Bounded Littlewood identities with fixed number of odd rows or odd columns

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Abstract

A Littlewood identity is an identity equating a sum of Schur functions with an infinite product. A bounded Littlewood identity is one where the sum is taken over the partitions with a bounded number of rows or columns. The price to pay is that the infinite product has to be replaced by a determinant. The focus of this article is on refinements of such bounded Littlewood identities where one also prescribes the number of odd-length rows or columns of the partitions. Goulden [*Discrete Math.* **99** (1992), 69–77] had given such a refinement in which the number of columns is bounded and the number of odd-length rows is prescribed. We provide refinements where the number of columns is bounded and the number of odd-length columns is prescribed. Furthermore, we present new formulations of such bounded Littlewood identities involving skewing operators. As corollaries we obtain non-standard formulas for numbers of standard Young tableaux with restricted shapes as above. In the last part of the article we discuss combinatorial interpretations of such identities in terms of up-down tableaux. As corollaries, we obtain identities between numbers of standard Young tableaux and numbers of (marked) vacillating tableaux.

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1 Introduction

It is common understanding nowadays, when one speaks of a *“Littlewood identity”*, to mean an identity that equates an infinite sum of *Schur functions* to an infinite product.

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(We refer the reader to Section 2 for all definitions, and to [19] and [23, Chapter 7] for in-depth introductions to the theory of symmetric functions.) The prototypical example is

$$\sum_{\lambda} s_\lambda(\mathbf{x}) = \frac{1}{\prod_i (1-x_i) \prod_{i<j}(1-x_i x_j)},$$

where $s_\lambda(\mathbf{x})$ denotes the Schur function indexed by the partition λ, even though this particular identity is due to Schur. (The reader may want to consult the introduction of [9] for a more detailed discussion of the history of Littlewood(-type) identities.) Other Littlewood identities provide product formulas for sums of Schur functions over subsets of the set of partitions, such as partitions with only even parts or self-conjugate partitions. A *bounded Littlewood identity* is one where the sum of Schur functions is over partitions with a restricted number of columns or rows. The theorem below contains the two classical bounded Littlewood identities. We refer the reader again to the introduction of [9] for attribution and slightly convoluted history of these identities.

Theorem 1.1 (TWO BOUNDED LITTLEWOOD IDENTITIES). For a nonnegative integer w, we have

$$\sum_{\lambda:\lambda_1 \leqslant 2w+1} s_\lambda(\mathbf{x}) = \sum_{k \geqslant 0} e_k(\mathbf{x}) \cdot \det_{1 \leqslant i,j \leqslant w} \left(f_{i-j}(\mathbf{x}) - f_{i+j}(\mathbf{x}) \right) \tag{1.1}$$

and

$$\sum_{\lambda:\lambda_1 \leqslant 2w} s_\lambda(\mathbf{x}) = \det_{1 \leqslant i,j \leqslant w} \left(f_{i-j}(\mathbf{x}) + f_{i+j-1}(\mathbf{x}) \right), \tag{1.2}$$

where $e_k(\mathbf{x})$ is the k-th elementary symmetric function and

$$f_r(\mathbf{x}) = \sum_{n \in \mathbb{Z}} e_n(\mathbf{x}) e_{n+r}(\mathbf{x}). \tag{1.3}$$

The focus of the present article is on *refinements* of the above bounded Littlewood identities and their combinatorics. In these refinements the number of odd rows or odd columns of the partitions over which the summation of Schur functions is taken is fixed. Here, an *odd row* (respectively *odd column*) means a row (respectively column) of odd length. Indeed, there is already a substantial literature on such refinements. It is our goal (1) to provide an overview of the relevant results that are scattered over the literature, (2) to exhibit the connections between the various existing results, and (3) to present new results, in particular combinatorial interpretations of the right-hand sides of such refined bounded Littlewood identities in terms of up-down tableaux.

In the remainder of this introduction, we provide an overview over the material that we present in this article. As already indicated, all relevant definitions are given in Section 2.

OVERVIEW — SECTION 3: COLUMN REFINEMENTS OF THEOREM 1.1. To begin with, we recall Goulden's (row) refinements [8] of the two bounded Littlewood identities in Theorem 1.1. In order to state these, let $r(\lambda)$ (respectively $c(\lambda)$) denote the number of odd rows (respectively odd columns) of λ.

Theorem 1.2 ([8, THEOREMS 2.4 AND 2.6]). For integers $w \geqslant 1$ and $k \geqslant 0$, we have

$$\sum_{\substack{\lambda:\lambda_1\leqslant 2w+1\\ r(\lambda)=k}} s_\lambda(\mathbf{x}) = e_k(\mathbf{x}) \det_{1\leqslant i,j\leqslant w} \left(f_{i-j}(\mathbf{x}) - f_{i+j}(\mathbf{x})\right) \tag{1.4}$$

and

$$\sum_{\substack{\lambda:\lambda_1\leqslant 2w\\ r(\lambda)=k}} s_\lambda(\mathbf{x}) = \det_{1\leqslant i,j\leqslant w} \left(\begin{cases} f_{i-j}(\mathbf{x}) - f_{i+j}(\mathbf{x}), & \text{if } 1 \leqslant i \leqslant w-1 \\ f_{i-j+k}(\mathbf{x}) - f_{i+j+k}(\mathbf{x}), & \text{if } i = w \end{cases}\right). \tag{1.5}$$

In particular,

$$\sum_{\substack{\lambda:\lambda_1\leqslant 2w+1\\ r(\lambda)=0}} s_\lambda(\mathbf{x}) = \sum_{\substack{\lambda:\lambda_1\leqslant 2w\\ r(\lambda)=0}} s_\lambda(\mathbf{x}) = \det_{1\leqslant i,j\leqslant w} \left(f_{i-j}(\mathbf{x}) - f_{i+j}(\mathbf{x})\right), \tag{1.6}$$

with $f_r(\mathbf{x})$ as given in (1.3).

We present here a refinement of Theorem 1.1 in which one keeps track of the number of odd *columns* of the partitions over which the sum is taken. For convenience, we put

$$e(\mathbf{x}) = \sum_{n\geqslant 0} e_n(\mathbf{x}), \qquad \overline{e}(\mathbf{x}) = \sum_{n\geqslant 0} (-1)^n e_n(\mathbf{x}).$$

Theorem 1.3. Let w be a nonnegative integer, and let u be an indeterminate.

(1) We have

$$\sum_{\lambda:\lambda_1\leqslant 2w} \left(u^{c(\lambda)} + u^{2w-c(\lambda)}\right) s_\lambda(\mathbf{x})$$
$$= \frac{1}{2}\sum_{k=0}^{w} \left(u^k + u^{2w-k}\right) \det_{1\leqslant i,j\leqslant w} \left(\begin{cases} f_{i-j}(\mathbf{x}) + f_{i+j-2}(\mathbf{x}), & \text{if } 1 \leqslant i \leqslant w-k \\ f_{i-j+1}(\mathbf{x}) + f_{i+j-1}(\mathbf{x}), & \text{if } w-k+1 \leqslant i \leqslant w \end{cases}\right) \tag{1.7}$$

and

$$\sum_{\lambda:\lambda_1\leqslant 2w} \left(u^{c(\lambda)} - u^{2w-c(\lambda)}\right) s_\lambda(\mathbf{x})$$
$$= e(\mathbf{x})\overline{e}(\mathbf{x})\sum_{k=0}^{w-1}\left(u^k - u^{2w-k}\right) \det_{1\leqslant i,j\leqslant w-1} \left(\begin{cases} f_{i-j}(\mathbf{x}) - f_{i+j}(\mathbf{x}), & \text{if } 1 \leqslant i \leqslant w-k-1 \\ f_{i-j+1}(\mathbf{x}) - f_{i+1j+11}(\mathbf{x}), & \text{if } w-k \leqslant i \leqslant w-1 \end{cases}\right). \tag{1.8}$$

(2) We have

$$\sum_{\lambda:\lambda_1\leqslant 2w+1} \left(u^{c(\lambda)} + u^{2w+1-c(\lambda)}\right) s_\lambda(\mathbf{x})$$
$$= e(\mathbf{x}) \cdot \sum_{k=0}^{w} \left(u^k + u^{2w+1-k}\right) \det_{1\leqslant i,j\leqslant w} \left(\begin{cases} f_{i-j}(\mathbf{x}) - f_{i+j-1}(\mathbf{x}), & \text{if } 1 \leqslant i \leqslant w-k \\ f_{i-j+1}(\mathbf{x}) - f_{i+j}(\mathbf{x}), & \text{if } w-k+1 \leqslant i \leqslant w \end{cases}\right) \tag{1.9}$$

and

$$\sum_{\lambda:\lambda_1\leqslant 2w+1}\left(u^{c(\lambda)}-u^{2w+1-c(\lambda)}\right)s_\lambda(\mathbf{x})$$
$$=\bar{e}(\mathbf{x})\cdot\sum_{k=0}^{w}\left(u^k-u^{2w+1-k}\right)\det_{1\leqslant i,j\leqslant w}\left(\begin{cases}f_{i-j}(\mathbf{x})+f_{i+j-1}(\mathbf{x}), & \text{if } 1\leqslant i\leqslant w-k\\ f_{i-j+1}(\mathbf{x})+f_{i+j}(\mathbf{x}), & \text{if } w-k+1\leqslant i\leqslant w\end{cases}\right). \tag{1.10}$$

Again, the series $f_r(\mathbf{x})$ are given in (1.3).

If we put $u=1$ in (1.7) and (1.9) and use Lemma 3.3 below and column operations, we recover the bounded Littlewood identities (1.2) and (1.1), respectively. (If we put $u=1$ in (1.8) and (1.10), then both sides become zero.) We note that the $u=0$ cases of Theorem 1.3 were first proved in [20, Theorem 2.3(3)] as classical group character identities of rectangular shape (see also Section 4). Cylindric versions were given in the authors' previous paper [9].

We prove Theorem 1.3 in Section 3. Our main tool is the *minor summation formula* of Ishikawa and Wakayama [10, Theorem 1]; see Theorem 3.1.

OVERVIEW — SECTION 4: EQUIVALENCE OF THEOREM 1.3 AND IDENTITIES FOR CLASSICAL GROUP CHARACTERS OF NEARLY RECTANGULAR SHAPE. As it turns out, the refined bounded Littlewood identities in Theorem 1.3 are equivalent to identities for classical group characters obtained by the third author in [15]. We explain this (non-trivial) equivalence in Section 4.

OVERVIEW — SECTION 5: SKEWING OPERATORS. Here we present new formulations of the bounded Littlewood identities in Theorem 1.1 and their refinements in Theorems 1.2 and 1.3 in terms of *skewing operators*. In order to state these, we recall that, given a symmetric function $f(\mathbf{x})$, the associated skewing operator $f^\perp$ is, by definition, the adjoint of multiplication by $f(\mathbf{x})$ with respect to the *Hall inner product*, that is, $\langle f^\perp r_1(\mathbf{x}), r_2(\mathbf{x})\rangle = \langle r_1(\mathbf{x}), f(\mathbf{x})r_2(\mathbf{x})\rangle$, for all symmetric functions $r_1(\mathbf{x})$ and $r_2(\mathbf{x})$. We refer the reader again to Section 2 for full definitions. In the theorems below, we only need the skewing operator associated with the first power-sum symmetric function $p_1(\mathbf{x})=x_1+x_2+\cdots$.

Theorem 1.4. For a nonnegative integer w,

$$\sum_{\lambda:\lambda_1\leqslant 2w+1}s_\lambda(\mathbf{x})=e(\mathbf{x})\det_{1\leqslant i,j\leqslant w}\left((p_1^\perp)^{i+j-2}(f_0(\mathbf{x})-f_2(\mathbf{x}))\right), \tag{1.11}$$
$$\sum_{\lambda:\lambda_1\leqslant 2w}s_\lambda(\mathbf{x})=\det_{1\leqslant i,j\leqslant w}\left((p_1^\perp)^{i+j-2}(f_0(\mathbf{x})+f_1(\mathbf{x}))\right). \tag{1.12}$$

We prove as well analogous formulations for the refinements in Theorems 1.2 and 1.3.

Theorem 1.5. For nonnegative integers w and k,

$$\sum_{\substack{\lambda:\lambda_1\leqslant 2w+1\\ r(\lambda)=k}}s_\lambda(\mathbf{x})=e_k(\mathbf{x})\det_{1\leqslant i,j\leqslant w}\left((p_1^\perp)^{i+j-2}(f_0(\mathbf{x})-f_2(\mathbf{x}))\right), \tag{1.13}$$

$$\sum_{\substack{\lambda:\lambda_1\leqslant 2w\\ r(\lambda)=k}} s_\lambda(\mathbf{x}) = \det_{1\leqslant i,j\leqslant w}\left((p_1^\perp)^{j-1}(f_{i+k\delta_{i,w}-1}(\mathbf{x}) - f_{i+k\delta_{i,w}+1}(\mathbf{x}))\right), \tag{1.14}$$

where $\delta_{i,j}=1$ if $i=j$ and $\delta_{i,j}=0$ otherwise.

