for the Waiter-Client setting: Theorem 7, Theorem 8 and Theorem 9.

Notation. We use standard notation. Given a graph G , we write $V(G)$ for its vertex set and $v(G) = |V(G)|$, and we write $G[X]$ for the subgraph induced on a vertex set X . Given a vertex v , we let $N_G(v)$ be the neighbourhood of v in G and set $d_G(v) = |N_G(v)|$ for the degree of v . Moreover, given any subset $A \subseteq V(G)$, we set $G - A = G[V(G) \setminus A]$ for short.

2 From hypergraphs to domination games

In order to describe our transference lemma, we make use of the following definitions.

Definition 10. Let $\mathcal{H}$ be a hypergraph, let G be a graph, let $a, b, s, t \in \mathbb{N}$. Then we write

- $\mathcal{H} \in M_i(a : b, s, t)$ if Maker has a strategy to claim a winning set of size at most s within at most t rounds in the $(a : b)$ Maker-Breaker game on $\mathcal{H}$, when Maker is the i^{th} player.
- $\mathcal{H} \in W(s, t)$ if Waiter has a strategy to claim a winning set of size at most s within at most t rounds in the unbiased Waiter-Client game on $\mathcal{H}$.
- $G \in D_{MB,i}(a : b, s, t)$ if Dominator has a strategy to occupy a dominating set of size at most s within at most t rounds in the $(a : b)$ Maker-Breaker domination game on G , when Dominator is the i^{th} player.
- $G \in D_{WC}(s, t)$ if Dominator has a strategy to occupy a dominating set of size at most s within at most t rounds in the unbiased Waiter-Client domination game on G .

Definition 11. Let $\mathcal{H} = (X, \mathcal{F})$ be a hypergraph, let $a \geq 1$ be an integer. Let $\mathcal{V}(\mathcal{H}) \subseteq \mathcal{P}(X)$ be the set of all vertex covers of $\mathcal{H}$, and for each $A \in \mathcal{V}(\mathcal{H})$, let X_A be a vertex set of size $4a(|X| + |\mathcal{V}(\mathcal{H})|)$, such that all the sets X_A and X are pairwise disjoint. We construct a graph $G = G(\mathcal{H}, a)$ as follows:

- its vertex set is $V(G) = X \cup (\bigcup_{A \in \mathcal{V}(\mathcal{H})} X_A)$, and

- its edge set is $E(G) = \{vw : (\exists A \in \mathcal{V}(\mathcal{H}) : v \in A \wedge w \in X_A)\} \cup \binom{X}{2}$.

Lemma 12 (Transference Lemma). *Let $\mathcal{H} = (X, \mathcal{F})$ be a hypergraph, let $a, b, s, t \in \mathbb{N}$, and let $G = G(\mathcal{H}, a)$ be constructed according to Definition 11. Then the following holds:*

$$(M) \mathcal{H} \in M_i(a : b, s, t) \Leftrightarrow G \in D_{MB,i}(a : b, s, t),$$

$$(W) \mathcal{H} \in W(s, t) \Leftrightarrow G \in D_{WC}(s, t).$$

Note that in (W) we stick only to the unbiased game as this is all we need for our later results; but a biased version would also be possible.

Before we prove Lemma 12 we first collect useful claims. From now on, let $\mathcal{H}$ and $G = G(\mathcal{H}, a)$ be fixed according to the assumptions of Lemma 12.

Claim 13. *Each $F \in \mathcal{F}$ is a dominating set of G .*

Proof. Let $v \in V(G) \setminus F$. If $v \in X$, then it is dominated by F since $G[X]$ is a complete graph. If $v \notin X$, then there is a set $A \in \mathcal{V}(\mathcal{H})$ such that $v \in X_A$, and therefore $A = N_G(v)$. Since A is a vertex-cover of $\mathcal{H}$, it holds that $A \cap F \neq \emptyset$, i.e. F contains a neighbour of v . $\square$

Claim 14. *Let $k := \min\{|F| : F \in \mathcal{F}\}$. The domination number of G is $\gamma(G) = k$.*

Proof. By Claim 13 we have $\gamma(G) \leq k$. Now, for contradiction, assume that there is dominating set D of size smaller than k . Then $A := X \setminus D$ is a vertex-cover of $\mathcal{H}$, since every hyperedge $F \in \mathcal{F}$ satisfies $|F| > |D|$. But then no vertex in X_A is dominated by $X \cap D$. So, if D is a dominating set, we then need $X_A \subseteq D \setminus X$. But this implies $|D| \geq |X_A| > 4|X| \geq 4k$, a contradiction. $\square$

Claim 15. *If $D \subseteq X$ is a dominating set in G , then there is some $F \in \mathcal{F}$ such that $F \subseteq D$.*

Proof. Let $D \subseteq X$ be a dominating set in G , and assume there is no $F \in \mathcal{F}$ such that $F \subseteq D$. Then each $F \in \mathcal{F}$ intersects $A := X \setminus D$ and hence A is a vertex-cover of $\mathcal{H}$. But then, no vertex in X_A has a neighbour in D , a contradiction. $\square$

With the above claims in hand, we can prove Lemma 12.

Proof of Lemma 12. (M) " $\Rightarrow$ " Assume that $\mathcal{H} \in M_i(a : b, s, t)$, i.e. Maker (being the i^{th} player) has a strategy $\mathcal{S}_M$ to claim a winning set of size at most s within at most t rounds in the $(a : b)$ Maker-Breaker game on $\mathcal{H}$. Then in the $(a : b)$ Maker-Breaker domination game on G , Dominator (being the i^{th} player) can restrict her moves to $G[X]$ and play according to the strategy $\mathcal{S}_M$. This way, Dominator claims a set $F \in \mathcal{F}$ of size at most s within at most t rounds, and thus by Claim 13, we have $G \in D_{MB,i}(a : b, s, t)$.

" $\Leftarrow$ " We do a proof by contraposition. Assume that $\mathcal{H} \notin M_i(a : b, s, t)$, and therefore Breaker has a strategy $\mathcal{S}_B$ in the $(a : b)$ Maker-Breaker game on $\mathcal{H}$ (where Maker is the i^{th} player) which prevents Maker from claiming a winning set of size at most s within the first t rounds. Then, in the $(a : b)$ Maker-Breaker domination game on G (where Dominator is the i^{th} player), Staller can apply the following strategy:

Stage I: As long as before Staller's move there is at least one free vertex in X , Staller plays on $G[X]$ only, and he plays according to Breaker's strategy $\mathcal{S}_B$. (If Dominator claims an element outside of X , Staller simply pretends that Dominator made an arbitrary move within X .)

Stage II: When Staller enters Stage II, all elements of X are claimed. For each $A \in \mathcal{V}(\mathcal{H})$, let $X'_A \subseteq X_A$ be the vertices which are still free at the end of Stage I. Then from now on, for each vertex that Staller can claim, Staller chooses a vertex from an arbitrary set X'_A in which he still has no vertex claimed.

Note that Stage I lasts at most $|X|$ rounds, and in the meantime Dominator cannot claim more than $a|X|$ vertices. Hence, for Stage II of Staller's strategy we have $|X'_A| \geq 4a|\mathcal{V}(\mathcal{H})|$ for every $A \in \mathcal{V}(\mathcal{H})$. Stage II lasts at most $|\mathcal{V}(\mathcal{H})|$ rounds, since Staller only wants to claim one vertex in each of the sets X'_A . Hence, during this whole stage Dominator does not claim more than $a|\mathcal{V}(\mathcal{H})|$ vertices, which implies that at any point in Stage II, there are at least $2a|\mathcal{V}(\mathcal{H})|$ free vertices in each set X'_A and thus Staller can always claim a free vertex as required by Stage II of the strategy.

It remains to prove that, by following the strategy, Staller prevents Dominator from getting a dominating set of G of size at most s within the first t rounds. For contradiction, assume the opposite, i.e. Dominator claims a dominating set D with $|D| \leq s$ within at most t rounds. Then $D' := D \cap X$ would be a dominating set by the following reason: Because of Stage II, for each $A \in \mathcal{V}(\mathcal{H})$, there is vertex $v \in X_A$ such that $v \notin D$. Hence, since $N_G(v) = A \subseteq X$, we get $D \cap A \neq \emptyset$ and therefore $D' \cap A \neq \emptyset$. Thus, D' dominates every vertex in X_A for every $A \in \mathcal{V}(\mathcal{H})$, and it also dominates every vertex in $X \setminus D' = X \setminus D$ since $G[X]$ is a complete graph.

