On increasing sequences formed by points from a random finite subset of a hypercube

Boris Pittel[a]

Submitted: Mar 06, 2026; Accepted: Jun 19, 2026; Published: Aug 28, 2026

Abstract

Consider S, a set of n points chosen uniformly at random and independently from the unit hypercube of dimension $t > 2$. Order S by using the Cartesian product of the t standard orders of $[0, 1]$. We determine a constant $\bar{x}(t) < e$ such that, with probability $\geqslant 1 - \exp(-\Theta(\varepsilon)n^{1/t})$, cardinality of a longest chain, i.e., a largest subset of comparable points, is at most $(\bar{x}(t) + \varepsilon)n^{1/t}$. The bound $\bar{x}(t)$ complements an explicit lower bound obtained by Bollobás and Winkler in 1988.

Mathematics Subject Classifications: 05A15, 05A16, 60C05

1 Introduction

Given two points $\mathbf{x}'$ and $\mathbf{x}''$ in R^t, we write $\mathbf{x}'' \geqslant \mathbf{x}'$ if each component of $\mathbf{x}''$ is not strictly below the corresponding component of $\mathbf{x}'$. This makes R^t a partially ordered set (poset). Consider a set $\mathcal{S}_n = \{\mathbf{X}_i\}_{i\in[n]}$, where $\mathbf{X}_i$ are chosen independently, and uniformly, in the t-dimensional unit cube $\{\mathbf{x} = (x_1, \dots, x_t) : x_i \in [0, 1]\}$. Equivalently, the nt components of $\mathbf{X}_1, \dots, \mathbf{X}_n$ are chosen uniformly, and independently of each other, in the interval $[0, 1]$. The partial order on R^t makes $\mathcal{S}_n$ a random poset.

This random poset was introduced by Winkler in 1985 [20]. He posed several open problems, subsequently studied in [3], [4], [5], [6], [14], [21], [22], [23]. In particular, Brightwell [5] proved that, with high probability, the width (cardinality of the largest *antichain*) of $\mathcal{S}_n$ is exactly of order $n^{1-1/t}$.

In this paper we contribute to analysis of the height of S_n, i.e., the length of the longest *chain* in $\mathcal{S}_n$. A chain is defined as an ordered sequence $S = \{\mathbf{X}_{i_1}, \mathbf{X}_{i_2}, \cdots, \mathbf{X}_{i_\ell}\}$ such that $\mathbf{X}_{i_j} \leqslant \mathbf{X}_{i_{j+1}}$, $1 \leqslant j \leqslant \ell - 1$; $\mathbf{X}_{i_0} := \overrightarrow{\mathbf{0}} = (0, \dots, 0)$. In 1961, Ulam [17] posed a problem to determine an asymptotic length of the longest

[a]Department of Mathematics, Ohio State University, Columbus, OH 43210 (pittel.1@osu.edu)

subsequence in the uniformly random permutation of $[n]$. It is easy to see that Ulam's problem is equivalent to the problem of the longest chain in the case $t=2$.

In 1977, the results of Logan and Shepp [7] and Vershik and Kerov [18], [19] provided a solution: with high probability, Ulam's parameter is asymptotic to $2n^{1/2}$. This breakthrough result was based on asymptotic analysis of the shape of a Young diagram of size n, chosen according to Plancherel distribution: the probability of a generic diagram λ is $d^2(\lambda)/n!$, where $d(\lambda)$ is $n!$ divided by the product of λ's hook lengths, which happens to be the number of the Young tableaux of shape λ. Combinatorially, the distribution arises from a Robinson–Schensted bijection (see [13], Knuth [12], Vol. 3, Theorem A), between $n!$ permutations and pairs of Young tableaux of the same shape λ, with a property that the size of the longest increasing subsequence in a permutation equals the size of the longest row in the associated λ. In 1991, Aldous and Diaconis [1] found an alternative proof of the $2n^{1/2}$ result based on Hammersley's interacting particles process.

Prior to [7], [18], and [19], Hammersley [8], [9] was already able to prove that the parameter in question scaled by $n^{1/2}$ converges, in probability, to a constant sandwiched between $\pi/8$ and e. That e is an upper bound for the limit in question followed from an observation that the expected number of chains of length ℓ is $(n)_\ell/(\ell!)^2$. The lower bound followed from an asymptotic analysis of an auxiliary process of growing a chain of points via a two-dimensional Poisson process. Convergence of the scaled length of a longest subsequence to a constant limit followed from a general result for subadditive ergodic processes, due to Kingman [11]. In 1991, Frieze [6] proved that the distance between the length of the longest increasing subsequence and its expected value is below n^α with probability sub-exponentially close to 1 for every $\alpha > 1/3$. In 1995, Talagrand [16] proved a stronger result: with probability sub-exponentially close to 1, the distance in question is $O(n^{1/4})$.