Theorem 1.6. For nonnegative integers w and k such that $0\leqslant k\leqslant w$, we have

$$\sum_{\substack{\lambda:\lambda_1\leqslant 2w+1\\ c(\lambda)=k}} s_\lambda(\mathbf{x}) + \sum_{\substack{\lambda:\lambda_1\leqslant 2w+1\\ c(\lambda)=2w+1-k}} s_\lambda(\mathbf{x}) \\ = e(\mathbf{x}) \det_{1\leqslant i,j\leqslant w}\left((p_1^\perp)^{j-1}(f_{i+\chi(i>w-k)-1}(\mathbf{x}) - f_{i+\chi(i>w-k)}(\mathbf{x}))\right), \tag{1.15}$$

$$\sum_{\substack{\lambda:\lambda_1\leqslant 2w+1\\ c(\lambda)=k}} s_\lambda(\mathbf{x}) - \sum_{\substack{\lambda:\lambda_1\leqslant 2w+1\\ c(\lambda)=2w+1-k}} s_\lambda(\mathbf{x}) \\ = \overline{e}(\mathbf{x}) \det_{1\leqslant i,j\leqslant w}\left((p_1^\perp)^{j-1}(f_{i+\chi(i>w-k)-1}(\mathbf{x}) + f_{i+\chi(i>w-k)}(\mathbf{x}))\right), \tag{1.16}$$

$$\sum_{\substack{\lambda:\lambda_1\leqslant 2w\\ c(\lambda)=k}} s_\lambda(\mathbf{x}) + \sum_{\substack{\lambda:\lambda_1\leqslant 2w\\ c(\lambda)=2w-k}} s_\lambda(\mathbf{x}) \\ = 2^{\chi(k=w)} \det_{1\leqslant i,j\leqslant w}\left((p_1^\perp)^{j-1} f_{i+\chi(i>w-k)-1}(\mathbf{x})\right), \tag{1.17}$$

and, for $0\leqslant k\leqslant w-1$,

$$\sum_{\substack{\lambda:\lambda_1\leqslant 2w\\ c(\lambda)=k}} s_\lambda(\mathbf{x}) - \sum_{\substack{\lambda:\lambda_1\leqslant 2w\\ c(\lambda)=2w-k}} s_\lambda(\mathbf{x}) \\ = e(\mathbf{x})\overline{e}(\mathbf{x}) \det_{2\leqslant i,j\leqslant w}\left((p_1^\perp)^{j-2}(f_{i+\chi(i>w-k)-2}(\mathbf{x}) - f_{i+\chi(i>w-k)}(\mathbf{x}))\right). \tag{1.18}$$

where $\chi(P)$ is defined to be 1 if a statement P is true and 0 otherwise.

The proofs of these three theorems are found in Section 5. They are based on elementary properties of the skewing operator $p_1^\perp$ when applied to the series $f_r(\mathbf{x})$ and row and column operations applied to the determinants in Theorems 1.1, 1.2, and 1.3.

OVERVIEW — SECTION 6: UNUSUAL FORMULAS FOR NUMBERS OF STANDARD YOUNG TABLEAUX OF BOUNDED WIDTH. In Section 6, we show that the formulas in Theorems 1.4, 1.5, and 1.6 can be used to derive formulas for the number of standard Young tableaux of bounded width, some of them known but derived only recently in a completely different manner, some of them new.

Let $\mathrm{SYT}_{n,w}$ denote the set of standard Young tableaux of size n with width at most w, where the *width* is the number of columns. In order to explain the context, we have to recall Gessel's classical result [3, Eq. (22), after application of the operator θ; cf. p. 277], which, as Gessel points out, is implicit in papers of Gordon and Houten [7, 6] and of Bender and Knuth [1].

Theorem 1.7. For a nonnegative integer w,

$$\sum_{n\geqslant 0}|\operatorname{SYT}_{n,2w+1}|\frac{x^n}{n!}=\exp(x)\det_{1\leqslant i,j\leqslant w}\left(I_{i-j}(2x)-I_{i+j}(2x)\right), \tag{1.19}$$

$$\sum_{n\geqslant 0}|\operatorname{SYT}_{n,2w}|\frac{x^n}{n!}=\det_{1\leqslant i,j\leqslant w}\left(I_{i-j}(2x)+I_{i+j-1}(2x)\right), \tag{1.20}$$

where

$$I_\alpha(2x)=\sum_{r\geqslant 0}\binom{2r+|\alpha|}{r}\frac{x^{2r+|\alpha|}}{(2r+|\alpha|)!}$$

is the modified Bessel function.

In [11, Theorem 10.7], the second author, Lee, and Oh found an integral expression for $|\operatorname{SYT}_{n,w}|$,

$$\int_{O(w+1)}(1-\det(X))(1+\operatorname{Tr}(X))^n d\mu(X),$$

and gave an explicit formula by evaluating the integral. In the first formula below, $\operatorname{Cat}(n)$ stands for the *n-th Catalan number* if n is a nonnegative integer, that is, $\operatorname{Cat}(n)=\frac{1}{n+1}\binom{2n}{n}$, and $\operatorname{Cat}(n)=0$ if n is not a nonnegative integer.

Theorem 1.8 ([11, Theorem 10.9]). For integers $w\geqslant 1$ and $n\geqslant 0$, we have

$$|\operatorname{SYT}_{n,2w+1}|=\sum_{\substack{t_0,t_1,\dots,t_w\in\mathbb{Z}_{\geqslant 0}\\ t_0+t_1+\cdots+t_w=n}}\binom{n}{t_0,t_1,\dots,t_w}\det_{1\leqslant i,j\leqslant w}\left(\operatorname{Cat}\left(\frac{t_i+2w-i-j}{2}\right)\right) \tag{1.21}$$

and

$$|\operatorname{SYT}_{n,2w}|=\sum_{\substack{t_1,\dots,t_w\in\mathbb{Z}_{\geqslant 0}\\ t_1+\cdots+t_w=n}}\binom{n}{t_1,\dots,t_w}\det_{1\leqslant i,j\leqslant w}\left(\binom{t_i+2w-i-j}{\lfloor (t_i+2w-i-j)/2\rfloor}\right). \tag{1.22}$$

We show in Section 6 that these formulas can conveniently be derived from the identities in Theorem 1.4. Similarly, from the identities in Theorem 1.5 we may derive analogous formulas for the number of standard Young tableaux with bounded width and prescribed number of odd rows. Let f^λ denote the number of standard Young tableaux of shape λ. In order to state the second of the formulas below, for nonnegative integers r and s, define $F(r,s):=\binom{r}{s}-\binom{r}{s-1}$. We also define $F(r,s/2):=0$ if s is an odd integer.

Theorem 1.9. For nonnegative integers k,w, and n, we have

$$\sum_{\substack{\lambda\vdash n\\ \lambda_1\leqslant 2w+1,\ r(\lambda)=k}}f^\lambda=\sum_{\substack{t_1,\dots,t_w\in\mathbb{Z}_{\geqslant 0}\\ t_1+\cdots+t_w=n-k}}\binom{n}{k,t_1,\dots,t_w}\det_{1\leqslant i,j\leqslant w}\left(\operatorname{Cat}\left(\frac{t_i+i+j-2}{2}\right)\right), \tag{1.23}$$

$$\sum_{\substack{\lambda\vdash n\\ \lambda_1\leqslant 2w,\ r(\lambda)=k}}f^\lambda=\sum_{\substack{t_1,\dots,t_w\in\mathbb{Z}_{\geqslant 0}\\ t_1+\cdots+t_w=n}}\binom{n}{t_1,\dots,t_w}\det_{1\leqslant i,j\leqslant w}\left(F\left(t_i+j-1,\frac{t_i-i-k\delta_{i,w}+j}{2}\right)\right). \tag{1.24}$$

The proofs of these identities are also found in Section 6.

OVERVIEW — SECTION 7: UP-DOWN TABLEAUX. The subject of Section 7 is combinatorial interpretations of the right-hand sides of the identities in Theorems 1.2 and 1.3. Combinatorial interpretations of these identities had been given before in [13, proof of Theorem 3, Eq. (2.2)] in terms of *non-crossing two-rowed arrays*. As opposed to that, our combinatorial interpretations here are in terms of (*marked*) *up-down tableaux*, see their definition in Definition 7.1 (with refinements in Definitions 7.6, 7.8, 7.9, 7.12, and 7.14). Theorems 7.5 and 7.7 present our combinatorial interpretations of the right-hand sides of the identities in Theorem 1.2. Theorems 7.11 and 7.15 then present our combinatorial interpretations of the right-hand sides of the identities in Theorem 1.3, rewritten in an equivalent form in Theorem 5.1. Our main tool to derive these results is the main theorem on *nonintersecting lattice paths*, due to Lindström [18, Lemma 1] (and later rediscovered, among others, by Gessel and Viennot [4, 5]), which allows one to interpret determinants as the ones on the right-hand sides of the identities in Theorems 1.2 and 1.3 as certain generating functions for nonintersecting lattice paths. We reformulate these families of nonintersecting lattice paths then in terms of (marked) up-down tableaux.

OVERVIEW — SECTION 8: STANDARD YOUNG TABLEAUX AND LATTICE WALKS. In the literature, there appear several results on equality of the number of standard Young tableaux satisfying certain conditions and the number of certain lattice walks. We restate here two, due to Zeilberger and to Eu, Fu, Hou, and Hsu, respectively.

Theorem 1.10 (ZEILBERGER [24]). The number $|\,\mathrm{SYT}_{n,2w+1}\,|$ is equal to the number of walks of length n in the region $\{(x_1,\dots,x_w) : x_1 \geqslant \dots \geqslant x_w \geqslant 0\}$ using steps in $\{\mathbf{0}, \pm\epsilon_1,\dots,\pm\epsilon_w\}$, where $\mathbf{0}=(0,\dots,0)$ and $\epsilon_i=(0,\dots,0,1,0,\dots,0)$, with the 1 in position i.

Theorem 1.11 (EU ET AL. [2]). The number $|\,\mathrm{SYT}_{n,2w}\,|$ is equal to the number of walks of length n in the region $\{(x_1,\dots,x_w) : x_1 \geqslant \dots \geqslant x_w \geqslant 0\}$ using steps in $\{\mathbf{0}, \pm\epsilon_1,\dots,\pm\epsilon_w\}$ such that zero steps $\mathbf{0}$ can only occur when $x_w=0$.

Moreover, this is also true with the refined condition that the number of odd rows is k and the number of zero steps is k.

While Zeilberger combines a few known results to obtain his theorem, Eu et al. provide bijective proofs of Theorem 1.11 *and* of Theorem 1.10. See however [9, Appendix B, proof of Corollary B.5] for more conceptual (bijective) proofs using growth diagrams.

Clearly, standard Young tableaux are the special case of semistandard Young tableaux where the entries are restricted to $1,2,\dots,n$, each appearing exactly once. Consequently, our results in Section 7 should imply identities between standard Young tableaux and the corresponding specializations of the (marked) up-down tableaux that were the combinatorial objects in that section. Section 8 is devoted to making the corresponding results explicit, which sometimes requires additional (combinatorial) arguments. Theorems 8.3 and 8.7 present our results, the former concerning standard Young tableaux with an odd bound on the number of columns, the latter with an even bound.

OVERVIEW — SECTION 9: FINAL QUESTIONS. We close our article by posing a few questions that are suggested by some of our results.

2 Definitions

In this section we collect definitions concerning partitions, tableaux and symmetric functions.

For a nonnegative integer n, we write $[n] = \{1, \dots, n\}$.

A *partition* of a nonnegative integer n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of positive integers, called *parts*, such that $\sum_{i \geqslant 1} \lambda_i = n$. This also includes the empty partition $()$, denoted by $\emptyset$, which, by definition, is the only partition of 0. We write $\lambda \vdash n$ to mean that λ is a partition of n. If λ is a partition of n into k parts, we write $|\lambda| = n$ and $\ell(\lambda) = k$, and say that λ has *size* n and *length* k. We denote by Par the set of all partitions. It is often convenient to identify a partition $(\lambda_1, \lambda_2, \dots, \lambda_k)$ with a sequence $(\lambda_1, \lambda_2, \dots, \lambda_k, 0, 0, \dots)$ or $(\lambda_1, \lambda_2, \dots, \lambda_k, 0, \dots, 0)$, where infinitely or finitely many zeros are appended at the end. Using this convention, we define $\lambda_i = 0$ for $i > \ell(\lambda)$.

The *Young diagram* of a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is the set $\{(i, j) \in \mathbb{Z}^2 : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant \lambda_i\}$. Each element (i, j) in a Young diagram is called a *cell*. The Young diagram of λ is visualized as a left-justified array of unit square cells with λ_i cells in the i-th row, $i = 1, 2, \dots, k$, from top to bottom. We identify λ with its Young diagram. The *conjugate* (or *transpose*) λ' of a partition λ is the partition whose Young diagram is given by $\{(j, i) : (i, j) \in \lambda\}$. For two partitions λ and μ we write $\mu \subseteq \lambda$ to mean that the Young diagram of μ is contained in that of λ. If $\mu \subseteq \lambda$, the *skew shape* λ/μ is the set-theoretic difference $\lambda - \mu$ of the Young diagrams of λ and μ. Each partition λ is also considered as the skew shape $\lambda/\emptyset$.

Given a partition λ, we denote by $r(\lambda)$ (respectively $c(\lambda)$) the number of rows (respectively columns) of odd length:

$$r(\lambda) = |\{i : \lambda_i \text{ is odd}\}|, \quad c(\lambda) = |\{j : \lambda'_j \text{ is odd}\}|.$$

A *tableau* of shape λ/μ is a filling of the cells in λ/μ with positive integers. For a tableau T of shape $\lambda = (\lambda_1, \lambda_2, \dots)$, the *size* and *width* of T are defined to be $|\lambda|$ and λ_1, respectively. A *semistandard Young tableau* (SSYT) is a tableau in which the entries along rows are weakly increasing and the entries along columns are strictly increasing. A *standard Young tableau* (SYT) is a semistandard Young tableau whose entries are the integers $1, 2, \dots, n$, where n is the size of the tableau.

A *symmetric function* in the variables $\mathbf{x} = \{x_1, x_2, \dots\}$ is a formal power series in these variables (in the sense that each monomial appearing in the expansion may only contain a finite number of variables) that is invariant under all permutations of the variables that leave almost all variables invariant. We denote the algebra of symmetric functions by Λ. (Strictly speaking, this is the *completion* $\hat{\Lambda}$ in the sense of [19, p. 19, Remarks 1]. However, we do not want to make this distinction here.)

In this paper we shall be concerned with the following symmetric functions.

For $n \geqslant 1$, the *n-th power-sum symmetric function* $p_n(\mathbf{x})$ is defined by

$$p_n(\mathbf{x}) = \sum_{i \geqslant 1} x_i^n.$$

The *n-th complete homogeneous symmetric function* $h_n(\mathbf{x})$ and the *n-th elementary symmetric function* $e_n(\mathbf{x})$ are defined by

$$h_n(\mathbf{x}) = \sum_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \quad \text{and} \quad e_n(\mathbf{x}) = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

respectively. By convention, we set $h_0(\mathbf{x}) = e_0(\mathbf{x}) = 1$ and define $h_n(\mathbf{x})$ and $e_n(\mathbf{x})$ to be zero for $n < 0$.

For any tableau T, let $\mathbf{x}^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$, where α_i is the number of i's in T. For a partition λ, the *Schur function* $s_\lambda(\mathbf{x})$ is defined by

$$s_\lambda(\mathbf{x}) = \sum_T \mathbf{x}^T, \tag{2.1}$$

where the sum is over all semistandard Young tableaux T of shape λ. The number of standard Young tableaux of shape λ is equal to the coefficient of $x_1 x_2 \cdots x_n$ in the Schur function $s_\lambda(\mathbf{x})$, where $n = |\lambda|$.