By Claim 15 it follows that there is some $F \in \mathcal{F}$ with $F \subseteq D' \subseteq D$, and note that $|F| \leq |D| \leq s$ by assumption. However, by following Stage I, Staller ensures that Dominator needs to play more than t rounds on X to occupy a winning set $F \in \mathcal{F}$ of size at most s , a contradiction.

(W) $\Rightarrow$ The proof is essentially the same as for (M).

$\Leftarrow$ The proof is very similar to the corresponding proof for (M). Assume that Client has a strategy $\mathcal{S}_C$ in the Waiter-Client game on $\mathcal{H}$ which prevents Waiter from getting a winning set of size at most s within the first t rounds. Now, in the Waiter-Client domination game on G , Staller can apply the following strategy which distinguishes three cases.

Case I: When Dominator offers two elements of X , then Staller distributes these vertices according to the strategy $\mathcal{S}_C$.

Case II: When Dominator offers a vertex $v \in X$ and a vertex $w \notin X$, then Staller claims v and gives w to Dominator. (For the strategy $\mathcal{S}_C$ in Case I, Client imagines that v was not played yet.)

Case III: When Dominator offers two vertices $v \in X_{A_1}$ and $w \in X_{A_2}$ with $A_1, A_2 \in \mathcal{V}(\mathcal{H})$, then Staller claims v if he does not have a vertex in X_{A_1} yet, and otherwise Staller claims w . The remaining vertex goes to Dominator.

By the strategy $\mathcal{S}_C$ in Case I, and since Dominator never gets an element of X in the other cases, Dominator needs more than t rounds in order to claim a winning set of $\mathcal{F}$ with size at most s . Moreover, by the strategy from Cases II and III, Staller makes sure to receive at least one vertex in each set X_A . For this note that for each $A \in \mathcal{V}(\mathcal{H})$ there can be at most $|X|$ rounds of Case II and at most $|\mathcal{V}(\mathcal{H})|$ rounds of Case III, in which a vertex $x \in X_A$ is offered while Staller still does not have a vertex in X_A , but Staller decides to claim the other vertex which was offered by Dominator.

Now, using these observation, the same discussion as at the end of the proof of (M) gives that Staller prevents Dominator from occupying a dominating set of G of size at most s within the first t rounds. $\square$

3 Constructions

3.1 A Maker-Breaker like game on directed multigraphs

As a first building block towards our constructions for Theorems 2 and 4–6, we define certain directed multigraphs $\vec{G}_t(b)$ and $\vec{H}_t(b)$, and also analyze a suitably defined positional game on these directed multigraphs. We start with the definition of $\vec{G}_t(b)$.

Definition 16. Let positive integers b, t be given. We iteratively define a directed multigraph $\vec{G}_t(b)$ with startpoint v_0 and endpoint v_t as follows:

1. Let $\vec{G}_1(b)$ consist of two vertices v_0 and v_1 , and one directed edge going from v_0 to v_1 .
2. Given $\vec{G}_i(b)$ for some $1 \leq i \leq t-1$, with endpoints v_0 and v_i , we obtain $\vec{G}_{i+1}(b)$ as follows:
 - (a) add a new vertex v_{i+1} ,
 - (b) add b copies of $\vec{G}_i(b)$, each starting in v_i and ending in v_{i+1} , such that these copies intersect only in v_i and v_{i+1} and they intersect the initial graph $\vec{G}_i(b)$ only in v_i .

For an example, see Figure 3.1. Note that by a simple induction on t it can be seen that all vertices of $\vec{G}_t(b)$ except from v_0 and v_t have outdegree b .

Definition 17. Let positive integers b, t with $t \geq 3$ be given. We construct $\vec{H}_t(b)$ as follows: We take $b+2$ vertices $x_0, \dots, x_{b+1}$. For every $i \in [b+1]$, we add $b+1$ different copies of $\vec{G}_{t-2}(b)$ with startpoint x_0 and endpoint x_i , such that these copies intersect only in x_0 and x_i and use new vertices otherwise.

We next define an auxiliary game on directed multigraphs and analyze how fast Maker can win this game on $\vec{G}_t(b)$ or $\vec{H}_t(b)$.

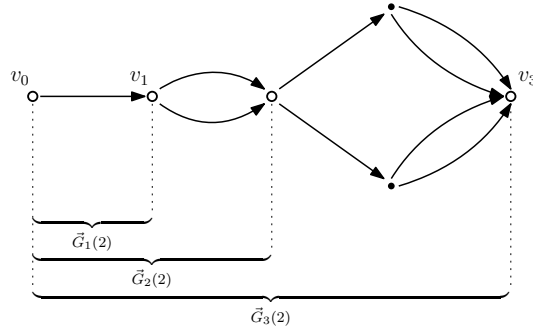


Figure 3.1: $\vec{G}_t(2)$ for $t \in \{1, 2, 3\}$

Definition 18. Let a directed multigraph $\vec{F}$ and a subset $S \subseteq V(\vec{F})$ be given. The game $\mathcal{G}(b, \vec{F}, S)$ is a $(1 : b)$ Maker-Breaker game played on the vertices and the directed edges of $\vec{F}$, where Maker is the first player, with the following adjustments:

- (a) Maker can only claim a directed edge (v, w) of $\vec{F}$ if she already claimed both vertices v, w .
- (b) At the beginning of the game, the vertices in S already belong to Maker.

Maker wins the game $\mathcal{G}(b, \vec{F}, S)$ if she manages to claim a directed edge; Breaker wins otherwise.

We note that for the description of the game above we could actually replace directed edges with simple edges. However, keeping these directions allows us later to define a partial ordering on the vertex set for the description and analysis of suitable strategies. We start with the following lemma.

Lemma 19. Let positive integers b, t be given. Let v_0 and v_t be the start- and endpoint of $\vec{G}_t(b)$. Then the following holds:

1. Maker has a strategy to win $\mathcal{G}(b, \vec{G}_t(b), \{v_0, v_t\})$ within t rounds.
2. Breaker has a strategy to win $\mathcal{G}(b, \vec{G}_t(b), \{v\})$ for any vertex v .
3. Breaker has a strategy to prevent Maker from winning $\mathcal{G}(b, \vec{G}_t(b), \{v_0, v_t\})$ within $t - 1$ rounds.

Proof. Throughout the proof, if x, y are vertices in $\vec{G}_t(b)$, we write $x \preccurlyeq y$ if there exists a directed path starting in x and ending in y . Note that this defines a partial ordering on $V(\vec{G}_t(b))$.

We first prove property (1) by induction on t . For $t = 1$, Maker already possesses v_0 and v_1 before the game starts, and hence she can win in the first round by claiming the directed edge (v_0, v_1) . For $t > 1$, Maker possesses v_0 and v_t before the game starts. For her first move, she claims the vertex v_{t-1} . As there are $b + 1$ copies of $\vec{G}_{t-1}(b)$, that have

their start- and endpoints in $\{v_0, v_{t-1}, v_t\}$ and which are pairwise disjoint outside of this set, and since Breaker's bias is b , after Breaker's first move there must be at least one copy of $\vec{G}_{t-1}(b)$ with no Breaker vertices, but with the start- and endpoint claimed by Maker. By induction Maker has a strategy to win within the next $t - 1$ rounds, hence proving the statement (1).

Next, consider statement (2). We prove by induction on the number of rounds that in the game $\mathcal{G}(b, \vec{G}_t(b), \{v\})$, Breaker can play in such a way that after each of his moves the following properties hold:

- (P1) There is at most one Maker's vertex x with free outgoing edges (where "free" means that it is not claimed by Breaker yet).
- (P2) For all vertices $y \neq z$ with $y \preceq z$: If y and z are claimed by Maker, then Breaker has claimed all outgoing edges at y .