In 1988 Bollobás and Winkler [4] studied the case $t > 2$. Extending Hammersley-Kingman's techniques, they proved that the length of the longest chain scaled by $n^{1/t}$ converges in probability to a constant in $\big[t^2/t!^{1/t}\Gamma(1/t), e\big)$, the left endpoint converging to the (excluded) right endpoint e as $t\to\infty$. In this paper we replace the upper bound e with a smaller, t-dependent, constant.

Associated with a chain S there are ℓ *intervals*:

$$[\overrightarrow{\mathbf{0}}, \mathbf{X}_{i_1}],\ [\mathbf{X}_{i_1}, \mathbf{X}_{i_2}], \dots, [\mathbf{X}_{i_{\ell-1}}, \mathbf{X}_{i_\ell}],$$

each interval consisting of the points $\mathbf{X}_j$, $(j \neq i_1, \dots, i_\ell)$, comparable to the interval's endpoints, and sandwiched strictly between them. We call a chain $\{\mathbf{X}_{i_1} \leqslant \mathbf{X}_{i_2} \leqslant \cdots \leqslant \mathbf{X}_{i_\ell}\}$ *maximal* if each of the ℓ intervals is empty. Clearly, the longest chains are maximal. (Of course, for a longest chain $\mathbf{X}_{i_1} \leqslant \cdots \leqslant \mathbf{X}_{i_\ell}$, the extra interval $[\mathbf{X}_{i_\ell}, \overrightarrow{\mathbf{1}}]$, $\overrightarrow{\mathbf{1}} := (1, \dots, 1)$, is empty too.)

Theorem 1. **(i)** *Let C_ℓ denote the total number of maximal chains of length $\ell = xn^{1/t}$, $x \in [e^{-\gamma}, e]$, $\gamma := -\Gamma'(1)$. $\mathbb{E}[C_\ell]$ is at most of order $n^{1/2}\Big(\big(\frac{e}{x}\big)^t q(x)\Big)^\ell$. $q(x)$ is defined as follows. For $x \in [e^{-\gamma}, e]$, $a > \gamma t$, and $b > 0$, introduce*

$$q(x;a,b) = \max\left\{\min_{u\in(0,1)} \frac{\Gamma^t(1-u)}{e^{au}}; \left(\frac{1+b}{e^b}\right)^t; \exp\left(-\frac{e^{-a}}{x^t(1+b)^t}\right)\right\},$$

so that $q(x;a,b) < 1$. Then $q(x) := \min_{a,b} q(x;a,b)$, which is attained at a unique point $(a(x), b(x))$ such that

$$q(x) = \min_{u\in(0,1)} \frac{\Gamma^t(1-u)}{e^{a(x)u}} = \left(\frac{1+b(x)}{e^{b(x)}}\right)^t = \exp\left(-\frac{e^{-a(x)}}{x^t(1+b(x))^t}\right) < 1; \qquad (1.1)$$

$q(x)$ increases with x. **(ii)** *Since $q(e) < 1$, $\bar{x} := \inf\big\{x \in [e^{-\gamma}, e] : \big(\frac{e}{x'}\big)^t q(x') < 1, \forall x' \geqslant x\big\}$ is strictly below e. Consequently*

$$\mathbb{E}[C_\ell] = O\big[\exp(-\Theta(\varepsilon)n^{1/t})\big], \text{ if } \ell = (\bar{x}+\varepsilon)n^{1/t}, \ \varepsilon > 0.$$

Therefore, the length of the longest chain is below $(\bar{x}+\varepsilon)n^{1/t}$ with probability $\geqslant 1 - O\big[\exp(-\Theta(\varepsilon)n^{1/t})\big]$.

Note. $\min_{u\in(0,1)} \Gamma^t(1-u)/(e^{au})$ in (1.1) makes tabulation of $q(x)$ problematic. Let us show existence of an easy-to tabulate $q^*(x) \in (q(x), 1)$ such the claim of Theorem 1 continues to hold with $q^*(x)$. Using $\Gamma(u) < u^{-1}$, $u \in (0,1)$, we get: for $a > t$,

$$\begin{aligned}\min_{u\in(0,1)} \frac{\Gamma^t(1-u)}{e^{au}} &\leqslant \min_{u\in(0,1)} \frac{(1-u)^{-t}}{e^{au}} \\ &= \left(\frac{a}{t}\right)^t \cdot e^{t-a} = \left(\frac{a}{t}e^{-a/t+1}\right)^t = e^t\left(e^{-a/t}\frac{a}{t}\right)^t.\end{aligned}$$