The Schur function satisfies determinantal formulas in terms of complete homogeneous symmetric functions and elementary symmetric functions. These formulas are known as the *Jacobi–Trudi identity* (cf. [19, p. 41, Eq. (3.4)] or [23, Theorem 7.16.1]) and the *Nägelsbach–Kostka identity* (cf. [19, p. 41, Eq. (3.5)] or [23, Corollary 7.16.2]). Explicitly, they are

$$s_\lambda(\mathbf{x}) = \det_{1 \leqslant i,j \leqslant p} \left(h_{\lambda_i - i + j}(\mathbf{x}) \right) = \det_{1 \leqslant i,j \leqslant q} \left(e_{\lambda'_i - i + j}(\mathbf{x}) \right), \tag{2.2}$$

where $p \geqslant \ell(\lambda)$ and $q \geqslant \lambda_1$.

The *Hall inner product* $\langle \cdot, \cdot \rangle$ on symmetric functions is defined by $\langle s_\lambda(\mathbf{x}), s_\mu(\mathbf{x}) \rangle = \delta_{\lambda,\mu}$ for all partitions λ and μ. For a given symmetric function $f(\mathbf{x}) \in \Lambda$, the *skewing operator* $f^\perp : \Lambda \to \Lambda$ is defined as the adjoint operator of multiplication by $f(\mathbf{x})$ with respect to the Hall inner product, that is,

$$\langle f^\perp r_1(\mathbf{x}), r_2(\mathbf{x}) \rangle = \langle r_1(\mathbf{x}), f(\mathbf{x}) r_2(\mathbf{x}) \rangle,$$

for all $r_1(\mathbf{x}), r_2(\mathbf{x}) \in \Lambda$.

3 Bounded Littlewood identities with a fixed number of odd columns: proof of Theorem 1.3

The purpose of this section is to prove Theorem 1.3, which contains refinements of the two bounded Littlewood identities in Theorem 1.1 according to the number of odd-length columns.

Our proof is based on the minor summation formula, which we recall in Subsection 3.1. We need however several further preparations before we are able to actually prove Theorem 1.3. These are formulas which reduce (certain) Pfaffians to determinants of matrices of half the size, and which are similar to formulas that Gordon had given in [6] — see

Lemma 3.2 in Subsection 3.2 —, a particular determinant simplification — see Lemma 3.3 in the same subsection —, and formulas for minors of certain special Pfaffians that serve as the starting point of the use of the minor summation formula — see Proposition 3.4 in Subsection 3.3. With these auxiliary results, Theorem 1.3 is finally established in Subsection 3.4.

3.1 The minor summation formula

We use the following notation for submatrices. Let $A = (a_{i,j})_{i \in I, j \in J}$ be a matrix and let $R = (r_1, \dots, r_p)$ and $S = (s_1, \dots, s_q)$ be sequences of row and column indices, respectively. We define

$$A_S^R = \left(a_{r_i, s_j}\right)_{1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q}.$$

If A is skew-symmetric, then we write A^R instead of A_R^R for brevity. We also define

$$(r_1, \dots, r_p) \sqcup (r'_1, \dots, r'_{p'}) = (r_1, \dots, r_p, r'_1, \dots, r'_{p'}).$$

By abusing notation, we write $A_S^{[m]}$ to mean that $[m] = (1, 2, \dots, m)$ is a sequence rather than the set $\{1, \dots, m\}$.

The minor summation formula of Ishikawa and Wakayama is the following.

Theorem 3.1 ([10, Theorem 1]). Let m be an even integer and p a positive integer (or infinity). For a $p \times p$ skew-symmetric matrix $A = (a_{r,s})_{1 \leqslant r,s \leqslant p}$ and an $m \times p$ matrix $M = (M_{i,r})_{1 \leqslant i \leqslant m, 1 \leqslant r \leqslant p}$, we have

$$\sum_K \operatorname{Pf}\left(A^K\right) \det\left(M_K^{[m]}\right) = \operatorname{Pf}\left(M\, A\, M^t\right) = \operatorname*{Pf}_{1 \leqslant i < j \leqslant m}\left(\sum_{r,s=1}^{p} a_{r,s} M_{i,r} M_{j,s}\right),$$

where $K = (k_1, \dots, k_m)$ runs over all increasing sequences $1 \leqslant k_1 < \cdots < k_m \leqslant p$ of integers.

3.2 Gordon-type reductions of Pfaffians to determinants, and a determinant lemma

The following Gordon-type formulas enable us to transform Pfaffians into determinants.

Lemma 3.2. If the quantities z_i, $i \in \mathbb{Z}$, satisfy $z_{-i} = -z_i$, then we have

$$\operatorname*{Pf}_{1 \leqslant i,j \leqslant 2w}(z_{j-i}) = \det_{1 \leqslant i,j \leqslant w}\left(z_{i-j+1} + z_{i-j+3} + \cdots + z_{i+j-1}\right) \tag{3.1}$$

$$= \frac{1}{2} \det_{1 \leqslant i,j \leqslant w}\left(\begin{cases} z_i - z_{i-2}, & \text{if } j = 1 \\ z_{i-j+1} - z_{i-j-1} + z_{i+j-1} - z_{i+j-3}, & \text{if } 2 \leqslant j \leqslant w \end{cases}\right) \tag{3.2}$$

$$= \det_{1 \leqslant i,j \leqslant w}\left(\sum_{k=1}^{2j-1}(z_{i-j+k} + z_{i-j+k-1})\right) \tag{3.3}$$

$$= \det_{1 \leqslant i,j \leqslant w}\left(\sum_{k=1}^{2j-1}(-1)^{k-1}(z_{i-j+k} - z_{i-j+k-1})\right). \tag{3.4}$$

Proof. By [6, Lemma 1], we have

$$\operatorname*{Pf}_{1\leqslant i,j\leqslant 2w}\Big(z_{j-i}\Big)=\det_{1\leqslant i,j\leqslant w}\Big(z_{|j-i|+1}+z_{|j-i|+3}+\cdots+z_{i+j-1}\Big).$$

If $i<j$, then we use the relations $z_{-r}+z_r=0$ to see that

$$\begin{aligned}&z_{i-j+1}+z_{i-j+3}+\cdots+z_{i+j-1}\\&=z_{i-j+1}+z_{i-j+3}+\cdots+z_{j-i-3}+z_{j-i-1}+z_{j-i+1}+z_{j-i+3}+\cdots+z_{i+j-3}+z_{i+j-1}\\&=z_{|j-i|+1}+z_{|j-i|+3}+\cdots+z_{i+j-3}+z_{i+j-1}.\end{aligned}$$

On the other hand, if $i\geqslant j$ then there is nothing to do. Hence we obtain (3.1).

The identities (3.2), (3.3) and (3.4) are deduced from (3.1) by elementary row and column operations.

We first prove (3.2). By subtracting the $(i-2)$-nd row from the i-th row for $i=w,w-1,\dots,3$, we obtain

$$\begin{aligned}A&:=\det_{1\leqslant i,j\leqslant w}\left(z_{i-j+1}+z_{i-j+3}+\cdots+z_{i+j-1}\right)\\&=\det_{1\leqslant i,j\leqslant w}\left(\begin{cases}z_{i-j+1}+z_{i-j+3}+\cdots+z_{i+j-1}, & \text{if } i\leqslant 2\\ z_{i+j-1}-z_{i-j-1}, & \text{if } i>2\end{cases}\right).\end{aligned}$$

Note that if $i=1$, then

$$z_{i-j+1}+z_{i-j+3}+\cdots+z_{i+j-1}=z_{-j+2}+z_{-j+4}+\cdots+z_{j-2}+z_j=z_j=\frac{1}{2}(z_j-z_{-j}),$$

and if $i=2$, then

$$\begin{aligned}z_{i-j+1}+z_{i-j+3}+\cdots+z_{i+j-1}&=z_{-j+3}+z_{-j+5}+\cdots+z_{j-3}+z_{j-1}+z_{j+1}\\&=z_{j-1}+z_{j+1}=z_{j+1}-z_{1-j}.\end{aligned}$$

Thus, we have

$$A=\frac{1}{2}\det_{1\leqslant i,j\leqslant w}\left(z_{i+j-1}-z_{i-j-1}\right).$$

By subtracting the $(j-2)$-nd column from the j-th column for $j=w,w-1,\dots,3$, we obtain

$$A=\frac{1}{2}\det_{1\leqslant i,j\leqslant w}\left(\begin{cases}z_{i+j-1}-z_{i-j-1}, & \text{if } j\leqslant 2\\ z_{i-j+1}-z_{i-j-1}+z_{i+j-1}-z_{i+j-3}, & \text{if } j>2\end{cases}\right),$$

which implies (3.2).

Equation (3.3) is derived from (3.1) by adding the $(i-1)$-st row to the i-th row for $i=w,w-1,\dots,2$, and then adding the $(j-1)$-st column to the j-th column for $j=w,w-1,\dots,2$. Similarly, Equation (3.4) is obtained by using subtraction instead of addition. ☐

In the last step of the proof of Theorem 1.3, we use the following determinant lemma.

Lemma 3.3. For $(n+1)$ row vectors $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n$ of length n and scalars $\beta_1, \dots, \beta_n$, $\gamma_1, \dots, \gamma_n$, we have

$$\det \begin{pmatrix} \beta_1 \mathbf{v}_0 + \gamma_1 \mathbf{v}_1 \\ \beta_2 \mathbf{v}_1 + \gamma_2 \mathbf{v}_2 \\ \vdots \\ \beta_n \mathbf{v}_{n-1} + \gamma_n \mathbf{v}_n \end{pmatrix} = \sum_{k=0}^{n} \prod_{i=1}^{k} \beta_i \prod_{j=k+1}^{n} \gamma_j \cdot \det \begin{pmatrix} \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_{k-1} \\ \mathbf{v}_{k+1} \\ \vdots \\ \mathbf{v}_n \end{pmatrix}.$$

Proof. For a subset I of $[n]$, let $V(I)$ be the $n \times n$ matrix whose i-th row is $\mathbf{v}_{i-1}$ if $i \in I$ and $\mathbf{v}_i$ if $i \notin I$. Then we have

$$\det \begin{pmatrix} \beta_1 \mathbf{v}_0 + \gamma_1 \mathbf{v}_1 \\ \beta_2 \mathbf{v}_1 + \gamma_2 \mathbf{v}_2 \\ \vdots \\ \beta_n \mathbf{v}_{n-1} + \gamma_n \mathbf{v}_n \end{pmatrix} = \sum_{I \subseteq [n]} \prod_{i \in I} \beta_i \prod_{j \notin I} \gamma_j \cdot \det V(I).$$

If there is an index p such that $p \notin I$ and $p+1 \in I$, then the p-th and $(p+1)$-st rows of $V(I)$ are identical, hence $\det V(I) = 0$. It follows that $\det V(I) = 0$ unless $I = \emptyset$ or $I = \{1, 2, \dots, k\}$ for some k with $1 \leqslant k \leqslant n$, which implies the formula in the lemma. ☐

3.3 Sub-Pfaffians and minors

Another key ingredient in the proof of Theorem 1.3 is the following sub-Pfaffian expressions of the weights $u^{c(\lambda)} \pm u^{m-c(\lambda)} = u^{r(\lambda')} \pm u^{m-r(\lambda')}$, where λ' is the conjugate of λ, $c(\lambda)$ is the number of odd columns of λ, and $r(\lambda')$ is the number of odd rows of λ', as before. For a partition μ of length $\leqslant m$, we put

$$I_m(\mu) = (\mu_m + 1, \mu_{m-1} + 2, \dots, \mu_1 + m).$$

Proposition 3.4. Let $A = (a_{i,j})$ be the skew-symmetric matrix with rows/columns indexed by the totally ordered set $\{0 < 0' < 1 < 2 < \cdots\}$, whose entries are given by

$$a_{0,0'} = 0, \quad a_{0,j} = 1 + u, \quad a_{0',j} = (-1)^{j-1}(1-u), \quad a_{i,j} = \begin{cases} 1 + u^2, & \text{if } j - i \text{ is odd,} \\ 2u, & \text{if } j - i \text{ is even,} \end{cases}$$

where i and j are positive integers and $i < j$.

(1) For a partition μ of length $\leqslant 2h$, we have

$$\operatorname{Pf} A^{I_{2h}(\mu)} = 2^{h-1} \left(u^{r(\mu)} + u^{2h - r(\mu)} \right),$$
$$\operatorname{Pf} A^{(0,0') \sqcup I_{2h}(\mu)} = 2^{h} \left(u^{r(\mu)} - u^{2h - r(\mu)} \right).$$

(2) For a partition μ of length $\leqslant 2h+1$, we have

$$\begin{aligned} \operatorname{Pf} A^{(0)\sqcup I_{2h+1}(\mu)} &= 2^h \left(u^{r(\mu)} + u^{2h+1-r(\mu)} \right), \\ \operatorname{Pf} A^{(0')\sqcup I_{2h+1}(\mu)} &= 2^h \left(u^{r(\mu)} - u^{2h+1-r(\mu)} \right). \end{aligned}$$

For the proof of this proposition, we need the following auxiliary result.

Lemma 3.5. Let n be an even integer, and let $N_n = (N_{i,j})_{1 \leqslant i,j \leqslant n}$ be the $n \times n$ skew-symmetric matrix with (i,j)-entry given by

$$N_{i,j} = \begin{cases} 1, & \text{if } j-i \text{ is odd,} \\ 0, & \text{if } j-i \text{ is even,} \end{cases}$$

for $i<j$. Then we have

$$\operatorname{Pf} N_n = 2^{n/2-1}.$$

Proof. If $n=2$, we have $\operatorname{Pf} N_n = 1$. Suppose $n \geqslant 4$ and proceed by induction on n. By subtracting the third row/column from the first row/column, and then by expanding the resulting Pfaffian along the first row/column, we see that $\operatorname{Pf} N_n = 2 \operatorname{Pf} N_{n-2}$. Hence the proof is completed by using the induction hypothesis. □

Proposition 3.4 is obtained from special cases of the following lemma.