Note that (P1) and (P2) are true before the game starts, as Maker possesses only the vertex v at that point. For induction, assume that (P1) and (P2) hold after the $(i - 1)$ -st round for some $i \in \mathbb{N}$, and let w be the vertex that Maker claims in round i . If all of Maker's vertices except from w have no free outgoing edges, Breaker simply claims all outgoing edges at w , and (P1) and (P2) follow immediately. Hence, using (P1) from the induction hypothesis, we may assume from now on that there is a unique vertex $w' \neq w$ that was claimed by Maker before the i -th round, and which still has free outgoing edges. If $w' \preceq w$, then Breaker claims all outgoing edges at w' . Then (P1) is maintained, as w is the only Maker's vertex that can have free outgoing edges. Also, due to this, if (P2) would fail to hold at the end of the i -th round, then we would need $w \preceq z$ for some Maker's vertex z that was claimed in an earlier round. But then also $w' \preceq z$ would need to hold by transitivity, while w' had free outgoing edges at the end of the $(i - 1)$ -st round, in contradiction to (P2) at the end of the $(i - 1)$ -st round. Hence, (P2) is maintained. If otherwise $w' \preceq w$ does not hold, then Breaker claims all outgoing edges at w . Then (P1) remains true, with w' staying the unique Maker's vertex with free outgoing edges. If (P2) would fail to hold at the end of the i -th round, then we would need $w' \preceq z$ for some Maker's vertex z . As (P2) was true at the end of the $(i - 1)$ -st round, z cannot be claimed until the $(i - 1)$ -st round, implying $z = w$, which however is in contradiction to the assumption of this case. Thus, (P2) holds again, finishing the argument that Breaker can always maintain both properties.

Finally, note that (P2) implies that Maker can never claim a directed edge (y, z) , since when y and z are claimed by Maker, Breaker claims all outgoing edges at y .

It remains to prove (3) about the game $\mathcal{G}(b, \vec{G}_t(b), \{v_0, v_t\})$. We again use induction on the number of rounds and prove that Breaker has a strategy that after his i -th move the following properties hold for all $i \in [t - 1]$:

- (P1') There is at most one Maker's vertex x with free outgoing edges.
- (P2') For all vertices $y \neq z$ with $y \preceq z$: If y and z are claimed by Maker and there is a directed path from y to z of length less than 2^{t-i-1} , then Breaker has claimed all outgoing edges at y .

At the start of the game (P1') and (P2') hold by definition. The only vertex with free outgoing edges is v_0 , and the shortest directed path between v_0 and v_t is by construction of length 2^{t-1} .

Now assume (P1') and (P2') hold after Breaker's $(i-1)$ -th move for some $i \in \mathbb{N}$. Let x be the single vertex with free outgoing edges claimed by Maker. For every other vertex z with $x \preceq z$ claimed by Maker we know that the shortest directed path from x to z is of length at least $2^{t-(i-1)-1}$, because otherwise Breaker would have claimed all outgoing edges of x by property (P2').

Let y be the vertex claimed by Maker in her i -th move. If $x \not\preceq y$, Breaker can claim all outgoing edges at y , and (P1') and (P2') hold as before. Otherwise, let ℓ be the length of a shortest directed path from x to y . If $\ell \geq 2^{t-i-1}$, Breaker again claims all outgoing edges at y , such that (P1') and (P2') hold true afterwards. Otherwise if $\ell < 2^{t-i-1}$, he claims all outgoing edges at x such that (P1') still trivially holds. For (P2'), it remains to show that there is no short directed path from y to some vertex z with $y \preceq z$ claimed by Maker. Assume there is such a path of length less than 2^{t-i-1} to a vertex z . Then there was a directed path of length less than $2^{t-(i-1)-1}$ from x to z after Breaker's $(i-1)$ -th move contradicting the assumption of (P2').

To conclude we observe that Maker can only win in her turn if there are two vertices y and z claimed by her such that there is a directed path of length 1 from y to z which is not blocked by Breaker. Thus, because (P2') holds throughout the game, Maker cannot win in less than t moves. $\square$

Lemma 20. *Let positive integers b, t with $t \geq 3$ be given. Let $\vec{H}_t(b)$ and the vertices $x_0, \dots, x_{b+1}$ be given according to Definition 17. Then the following holds for the game $\mathcal{G}(b, \vec{H}_t(b), \emptyset)$:*

1. *Maker has a strategy to win within t rounds.*
2. *Breaker has a winning strategy if he can claim a single vertex before Maker starts with her first move.*
3. *Breaker has a strategy to prevent Maker from winning within $t-1$ rounds.*

Proof. We first prove (1). Maker claims x_0 in her first move. Since Breaker can only claim b elements in his move, there has to be an $i \in [b+1]$ such that Breaker claimed no vertices from the copies of $\vec{G}_{t-2}(b)$ that start in x_0 and end in x_i . Then Maker claims x_i , and analogously we have that after Breaker's next move there must be a copy of $\vec{G}_{t-2}(b)$ from x_0 to x_i that Breaker did not touch yet. By Lemma 19(1), playing only on this copy, Maker then has a strategy to win the game within the next $t-2$ rounds, proving (1).

To prove (2), Breaker claims x_0 as the single element before Maker's first move. Afterwards, we may pretend that Maker already claimed all vertices $x_1 \dots x_{b+1}$; then every free vertex belongs to a unique copy of $\vec{G}_{t-2}(b)$. On these copies, Breaker can then separately apply the strategy guaranteed by Lemma 19(2), and hence win the game.

It remains to prove (3). We distinguish two cases depending on Maker's first move. First, we consider the case when Maker does not take x_0 in her first move, and show that Breaker actually wins the game in that case.

Claim 21. *If Maker does not claim x_0 in her first move, Breaker has a strategy to win the game.*

Proof. Independent of Maker's first move (except x_0), Breaker will claim x_0 in his first move (and arbitrary other vertices or edges if $b > 1$). Afterwards, Breaker claims in each of his moves all outgoing edges of a vertex claimed by Maker which is possible because the outdegrees are bounded by b . By doing so, he makes sure that after his move there will be at most one vertex claimed by Maker with free outgoing edges. If Maker ever claims a vertex without outgoing edges, i.e. a vertex from $\{x_1, \dots, x_{b+1}\}$, there will be no Maker's vertices with free outgoing edges after Breaker's next move, and after all further Breaker moves as well. This is even the case if Maker claims a vertex from $\{x_1, \dots, x_{b+1}\}$ in her first move. Since Maker can only win if there is a Maker's vertex with free outgoing edges at the start of her turn, she will not be able to win anymore if she ever claims a vertex from $\{x_1, \dots, x_{b+1}\}$. Thus, we assume she never claims a vertex from $\{x_0, \dots, x_{b+1}\}$. Each vertex $v \notin \{x_0, \dots, x_{b+1}\}$ belongs to a unique copy $\vec{G}$ of $\vec{G}_{t-2}(b)$. When Maker claims v in any turn besides the first, Breaker answers on $\vec{G}$ according to his strategy from Lemma 19(2) which allows him to win on $\vec{G}$ even if Maker already claimed a vertex of $\vec{G}$ in her first move. Moreover, in the strategy of Lemma 19(2), Breaker claims in each of his turns all outgoing edges of a Maker's vertex as required. $\square$

Hence, from now on we can assume that Maker's first move is to claim x_0 . In this case, Breaker could actually skip his first move, so he claims arbitrary edges or vertices and pretends that he did not claim anything yet. Whenever his strategy leads him to claim an element he already possesses, he claims an arbitrary vertex instead. We distinguish two cases depending on Maker's second move. If Maker claims x_i for some $i \in [b+1]$ in her second move, then afterwards Breaker can pretend that Maker actually claimed $x_1, \dots, x_{b+1}$ and applies the strategy from Lemma 19(3) to every copy of $\vec{G}_{t-2}(b)$, to ensure that Maker needs at least $t-2$ further rounds for winning, as required. In this case, Breaker's second move is again arbitrary and could be skipped. Otherwise, assume that Maker claims a vertex $v \notin \{x_0, \dots, x_{b+1}\}$ in her second move, i.e. a vertex v that again belongs to a unique copy $\vec{G}_v$ of $\vec{G}_{t-2}(b)$ from x_0 to a vertex x_i , $i \in [b+1]$. Whenever Maker plays on $\vec{G}_v$ (including v in her second move, and potentially x_i in a future move), Breaker answers with his strategy from Lemma 19(2), preventing her from winning on $\vec{G}_v$. Whenever Maker plays on any other copy $\vec{G}$ of $\vec{G}_{t-2}(b)$ from x_0 to some x_i , $i \in [b+1]$, Breaker pretends that Maker already claimed the corresponding x_i , and responds on $\vec{G}$ according to his strategy from Lemma 19(3) making sure that Maker cannot win on $\vec{G}$ in less than $t-2$ rounds. Thus, she cannot win in less than t rounds as required, since she already claimed x_0 and v in her first two moves. $\square$

3.2 From digraphs to hypergraphs

Getting slightly closer to our final constructions, we now start creating suitable hypergraphs based on the directed multigraphs $\vec{H}_t(b)$.