In this display, just like the top left function of $a \geqslant \gamma t$, the bottom right function of $a \geqslant t$ also decreases from 1 to 0. No changes in the proof of Theorem 1 are needed to get a weaker version with $q(x)$ replaced by $q^*(x) > q(x)$, satisfying

$$q^*(x) = e^t\left(e^{-a^*(x)/t}\frac{a^*(x)}{t}\right)^t = \left(\frac{1+b^*(x)}{e^{b^*(x)}}\right)^t = \exp\left(-\frac{e^{-a^*(x)}}{x^t(1+b^*(x))^t}\right) < 1; \qquad (1.2)$$

here $a^*(x) > t$, of course. This time it follows that

$$-\frac{a^*(x)}{t}e^{-a^*(x)/t} = -\frac{t}{e}\cdot\frac{1+b^*(x)}{e^{b^*(x)}} \Longrightarrow -\frac{a^*(x)}{t} = W\!\left(-\frac{t}{e}\cdot\frac{1+b^*(x)}{e^{b^*(x)}}\right),$$

where $W(\cdot)$ is a classic Lambert function, the inverse of ze^z on $[-e^{-1}, \infty)$, supported by the mathematical software, like Maple. Using this formula for $-a^*(x)/t$, and the second equation in (1.2) we compute $b^*(x)$ via command "fsolve", thus computing $q^*(x) = \left[(1+b^*(x))/e^{b^*(x)}\right]^t$.

To conclude, we refer the reader to Steele [15] for a highly readable, meticulously written survey of a rich collection of papers on lengths of increasing subsequences in a variety of combinatorial settings.

2 Proof of Theorem 1

By symmetry, with $\mathbf{x}_0 = \overrightarrow{\mathbf{0}}$,

$$\mathbb{E}[C_\ell] = (n)_\ell \int\limits_{\mathbf{0}\leqslant\mathbf{x}(1)\leqslant\mathbf{x}(2)\leqslant\ldots\leqslant\mathbf{x}(\ell)\leqslant\mathbf{1}} \left(1-\sum_{i=1}^{\ell}\prod_{j=1}^{t}\bigl(x_j(i)-x_j(i-1)\bigr)\right)^{n-\ell}\prod_{i=1}^{l} d\mathbf{x}(i).$$

Explanation: $(n)_\ell$ is the total number of ways to select an ordered sequence of points $\{\mathbf{X}(i)\}_{i\in[\ell]}$ in the cube $[0,1]^t$. The integral is the probability that this sequence is *maximal*, so that it is increasing and the *first* ℓ intervals $[\mathbf{X}(i-1),\mathbf{X}(i)]$ are empty; $\mathbf{X}(0) := \overrightarrow{\mathbf{0}}$. Indeed, conditioned on $\{\mathbf{X}(i)=\mathbf{x}(i)\}_{i\in[\ell]}$, the integrand is the probability that none of the remaining $(n-\ell)$ points $\mathbf{X}(k)$ belongs to any of the ℓ intervals in question.

Introduce the ℓ increments $\mathbf{y}(i)=\mathbf{x}(i)-\mathbf{x}(i-1)$, $i\in[1,\ell]$, so that $\sum_{i=1}^{\ell}\mathbf{y}(i)\leqslant\mathbf{1}$. Switching from the variables $\mathbf{x}(i)$, $i\in[\ell]$, to $\mathbf{y}(i)=\{y_j(i)\}_{j\in[t]}$, $i\in[\ell]$, we rewrite the above formula:

$$\mathbb{E}[C_\ell] = (n)_\ell \int\limits_{\substack{\mathbf{y}_1,\ldots,\mathbf{y}_\ell\geqslant\mathbf{0},\\ \mathbf{y}_1+\cdots+\mathbf{y}_\ell\leqslant\mathbf{1}}} \left(1-\sum_{i\in[\ell]}\prod_{j\in[t]} y_j(i)\right)^{n-\ell}\prod_{i=1}^{\ell} d\mathbf{y}(i).$$

Our task is to upper bound this $t\ell$-dimensional integral. As the first step, we use the arithmetic/geometric means inequality in combination with $1-z\leqslant e^{-z}$, ($z\in[0,1]$), and bound the integrand by

$$\left(1-\ell\prod_{i\in[\ell]}\prod_{j\in[t]} y_j^{1/\ell}(i)\right)^{n-\ell}\leqslant\exp\left(-\ell(n-\ell)\prod_{j\in[t]}\prod_{i\in[\ell]} y_j^{1/\ell}(i)\right).$$

Probabilistically,

$$\mathbb{E}[C_\ell]\leqslant(n)_\ell\,\mathbb{E}\left[\exp\left(-\ell(n-\ell)\prod_{j\in[t]}\prod_{i\in[\ell]} Y_j^{1/\ell}(i)\right)\cdot\mathbb{I}\left(\sum_{i\in[\ell]}\mathbf{Y}(i)\leqslant\overrightarrow{\mathbf{1}}\right)\right], \tag{2.1}$$

where $t\ell$ random variables $\{Y_j(i)\}_{i\in[\ell],j\in[t]}$ are independent $(0,1)$-Uniforms, and $\mathbf{Y}(i)=\{Y_j(i)\}_{j\in[t]}$.