Lemma 3.6. Let n be an even integer. Let $\mathbf{u} = (u_1, \dots, u_n)$ be a sequence of indeterminates and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{1,-1\}^n$. We define $M^{\varepsilon}(\mathbf{u}) = \big(m_{ij}\big)_{1 \leqslant i,j \leqslant n}$ to be the skew-symmetric matrix with (i,j)-entry, $i<j$, given by

$$m_{ij} = \begin{cases} 1 + u_i u_j, & \text{if } \varepsilon_i \varepsilon_j = (-1)^{i+j+1}, \\ u_i + u_j, & \text{if } \varepsilon_i \varepsilon_j = (-1)^{i+j}. \end{cases}$$

Then we have

$$\operatorname{Pf} M^{\varepsilon}(\mathbf{u}) = 2^{n/2-1} \left(\prod_{i:\varepsilon_i=+1} u_i + \prod_{i:\varepsilon_i=-1} u_i \right).$$

Proof. For a subset I of $[n]$, we compute the coefficient of $u^I = \prod_{i \in I} u_i$ in $\operatorname{Pf} M^{\varepsilon}(\mathbf{u})$. This coefficient is equal to the Pfaffian of the matrix $M[I]$ whose entries are given as follows:

(a) If $\varepsilon_i \varepsilon_j = (-1)^{i+j+1}$, then

$$M[I]_{i,j} = \begin{cases} 1, & \text{if } (i,j) \in (I \times I) \cup (I^c \times I^c), \\ 0, & \text{if } (i,j) \in (I^c \times I) \cup (I \times I^c). \end{cases}$$

(b) If $\varepsilon_i \varepsilon_j = (-1)^{i+j}$, then

$$M[I]_{i,j} = \begin{cases} 0, & \text{if } (i,j) \in (I \times I) \cup (I^c \times I^c), \\ 1, & \text{if } (i,j) \in (I^c \times I) \cup (I \times I^c). \end{cases}$$

Here I^c denotes the complement of I in $[n]$. Then it suffices to show that

$$\operatorname{Pf} M[I] = \begin{cases} 2^{n/2-1}, & \text{if } I = E^+ \text{ or } E^-, \\ 0, & \text{otherwise,} \end{cases}$$

where we put

$$E^+ = \{i : \varepsilon_i = 1\}, \quad E^- = \{i : \varepsilon_i = -1\}.$$

First we consider the case where $I = E^+$ or $I = E^-$. In this case, we see that $M[I] = N_n$, thus we have $\operatorname{Pf} M[I] = 2^{n/2-1}$ by Lemma 3.5.

Next we prove that $\operatorname{Pf} M[I] = 0$ unless $I = E^+$ or E^-. Given a subset $K \subseteq [n]$, we put

$$C(K) = \{i \in [n-1] : (i, i+1) \in (K \times K) \cup (K^c \times K^c)\}.$$

Then it is clear that $C(K) = C(L)$ if $K = L^c$. Conversely, if $C(K) = C(L)$, then we can prove, by induction on k, that $[k] \cap K = [k] \cap L$, for $k = 1, 2, \dots, n$, or $[k] \cap K = [k] \cap L^c$, for $k = 1, 2, \dots, n$. Hence we have $K = L$ or $K = L^c$.

Suppose $I \neq E^+$ or E^-, i.e., $C(I) \neq C(E^+) = C(E^-)$. Then there is an index $i \in [n-1]$ satisfying one of the following conditions:

(i) $(i, i+1) \in (I \times I) \cup (I^c \times I^c)$ and $\varepsilon_i = -\varepsilon_{i+1}$;

(ii) $(i, i+1) \in (I \times I^c) \cup (I^c \times I)$ and $\varepsilon_i = \varepsilon_{i+1}$.

In each of these cases, we can check that the i-th row/column of $M[I]$ is identical with the $(i+1)$-st row/column. Hence we obtain $\operatorname{Pf} M[I] = 0$. □

Proof of Proposition 3.4. Given a partition μ of length $\leqslant m$, we define $(\varepsilon_1, \dots, \varepsilon_m)$ by

$$\varepsilon_i = (-1)^{\mu_{m+1-i}}, \qquad \text{for } 1 \leqslant i \leqslant m.$$

Then we have $r(\mu) = |\{i : \varepsilon_i = -1\}|$ and $m - r(\mu) = |\{i : \varepsilon_i = 1\}|$. It is straightforward to check that

$$\begin{aligned} A^{I_{2h}(\mu)} &= M^{(\varepsilon_1, \dots, \varepsilon_{2h})}(u, \dots, u), \\ A^{(0,0') \sqcup I_{2h}(\mu)} &= M^{(1,1,\varepsilon_1, \dots, \varepsilon_{2h})}(1, -1, u, \dots, u), \\ A^{(0) \sqcup I_{2h+1}(\mu)} &= M^{(1,\varepsilon_1, \dots, \varepsilon_{2h+1})}(1, u, \dots, u), \\ A^{(0') \sqcup I_{2h+1}(\mu)} &= M^{(1,\varepsilon_1, \dots, \varepsilon_{2h+1})}(-1, u, \dots, u). \end{aligned}$$

Hence the proof is completed by applying Lemma 3.6. □

The next lemma collects several special instances of the Jacobi–Trudi identity (2.2).

Lemma 3.7. Let T be the following matrix with rows indexed by $0, 0', 1, \dots, m$ and columns indexed by $0, 0', 1, 2, \dots$:

$$T = \begin{pmatrix} 1 & 0 & O \\ 0 & 1 & O \\ O & O & (e_{r-i})_{1 \leqslant i \leqslant m, r \geqslant 1} \end{pmatrix},$$

where each O is a block matrix consisting of zeros. Then the following properties hold.

(1) For an increasing subsequence K of $(1, 2, \dots)$ of length m, we have

$$\det T_K^{[m]} = \begin{cases} s_{\mu'}(\mathbf{x}) & \text{if } K = I_m(\mu) \text{ for some partition } \mu \text{ of length} \leqslant m, \\ 0, & \text{otherwise.} \end{cases}$$

(2) For an increasing subsequence K of $(0, 0', 1, 2, \dots)$ of length $m + 2$, we have

$$\det T_K^{(0,0')\sqcup[m]} = \begin{cases} s_{\mu'}(\mathbf{x}), & \text{if } K = (0, 0') \sqcup I_m(\mu) \text{ for some partition } \mu \text{ of length} \leqslant m, \\ 0, & \text{otherwise.} \end{cases}$$

(3) For an increasing subsequence K of $(0, 1, 2, \dots)$ of length $m + 1$, we have

$$\det T_K^{(0)\sqcup[m]} = \begin{cases} s_{\mu'}(\mathbf{x}), & \text{if } K = (0) \sqcup I_m(\mu) \text{ for some partition } \mu \text{ of length} \leqslant m, \\ 0, & \text{otherwise.} \end{cases}$$

(4) For an increasing subsequence K of $(0', 1, 2, \dots)$ of length $m + 1$, we have

$$\det T_K^{(0')\sqcup[m]} = \begin{cases} s_{\mu'}(\mathbf{x}), & \text{if } K = (0') \sqcup I_m(\mu) \text{ for some partition } \mu \text{ of length} \leqslant m, \\ 0, & \text{otherwise.} \end{cases}$$

3.4 Proof of Theorem 1.3

Now we are in the position to give a proof of Theorem 1.3. Throughout the proof, let $(\alpha_r)_{r\in\mathbb{Z}}$ and $(c_r)_{r\in\mathbb{Z}}$ be the sequences given by

$$\alpha_r = \begin{cases} 1 + u^2, & \text{if } r \text{ is odd,} \\ 2u, & \text{if } r \text{ is even,} \end{cases} \qquad r > 0, \qquad \alpha_0 = 0, \quad \alpha_{-r} = -\alpha_r,$$

and

$$c_r = \sum_{k\in\mathbb{Z}} \alpha_k f_{r-k},$$

where we recall from (1.3) that $f_r = f_r(\mathbf{x}) = \sum_{i\in\mathbb{Z}} e_i(\mathbf{x}) e_{i+r}(\mathbf{x})$, which implies $f_{-r} = f_r$. Note that the (r, s)-entry of the skew-symmetric matrix A given in Proposition 3.4 is equal to α_{s-r} for positive integers r and s, and $c_{-r} = -c_r$. Moreover, we have

$$\sum_{r,s\geqslant 1} \alpha_{s-r} e_{r-i}(\mathbf{x}) e_{s-j}(\mathbf{x}) = c_{j-i}. \tag{3.5}$$

The proof of the identities in Theorem 1.3 works as follows:

(a) We use Proposition 3.4 and Lemma 3.7 and apply the minor summation formula (Theorem 3.1) to express the left-hand side as a Pfaffian.

(b) We use one of the Gordon-type identities in Lemma 3.2 to transform the Pfaffian obtained in (a) into a determinant.

(c) We perform row operations with the determinant obtained in (b) and use Lemma 3.3 to obtain the right-hand side.

Proof of Theorem 1.3. First we give a proof of (1.7). By applying the minor summation formula to the matrices A and T in Proposition 3.4 and Lemma 3.7, where $m = 2w$, with the 0-th and $0'$-th rows/columns removed, we have

$$\begin{aligned}\sum_{\lambda:\lambda_1 \leqslant 2w} \left(u^{c(\lambda)} + u^{2w-c(\lambda)}\right) s_\lambda(\mathbf{x}) &= \sum_{\mu:\ell(\mu)\leqslant 2w} \left(u^{r(\mu)} + u^{2w-r(\mu)}\right) s_{\mu'}(\mathbf{x}) \\ &= 2^{1-w} \sum_{\mu:\ell(\mu)\leqslant 2w} \operatorname{Pf} A^{I_{2w}(\mu)} \det T^{[2w]}_{I_{2w}(\mu)} \\ &= 2^{1-w} \sum_{K} \operatorname{Pf} A^{K} \det T^{[2w]}_{K} \\ &= 2^{1-w} \operatorname*{Pf}_{1\leqslant i,j\leqslant 2w} \left(\sum_{r,s\geqslant 1} A_{r,s} T_{i,r} T_{j,s} \right) \\ &= 2^{1-w} \operatorname*{Pf}_{1\leqslant i,j\leqslant 2w} \left(c_{j-i}\right),\end{aligned}$$

where K runs over all increasing subsequences of $2w$ positive integers. Then we use Identity (3.2) to obtain

$$\operatorname*{Pf}_{1\leqslant i,j\leqslant 2w} \left(c_{j-i}\right) = \frac{1}{2} \det_{1\leqslant i,j\leqslant w} \left(\begin{cases} c_i - c_{i-2}, & \text{if } j = 1 \\ c_{i-j+1} - c_{i-j-1} + c_{i+j-1} - c_{i+j-3}, & \text{if } 2 \leqslant j \leqslant w \end{cases} \right).$$

Since $c_r = \sum_{k\geqslant 1} \alpha_k \left(f_{r-k} - f_{r+k}\right)$ with $\alpha_{2p-1} = 1 + u^2$ and $\alpha_{2p} = 2u$ for $p \geqslant 1$, we have

$$c_r - c_{r-2} = 2(1+u^2) f_{r-1} + 2u(f_{r-2} + f_r).$$

Hence, by taking the factor 2 out of each column except for the first column, we obtain

$$\begin{aligned}&\operatorname*{Pf}_{1\leqslant i,j\leqslant 2w} \left(c_{j-i}\right) \\ &= \frac{1}{2} \cdot 2^{w-1} \det_{1\leqslant i,j\leqslant w} \left(\begin{cases} (1+u^2)(f_{1-j} + f_{j-1}) + 2u(f_{2-j} + f_j), & \text{if } i = 1 \\ (1+u^2)(f_{i-j} + f_{i+j-2}) + u(f_{i-j-1} + f_{i+j-3} + f_{i-j+1} + f_{i+j-1}), & \text{if } i \geqslant 2 \end{cases} \right).\end{aligned}$$

In the last determinant, we subtract the first row multiplied by $u/(1+u^2)$ from the second row, and then subtract the $(i-1)$-st row multiplied by $u(1+u^{2i-4})/(1+u^{2i-2})$ from the i-th row for $i = 3, 4, \ldots, w$. Then we see that the last determinant equals

$$\det_{1\leqslant i,j\leqslant w} \left(\begin{cases} (1+u^2)(f_{1-j} + f_{j-1}) + 2u(f_{2-j} + f_j), & \text{if } i = 1 \\ \dfrac{1+u^{2i}}{1+u^{2i-2}}(f_{i-j} + f_{i+j-2}) + u(f_{i-j+1} + f_{i+j-1}), & \text{if } i \geqslant 2 \end{cases} \right).$$

Now the proof of (1.7) is completed by applying Lemma 3.3 with $\beta_1 = 1 + u^2$, $\beta_i = (1 + u^{2i})/(1 + u^{2i-2})$ for $2 \leqslant i \leqslant w$, $\gamma_1 = 2u$, $\gamma_i = u$ for $2 \leqslant i \leqslant w$, and the row vectors $\mathbf{v}_i = (f_{i-j+1} + f_{i+j-1})_{1 \leqslant j \leqslant w}$ for $0 \leqslant i \leqslant w$.

Next we prove (1.8). By applying the minor-summation formula to the matrices A and T in Proposition 3.4 and Lemma 3.7 with $m = 2w$, we obtain

$$\sum_{\lambda : \lambda_1 \leqslant 2w} \left(u^{c(\lambda)} - u^{2w - c(\lambda)} \right) s_\lambda(\mathbf{x}) = 2^{-w} \operatorname{Pf} \left(TAT^t \right).$$

The entries of the skew-symmetric matrix $Q = TAT^t$ are given by

$$Q_{0,0'} = 0, \quad Q_{0,j} = (1+u) e(\mathbf{x}), \quad Q_{0',j} = (-1)^{j-1} (1-u) \overline{e}(\mathbf{x}), \quad Q_{i,j} = c_{j-i},$$

where $1 \leqslant i, j \leqslant 2w$, and the last equality follows from (3.5) and the fact that $T_{k,0} = T_{k,0'} = 0$ for $k \geqslant 1$.