Definition 22. Let $m, b, s, t \in \mathbb{N}$ with $s \geq 2m + 1$, $m \leq b$, and $t \geq \lceil \frac{s}{m} \rceil$. We define an s -uniform hypergraph $\mathcal{H}(m, b, s, t)$ and a family $\mathcal{V}$ of associated vertex subsets as follows:

- (a) Let $s \leq 3m$. Then take $\vec{H} := \vec{H}_t(b)$. Replace all vertices x of $\vec{H}$ with pairwise disjoint sets V_x of size m each, and replace every directed edge (x, y) with a hyperedge that contains V_x and V_y plus $s - 2m$ further vertices that are not used for any other hyperedge. Call the resulting s -uniform hypergraph $\mathcal{H}(m, b, s, t)$, and set $\mathcal{V} = \{V_x : x \in V(\vec{G})\}$ to be its family of associated sets.
- (b) Let $s > 3m$, the hypergraph $\mathcal{H}(m, b, s, t)$ is constructed as follows: we take a set V of m vertices and $b+1$ copies of $\mathcal{H}(m, b, s-m, t-1)$ that are disjoint from each other and from V . Then we add V to every hyperedge of every copy of $\mathcal{H}(m, b, s-m, t-1)$. Call the resulting s -uniform hypergraph $\mathcal{H}(m, b, s, t)$. Moreover, let $\mathcal{V}$ contain V and all associated sets from all the $b+1$ copies of $\mathcal{H}(m, b, s-m, t-1)$ used in this construction.

Figure 3.2 shows how a copy of $\vec{G}_3(2)$ contained in $\vec{H}_5(2)$ is transformed into a subhypergraph of $\mathcal{H}(5, 2, 14, 5)$, according to the above definition.

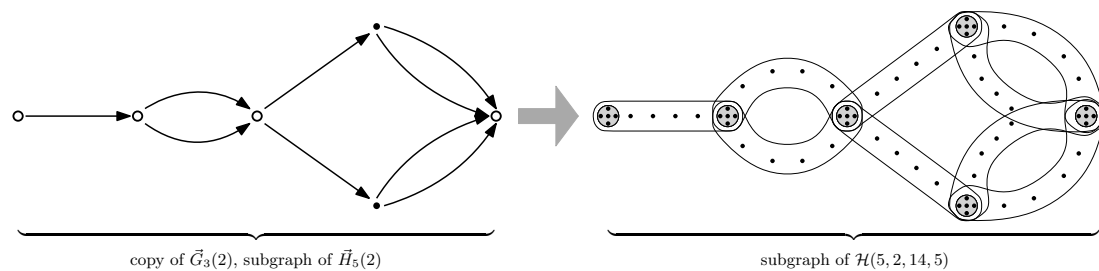


Figure 3.2: Replacing directed edges with hyperedges in Definition 22(a)

The following can be seen with an easy induction on s .

Observation 23. Let $m, b, s, t \in \mathbb{N}$ with $s \geq 2m + 1$, $m \leq b$, and $t \geq \lceil \frac{s}{m} \rceil$. The hypergraph $\mathcal{H}(m, b, s, t) = (X, \mathcal{F})$ fulfils the following properties:

1. For any $V \in \mathcal{V}$ and any $F \in \mathcal{F}$ we have that $V \subseteq F$ or $V \cap F = \emptyset$.
2. For each vertex $x \in X \setminus \bigcup_{V \in \mathcal{V}} V$ we have that x is in at most one hyperedge from $\mathcal{F}$.

Due to these properties we can allow ourselves to make some simple assumptions on Maker's strategy when playing on $(X, \mathcal{F})$, as summarized by the following proposition.

Proposition 24. Let $m, b \in \mathbb{N}$ with $m \leq b$, and let a hypergraph $\mathcal{H} = (X, \mathcal{F})$ with a collection $\mathcal{V}$ of disjoint vertex-sets fulfilling the following properties be given:

1. $|F| > m$ for all $F \in \mathcal{F}$.

2. $|V| = m$ for all $V \in \mathcal{V}$.
3. For any $V \in \mathcal{V}$ and any $F \in \mathcal{F}$ we have that $V \subseteq F$ or $V \cap F = \emptyset$.
4. For each vertex $x \in X \setminus \bigcup_{V \in \mathcal{V}} V$ we have that x is in at most one hyperedge from $\mathcal{F}$.

If Maker has a winning strategy playing an $(m : b)$ -game on $\mathcal{H}$, then she also has a winning strategy playing an $(m : b)$ -game on $\mathcal{H}$ where in each of her moves she either claims all vertices of a set $V \in \mathcal{V}$, or fully claims all remaining vertices of a winning set $F \in \mathcal{F}$ to immediately win the game.

Proof. Let M_i denote the set of vertices Maker takes in the i -th round of the game when she plays her winning strategy, and let $\mathcal{F}_i$ denote all hyperedges which are not touched by Breaker before the i -th round. Let j be the smallest integer such that $M_j \notin \mathcal{V}$. Assume Maker did not win after j moves. We aim to show that Breaker can claim a set B_j such that for all $F \in \mathcal{F}_j$ with $F \cap M_j \neq \emptyset$ we also have $F \cap B_j \neq \emptyset$. Note that $|B_j| \geq |M_j|$ since $b \geq m$. Thus, it suffices to show that for each $x \in M_j$ Breaker can claim a vertex $y \in X$ such that $y \in F$ for all $F \in \mathcal{F}_j$ with $x \in F$. If $x \notin \bigcup_{V \in \mathcal{V}} V$, then there is at most one $F \in \mathcal{F}_j$ with $x \in F$ by (4). Since Maker did not win with her j -th move, there are still vertices of F available, and Breaker can claim an arbitrary vertex of those. Otherwise, there is $V \in \mathcal{V}$ with $x \in V$. Since $M_j \notin \mathcal{V}$, and $M_i \in \mathcal{V}$ for all $i < j$, Maker cannot have claimed all vertices of V yet. If Breaker has not already claimed a vertex of V , he can take an arbitrary $y \in V \setminus M_j$. By (3), $y \in F$ for all $F \in \mathcal{F}_j$ with $x \in F$ as required.

Hence, Maker is not able to complete any winning set that contains a vertex from M_j , and she also wins the game if she skips her j -th move. Repeating this argument gives the desired winning strategy. $\square$

In particular, we can deduce the following main lemma of this section.

Lemma 25. *Let $m, b, s, t \in \mathbb{N}$ with $s \geq 2m + 1$, $m \leq b$ and $t \geq \lceil \frac{s}{m} \rceil$, and let $\mathcal{H} = (\mathcal{X}, \mathcal{F}) := \mathcal{H}(m, b, s, t)$ be given by Definition 22. Then the following holds for the $(m : b)$ Maker-Breaker game on $\mathcal{H}$:*

1. *Maker has a strategy as the starting player to win in t rounds.*
2. *Breaker has a winning strategy if he can claim a single vertex before Maker starts with her first move.*
3. *Breaker has a strategy to prevent Maker from winning within $t - 1$ rounds, even if Maker starts the game.*

Proof. We prove the statement by induction on s .