To proceed, for each $j \in [t]$ switch to $s_j = \sum_{i\in[\ell]} y_j(i)$, $v_j(i) = y_j(i)/s_j$, $i < \ell$, and *define* $v_j(\ell) = 1 - \sum_{i<\ell} v_j(i)$. In the new variables,

$$\prod_{j\in[t]}\prod_{i\in[\ell]} y_j^{1/\ell}(i) = \Pi(\mathbf{v})\cdot\prod_{j\in[t]} s_j, \quad \Pi(\mathbf{v}) := \prod_{j\in[t]}\prod_{i\in[\ell]}(v_j(i))^{1/\ell}. \tag{2.2}$$

An admissible $\{s_j, v_j(i)\}_{j\in[t],i\in[\ell]}$ meets conditions $s_j \leqslant 1$, $\sum_i v_j(i) = 1$.

For each $j \in [t]$, the Jacobian matrix of $\{y_j(i)\}_{i\in[\ell]}$, with respect to $\{\{v_j(i)\}_{i\in[\ell-1]}, s_j\}$, is an $\ell\times\ell$ matrix, consisting of $(\ell-1)\times(\ell-1)$ identity matrix multiplied by s_j, which is augmented at the bottom by the row $\{-s_j,\dots,-s_j\}$, with the ℓ-high column $\{v_j(1),\dots,v_j(\ell)\}^T$ attached on the right side of the resulting $\ell\times(\ell-1)$ sub-matrix. By adding the first $(\ell-1)$ rows to the bottom row, we get an upper-triangular ℓ matrix with diagonal $\{s_j,\dots,s_j,1\}$, whose determinant equals $s_j^{\ell-1}$. Thus, the *full* Jacobian determinant equals $\prod_{j\in[t]} s_j^{\ell-1}$.

Probabilistically, it follows that the joint density of $\{S_j := \sum_{i\in[\ell]} Y_j(i)$, $V_j(i) := Y_j(i)/S_j\}_{j\in[t],i\in[\ell-1]}$ is

$$\prod_{j\in[t]} s_j^{\ell-1}\cdot\mathbb{I}\Bigg(s_j\leqslant 1, \sum_{i\in[\ell-1]} v_j(i)\leqslant 1\Bigg) = \prod_{j\in[t]}\frac{s_j^{\ell-1}\mathbb{I}(s_j\leqslant 1)}{(\ell-1)!}\cdot g\big(\{v_j(i)\}_{i\in[\ell-1]}\big).$$

Here, for each $j\in[t]$,

$$g(\{v_j(i)\}_{i\in[\ell-1]}) := (\ell-1)!\cdot\mathbb{I}\Bigg(\sum_{i\in[\ell-1]} v_j(i)\leqslant 1\Bigg)$$

is the joint density of the *first* $(\ell-1)$ components of $\mathbf{L}_j = \{L_j(i)\}_{i\in[\ell]}$, where $L_j(i)$ are the lengths of ℓ consecutive subintervals of $[0,1]$ obtained by selecting $(\ell-1)$ points independently and uniformly at random in $[0,1]$. The product of these densities signifies that ℓ-dimensional vectors $\mathbf{L}_1,\dots,\mathbf{L}_t$ are independent. So, the expectation on the the RHS of (2.1) is

$$\begin{aligned}&\frac{1}{((\ell-1)!)^t}\int\limits_{\{v_j(i)\}}\prod_{j\in[t]} g\big(\{v_j(i)\}_{i\in[\ell-1]}\big)\\ &\quad\times\Bigg(\int\limits_{\{s_j\}}\exp\Bigg(-\ell(n-\ell)\cdot\Pi(\mathbf{v})\prod_{j\in[t]} s_j\Bigg)\prod_{j\in[t]}\mathbb{I}(s_j\leqslant 1)s_j^{\ell-1}\,d\mathbf{s}\Bigg)\prod_{j\in[t]}\prod_{i\in[\ell-1]} d\mathbf{v}_j(i);\end{aligned} \tag{2.3}$$

here $\sum_{i\in[\ell-1]} v_j(i)\leqslant 1$, $v_j(\ell) = 1-\sum_{i\in[\ell-1]} v_j(i)$, $s_j\leqslant 1$, $(j\in[t])$.