By subtracting the $(i-1)$-st row/column from the i-th row/column for $i = 2w, 2w-1, \dots, 2$, and by subsequently expanding the resulting Pfaffian along the 0-th row/column, we obtain

$$\operatorname{Pf} Q = -(1+u) e(\mathbf{x}) \cdot \operatorname{Pf} Q',$$

where the entries of Q' are given by

$$Q'_{0',j} = 2(-1)^j \overline{e}(\mathbf{x}), \quad Q'_{i,j} = 2c_{j-i} - c_{j-i+1} - c_{j-i-1}.$$

By adding the $(i-1)$-st row/column to the i-th row/column for $i = 2w, 2w-1, \dots, 3$, and by subsequently expanding the resulting Pfaffian along the $0'$-th row/column, we have

$$\operatorname{Pf} Q = (1+u) e(\mathbf{x}) \cdot 2(1-u) \overline{e}(\mathbf{x}) \cdot \operatorname*{Pf}_{3 \leqslant i,j \leqslant 2w} \left(2c_{j-i} - c_{j-i+2} - c_{j-i-2} \right).$$

By using the Gordon-type formula (3.1), we obtain

$$\begin{aligned} \operatorname*{Pf}_{3 \leqslant i,j \leqslant 2w} \left(2c_{j-i} - c_{j-i+2} - c_{j-i-2} \right) &= \det_{1 \leqslant i,j \leqslant w-1} \left(c_{i-j+1} - c_{i-j-1} - c_{i+j+1} + c_{i+j-1} \right) \\ &= 2^{w-1} \det_{1 \leqslant i,j \leqslant w-1} \Big((1+u^2)(f_{i-j} - f_{i+j}) \\ &\qquad + u(f_{i-j-1} + f_{i-j+1} - f_{i+j-1} - f_{i+j+1}) \Big). \end{aligned}$$

By subtracting the $(i-1)$-st row multiplied by $u(1-u^{2i-2})/(1-u^{2i})$ from the i-th row for $i = 2, 3, \dots, w-1$, we see that the last determinant equals

$$\det_{1 \leqslant i,j \leqslant w-1} \left(\frac{1 - u^{2i+2}}{1 - u^{2i}} (f_{i-j} - f_{i+j}) + u(f_{i-j+1} - f_{i+j+1}) \right).$$

Now we apply Lemma 3.3 with $\beta_i = (1 - u^{2i+2})/(1 - u^{2i})$, $\gamma_i = u$, for $1 \leqslant i \leqslant w-1$, and $\mathbf{v}_i = (f_{i-j+1} - f_{i+j+1})_{1 \leqslant j \leqslant w-1}$, for $0 \leqslant i \leqslant w-1$, to obtain (1.8).

Now we turn to (1.9). We apply the minor summation formula to the matrices A and T in Proposition 3.4 and Lemma 3.7, where $m = 2w+1$, with the $0'$-th row/column removed to obtain

$$\sum_{\lambda:\lambda_1 \leqslant 2w+1} \left(u^{c(\lambda)} + u^{2w+1-c(\lambda)}\right) s_\lambda(\mathbf{x}) = 2^{-w} \operatorname{Pf} \begin{pmatrix} 0 & ((1+u)e(\mathbf{x}))_{1 \leqslant j \leqslant 2w+1} \\ * & (c_{j-i})_{1 \leqslant i,j \leqslant 2w+1} \end{pmatrix}.$$

By subtracting the $(i-1)$-st row/column from the i-th row/column for $i = 2w+1, 2w, \ldots, 2$, and by subsequently expanding the resulting Pfaffian along the 0-th row/column, we see that

$$\operatorname{Pf} \begin{pmatrix} 0 & ((1+u)e(\mathbf{x}))_{2 \leqslant j \leqslant 2w+1} \\ * & (c_{j-i})_{2 \leqslant i,j \leqslant 2w+1} \end{pmatrix} = (1+u)e(\mathbf{x}) \cdot \operatorname*{Pf}_{2 \leqslant i,j \leqslant 2w+1} \left(2c_{j-i} - c_{j-i+1} - c_{j-i-1}\right).$$

By using (3.3), we have

$$\begin{aligned} &\operatorname*{Pf}_{2 \leqslant i,j \leqslant 2w+1} \left(2c_{j-i} - c_{j-i+1} - c_{j-i-1}\right) \\ &= 2^w \det_{1 \leqslant i,j \leqslant w} \left((1-u+u^2)(f_{i-j} - f_{i+j-1}) + u(f_{i-j-1} + f_{i-j} + f_{i-j+1} - f_{i+j-2} - f_{i+j-1} - f_{i+j})\right). \end{aligned}$$

Finally we subtract the $(i-1)$-st row multiplied by $u(1+u^{2i-3})/(1+u^{2i-1})$ from the i-th row for $i = w, w-1, \ldots, 2$ to obtain

$$\begin{aligned} &\det_{1 \leqslant i,j \leqslant w} \left((1-u+u^2)(f_{i-j} - f_{i+j-1}) + u(f_{i-j-1} + f_{i-j} + f_{i-j+1} - f_{i+j-2} - f_{i+j-1} - f_{i+j})\right) \\ &\qquad = \det_{1 \leqslant i,j \leqslant w} \left(\frac{1+u^{2i+1}}{1+u^{2i-1}}(f_{i-j} - f_{i+j-1}) + u(f_{i-j+1} - f_{i+j})\right). \end{aligned}$$

The proof of (1.9) is completed by applying Lemma 3.3.

The proof of (1.10) is the same as (1.9) up to a sign, so we omit it. □

4 Equivalence of Theorem 1.3 and identities for classical group characters of nearly rectangular shape

In this section, we explain that Theorem 1.3 is equivalent to the nearly rectangular character identities for the even orthogonal Lie algebra $\mathfrak{so}_{2n}$ (Theorem 4.1 below) obtained by the third author in [15].

The finite dimensional irreducible representations of $\mathfrak{so}_{2n}$ are parameterized by their highest weights λ, where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a sequence of integers or of half-integers such that

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_{n-1} \geqslant |\lambda_n|. \tag{4.1}$$

Let $\mathbf{x}_n = (x_1, \ldots, x_n)$. We denote by $\mathrm{so}_\lambda(\mathbf{x}_n)$ the character of the irreducible representation with highest weight λ.

The third author used tableau descriptions of the special orthogonal characters that are derived from Lakshmibai–Seshadri paths to prove the following character identities.

Theorem 4.1 ([15, Theorem 2, Eq. (3.7)]). If b is a nonnegative integer or half-integer and $0 \leqslant k \leqslant 2b$, then we have

$$\mathrm{so}_{(b^{n-1},b-k)}(\mathbf{x}_n) = (x_1 \cdots x_n)^{-b} \sum_{\substack{\lambda \subseteq ((2b)^n) \\ c(((2b)^n)/\lambda)=k}} s_\lambda(\mathbf{x}_n). \tag{4.2}$$

In words: λ runs over all partitions contained in the $n \times (2b)$ rectangle $((2b)^n)$ such that the skew diagram $((2b)^n)/\lambda$ has exactly k columns of odd length.

For a sequence λ satisfying (4.1), we put

$$\lambda^{\#} = (\lambda_1, \dots, \lambda_{n-1}, -\lambda_n),$$

and define

$$\mathrm{o}_\lambda(\mathbf{x}_n) = \begin{cases} \mathrm{so}_\lambda(\mathbf{x}_n) + \mathrm{so}_{\lambda^{\#}}(\mathbf{x}_n), & \text{if } \lambda_n \neq 0, \\ \mathrm{so}_\lambda(\mathbf{x}_n), & \text{if } \lambda_n = 0, \end{cases}$$

$$\overline{\mathrm{o}}_\lambda(\mathbf{x}_n) = \begin{cases} \mathrm{so}_\lambda(\mathbf{x}_n) - \mathrm{so}_{\lambda^{\#}}(\mathbf{x}_n), & \text{if } \lambda_n \neq 0, \\ 0, & \text{if } \lambda_n = 0. \end{cases}$$

Then these Laurent polynomials can be described in terms of elementary symmetric polynomials $e_r(\mathbf{x}_n^{\pm 1}) = e_r(x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1})$. Given a partition λ of length $\leqslant n$, we write $\lambda + 1/2 = (\lambda_1 + 1/2, \dots, \lambda_n + 1/2)$.

Proposition 4.2 (Cf. [12, 22]).

(1) For a partition λ of length $\leqslant n$, we have

$$\mathrm{o}_\lambda(\mathbf{x}_n) = \frac{1}{2} \det_{1 \leqslant i,j \leqslant \lambda_1} \left(e_{\lambda'_i - i + j}(\mathbf{x}_n^{\pm 1}) + e_{\lambda'_i - i - j + 2}(\mathbf{x}_n^{\pm 1}) \right),$$

and, if $\ell(\lambda) = n$, we have

$$\overline{\mathrm{o}}_\lambda(\mathbf{x}_n) = \prod_{i=1}^{n} \left(x_i - x_i^{-1} \right) \cdot \det_{2 \leqslant i,j \leqslant \lambda_1} \left(e_{\lambda'_i - i + j}(\mathbf{x}_n^{\pm 1}) - e_{\lambda'_i - i - j}(\mathbf{x}_n^{\pm 1}) \right).$$

(2) For a partition λ of length $\leqslant n$, we have

$$\mathrm{o}_{\lambda+1/2}(\mathbf{x}_n) = \prod_{i=1}^{n} \left(x_i^{1/2} + x_i^{-1/2} \right) \cdot \det_{1 \leqslant i,j \leqslant \lambda_1} \left(e_{\lambda'_i - i + j}(\mathbf{x}_n^{\pm 1}) - e_{\lambda'_i - i - j + 1}(\mathbf{x}_n^{\pm 1}) \right),$$

$$\overline{\mathrm{o}}_{\lambda+1/2}(\mathbf{x}_n) = \prod_{i=1}^{n} \left(x_i^{1/2} - x_i^{-1/2} \right) \cdot \det_{1 \leqslant i,j \leqslant \lambda_1} \left(e_{\lambda'_i - i + j}(\mathbf{x}_n^{\pm 1}) + e_{\lambda'_i - i - j + 1}(\mathbf{x}_n^{\pm 1}) \right).$$

Now we prove the equivalence between Theorems 1.3 and 4.1. Note that a symmetric function identity $f(\mathbf{x}) = g(\mathbf{x})$ holds for infinitely many variables $\mathbf{x}$ if and only if $f(\mathbf{x}_n) = g(\mathbf{x}_n)$ for all positive integers n.

First we show that Equations (1.7) and (1.8) are equivalent to Theorem 4.1 in the case where b is a positive integer. Under the specialization $x_{n+1} = x_{n+2} = \cdots = 0$, we have

$$f_r(\mathbf{x}_n) = \sum_i e_i(\mathbf{x}_n) \cdot e_{r+i}(\mathbf{x}_n) = \sum_i e_i(\mathbf{x}_n) \cdot (x_1 \cdots x_n) e_{n-r-i}(\mathbf{x}_n^{-1}) = (x_1 \cdots x_n) e_{n-r}(\mathbf{x}_n^{\pm 1}),$$

where $\mathbf{x}_n^{-1} = (x_1^{-1}, \dots, x_n^{-1})$, and

$$e(\mathbf{x}_n) = (x_1 \cdots x_n)^{1/2} \prod_{i=1}^n \left(x_i^{1/2} + x_i^{-1/2} \right), \overline{e}(\mathbf{x}_n) = (-1)^n (x_1 \cdots x_n)^{1/2} \prod_{i=1}^n \left(x_i^{1/2} - x_i^{-1/2} \right).$$

Comparing the determinants on the right-hand sides of (1.7) and (1.8) with the Jacobi–Trudi-type identities in Proposition 4.2(1) with $\lambda = (w^{n-1}, w-k)$, we obtain

$$\begin{aligned} &\frac{1}{2} \det_{1 \leqslant i,j \leqslant w} \left(\begin{cases} f_{i-j}(\mathbf{x}_n) + f_{i+j-2}(\mathbf{x}_n), & \text{if } 1 \leqslant i \leqslant w-k \\ f_{i-j+1}(\mathbf{x}_n) + f_{i+j-1}(\mathbf{x}_n), & \text{if } w-k+1 \leqslant i \leqslant w \end{cases} \right) \\ &= (x_1 \cdots x_n)^w \cdot \mathrm{o}_{(w^{n-1}, w-k)}(\mathbf{x}_n) \end{aligned}$$

and

$$\begin{aligned} &e(\mathbf{x}_n)\overline{e}(\mathbf{x}_n) \cdot \det_{1 \leqslant i,j \leqslant w-1} \left(\begin{cases} f_{i-j}(\mathbf{x}_n) - f_{i+j}(\mathbf{x}_n), & \text{if } 1 \leqslant i \leqslant w-k-1, \\ f_{i-j+1}(\mathbf{x}_n) - f_{i+j+1}(\mathbf{x}_n), & \text{if } w-k \leqslant i \leqslant w-1 \end{cases} \right) \\ &= (-1)^n (x_1 \cdots x_n)^w \cdot \overline{\mathrm{o}}_{(w^{n-1}, w-k)}(\mathbf{x}_n). \end{aligned}$$

By the definition of $\mathrm{o}_\lambda(\mathbf{x}_n)$ and $\overline{\mathrm{o}}_\lambda(\mathbf{x}_n)$, we have

$$\begin{aligned} \sum_{k=0}^{w} \left(u^k + u^{2w-k} \right) \mathrm{o}_{(w^{n-1}, w-k)}(\mathbf{x}_n) &= \sum_{k=0}^{2w} \left(u^k + u^{2w-k} \right) \mathrm{so}_{(w^{n-1}, w-k)}(\mathbf{x}_n), \\ \sum_{k=0}^{w-1} \left(u^k - u^{2w-k} \right) \overline{\mathrm{o}}_{(w^{n-1}, w-k)}(\mathbf{x}_n) &= \sum_{k=0}^{2w} \left(u^k - u^{2w-k} \right) \mathrm{so}_{(w^{n-1}, w-k)}(\mathbf{x}_n). \end{aligned}$$

Hence (1.7) and (1.8) become

$$\begin{aligned} (x_1 \cdots x_n)^{-w} \sum_{\lambda_1 \leqslant 2w} (u^{c(\lambda)} + u^{2w-c(\lambda)}) s_\lambda(\mathbf{x}_n) &= \sum_{k=0}^{2w} \left(u^k + u^{2w-k} \right) \mathrm{so}_{(w^{n-1}, w-k)}(\mathbf{x}_n), \\ (-1)^n (x_1 \cdots x_n)^{-w} \sum_{\lambda_1 \leqslant 2w} (u^{c(\lambda)} - u^{2w-c(\lambda)}) s_\lambda(\mathbf{x}_n) &= \sum_{k=0}^{2w} \left(u^k - u^{2w-k} \right) \mathrm{so}_{(w^{n-1}, w-k)}(\mathbf{x}_n). \end{aligned}$$

This pair of identities is equivalent to

$$\sum_{k=0}^{2w} u^k \mathrm{so}_{(w^{n-1},w-k)}(\mathbf{x}_n) = (x_1 \cdots x_n)^{-w} \begin{cases} \sum_{\lambda_1 \leqslant 2w} u^{c(\lambda)} s_\lambda(\mathbf{x}_n), & \text{if } n \text{ is even,} \\ \sum_{\lambda_1 \leqslant 2w} u^{2w-c(\lambda)} s_\lambda(\mathbf{x}_n), & \text{if } n \text{ is odd.} \end{cases}$$

If $\lambda \subseteq ((2w)^n)$, then

$$c((2w)^n/\lambda) = \begin{cases} c(\lambda), & \text{if } n \text{ is even,} \\ 2w - c(\lambda), & \text{if } n \text{ is odd.} \end{cases}$$

Therefore we obtain

$$\sum_{k=0}^{2w} u^k \mathrm{so}_{(w^{n-1},w-k)}(\mathbf{x}_n) = (x_1 \cdots x_n)^{-w} \sum_{\lambda \subseteq ((2w)^n)} u^{c((2w)^n/\lambda)} s_\lambda(\mathbf{x}_n),$$

which is the generating function expression of the third author's formula (4.2) in the case where $b = w$ is a positive integer.