Assume first that $s \leq 3m$. Let $\mathcal{V}$ denote the family of associated vertex sets for $\mathcal{H}(m, b, s, t)$, which is constructed from $\vec{H}_t(b)$ such that $\mathcal{V} = \{V_x : x \in V(\vec{H}_t(b))\}$. By Proposition 24 we can assume that Maker always claims all vertices of a set $V \in \mathcal{V}$, or fully claims all remaining vertices of a winning set $F \in \mathcal{F}$ to win the game. This can

be represented as Maker claiming vertices of $\vec{H}_t(b)$, or claiming a directed edge (v, w) to immediately win the game on $\vec{H}_t(b)$. Moreover, whenever Breaker claims a vertex x of $\mathcal{H}(m, b, s, t)$, this can be represented as Breaker claiming a vertex from $\vec{H}_t(b)$ (if $x \in V$ for some $V \in \mathcal{V}$) or an edge of $\vec{H}_t(b)$ (if x does not belong to a set $V \in \mathcal{V}$), and vice versa. Therefore, the statements (1)–(3) in Lemma 25 follow from the properties (1)–(3) in Lemma 20.

Assume then that $s > 3m$. To prove (1), Maker claims in her first move all vertices of the set V described in part (b) of Definition 22. After Breaker's move, there has to be a copy $\mathcal{H}'$ of $\mathcal{H}(m, b, s - m, t - 1)$ which is not touched by Breaker yet, and thus, Maker wins on $\mathcal{H}'$ in $t - 1$ further rounds by induction. To prove (2), Breaker chooses for his first single element any vertex from the set V described in part (b) of Definition 22. Since every vertex of V is part of every hyperedge, Maker cannot win anymore.

Finally, we consider (3). Maker has to claim all vertices of V in her first move. Otherwise, Breaker would be able to claim a single vertex from V , and thus wins as already explained. Afterwards, Breaker can skip his first move and still achieve his goal by playing according to the strategy promised by (3) given by the induction hypothesis to prevent Maker from winning in less than $t - 2$ further rounds on any copy of $\mathcal{H}(m, b, s - m, t - 1)$. Hence, in total Maker does not win within $t - 1$ rounds. $\square$

Remark 26. Note that the strategies given for property (3) in Lemma 20 and Lemma 25 have the following property: depending on Maker's first move, Breaker can either skip his first move and still ensure that the game lasts at least $t - 1$ rounds, or Breaker can prevent Maker from winning.

Having the last lemma in our hands, we are now ready to prove our main theorems.

3.3 Proof of Theorem 2

Let m, b, s, s', t, t' be given as stated in Theorem 2. We use the following construction: $\mathcal{H}$ consists disjointly of b copies of $\mathcal{H}(m, b, s, t)$ denoted by $\mathcal{H}_i, i \in [b]$, and a single copy of $\mathcal{H}(m, b, s', t')$ denoted by $\mathcal{H}_{b+1}$. By Observation 23 we see that $\mathcal{H}$ fulfils the requirements of Proposition 24, and hence we can assume that Maker in each of her moves claims only elements that belong to the same winning set when playing on $\mathcal{H}$.

First, if Maker starts the game, she has a strategy to claim a winning set of size s in t rounds on $\mathcal{H}_1$. She cannot claim a smaller winning set since every hyperedge has at least s vertices. Additionally, Maker cannot win faster than within t rounds on any single copy $\mathcal{H}_i, i \in [b + 1]$ due to Property (3) of Lemma 25. Because of Proposition 24, Maker claims in each of her move only elements from a single $\mathcal{H}_i, i \in [b + 1]$, and she cannot win faster on $\mathcal{H}$ than on a single copy $\mathcal{H}_i, i \in [b + 1]$.

Second, when Breaker starts the game, he can claim a single vertex from $\mathcal{H}_i$ for each $i \in [b]$, and thereby is able to prevent Maker from winning on those copies because Maker always needs to play in a single copy $\mathcal{H}_i, i \in [b]$ due to Proposition 24, and Breaker can respond on that copy by the strategy guaranteed by Lemma 25(2). Afterwards, Maker can win on $\mathcal{H}_{b+1}$ within t' rounds (but not faster) claiming a winning set of size s' (but not smaller), again by Lemma 25. $\square$

3.4 Proof of Theorem 4

Let m, b, r, s, s', t, t' be given as stated in Theorem 4. Let $\mathcal{H}$ be the hypergraph constructed in the previous proof, and obtain $\mathcal{H}'$ by adding a single hyperedge F_γ with $|F_\gamma| = r$ on a new set of r vertices. As before, using Proposition 24, we can assume that within a single round Maker only claims elements that belong to the same winning set. But then, since $r > m$, the additional hyperedge F_γ does not help Maker claim a winning set, and hence $\mathcal{H}'$ also satisfies the properties (1) and (2) from Theorem 2. Next, let G be obtained from Lemma 12 with input $\mathcal{H}'$. First, $\gamma(G) = r$, due to Claim 14 and since the smallest winning set of $\mathcal{H}'$ is F_γ . Second, due to Lemma 12 and the properties of $\mathcal{H}'$, we obtain $s_{MB}(G, m : b) = s$, $\gamma_{MB}(G, m : b) = t$, $s'_{MB}(G, m : b) = s'$ and $\gamma'_{MB}(G, m : b) = t'$. $\square$

3.5 Proof of Theorem 5

Let m, b, s, t be given as stated in Theorem 5. We first construct a hypergraph $\mathcal{H}$ in the following way: We take a copy of $\mathcal{H}(m, b, s, \lceil \frac{t}{m} \rceil + 1)$ on a vertex set X , and add a hyperedge for every subset of X of size t . Then every winning set in the game on $\mathcal{H}$ is either of size s or of size t . According to Lemma 25 Maker can claim a winning set of size s in $\lceil \frac{t}{m} \rceil + 1$ rounds, but not faster. Moreover, after $\lceil \frac{t}{m} \rceil$ rounds, Maker occupies a winning set of size t , independent of how she plays, because every set of t vertices forms a winning set.

Next, let G be the result of calling Lemma 12 with input $\mathcal{H}$. We aim to show that G satisfies the required properties. By Lemma 12 we immediately have $s_{MB}(G, m : b) = s$. Hence, it remains to show that Dominator cannot claim a dominating set of size s before having a minimal dominating set of size t . To do so, fix any Dominator's strategy. Then let Staller play on G according to the strategy given in the proof of Lemma 12, where $\mathcal{S}_B$ is a Breaker's strategy that prevents Maker's win on $\mathcal{H}(m, b, s, \lceil \frac{t}{m} \rceil + 1)$ for $\lceil \frac{t}{m} \rceil$ rounds. Assume that Dominator at some point occupies a dominating set D of size s . As explained in the proof of Lemma 12, the set $D' = D \cap X$ must then be a dominating set of G , and hence must contain a winning set F of $\mathcal{H}$, because of Claim 15. However, as before, by using the strategy $\mathcal{S}_B$ and since $|F| \leq |D'| \leq s$, Dominator needs to play at least $\lceil \frac{t}{m} \rceil + 1$ rounds on X in order to occupy such a set D' , and thus claim at least $t + m$ elements of X . The set of the first t vertices Dominator claims on X then forms a winning set of $\mathcal{H}$ and hence a dominating set of G by Claim 13. This dominating set must be minimal, as it would otherwise need to contain a dominating set of size s ; but Dominator cannot occupy such a set within $\lceil \frac{t}{m} \rceil$ rounds on X . $\square$

3.6 Proof of Theorem 6

Let m, b, s, s', t, t' be given as stated in Theorem 6. We construct a hypergraph $\mathcal{H}$ as the disjoint union of a copy $\mathcal{H}_1$ of $\mathcal{H}(m, b, s', t)$ and a copy $\mathcal{H}_2$ of $\mathcal{H}(m, b, s, t')$. Let G be the result of calling Lemma 12 with input $\mathcal{H}$. We aim to show that G fulfills the required properties. First note that $\mathcal{H}$ fulfills the requirements of Proposition 24. Thus, Maker can claim a winning set of size s' in t rounds but not faster by playing according to Lemma 25

on $\mathcal{H}_1$. In the same way, she can claim a winning set of size s in t' rounds but not faster by playing on $\mathcal{H}_2$. Thus, we have $\gamma_{MB}(G, m : b) = t$ by Lemma 12 and $s_{MB}(G, m : b) = s$ by the same lemma and the fact, that $\mathcal{H}$ does not contain any winning sets of size smaller than s .