To evaluate the innermost integral, switch to the variables

$$z_1 = s_1, z_2 = s_1 s_2, \quad \dots \quad, z_{t-1} = s_1\cdots s_{t-1}, z = s_1\cdots s_t.$$

The new variables meet the conditions $1 \geqslant z_1 \geqslant \cdots \geqslant z_{t-1} \geqslant z$. The Jacobian of the variables s_j, with respect to $z_1, \ldots, z_{t-1}, z$ is $1/z_1 \times \cdots \times 1/z_{t-1}$. So, the new integrand is

$$z^{\ell-1} \exp\biggl(-\ell(n-\ell) \cdot \Pi(\mathbf{v}) z\biggr) \cdot \prod_{j \in [t-1]} z_j^{-1} \cdot \mathbb{I}(1 \geqslant z_1 \geqslant z_2 \geqslant \cdots \geqslant z_{t-1} \geqslant z).$$

Integrating this function with respect to $z_1, \ldots, z_{t-1}$, first over $z_1 \in [z_2, 1]$, second over $z_2 \in [z_3, 1]$,..., and last over $z_{t-1} \in [z, 1]$, we see that the innermost integral simplifies to

$$\frac{1}{(t-1)!} \int_0^1 z^{\ell-1} \exp\left[-\ell(n-\ell) \cdot \Pi(\mathbf{v}) z\right] \log^{t-1}(1/z)\, dz.$$

Hence, recalling the definition of $g(\{v(i)\}_{i \in [\ell-1]})$, and setting $\mathbf{L} := \{\mathbf{L}_j\}_{j \in [t]}$, we rewrite (2.3) this way:

$$\frac{\ell^t}{((\ell)!)^t\, (t-1)!}\, \mathbb{E}\left[\int_0^1 z^{\ell-1} \exp\left[-\ell(n-\ell) \cdot \Pi(\mathbf{L}) z\right] \log^{t-1}(1/z)\, dz\right].$$

In summary, (2.1) becomes

$$\begin{aligned} \mathbb{E}[C_\ell] \leqslant{} & \frac{(n)_\ell\, \ell^t}{(\ell!)^t (t-1)!} \\ & \times \mathbb{E}\left[\int_0^1 z^{\ell-1} \exp\left[-\ell(n-\ell) \cdot \Pi(\mathbf{L}) z\right] \log^{t-1}(1/z)\, dz\right]. \end{aligned} \tag{2.4}$$

To sharply estimate the RHS expectation, we use a classic property of the uniformly random partition of $[0,1]$, (Karlin and Taylor [10]): the lengths $L(1), \ldots, L(\ell)$ of the subintervals of $[0,1]$ obtained by throwing $\ell - 1$ points into $[0,1]$ uniformly and independently have the same joint distribution as $\bigl\{W(1)/S(\ell), \ldots, W(\ell)/S(\ell)\bigr\}$, where $W(i)$ are independent exponentials with mean 1, and $S(\ell) = \sum_{i \in [\ell]} W(i)$. Then, introducing $t\ell$ independent exponentials $W_j(i)$, and using $\overset{\mathcal{D}}{\equiv}$ to indicate equality of two distributions in question, we have: by the definition of $\Pi(\cdot)$ in (2.2),

$$\Pi(\mathbf{L}) \overset{\mathcal{D}}{\equiv} \Pi(\mathbf{W}) = \frac{\prod_{j,i} W_j^{1/\ell}(i)}{\prod_j S_j(\ell)}, \quad S_j(\ell) = \sum_i W_j(i). \tag{2.5}$$

(1) Using Chernoff-type bound for the i.i.d. summands $\log W_j(i)$, we have: for $a > 0$,

$$\begin{aligned} \mathbb{P}\biggl(\ell^{-1} \sum_{j \in [t], i \in [\ell]} \log W_j(i) \leqslant -a\biggr) &= \mathbb{P}\biggl(\sum_{j \in [t], i \in [\ell]} \log W_j(i) \leqslant -a\ell\biggr) \\ &\leqslant \inf_{u \in (0,1)} \frac{\mathbb{E}\Bigl[\exp\Bigl(-u \sum_{j \in [t], i \in [\ell]} \log W_j(i)\Bigr)\Bigr]}{\exp(a\ell u)} = \inf_{u \in (0,1)} \frac{\bigl(\mathbb{E}[e^{-u \log W}]\bigr)^{t\ell}}{e^{a\ell u}}; \end{aligned}$$

here

$$\mathbb{E}[e^{-u\log W}] = \int_0^\infty e^{-u\log w}e^{-w}\,dw = \int_0^\infty w^{-u}e^{-w}\,dw = \Gamma(1-u).$$

Therefore

$$\mathbb{P}\biggl(\ell^{-1}\sum_{j\in[t],i\in[\ell]}\log W_j(i)\leqslant -a\biggr)\leqslant\biggl(\inf_{u\in(0,1)}\frac{\Gamma^t(1-u)}{e^{au}}\biggr)^{\ell}.$$