By using the Jacobi–Trudi-type identities in Proposition 4.2(2), we can show that the pair of identities (1.9) and (1.10) is equivalent to the third author's formula (4.2) in the case where b is a positive half-integer.

Remark 4.3. Along a similar argument, we can see that Goulden's original identity (1.5) is equivalent to the nearly rectangular character identity for the symplectic Lie algebra [15, Theorem 2, Eq. (3.6)].

5 Bounded Littlewood identities and skewing operators: proofs of Theorems 1.4, 1.5, and 1.6

In this section, we provide the proofs of the formulations of the (refined) bounded Littlewood identities in terms of skewing operators, given in Theorems 1.4, 1.5, and 1.6.

We will use the following restatement of Theorem 1.3.

Theorem 5.1. For nonnegative integers $0 \leqslant k \leqslant w$, we have

$$\sum_{\substack{\lambda:\lambda_1 \leqslant 2w+1 \\ c(\lambda)=k}} s_\lambda(\mathbf{x}) + \sum_{\substack{\lambda:\lambda_1 \leqslant 2w+1 \\ c(\lambda)=2w+1-k}} s_\lambda(\mathbf{x}) = e(\mathbf{x}) \det_{1 \leqslant i,j \leqslant w} \left(f_{i+\chi(i>w-k)-j}(\mathbf{x}) - f_{i+\chi(i>w-k)+j-1}(\mathbf{x}) \right), \tag{5.1}$$

$$\sum_{\substack{\lambda:\lambda_1 \leqslant 2w+1 \\ c(\lambda)=k}} s_\lambda(\mathbf{x}) - \sum_{\substack{\lambda:\lambda_1 \leqslant 2w+1 \\ c(\lambda)=2w+1-k}} s_\lambda(\mathbf{x}) = \overline{e}(\mathbf{x}) \det_{1 \leqslant i,j \leqslant w} \left(f_{i+\chi(i>w-k)-j}(\mathbf{x}) + f_{i+\chi(i>w-k)+j-1}(\mathbf{x}) \right), \tag{5.2}$$

$$\sum_{\substack{\lambda:\lambda_1\leqslant 2w\\ c(\lambda)=k}} s_\lambda(\mathbf{x}) + \sum_{\substack{\lambda:\lambda_1\leqslant 2w\\ c(\lambda)=2w-k}} s_\lambda(\mathbf{x}) = \left(\frac{1}{2}\right)^{\chi(k<w)} \det_{1\leqslant i,j\leqslant w}\left(f_{i+\chi(i>w-k)-j}(\mathbf{x}) + f_{i+\chi(i>w-k)+j-2}(\mathbf{x})\right), \tag{5.3}$$

and, for $0 \leqslant k \leqslant w-1$,

$$\sum_{\substack{\lambda:\lambda_1\leqslant 2w\\ c(\lambda)=k}} s_\lambda(\mathbf{x}) - \sum_{\substack{\lambda:\lambda_1\leqslant 2w\\ c(\lambda)=2w-k}} s_\lambda(\mathbf{x}) = e(\mathbf{x})\overline{e}(\mathbf{x}) \det_{2\leqslant i,j\leqslant w}\left(f_{i+\chi(i>w-k)-j}(\mathbf{x}) - f_{i+\chi(i>w-k)+j-2}(\mathbf{x})\right). \tag{5.4}$$

We begin with an auxiliary result that describes the action of the skewing operator $p_1^\perp$ on the series $f_i(\mathbf{x})$ given in (1.3). We remind the reader that the definition of the skewing operators is given at the end of Section 2.

Lemma 5.2. For integers i and $j \geqslant 0$, we have

$$(p_1^\perp)^j f_i(\mathbf{x}) = \sum_{r=0}^{j} \binom{j}{r} f_{i-j+2r}(\mathbf{x}). \tag{5.5}$$

In particular, $p_1^\perp f_i(\mathbf{x}) = f_{i-1}(\mathbf{x}) + f_{i+1}(\mathbf{x})$.

Proof. For the case $j=0$ the claim is obvious. Now we use induction on $j \geqslant 1$. For $j=1$, since $p_1^\perp$ is a derivation, we have

$$\begin{aligned} p_1^\perp f_i(\mathbf{x}) &= \sum_{n\in\mathbb{Z}} \left((p_1^\perp e_n(\mathbf{x}))e_{n+i}(\mathbf{x}) + e_n(\mathbf{x})(p_1^\perp e_{n+i}(\mathbf{x}))\right)\\ &= \sum_{n\in\mathbb{Z}} (e_{n-1}(\mathbf{x})e_{n+i}(\mathbf{x}) + e_n(\mathbf{x})e_{n+i-1}(\mathbf{x}))\\ &= f_{i+1}(\mathbf{x}) + f_{i-1}(\mathbf{x}). \end{aligned}$$

To get the second line, we have used that $p_k^\perp e_n(\mathbf{x}) = (-1)^{k-1}e_{n-k}(\mathbf{x})$, as is well known [19, p. 76].

Suppose that (5.5) holds for some $j \geqslant 1$. Then,

$$\begin{aligned} (p_1^\perp)^{j+1} f_i(\mathbf{x}) &= (p_1^\perp)^j (f_{i+1}(\mathbf{x}) + f_{i-1}(\mathbf{x}))\\ &= \sum_{r=0}^{j}\binom{j}{r} f_{(i+1)-j+2r}(\mathbf{x}) + \sum_{r=0}^{j}\binom{j}{r} f_{(i-1)-j+2r}(\mathbf{x})\\ &= \sum_{r=1}^{j+1}\binom{j}{r-1} f_{i-(j+1)+2r}(\mathbf{x}) + \sum_{r=0}^{j}\binom{j}{r} f_{i-(j+1)+2r}(\mathbf{x})\\ &= \sum_{r=0}^{j+1}\binom{j+1}{r} f_{i-(j+1)+2r}(\mathbf{x}) \end{aligned}$$

so that (5.5) holds for all $j \geqslant 0$. □

This lemma has the following consequence.

Lemma 5.3. For any integers i and $j \geqslant 1$,

$$(p_1^{\perp})^{j-1}(f_{i-1}(\mathbf{x}) \pm f_{i+1}(\mathbf{x})) = f_{i-j}(\mathbf{x}) \pm f_{i+j}(\mathbf{x}) + \sum_{r=1}^{j-1} \binom{j-1}{r} (f_{i-(j-2r)}(\mathbf{x}) \pm f_{i+(j-2r)}(\mathbf{x})), \tag{5.6}$$

$$(p_1^{\perp})^{j-1}(f_{i-1}(\mathbf{x}) \pm f_{i}(\mathbf{x})) = f_{i-j}(\mathbf{x}) \pm f_{i+j-1}(\mathbf{x}) + \sum_{r=1}^{j-1} \binom{j-1}{r} (f_{i-(j-2r)}(\mathbf{x}) \pm f_{i+(j-2r)-1}(\mathbf{x})). \tag{5.7}$$

Proof. By Lemma 5.2, the left-hand side of (5.6) is equal to

$$\begin{aligned}
&\sum_{r=0}^{j-1} \binom{j-1}{r} f_{(i-1)-(j-1)+2r}(\mathbf{x}) \pm \sum_{r=0}^{j-1} \binom{j-1}{r} f_{(i+1)-(j-1)+2r}(\mathbf{x}) \\
&= \sum_{r=0}^{j-1} \binom{j-1}{r} f_{i-j+2r}(\mathbf{x}) \pm \sum_{r=0}^{j-1} \binom{j-1}{j-1-r} f_{i+j-2r}(\mathbf{x}).
\end{aligned}$$

which is equal to the right-hand side of (5.6). The second identity (5.7) can be proved similarly. ☐

With Lemma 5.3, we are now in the position to prove Theorems 1.4–1.6.

Proof of Theorem 1.4. By (5.6) we have

$$(p_1^{\perp})^{j-1}(f_{i-1}(\mathbf{x}) - f_{i+1}(\mathbf{x})) = f_{i-j}(\mathbf{x}) - f_{i+j}(\mathbf{x}) + \sum_{r=1}^{j-1} \binom{j-1}{r} (f_{i-(j-2r)}(\mathbf{x}) - f_{i+(j-2r)}(\mathbf{x})).$$

Note that, if $1 \leqslant r \leqslant j-1$, then $f_{i-(j-2r)}(\mathbf{x}) - f_{i+(j-2r)}(\mathbf{x})$ is equal to either 0 or $\pm(f_{i-j'}(\mathbf{x}) - f_{i+j'}(\mathbf{x}))$ for some j' with $1 \leqslant j' \leqslant j-1$. Hence the matrix $((p_1^{\perp})^{j-1}(f_{i-1}(\mathbf{x}) - f_{i+1}(\mathbf{x})))_{1 \leqslant i,j \leqslant w}$ can be obtained from the matrix $(f_{i-j}(\mathbf{x}) - f_{i+j}(\mathbf{x}))_{1 \leqslant i,j \leqslant w}$ by adding a multiple of column j' to column j for $1 \leqslant j' \leqslant j-1$. By (1.1), this implies that

$$\sum_{\lambda : \lambda_1 \leqslant 2w+1} s_\lambda(\mathbf{x}) = e(\mathbf{x}) \det_{1 \leqslant i,j \leqslant w} (f_{i-j}(\mathbf{x}) - f_{i+j}(\mathbf{x})) = e(\mathbf{x}) \det_{1 \leqslant i,j \leqslant w} \left((p_1^{\perp})^{j-1}(f_{i-1}(\mathbf{x}) - f_{i+1}(\mathbf{x}))\right). \tag{5.8}$$

Again, by (5.6) we also obtain

$$(p_1^{\perp})^{i-1}(f_0(\mathbf{x}) - f_2(\mathbf{x})) = f_{1-i}(\mathbf{x}) - f_{1+i}(\mathbf{x}) + \sum_{r=1}^{i-1} \binom{i-1}{r} (f_{1-(i-2r)}(\mathbf{x}) - f_{1+(i-2r)}(\mathbf{x})).$$

Since the operator $(p_1^{\perp})^{j-1}$ is linear, by the same argument with row operations we have

$$\det_{1 \leqslant i,j \leqslant w} \left((p_1^{\perp})^{j-1}(f_{1-i}(\mathbf{x}) - f_{1+i}(\mathbf{x}))\right) = \det_{1 \leqslant i,j \leqslant w} \left((p_1^{\perp})^{j-1}(p_1^{\perp})^{i-1}(f_0(\mathbf{x}) - f_2(\mathbf{x}))\right). \tag{5.9}$$

Since $f_{1-i}(\mathbf{x}) = f_{i-1}(\mathbf{x})$, we obtain the first identity (1.11) from (5.8) and (5.9).

Now we prove the second identity, Equation (1.12). By (5.7) we have

$$(p_1^\perp)^{j-1}(f_{i-1}(\mathbf{x})+f_i(\mathbf{x})) = f_{i-j}(\mathbf{x})+f_{i+j-1}(\mathbf{x})+\sum_{r=1}^{j-1}\binom{j-1}{r}(f_{i-(j-2r)}(\mathbf{x})+f_{i+(j-2r)-1}(\mathbf{x})).$$

For r with $1 \leqslant r \leqslant j-1$, let $j' := j-2r$ if $1 \leqslant r < \lfloor j/2 \rfloor$, and $j' := 2r-j+1$ if $\lfloor j/2 \rfloor \leqslant r \leqslant j-1$. Then $f_{i-(j-2r)}(\mathbf{x}) + f_{i+(j-2r)-1}(\mathbf{x}) = f_{i-j'}(\mathbf{x}) + f_{i+j'-1}(\mathbf{x})$ and $1 \leqslant j' \leqslant j-1$. By (1.2), this implies that

$$\sum_{\lambda:\lambda_1\leqslant 2w} s_\lambda(\mathbf{x}) = \det_{1\leqslant i,j\leqslant w}(f_{i-j}(\mathbf{x})+f_{i+j-1}(\mathbf{x})) = \det_{1\leqslant i,j\leqslant w}\left((p_1^\perp)^{j-1}(f_{i-1}(\mathbf{x})+f_i(\mathbf{x}))\right). \quad (5.10)$$

Again, by (5.7) we obtain

$$(p_1^\perp)^{i-1}(f_{-1}(\mathbf{x})+f_0(\mathbf{x})) = f_{-i}(\mathbf{x})+f_{i-1}(\mathbf{x})+\sum_{r=1}^{i-1}\binom{i-1}{r}(f_{-(i-2r)}(\mathbf{x})+f_{(i-2r)-1}(\mathbf{x})).$$

By the same argument one can show that

$$\det_{1\leqslant i,j\leqslant w}\left((p_1^\perp)^{j-1}(f_{-i}(\mathbf{x})+f_{i-1}(\mathbf{x}))\right) = \det_{1\leqslant i,j\leqslant w}\left((p_1^\perp)^{j-1}(p_1^\perp)^{i-1}(f_{-1}(\mathbf{x})+f_0(\mathbf{x}))\right). \quad (5.11)$$

Since $f_{-i}(\mathbf{x}) = f_i(\mathbf{x})$, Identity (1.12) follows from (5.10) and (5.11). ☐

Proof of Theorem 1.5. The first identity, Equation (1.13), can be proved by the same argument as in the proof of (1.11).