Now, consider the following Breaker strategy on $\mathcal{H}$: Whenever Maker plays on some $\mathcal{H}_i$, Breaker will answer on that same $\mathcal{H}_i$ with the strategy given in either Lemma 25(2) or (3), depending on Maker's first move. Let $i \in [2]$ and $j \in [2] \setminus \{i\}$. Assume Maker's first move is played on $\mathcal{H}_i$. Breaker responds on $\mathcal{H}_i$ according to the strategy given in the proof of Lemma 25(3) which by Remark 26 means that he will either prevent Maker from winning at all on $\mathcal{H}_i$, or he can skip his first move, depending on Maker's first move. If he can skip his first move, he instead claims a single element of $\mathcal{H}_j$ according to the strategy from Lemma 25(2) and keeps responding according to that strategy whenever Maker plays on $\mathcal{H}_j$. Otherwise if he cannot skip his first move on $\mathcal{H}_i$, he will respond on both $\mathcal{H}_i$ and $\mathcal{H}_j$ according to Lemma 25(3) whenever Maker plays on the respective hypergraph. Note that by applying this strategy, Breaker prevents Maker from winning on either $\mathcal{H}_1$ or $\mathcal{H}_2$, again depending on Maker's first move.

By following this strategy, Breaker ensures that Maker does not claim a winning set of size smaller than s because such a set does not exist, and that she cannot win within $t - 1$ rounds because Breaker either wins on $\mathcal{H}_1$ or prevents a Maker win for at least $t - 1$ rounds by Lemma 25(3). In the same way, Maker cannot win on $\mathcal{H}_2$ within $t' - 1$ rounds. Additionally, Maker can only win on one of the copies, $\mathcal{H}_1$ or $\mathcal{H}_2$. If she wins within less than t' rounds, she has to do so on $\mathcal{H}_1$, but then she cannot win on $\mathcal{H}_2$. Thus, the smallest winning set she can achieve is of size s' in that case since $\mathcal{H}_1$ does not contain any winning sets of smaller size.

Regarding the domination game on G , Staller can play according to the strategy given in the proof of Lemma 12 with $\mathcal{S}_B$ being the described strategy on $\mathcal{H}$. By doing so, Staller can ensure the same properties as Breaker in the game on $\mathcal{H}$ following a similar argument as in the proof of Theorem 5. $\square$

3.7 Proof of Theorem 1

Let $B = \{b_1, b_2, \dots, b_\ell\}$ and assume $b_1 < b_2 < \dots < b_\ell$. For each $i \in [\ell + 1]$, we first take a disjoint set V_i of vertices such that the following size condition holds (where we set $b_0 := -1$):

$$|V_i| = b_j \Leftrightarrow b_{j-1} + 1 < i \leq b_j + 1.$$

Then define $\mathcal{H} = (X, \mathcal{F})$ as the hypergraph which satisfies

$$X = \bigcup_{i \in [\ell+1]} V_i \text{ and } \mathcal{F} = \{F \subseteq X : |F \cap V_i| = 1 \text{ for every } i \in [\ell + 1]\}.$$

We first prove that Breaker wins the $(b : b)$ game on $\mathcal{H}$ if $b \in B$: Let $b = b_j$ for some $j \in [\ell]$. We note that $|V_i| \leq b$ for every $i \leq b + 1$. After Maker's first move, at least one of the sets V_i with $i \leq b + 1$ does not contain a vertex claimed by Maker, and thus

Breaker can fully claim such a set V_i in her first move. As every hyperedge of $\mathcal{H}$ intersect V_i , Breaker immediately wins the game.

Next we prove that Maker wins the $(b : b)$ game on $\mathcal{H}$ if $b \notin B$: In her first move, Maker claims exactly one vertex of each of the sets V_i with $i \leq b$. Note that $|V_i| > b$ for every $i > b$. Hence, from now on, whenever Breaker claims an element in one of these sets V_i of which Maker still does not have a vertex, Maker can immediately afterwards claim an element of the same set V_i . By following this strategy, Maker manages to claim a vertex in each of the sets V_i . These vertices form a hyperedge of $\mathcal{H}$, and hence Maker wins. $\square$

4 Results for Waiter-Client domination games

4.1 Dominating trees in Waiter-Client games

Before we prove Theorem 7, we first consider a suitable residue graph $R(G)$ for any graph G , and with Corollary 28 we provide a Waiter-Client version of Theorem 4.4 in [23]. We start with the following lemma.

Lemma 27. *Let G be a graph, and let $vw \in E(G)$ such that $d_G(v) = 1$ and $d_G(w) = 2$. Then*

$$\gamma_{WC}(G) = 1 + \gamma_{WC}(G - \{v, w\}) \quad \text{and} \quad s_{WC}(G) = 1 + s_{WC}(G - \{v, w\}).$$

Proof. We start with the discussion on $\gamma_{WC}(G)$. Let $\mathcal{S}_W$ be a strategy for Dominator which ensures her win in the Waiter-Client domination game on $G - \{v, w\}$ within $\gamma_{WC}(G - \{v, w\})$ rounds. Moreover, let $\mathcal{S}_C$ be a strategy for Staller which ensures that Dominator does not win the Waiter-Client domination game on $G - \{v, w\}$ within less than $\gamma_{WC}(G - \{v, w\})$ rounds.

In order to prove $\gamma_{WC}(G) \leq 1 + \gamma_{WC}(G - \{v, w\})$, consider the following strategy for Dominator. In the first round, Dominator offers v and w , and no matter what Staller does, both vertices are dominated by Dominator. Afterwards, Dominator follows the strategy $\mathcal{S}_W$ in order to dominate the remaining vertices. In total, this takes no more than $1 + \gamma_{WC}(G - \{v, w\})$ rounds.

In order to prove $\gamma_{WC}(G) \geq 1 + \gamma_{WC}(G - \{v, w\})$, consider the following strategy for Staller. As long as Dominator does not offer a vertex from $\{v, w\}$, Staller plays according $\mathcal{S}_C$. If Dominator offers $\{v, w\}$ in the same round, then Staller claim w and gives v to Dominator; afterwards Staller continues according to $\mathcal{S}_C$. This way, v, w do not help Dominator to dominate a vertex in $G - \{v, w\}$, and hence, Dominator needs at least $1 + \gamma_{WC}(G - \{v, w\})$ rounds for winning. Otherwise, i.e. if v and w are offered in different rounds, then Staller claims both v and w for herself. In this case, Dominator loses as she cannot dominate v .

Next, we consider $s_{WC}(G)$. In the above argument, replace $\mathcal{S}_W$ with a strategy for Dominator which ensures that in the Waiter-Client domination game on $G - \{v, w\}$ she can get a dominating set of size $s_{WC}(G - \{v, w\})$, and replace $\mathcal{S}_C$ with a strategy for Staller

which ensures that in the Waiter-Client domination game on $G - \{v, w\}$ Dominator does not get a dominating set of size $s_{WC}(G - \{v, w\}) - 1$. Then, $s_{WC}(G) = 1 + s_{WC}(G - \{v, w\})$ can be deduced analogously. $\square$

Now, as in [23], we obtain $R(G)$ by iteratively deleting as long as possible adjacent vertices v, w such that v is a leaf in the current graph and w is a neighbour of v with degree 2. Note that $R(G)$ is well-defined by Lemma 4.1 in [23]. Moreover, the following is an easy consequence of Lemma 27.

Corollary 28. *For every graph G , we have*

$$\gamma_{WC}(G) = \frac{v(G) - v(R(G))}{2} + \gamma_{WC}(R(G)) \quad \text{and} \quad s_{WC}(G) = \frac{v(G) - v(R(G))}{2} + s_{WC}(R(G)).$$

Finally, we can use this to prove Theorem 7.

Proof of Theorem 7. Let T be a tree and let $R(T)$ be its residue. By Lemma 4.2 in [23], T has a perfect matching if and only if $R(T)$ has a perfect matching. Moreover, by Lemma 21 in [15], $R(T)$ is either (a) an isolated vertex, (b) an edge or (c) contains a vertex x with at least two leaves as neighbours.