Since $\Gamma'(1)=-\gamma$, where γ (Euler-Mascheroni constant) is $0.57721\dots$), the infimum is strictly below 1 if $a>\gamma t$, which we assume from now. Since $\Gamma(x)$ is log-convex (Andrews, Askey, and Roy [2], Corollary 1.2.6), and $\Gamma(+0)=\infty$, the infimum is attained at a single stationary point $u(a)$ of $e^{-au}\,\Gamma^t(1-u)$ in $(0,1)$, i.e. the a-dependent root of

$$\bigl(t\log\Gamma(1-u)-au\bigr)'=-\,t\frac{\Gamma'(v)}{\Gamma(v)}\bigg|_{v=1-u}-a=0.$$

Hence

$$\mathbb{P}\biggl(\ell^{-1}\sum_{j\in[t],i\in[\ell]}\log W_j(i)\leqslant -a\biggr)\leqslant\rho^{\ell}(a),\quad \rho(a):=\min_{u\in(0,1)}\frac{\Gamma^t(1-u)}{e^{au}}. \tag{2.6}$$

$\rho(a)$ decreases as a function of a, $\rho(\infty)=0$ and $\rho(\gamma t+)=1$ since $\Gamma'(1)=-\gamma$.

(2) Furthermore, $S_j(\ell)$, $j\in[t]$, are i.i.d., with density $e^{-y}y^{\ell-1}/\Gamma(\ell)$, since for each $j\in[t]$, $S_j(j)=\sum_{i\in[\ell]}W_j(i)$, and $\{W_j(i)\}_{j\in[t],i\in[\ell]}$ are independent Exponentials with mean 1. Picking $b>0$, we bound, again via Chernoff-type method,

$$\begin{aligned}\mathbb{P}\biggl(\sum_{j\in[t]}\log S_j(\ell)\geqslant t\log(\ell(1+b))\biggr)&\leqslant\frac{\bigl(\mathbb{E}\bigl[e^{u\log S_1(\ell)}\bigr]\bigr)^t}{\exp\bigl[ut\log(\ell(1+b)\bigr])}\\&=(\ell(1+b))^{-ut}\biggl(\int_0^\infty e^{-y}\frac{y^{\ell-1}}{\Gamma(\ell)}y^u\,dy\biggr)^t\\&=\biggl[(\ell(1+b))^{-u}\frac{\Gamma(u+\ell)}{\Gamma(\ell)}\biggr]^t,\quad\forall u>0.\end{aligned}\tag{2.7}$$

Next, by Stirling-type formula, $\Gamma(\eta)=\Theta\bigl(\eta^{1/2}(\eta/e)^{\eta}\bigr)$. It is easy to show that the function $\ell(1+b))^{-ut}\cdot\bigl[(u+\ell)/e\bigr]^{u+\ell}$ attains its minimum at $u=\ell b$, and the minimum is $\bigl(\frac{1+b}{e^b}\bigr)^{\ell}$. Using (2.7), with $\Gamma(\cdot)$ replaced by its Stirling approximation, for $u=\ell b$, we obtain

$$\mathbb{P}\biggl(\sum_j\log S_j(\ell)\geqslant t\log(\ell(1+b))\biggr)=O\biggl[\biggl(\frac{1+b}{e^b}\biggr)^{\ell t}\biggr]. \tag{2.8}$$

Now, by (2.5), $\Pi(\mathbf{L})\geqslant 0$ and

$$\Pi(\mathbf{L})\geqslant\frac{e^{-a}}{\bigl[\ell(1+b)\bigr]^t},\quad\text{if}\ \prod_{j,i}W_j^{1/\ell}(i)>e^{-a},\ \prod_j S_j(\ell)<\bigl[\ell(1+b)\bigr]^t, \tag{2.9}$$

and, by (2.6) –(2.8), we have

$$\mathbb{P}\left(\prod_{j,i} W_j^{1/\ell}(i) \leqslant e^{-a} \text{ or } \prod_j S_j(\ell) \geqslant \big[\ell(1+b)\big]^t\right) = O\left(\rho^\ell(a) + \left(\frac{1+b}{e^b}\right)^{\ell t}\right). \tag{2.10}$$

Combining (2.9) and (2.10), we bound the expectation in (2.4) by

$$O\left[\rho^\ell(t) + \left(\frac{1+b}{e^b}\right)^{\ell t}\right] \int_0^1 z^{\ell-1} \log^{t-1}(1/z)\, dz \\ + \int_0^1 z^{\ell-1} \exp\left(-e^{-a} \frac{\ell(n-\ell)}{\ell^t(1+b)^t}\, z\right) \cdot \log^{t-1}(1/z)\, dz. \tag{2.11}$$