For the second identity, observe that (5.6) gives

$$\begin{aligned}(p_1^\perp)^{j-1}(f_{i+k\delta_{i,w}-1}(\mathbf{x}) - f_{i+k\delta_{i,w}+1}(\mathbf{x})) &= f_{i+k\delta_{i,w}-j}(\mathbf{x}) - f_{i+k\delta_{i,w}+j}(\mathbf{x}) \\ &+ \sum_{r=1}^{j-1}\binom{j-1}{r}(f_{i+k\delta_{i,w}-(j-2r)}(\mathbf{x}) - f_{i+k\delta_{i,w}+(j-2r)}(\mathbf{x})).\end{aligned}$$

Therefore, we can apply column operations in (1.5) to obtain

$$\begin{aligned}\sum_{\substack{\lambda:\lambda_1\leqslant 2w\\ r(\lambda)=k}} s_\lambda(\mathbf{x}) &= \det_{1\leqslant i,j\leqslant w}\left(f_{i+k\delta_{i,w}-j}(\mathbf{x}) - f_{i+k\delta_{i,w}+j}(\mathbf{x})\right) \\ &= \det_{1\leqslant i,j\leqslant w}\left((p_1^\perp)^{j-1}(f_{i+k\delta_{i,w}-1}(\mathbf{x}) - f_{i+k\delta_{i,w}+1}(\mathbf{x}))\right),\end{aligned}$$

which concludes the proof. ☐

Proof of Theorem 1.6. We apply column operations in Theorem 5.1 as in the proof of Theorem 1.5. We only give a proof of (1.17). By (5.3) and Lemmas 5.2 and 5.3, we obtain

$$\sum_{\substack{\lambda:\lambda_1\leqslant 2w\\ c(\lambda)=k}} s_\lambda(\mathbf{x}) + \sum_{\substack{\lambda:\lambda_1\leqslant 2w\\ c(\lambda)=2w-k}} s_\lambda(\mathbf{x}) = \left(\frac{1}{2}\right)^{\chi(k<w)} \det_{1\leqslant i,j\leqslant w}\left(f_{i+\chi(i>w-k)-j}(\mathbf{x}) + f_{i+\chi(i>w-k)+j-2}(\mathbf{x})\right)$$

$$= 2^{\chi(k=w)} \det_{1\leqslant i,j\leqslant w}\left(\begin{cases} f_{i+\chi(i>w-k)-1}(\mathbf{x}), & \text{if } j=1\\ f_{(i-1)+\chi(i>w-k)-(j-1)}(\mathbf{x}) + f_{(i-1)+\chi(i>w-k)+(j-1)}(\mathbf{x}), & \text{if } 2\leqslant j\leqslant w\end{cases}\right)$$

$$= 2^{\chi(k=w)} \det_{1\leqslant i,j\leqslant w}\left(\begin{cases} f_{i+\chi(i>w-k)-1}(\mathbf{x}), & \text{if } j=1\\ (p_1^\perp)^{j-2}(f_{i+\chi(i>w-k)-2}(\mathbf{x}) - f_{i+\chi(i>w-k)}(\mathbf{x})), & \text{if } 2\leqslant j\leqslant w\end{cases}\right)$$

$$= 2^{\chi(k=w)} \det_{1\leqslant i,j\leqslant w}\left((p_1^\perp)^{j-1} f_{i+\chi(i>w-k)-1}(\mathbf{x})\right),$$

completing the proof. □

6 Formulas for the number of standard Young tableaux of bounded width: proofs of Theorems 1.8 and 1.9

Here we prove the formulas for the number of standard Young tableaux of bounded width in Theorems 1.8 and 1.9, the former being an alternative to Theorem 1.7, the latter being a refinement.

The important tool here is the ring homomorphism $\theta : \Lambda \to \mathbb{Q}[[x]]$, defined by $\theta(p_1(\mathbf{x})) = x$ and $\theta(p_n(\mathbf{x})) = 0$ for $n > 1$. Gessel [3, Theorem 1] showed that, for $f(\mathbf{x}) \in \Lambda$,

$$\theta(f(\mathbf{x})) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!},$$

where a_n is the coefficient of $x_1x_2\cdots x_n$ in $f(\mathbf{x})$. In particular, we have $\theta(e_n(\mathbf{x})) = x^n/n!$, which implies

$$\theta(f_n(\mathbf{x})) = I_n(2x) = \sum_{r\geqslant 0}\binom{|n|+2r}{r}\frac{x^{|n|+2r}}{(|n|+2r)!}. \tag{6.1}$$

Moreover, by definition, if $\lambda \vdash n$, we have

$$\theta(s_\lambda(\mathbf{x})) = f^\lambda \frac{x^n}{n!}, \tag{6.2}$$

where f^λ denotes the number of SYTs of shape λ. If, with this knowledge, one applies θ to both sides of (1.1) and (1.2), then one obtains (1.19) and (1.20), respectively.

The property of the operator θ that is most relevant in our context is that θ intertwines $p_1^\perp$ and the differential operator d/dx. More precisely, for $f(\mathbf{x}) \in \Lambda$, we have

$$\theta(p_1^\perp f(\mathbf{x})) = \frac{d}{dx}\theta(f(\mathbf{x})), \tag{6.3}$$

as is not very hard to check.

Proof of Theorem 1.8. By (6.1), (6.2) and (6.3), application of θ to Theorem 1.4 yields that

$$\sum_{n\geqslant 0} |\operatorname{SYT}_{n,2w+1}| \frac{x^n}{n!} = \exp(x) \det_{1\leqslant i,j\leqslant w}\left(\left(\frac{d}{dx}\right)^{i+j-2}(I_0(2x)-I_2(2x))\right),$$

and

$$\sum_{n\geqslant 0} |\operatorname{SYT}_{n,2w}| \frac{x^n}{n!} = \det_{1\leqslant i,j\leqslant w}\left(\left(\frac{d}{dx}\right)^{i+j-2}(I_0(2x)+I_1(2x))\right).$$

From the fact that

$$I_0(2x)-I_2(2x) = \sum_{n\geqslant 0}\operatorname{Cat}(n/2)\frac{x^n}{n!} \qquad \text{and} \qquad I_0(2x)+I_1(2x) = \sum_{n\geqslant 0}\binom{n}{\lfloor n/2\rfloor}\frac{x^n}{n!},$$

extraction of coefficients of $\frac{x^n}{n!}$ on both sides of the above equations finishes the proof. ☐

Proof of Theorem 1.9. The first identity (1.23) can be proved similarly as in the proof of Theorem 1.8. For the second identity (1.24), we use the fact that for a positive integer α,

$$I_{\alpha-1}(2x)-I_{\alpha+1}(2x) = \sum_{s\geqslant 0} F(s+\alpha-1, s/2)\frac{x^{s+\alpha-1}}{(s+\alpha-1)!}.$$

By applying θ to (1.14), we obtain

$$\begin{aligned}\sum_{n\geqslant 0}\left(\sum_{\substack{\lambda\vdash n\\ \lambda_1\leqslant 2w,\ r(\lambda)=k}} f^\lambda\right)\frac{x^n}{n!} &= \det_{1\leqslant i,j\leqslant w}\left(\left(\frac{d}{dx}\right)^{j-1}(I_{i+k\delta_{i,w}-1}(2x)-I_{i+k\delta_{i,w}+1}(2x))\right)\\ &= \det_{1\leqslant i,j\leqslant w}\left(\sum_{s\geqslant 0}F(i+k\delta_{i,w}+s-1, s/2)\frac{x^{i+k\delta_{i,w}+s-j}}{(i+k\delta_{i,w}+s-j)!}\right).\end{aligned}$$

The proof is completed by extracting the coefficient of $x^{t_i}/t_i!$ in the i-th row, $1\leqslant i\leqslant w$, of the determinant on the right-hand side, where $t_1+t_2+\cdots+t_w=n$. ☐

7 Combinatorial interpretations using up-down tableaux

In this section, we give combinatorial interpretations for the right-hand sides of the identities in Theorems 1.2 and 1.3 in terms of (marked) up-down tableaux. Our main results are Theorems 7.5, 7.7, and 7.15.

7.1 Basic definitions and preliminaries

We begin with basic definitions and auxiliary results. We follow the notation in [9, Section 7].

Definition 7.1. Let μ and ν be partitions. A *w-up-down tableau* T of length $2n$ from μ to ν is a sequence $(T_0, T_1, \dots, T_{2n})$ of partitions satisfying the following properties:

(i) $\mu = T_0 \subseteq T_1 \supseteq T_2 \subseteq T_3 \supseteq T_4 \subseteq \cdots \subseteq T_{2n-1} \supseteq T_{2n} = \nu$;

(ii) each pair (T_{i-1}, T_i) differs by a vertical strip (that is, by a collection of cells which contains at most one cell in each row), for $i = 1, 2, \dots, 2n$;

(iii) each T_i has at most w rows, for $i = 0, 1, \dots, 2n$.

The weight of T is defined by

$$\omega(T) = \prod_{i=1}^{n} x_i^{-|T_{2i-2}|+2|T_{2i-1}|-|T_{2i}|}.$$

It should be noted that the exponent $-|T_{2i-2}| + 2|T_{2i-1}| - |T_{2i}|$ is the sum of of the differences in sizes of (T_{2i-2}, T_{2i-1}) and of (T_{2i-1}, T_{2i}), for $i = 1, 2, \dots, n$.

We denote the set of w-up-down tableaux of length $2n$ from μ to ν by $\mathrm{UD}_n(w; \mu \to \nu)$.

We will identify $T = (T_0, T_1, \dots, T_{2n}) \in \mathrm{UD}_n(w; \mu \to \nu)$ with the family of nonintersecting paths $\mathbf{P} = (P_1, \dots, P_w)$, where P_i is the path from $(\mu_{w+1-i}+i, 0)$ to $(\nu_{w+1-i}+i, 2n)$ consisting of the points $((T_j)_{w+1-i} + i, j)$ for $j = 0, 1, \dots, 2n$. For example, the sequence

$$\begin{aligned}((1,1,0,0),\ (1,1,0,0),\ (1,0,0,0),\ (2,1,1,1),\ (2,1,1,0),\ (3,2,2,0),&\\ (2,2,1,0),\ (2,2,1,1),\ (1,1,1,1),\ (1,1,1,1),\ (0,0,0,0))&\end{aligned}$$

forms a 4-up-down tableau in $\mathrm{UD}_5(4; (1,1,0,0) \to (0,0,0,0))$ and is identified with the family $\mathbf{P}$ in Figure 1.

Definition 7.2. Let $L(u \to v)$ denote the set of lattice paths from u to v with steps from the set

$$S = \{(i,j) \to (i,j+1),\ (i,2j-2) \to (i+1,2j-1),\ (i,2j-1) \to (i-1,2j) : i,j \in \mathbb{Z}\}.$$

We define the weight of a vertical step $(i,j) \to (i,j+1)$ to be 1 and the weight of a forward (respectively backward) diagonal step $(i,2j-2) \to (i+1,2j-1)$ (respectively $(i,2j-1) \to (i-1,2j)$) to be x_j. The weight of a path P equals the product of the weights of its steps. For example, the weight of the left-most path in Figure 1 equals $x_2^2 x_4 x_5$.

Let $L(a,b; u \to v)$ denote the set of paths $P \in L(u \to v)$ for which every point $(i,2j)$ of even height in P satisfies $a \leqslant i \leqslant b$. We also write $L_t(u \to v) = L(t, \infty; u \to v)$.

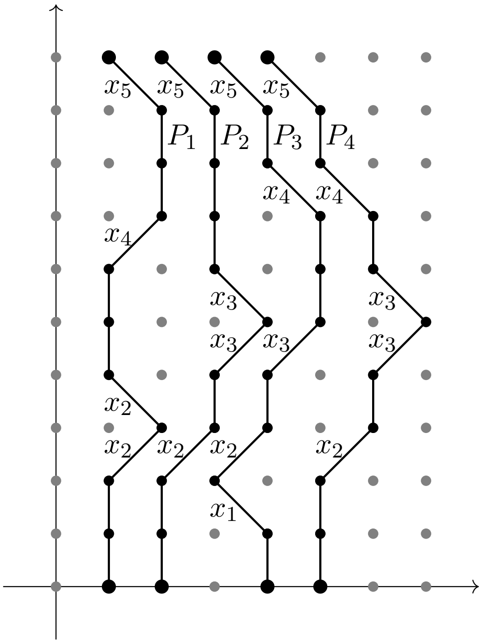


Figure 1: An example of a family $\mathbf{P} = (P_1, P_2, P_3, P_4)$ of nonintersecting paths.

Figure 2: An odd-branch point u (left) and an even-branch point v (right).

Definition 7.3. An *odd-branch point* of a lattice path P is a point $(1, 2j-1)$ with the property that P passes through $(1, 2j-2)$, $(1, 2j-1)$, and $(1, 2j)$. An *even-branch point* of a lattice path P is a point $(1, 2j-2)$ with the property that P passes through $(1, 2j-2)$, $(2, 2j-1)$, and $(2, 2j)$. See Figure 2.

For $t \geqslant 1$, an *odd-marked lattice path* (respectively *even-marked lattice path*) is a pair (P, M) of a lattice path $P \in L_t(u \to v)$ and a set M of integers such that if $j \in M$, then $(1, 2j-1)$ (respectively $(1, 2j-2)$) is an odd-branch (respectively even-branch) point of P. Let $L_t^o(u \to v)$ (respectively $L_t^e(u \to v)$) denote the set of odd-marked (respectively even-marked) lattice paths (P, M) with $P \in L_t(u \to v)$.

The lemma below collects basic generating functions for various sets of lattice paths that we shall need in the sequel.