In case (a), T has an odd number of vertices and hence no perfect matching. Moreover, $\gamma_{WC}(R(T)) = \infty$ and $s_{WC}(R(T)) = \infty$, as by the definition of Waiter-Client games, Staller obtains the unique vertex of $R(T)$ when playing on $R(T)$ only. Thus, $\gamma_{WC}(T) = \infty$ and $s_{WC}(T) = \infty$ by Corollary 28.

In case (b), $R(T)$ has a perfect matching and hence T has a perfect matching. Moreover, as $\gamma_{WC}(R(T)) = s_{WC}(R(T)) = 1$, Corollary 28 implies $\gamma_{WC}(T) = s_{WC}(T) = \frac{v(T)}{2}$.

In case (c), $R(T)$ does not have a perfect matching and hence T does not either. Staller wins the Waiter-Client domination game on $R(T)$, as a simple case distinction shows that he can claim the vertex x and at least one of its leaf neighbours. This leaf neighbour then cannot be dominated by Dominator. In particular, $\gamma_{WC}(R(T)) = \infty$ and $s_{WC}(R(T)) = \infty$, and hence $s_{WC}(T) = \infty$ and $\gamma_{WC}(T) = \infty$ by Corollary 28. $\square$

4.2 Dominating cycles in Waiter-Client games

Proof of Theorem 8. Let the vertices of C_n be labeled $v_1, v_2, \dots, v_n$ such that $v_i v_j$ forms an edge if $i - j \equiv \pm 1 \pmod{n}$. We first show $\gamma_{WC}(C_n) \leq \lfloor \frac{n}{2} \rfloor$ and $s_{WC}(C_n) \leq \lfloor \frac{n}{2} \rfloor$, by describing a Dominator strategy. In the first round, let Dominator offer $\{v_{n-1}, v_n\}$. By symmetry we can assume that Dominator receives the vertex v_{n-1} . Afterwards let k be the largest even integer with $k \leq n - 2$. Playing on the path $(v_1, \dots, v_k)$, Dominator can dominate all its vertices within $\frac{k}{2}$ rounds by Theorem 7. As v_{n-2}, v_{n-1}, v_n are dominated by the vertex v_{n-1} from the first round, Dominator has a dominating set then. In total, she used $\frac{k}{2} + 1 = \lfloor \frac{n}{2} \rfloor$ rounds.

Next we show $\gamma_{WC}(C_n) \geq \lfloor \frac{n}{2} \rfloor$. This inequality is easy for $n \in \{3, 4, 5\}$, as in these cases the domination number of C_n equals $\lfloor \frac{n}{2} \rfloor$. Hence, from now on we can assume that $n \geq 6$. We consider two cases depending on the first move of Dominator.

Assume first that in the first round, Dominator offers two vertices which are not adjacent; by symmetry we may assume that she offers v_1 and v_i with $3 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1$. Then Staller can keep v_1 and give v_i to Dominator. Afterwards, Staller can answer almost arbitrarily to Dominator's offers, as long as he ensures the following:

- (a) If Dominator offers v_n and v_2 in different rounds, Staller keeps both of these vertices.
- (b) If Dominator offers v_n and v_2 in the same round, Staller keeps v_n and gives v_2 to Dominator.
- (c) If Dominator offers v_{n-1} together with a vertex $w \notin \{v_2, v_n\}$, then Staller keeps v_{n-1} .

It is easy to check that Staller can ensure all three conditions (a)–(c) to hold. If Dominator offers v_2, v_n in different rounds, then by (a), the vertex v_1 cannot be dominated by Dominator. Otherwise, by (b) and (c), the vertex v_n cannot be dominated by Dominator. In any case, Dominator loses.

Assume then that in the first round, Dominator offers two adjacent vertices, w.l.o.g. the vertices v_1 and v_2 . Then Staller keeps v_2 and gives v_1 to Dominator. Afterwards, Staller answers to Dominator's offers as follows:

- (a) If Dominator offers $\{v_{2i-1}, v_{2i}\}$ for some $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$, then Staller keeps v_{2i} and gives v_{2i-1} to Dominator.
- (b) Otherwise, if Dominator offers a pair of vertices different from those in (a), then Staller keeps the vertex of smaller index and gives the other to Dominator.

Note that the first round was already played according to (a). If throughout the game, Dominator always offers pairs as described in (a), she gets a dominating set, but then she plays $\lfloor \frac{n}{2} \rfloor$ rounds until she wins, and there is no dominating set with less than $\lfloor \frac{n}{2} \rfloor$ vertices contained in Dominator's vertex set. Otherwise, Dominator loses by the following reason: Let $j \leq \lfloor \frac{n}{2} \rfloor - 1$ be the smallest index such that Dominator does not offer $\{v_{2j-1}, v_{2j}\}$ in the same round. Then by following (b), Staller claims both v_{2j-1}, v_{2j} for himself. Moreover, as v_{2j-3}, v_{2j-2} were offered in the same round (by minimality of j), Staller also keeps v_{2j-2} (because of (a) or the first round), and hence, Dominator cannot dominate v_{2j-1} . $\square$

4.3 Comparing Maker-Breaker and Waiter-Client domination games

Proof of Theorem 9. Due to Lemma 12, it is enough to find a hypergraph $\mathcal{H}_{s,t}$ such that

- (i) s is the smallest number of rounds in which Maker wins the unbiased Maker-Breaker game on $\mathcal{H}_{s,t}$ when Breaker starts, and
- (ii) t is the smallest number of rounds in which Waiter wins the unbiased Waiter-Client game on $\mathcal{H}_{s,t}$.

Case 1: We start with the case $s \geq t$ (for which it is actually enough to require that $t \geq 3$) and prepare for the desired construction with the following claim.

Claim 29. *For every $t \geq 3$ there is a hypergraph $\mathcal{H}_t$ such that Maker loses on $\mathcal{H}_t$, and such that t is the smallest number of rounds within which Waiter wins on $\mathcal{H}_t$.*

Proof. Let $a_i, b_i \notin [2t-6]$ with $i \in [4]$ be distinct elements, and define $\mathcal{H}_t = (X_t, \mathcal{F}_t)$ such that $X_t := \{a_i, b_i : i \in [4]\} \cup [2t-6]$ and $\mathcal{F}_t := \{\{a_i, b_i\} \cup M : i \in [4], M \in \binom{[2t-6]}{t-3}\}$. By playing a pairing strategy for the pairs $\{a_i, b_i\}$, Breaker can prevent Maker from winning. Waiter can win within t rounds as follows: In the first two rounds, Waiter offers a_1, a_2, a_3, a_4 and w.l.o.g. Waiter gets a_1, a_2 . Then she offers b_1, b_2 and w.l.o.g. receives b_1 . For the next $t-3$ rounds, Waiter offers the elements of $[2t-6]$ in an arbitrary way, and completes a winning set of the form $\{a_1, b_1\} \cup M$ with $M \in \binom{[2t-6]}{t-3}$. Moreover, Client can prevent Waiter from winning in less than t rounds as follows: In each round, in which Waiter offers at least one element in $[2t-6]$, Client gives an element of $[2t-6]$ to Waiter. Note that Waiter needs to play at least $t-3$ such rounds. In order to complete a winning set, Waiter moreover needs a pair $\{a_i, b_i\}$. W.l.o.g. let a_1 be the first element from $A := \{a_i, b_i : i \in [4]\}$ that Waiter occupies (by offering a pair in A), then when Waiter offers the next pair in A , Client only needs to make sure that Waiter does not get b_1 . This way, Waiter needs to offer at least 3 pairs in A until she wins, and hence needs to play at least t rounds in total. $\square$

Next, we set $\mathcal{H}_{s,t} := \mathcal{H}_t \dot{\cup} K_{2s}^{(s)}$ where $K_{2s}^{(s)}$ is the s -uniform complete hypergraph on $2s$ vertices. By Claim 29, the elements of $\mathcal{H}_t$ do not help Maker in winning, but Maker can obviously win within s rounds (but not faster) by playing on $K_{2s}^{(s)}$. This shows (i). By Claim 29, Waiter can win on $\mathcal{H}_{s,t}$ within t rounds by playing on $\mathcal{H}_t$ only (and here she cannot be faster). Moreover, the vertices of $K_{2s}^{(s)}$ do not help her to be faster, as every winning set in $K_{2s}^{(s)}$ has size $s \geq t$. This proves (ii).