The top integral in (2.11) is bounded since $z \leqslant 1$. Since $\frac{\ell^2}{\ell^t} \leqslant 1$, the bottom integral is of order

$$\int_0^1 z^{\ell-1} \exp\left(-e^{-a} \frac{\ell n}{\ell^t(1+b)^t}\, z\right) \cdot \log^{t-1}(1/z)\, dz \\ \leqslant \max_{z\in[0,1]} z^{\ell-1} \exp\left(-e^{-a} \frac{\ell n}{\ell^t(1+b)^t}\, z\right) \times \int_0^1 \log^{t-1}(1/z)\, dz.$$

The second line integral is bounded, and the outside maximum is attained at $z = 1$, provided that $\ell \geqslant e^{-\gamma} n^{1/t}$ and n is large. Indeed, the function in question increases for $z \leqslant (1-1/\ell)e^a(1+b)^t(\ell^t/n)$; since $a > \gamma t$, the bound for z exceeds 1 for $\ell \geqslant e^{-\gamma} n^{1/t}$, and n is sufficiently large. The resulting bound together with (2.11) imply that, for $x := \ell/n^{1/t} \geqslant e^{-\gamma}$, $a > \gamma t$, and $b > 0$, the expectation in (2.4) is at most of order

$$\rho^\ell(a) + \left(\frac{1+b}{e^b}\right)^{\ell t} + \left[\exp\left(-\frac{e^{-a}}{x^t(1+b)^t}\right)\right]^\ell.$$

It remains to determine $a(x)$ and $b(x)$ that minimize this bound. As functions of $b \in [0, \infty]$, $\left(\frac{1+b}{e^b}\right)^t$ decreases from 1 to 0, while $\exp\left(-\frac{e^{-a}}{x^t(1+b)^t}\right)$ increases to 1. So, there exists a unique root $b(a, x)$ of

$$\left(\frac{1+b}{e^b}\right)^t = \exp\left(-\frac{e^{-a}}{x^t(1+b)^t}\right), \quad b > 0. \tag{2.12}$$

Clearly, $b(a, x)$ is the minimum point of $\max\left\{\left(\frac{1+b}{e^b}\right)^t, \exp\left(-\frac{e^{-a}}{x^t(1+b)^t}\right)\right\}$. Now, for

$$H(x, a, b) := t\big[\log(1+b) - b\big] + \frac{e^{-a}}{x^t(1+b)^t},$$

we have $H(x, a, b(a, x)) = 0$, $H'_b < 0$, $H'_a < 0$, and $H'_x < 0$. By implicit differentiation, $b(a, x)$ decreases to zero as a function of a and x. Finally, we choose $a = a(x) > \gamma t$ such that

$$\rho(a) = \left(\frac{1+b(a,x)}{e^{b(a,x)}}\right)^t; \tag{2.13}$$

$a(x)$ exists since, as functions of $a \in (\gamma t, \infty)$, $\rho(a)$ decreases from 1 to 0, while $\left(\frac{1+b(a,x)}{e^{b(a,x)}}\right)^t$ increases to 1. And $a(x)$ minimizes $\max\left\{\rho(a), \left(\frac{1+b(a,x)}{e^{b(a,x)}}\right)^t\right\}$. Now, for

$$K(x,a,b(a,x)) := \rho(a) - \left(\frac{1+b(a,x)}{e^{b(a,x)}}\right)^t,$$

we have

$$\begin{aligned} K'_a(x,a,b(x,a)) = K'_a(x,a,u)\big|_{u=b(x,a)} + K'_u(x,a,u)\big|_{u=b(a,x)} b'_a(a,x) \\ = \rho'_a(a) + b(x,a)e^{-b(x,a)} b'_a(a,x) < 0, \end{aligned}$$

since $\rho'_a(a) < 0$ and $b'_a(a,x) < 0$, and

$$\begin{aligned} K'_x(x,a,b(x,a)) = K'_x(x,a,u)\big|_{u=b(x,a)} + K'_u(x,a,u)\big|_{u=b(a,x)} b'_x(a,x) \\ = b(x,a)e^{-b(x,a)} b'_x(a,x) < 0. \end{aligned}$$

Again by implicit differentiation, $a(x)$, the root of $K(x,a,b(x,a)) = 0$, decreases as a function of x.

In summary, we proved that $(a(x), b(x))$, $(b(x) := b(x,a(x)))$, is a unique minimum point of

$$q(x;a,b) := \max\left\{\rho(a), \left(\frac{1+b}{e^b}\right)^t, \exp\left(-\frac{e^{-a}}{x^t(1+b)^t}\right)\right\},$$

and, denoting $q(x) = q(x; a(x), b(x))$,

$$q(x) = \rho(a(x)) = \left(\frac{1+b(x)}{e^{b(x)}}\right)^t = \exp\left(-\frac{e^{-a(x)}}{x^t(1+b(x))^t}\right).$$

Since both $\rho(a)$ and $a(x)$ are decreasing, $q(x) = \rho(a(x))$ is increasing.