Case 2: We then consider the case $s < t$ and prepare for the desired construction with the following claim. For the proof, note that a Client-Waiter game on a hypergraph $\mathcal{H}$ is played the same way as a Waiter-Client game with the difference that Client wins if and only if he obtains a winning set. (Again, if there is only one element in the last round, then it is given to Client.)

Claim 30. *For every $s \geq 7$ there is a hypergraph $\mathcal{H}'_s$ such that Waiter loses on $\mathcal{H}'_s$, and such that s is the smallest number of rounds within which Maker wins on $\mathcal{H}'_s$ (if Breaker starts).*

Proof. In [29] (see Construction 5), there is a hypergraph $G_{CP} = (X_{CP}, \mathcal{F}_{CP})$ on 15 vertices given with the following properties:

- (i') Breaker wins the Maker-Breaker game on G_{CP} , if Maker is the first player (Proposition 6 in [29]), and this takes at most 7 rounds.
- (ii') Client wins the Client-Waiter game on G_{CP} (Proposition 7 in [29]).

Now, consider the transversal hypergraph $\mathcal{H}_0 = (X_{CP}, \mathcal{F}_0)$, where

$$\mathcal{F}_0 := \{F \subseteq X_{CP} : (\forall F' \in \mathcal{F}_{CP} : F \cap F' \neq \emptyset)\}.$$

Then the above properties can be reformulated as follows:

- (i'') Maker wins the Maker-Breaker game on $\mathcal{H}_0$, if Breaker is the first player, and this takes at most 7 rounds.
- (ii'') Client wins the Waiter-Client game on $\mathcal{H}_0$.

Let $r \leq 7$ be the smallest number of rounds within which Maker can always win on $\mathcal{H}_0$. W.l.o.g assume that $X_{CP} \cap \mathbb{N} = \emptyset$. We now consider the hypergraph $\mathcal{H}'_s = (X_s, \mathcal{F}_s)$ with $X_s := X_{CP} \cup [2s - 2r + 7]$, and

$$\mathcal{F}_s := \left\{ F \cup M : F \in \mathcal{F}_0, M \in \binom{[2s - 2r + 7]}{s - r} \right\}.$$

Then Client wins on $\mathcal{H}'_s$ as follows: Whenever Waiter offers two elements in X_{CP} , Client plays according to the strategy promised by (ii''), and in each other case she never gives an element of X_{CP} to Waiter. This way, Waiter never occupies a set from $\mathcal{F}_0$ and in particular no set in $\mathcal{F}_s$. Maker wins on $\mathcal{H}'_s$ within s rounds as follows: She first plays r rounds on X_{CP} to occupy a set $F \in \mathcal{F}_0$, which is possible by (i''), and then she plays $s - r$ further rounds in which she claims arbitrary elements of $[2s - 2r + 7]$, which is possible as before that at most 7 rounds were played. Moreover, Breaker can prevent Maker from winning faster by the following reason: Maker needs to claim a set $F \cup M$ with $F \in \mathcal{F}_0$ and $M \in \binom{[2s - 2r + 7]}{s - r}$. By definition of r , Breaker can play so that for a suitable set F , Maker needs to play at least r rounds in X_{CP} , and additionally Maker needs $s - r$ elements outside X_{CP} for a suitable set M . $\square$

Using the above claim, we set $\mathcal{H}_{s,t} := \mathcal{H}'_s \dot{\cup} K_{2t}^{(t)}$. Now, showing that (i) and (ii) hold can be done analogously to Case 1. This completes the proof of Theorem 9. $\square$

5 Concluding remarks and open problems

Optimality. Let us first look at the conditions assumed in Theorem 2 and Theorem 4. Since Maker as first player can always achieve what Maker as second player can do, the inequalities $s' \geq s$ and $t' \geq t$ need to be assumed in both theorems. Similarly, since claiming a set of size x with bias m takes at least $\lceil \frac{x}{m} \rceil$ rounds, the conditions $t \geq \lceil \frac{s}{m} \rceil$ and $t' \geq \lceil \frac{s'}{m} \rceil$ need to be assumed as well. Additionally, assuming $s \geq r$ in Theorem 4 comes from the fact that Maker cannot claim a winning set which is smaller than the smallest winning set, and $m \leq r - 1$ is assumed since otherwise Maker would be able to win the game in the first round. The only conditions which may not be optimal in both theorems, but which are required for our constructions, are $s \geq 2m + 1$ and $m \leq b$. However, we note that these conditions are still optimal for $m = 1$, which can be seen as follows. The inequality $1 = m \leq b$ is clearly satisfied then. If we would have $s \leq 2m$ in Theorem 2, then $s \in \{1, 2\}$. For property (1) in Theorem 2 being true, the hypergraph $\mathcal{H}$ would need to contain at least one winning set of size at most $s \leq 2$. If there is a winning set of size 1, Maker can win in the first round and hence, property (1) cannot be true for arbitrary t .

Otherwise, let G be the graph induced by all hyperedges of $\mathcal{H}$ of size 2. If the maximum degree $\Delta(G)$ of G satisfies $\Delta(G) \geq b + 1$, then Maker can occupy an edge of G within 2 rounds, and therefore, t cannot be chosen arbitrarily in property (1). If instead $\Delta(G) \leq b$, then in each of his moves Breaker can block all edges which contain the vertex previously claimed by Maker, in contradiction to Maker being able to occupy a winning set of size 2. The discussion for Theorem 4 is analogous. Moreover, a similar discussion shows that, for general $m \in \mathbb{N}$, the assumption $s \geq 2m + 1$ cannot be replaced with $s \geq m$. To close this gap is an open problem.

Problem 31. Can the condition $s \geq 2m + 1$ in Theorem 2 and Theorem 4 be replaced with $s \geq 2m$? If yes, what is the best bound on s that can be assumed here?

Moreover, the above discussion can be modified to show that $s \geq 2m$ is actually not possible if the construction should fulfill the requirements of Proposition 24. Thus, Maker's strategy to win on a construction that positively answers Problem 31 would necessarily include her claiming elements from different winning sets in some of her moves. Similarly, Proposition 24 does not work if $m > b$. In fact, if $m > b$, Maker can win on each s -uniform hypergraph $\mathcal{H}$ that contains a large enough matching in a number of moves depending on s using a box game argument. Therefore, if the number of required moves should be arbitrarily large compared to s , $\mathcal{H}$ cannot contain a large matching in that case.

Waiter-Client domination games. In this paper we give first results on Waiter-Client domination games. It seems natural to study analogues of the questions discussed for Maker-Breaker domination games in e.g. [1, 6, 8, 13, 14, 15]. For instance:

- Is deciding the winner of the Waiter-Client domination game PSPACE-complete?
- What can be said about biased Waiter-Client domination games?
- What happens if Waiter wants to occupy a total dominating set?

Random graphs. As already observed in [1], it is not hard to show that with high probability, if G is a Binomial random graphs $G_{n,p}$ with constant edge probability $p \in (0, 1)$ or a random geometric graphs $\mathcal{G}_2(n, r)$ with a radius $r \in [0, 1]$, the parameters $\gamma_{MB}(G)$ and $\gamma'_{MB}(G)$ asymptotically equal the domination number of G . We conjecture that in these cases the two game parameters γ_{MB} and s_{MB} are equal.

Conjecture 32. Let G be a Binomial random graph $G_{n,p}$ with constant edge probability $p \in (0, 1)$, then $\gamma_{MB}(G, b) = s_{MB}(G, b)$ holds with high probability.

Conjecture 33. Let G be a random geometric graph $\mathcal{G}_2(n, r)$ with constant radius $r \in (0, 1)$, then $\gamma_{MB}(G, b) = s_{MB}(G, b)$ holds with high probability.

Acknowledgements

This topic was started in connection with a Bachelor thesis of the first author at the Hamburg University of Technology. He wants to thank Anusch Taraz and his research group for the supervision. Moreover, this research was started when the second and fourth author were supported by Deutsche Forschungsgemeinschaft (Project CL 903/1-1). Additionally, the research of the fourth author was partly supported by the ANR project P-GASE (ANR-21-CE48-0001-01).

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