Now, in (2.4), $(n)_\ell \ell^t/(\ell!)^t$ is of order $\left[n/(\ell/e)^t\right]^\ell \ell^{t/2}$. So, the previous discussion shows that for $\ell = xn^{1/t}$, $E[C_\ell]$ is at most of order $n^{1/2}\left(\left(\frac{e}{x}\right)^t q(x)\right)^\ell$. Introducing $\bar{x} = \inf\{x \in [e^{-\gamma}, e] : (e/x)^t q(x) < 1\}$, we see that $\bar{x} < e$, since $q(e) < 1$. And we conclude that $\mathbb{E}[C_\ell] = O\left[\exp(-\Theta(\varepsilon)n^{1/t})\right]$, if $\ell \geqslant (\bar{x}+\varepsilon)n^{1/t}$. The proof of Theorem 1 is complete.

Acknowledgment. I am grateful to a reviewer for an expert reading of the manuscript and asking probing, consequential questions.

References

[1] D. Aldous and P. Diaconis, *Hammersley's interacting particle process and longest increasing subsequences*, Technical Report, Dept. Stat., U.C. Berkeley, CA (1993).

[2] G. E. Andrews, R. Askey, and R. Roy, *Special functions*, Encyclopedia Math. Appl. Cambridge (1999).

[3] B. Bollobas and G. Brightwell, *The height of a random partial order: concentration of measure*, Ann. Appl. Probab. **2** (1992), 1009–1018.

[4] B. Bollobás and P. Winkler, *The longest chain among random points in Euclidean space*, Proc, Amer. Math. Soc. **103** (1988), 347–353.

[5] G. Brightwell, *Random k-dimensional orders: width and number of linear extensions*, Order **9** (1992), 333–342.

[6] A. M. Frieze, *On the length of the longest monotone subsequence in a random permutation*, Ann. Appl. Prob. **1** (1991), 301-305.

[7] B. F. Logan and L. A. Shepp, *A variational problem for random Young tableaux*, Adv. in Math. **26** (1977), 206–222.

[8] J. M. Hammersley, *A few seedlings of research*, Proc. Sixth Berkeley Sympos. Math. Stat. Prob., Univ. of California Press, Berkeley (1972), 345–394.

[9] J. M. Hammersley, *Postulates for subadditive processes*, Ann. Probab. **2** (1973), 903.

[10] S. Karlin and H. M. Taylor, *A second course in stochastic processes*, Acad. Press (1981).

[11] J. F. C. Kingman, *Subadditive ergodic theory*, Ann. Probab. **1** (1973), 883–909.

[12] D. E. Knuth, *The Art of Computer Programming*, Vol 3, Sorting and Searching, 2nd Edition, Section 5.2.2, Addison-Wesley (1998).

[13] C. Schensted, *Longest increasing and decreasing subsequences*, Canad, J. Math. **13** (1961), 179–191.

[14] J. Spencer (1991) Nonconvergence m the theory of random orders, Order 7, 341-348.

[15] J. M. Steele, *Variations on the monotone subsequence theme of Erdős and Szekeres*, Discrete probability and algorithms, IMA Vol. Math. Appl., Springer-Verlag **72** (1993) 111–131.

[16] M. Talagrand, *Concentration of measure and isoperimetric inequalities in product spaces*, Institute Des Hautes Études Scientifiques, Publications Mathématiques **81** (1995), 73–205.

[17] S. M. Ulam, *Monte Carlo calculations in problems of mathematical physics*, Modern Mathematics for the Engineer (E. F. Beckenbach, Ed.), McGraw-Hill (1961).

[18] A. M. Versik and S. V. Kerov, *Asymptotics of the Plancherel measure of the symmetric group and the limiting form of Young tableaux*, Dokl. Akad. Nauk SSSR **233** (1977), 1024–1028.

[19] A. M. Versik and S. V. Kerov, *Asymptotics of the largest and the typical dimensions of irreducible representations of the symmetric group*, Funct. Anal. Appl. **19** (1985), 21–31.

[20] P. M. Winkler, *Random orders*, Order **1** (1985), 317-335.

[21] P. M. Winkler, *Connectedness and diameter for random orders of fixed dimension*, Order **2** (1985), 165–171.

[22] P. M. Winkler, *A counterexample in the theory of random orders*, Order 5 (1989), 363–368.

[23] P M. Winkler, *Random orders of dimension 2*, Order 7 (1991), 329–339.