Lemma 7.4. For positive integers i, j, and n, we have

$$f_{i-j}(x_1, \dots, x_n) - f_{i+j}(x_1, \dots, x_n) = \sum_{P \in L_1((i,0)\to(j,2n))} \omega(P), \tag{7.1}$$

$$f_{i-j}(x_1, \dots, x_n) - f_{i+j-1}(x_1, \dots, x_n) = \sum_{(P,M) \in L_1^o((i,0)\to(j,2n))} (-1)^{|M|} \omega(P)\omega(M), \tag{7.2}$$

$$f_{i-j}(x_1, \dots, x_n) + f_{i+j-1}(x_1, \dots, x_n) = \sum_{(P,M) \in L_1^o((i,0)\to(j,2n))} \omega(P)\omega(M), \tag{7.3}$$

$$f_{i-j}(x_1, \dots, x_n) - f_{i+j-2}(x_1, \dots, x_n) = \sum_{P \in L_2((i,0)\to(j,2n))} \omega(P), \tag{7.4}$$

$$f_{i-j}(x_1, \dots, x_n) + f_{i+j-2}(x_1, \dots, x_n) = \sum_{(P,M) \in L_1^e((i,0)\to(j,2n))} 2^{\chi(j=1)} \omega(P), \tag{7.5}$$

where $\omega(M) = \prod_{j \in M} x_j$.

Proof. Identity (7.1) is obtained from Equation (7.3) in [9] by taking the limit $N \to \infty$ and using the fact $f_{-i+j} = f_{i-j}$. Similarly, Equation (7.3) is obtained from Equation (7.11) in [9]. Identity (7.4) is obtained from (7.1) with i and j replaced by $i-1$ and $j-1$, respectively. Identity (7.2) follows from (7.3) by replacing every x_t by $-x_t$.

Identity (7.5) can be proved similarly as in the proof of [9, Eq. (7.3)]. The idea is that we find the smallest t such that the path P visits $(1, 2t-2), (1, 2t-1)$, and $(0, 2t)$. Let $P = P'P''$, where P' and P'' are the subpaths of P obtained by dividing it at $(1, 2t-2)$. Update P to be the path $P'\mathfrak{R}(P'')$ and add t to M, where $\mathfrak{R}$ is the map in [9, proof of Lemma 7.3]. Repeat this process until there is no such t. Then we obtain $(P, M) \in L_1^e((i,0) \to (j, 2n))$. □

7.2 Combinatorial interpretation for Theorem 1.2

Our combinatorial interpretation of the right-hand side of (1.5) is the following.

Theorem 7.5. For integers $0 \leqslant k \leqslant w$, we have

$$\sum_{\substack{\lambda:\lambda_1 \leqslant 2w \\ r(\lambda)=k}} s_\lambda(\mathbf{x}_n) = \sum_{T \in \mathrm{UD}_n(w;(k,0^{w-1}) \to (0^w))} \omega(T). \tag{7.6}$$

Proof. This is in fact the special case of a more general result; cf. [14, Eq. (1.1)]. For the convenience of the reader, we provide the (specialized) argument. Namely, by (1.5) and (7.1), we have

$$\sum_{\substack{\lambda:\lambda_1 \leqslant 2w \\ r(\lambda)=k}} s_\lambda(\mathbf{x}_n) = \det_{1 \leqslant i,j \leqslant w} \left(f_{i+k\delta_{i,w}-j}(\mathbf{x}_n) - f_{i+k\delta_{i,w}+j}(\mathbf{x}_n) \right) = \sum_{\mathbf{P} \in X} \mathrm{sgn}(\mathbf{P}) \prod_{i=1}^{w} \omega(P_i), \tag{7.7}$$

where X is the set of families $\mathbf{P} = (P_1, \dots, P_w)$ of lattice paths $P_i \in L_1((i + k\delta_{i,w}, 0) \to (\sigma(i), 2n))$ for some permutation σ on $[w]$, and $\mathrm{sgn}(\mathbf{P}) = \mathrm{sgn}(\sigma)$. By applying the Lindström–Gessel–Viennot lemma [18, Lemma 1] to (7.7), we have

$$\det_{1 \leqslant i,j \leqslant w} \left(f_{i+k\delta_{i,w}-j}(\mathbf{x}_n) - f_{i+k\delta_{i,w}+j}(\mathbf{x}_n) \right) = \sum_{\mathbf{P} \in Y} \prod_{i=1}^{w} \omega(P_i), \tag{7.8}$$

where Y is the set of families $\mathbf{P} = (P_1, \dots, P_w)$ of nonintersecting lattice paths $P_i \in L_1((i + k\delta_{i,w}, 0) \to (i, 2n))$. Therefore, by (1.5), (7.8), and by identifying $\mathbf{P}$ as a w-up-down tableau T, we obtain

$$\sum_{\substack{\lambda:\lambda_1 \leqslant 2w \\ r(\lambda)=k}} s_\lambda(\mathbf{x}_n) = \sum_{T \in \mathrm{UD}_n(w;(0^w) \to (k,0^{w-1}))} \omega(T). \tag{7.9}$$

By the symmetry of the variables $x_1, \dots, x_n$, (7.9) is equivalent to (7.6). ☐

For our upcoming combinatorial interpretation of the right-hand side of (1.4) we need one more definition.

Definition 7.6. Let $\mathrm{MUD}_n(w; \mu \to \nu)$ denote the set of pairs (T, S) such that $T \in \mathrm{UD}_n(w; \mu \to \nu)$ and $S \subseteq [n]$. We call an element $(T, S) \in \mathrm{MUD}_n(w; \mu \to \nu)$ a *marked w-up-down tableau.*

Here is now our combinatorial interpretation of the right-hand side of (1.4).

Theorem 7.7. For integers $0 \leqslant k \leqslant w$, we have

$$\sum_{\substack{\lambda:\lambda_1 \leqslant 2w+1 \\ r(\lambda)=k}} s_\lambda(\mathbf{x}_n) = \sum_{\substack{(T,S) \in \mathrm{MUD}_n(w;(0^w) \to (0^w)) \\ |S|=k}} \omega(T)\omega(S), \tag{7.10}$$

where $\omega(S) = \prod_{j \in S} x_j$.

Proof. By (7.1), we have

$$\det_{1\leqslant i,j\leqslant w}\left(f_{i-j}(\mathbf{x}_n)-f_{i+j}(\mathbf{x}_n)\right)=\sum_{\mathbf{P}\in X}\operatorname{sgn}(\mathbf{P})\prod_{i=1}^{w}\omega(P_i), \tag{7.11}$$

where X is the set of families $\mathbf{P}=(P_1,\ldots,P_w)$ of lattice paths $P_i\in L_1((i,0)\to(\sigma(i),2n))$ for some permutation σ on $[w]$, and $\operatorname{sgn}(\mathbf{P})=\operatorname{sgn}(\sigma)$. By applying the Lindström–Gessel–Viennot lemma [18, Lemma 1] to (7.11) and multiplying $e_k(\mathbf{x}_n)$ on both sides, we have

$$e_k(\mathbf{x}_n)\det_{1\leqslant i,j\leqslant w}\left(f_{i-j}(\mathbf{x}_n)-f_{i+j}(\mathbf{x}_n)\right)=\sum_{(\mathbf{P},S)\in Y}\omega(S)\prod_{i=1}^{w}\omega(P_i), \tag{7.12}$$

where Y is the set of $(\mathbf{P},S)$ such that $\mathbf{P}=(P_1,\ldots,P_w)$ is a family of nonintersecting lattice paths $P_i\in L_1((i,0)\to(i,2n))$ and $S\subseteq[n]$ with $|S|=k$. Therefore, by (1.4), (7.12), and by identifying $\mathbf{P}$ as a w-up-down tableau T, we obtain (7.10). ☐

7.3 Combinatorial interpretation for Theorem 1.3 (odd bound case)

The goal of the remainder of this section is to provide combinatorial interpretations of the right-hand sides of the identities in Theorem 1.3, in its equivalent form given in Theorem 5.1.

Definition 7.8. Let $T\in\mathrm{UD}_n(w;\mu\to\nu)$. For an integer $j\geqslant 1$, we say that $2j-1$ is a *length-peak* of T if $\ell(T_{2j-2})<\ell(T_{2j-1})>\ell(T_{2j})$. Furthermore, we say that the length-peak $2j-1$ is *full* if $\ell(T_{2j-1})=w$, and *non-full* otherwise. We define

$$\widehat{\mathrm{UD}}_n(w;\mu\to\nu)=\{T\in\mathrm{UD}_n(w;\mu\to\nu):\text{every length-peak of }T\text{, if any, is full}\}.$$

Definition 7.9. Let $\mathrm{MUD}_n^o(w;\mu\to\nu)$ denote the set of pairs (T,S) of $T\in\widehat{\mathrm{UD}}_n(w;\mu\to\nu)$ and $S\subseteq[n]$ satisfying the following conditions:

- if $2j-1$ is a length-peak of T, then $j\in S$;
- if $\ell(T_{2j-1})<w$, then $j\notin S$.

The following lemma concerns the right-hand sides of (5.1) and (5.2), that is, the identities with an odd bound on the number of columns of the partitions over which the sum is taken on the left-hand side.

Lemma 7.10. We have

$$\begin{aligned}
&e(\mathbf{x}_n)\det_{1\leqslant i,j\leqslant w}\left(f_{i+\chi(i>w-k)-j}(\mathbf{x}_n)-f_{i+\chi(i>w-k)+j-1}(\mathbf{x}_n)\right)\\
&=\sum_{(T,S)\in\mathrm{MUD}_n^o(w;(1^k,0^{w-k})\to(0^w))}\omega(T)\omega(S), \qquad (7.13)\\
&\overline{e}(\mathbf{x}_n)\det_{1\leqslant i,j\leqslant w}\left(f_{i+\chi(i>w-k)-j}(\mathbf{x}_n)+f_{i+\chi(i>w-k)+j-1}(\mathbf{x}_n)\right)
\end{aligned}$$

$$= \sum_{(T,S)\in \mathrm{MUD}_n^o(w;(1^k,0^{w-k})\to(0^w))} (-1)^{|S|}\omega(T)\omega(S), \tag{7.14}$$

where $\omega(S) = \prod_{j\in S} x_j$.

Proof. By (7.2), we have

$$\det_{1\leqslant i,j\leqslant w} \left(f_{i+\chi(i>w-k)-j}(\mathbf{x}_n) - f_{i+\chi(i>w-k)+j-1}(\mathbf{x}_n)\right) = \sum_{\mathbf{Q}\in X} \mathrm{sgn}(\mathbf{Q}) \prod_{i=1}^{w} (-1)^{|M_i|}\omega(P_i)\omega(M_i), \tag{7.15}$$

where X is the set of tuples $\mathbf{Q} = (Q_1, \dots, Q_w)$ of odd-marked lattice paths

$$Q_i = (P_i, M_i) \in L_1^o((i + \chi(i > w - k), 0) \to (\sigma(i), 2n)),$$

for some permutation σ on $[w]$, and $\mathrm{sgn}(\mathbf{Q}) = \mathrm{sgn}(\sigma)$. By applying the Lindström–Gessel–Viennot lemma [18, Lemma 1] to (7.15) and multiplying $e(\mathbf{x}_n)$ on both sides, we have

$$\begin{aligned} &e(\mathbf{x}_n) \det_{1\leqslant i,j\leqslant w} \left(f_{i+\chi(i>w-k)-j}(\mathbf{x}_n) - f_{i+\chi(i>w-k)+j-1}(\mathbf{x}_n)\right) \\ &= \sum_{(\mathbf{Q},S)\in Y} \omega(S) \prod_{i=1}^{w} (-1)^{|M_i|}\omega(P_i)\omega(M_i), \end{aligned} \tag{7.16}$$

where Y is the set of $(\mathbf{Q}, S)$ such that $\mathbf{Q} = (Q_1, \dots, Q_w)$ is a family of nonintersecting odd-marked lattice paths

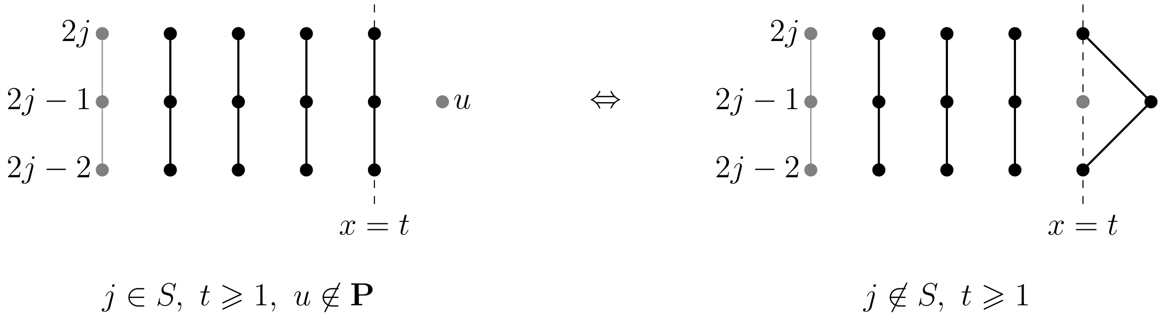
$$Q_i = (P_i, M_i) \in L_1^o((i + \chi(i > w - k), 0) \to (i, 2n))$$

and $S \subseteq [n]$. Observe that, if $(\mathbf{Q}, S) \in Y$ with $Q_i = (P_i, M_i)$, then $M_i = \emptyset$ for all $i \neq 1$.

Suppose that $(\mathbf{Q}, S) \in Y$. Let j be the smallest integer such that P_1 contains $(1, 2j-1)$ and $j \in (M_1 \cup S) \setminus (M_1 \cap S)$. If such j exists, let $(\mathbf{Q}', S')$ be the pair obtained from $(\mathbf{Q}, S)$ by moving the integer j from M_1 to S or vice versa, otherwise let $(\mathbf{Q}', S') = (\mathbf{Q}, S)$. Then the map $(\mathbf{Q}, S) \mapsto (\mathbf{Q}', S')$ is a sign-reversing involution on Y whose fixed points are the pairs $(\mathbf{Q}, S) \in Y$ with $Q_i = (P_i, M_i)$ such that, if P_1 contains $(1, 2j - 1)$, then $j \in M_1 \cap S$ or $j \notin M_1 \cup S$, and if P_1 does not contain $(1, 2j - 1)$, then $j \notin M_1$. Observe that, if $(\mathbf{Q}, S)$ is a fixed point, then M_1 is completely determined by P_1 and S. Therefore, by identifying $(P_1, \dots, P_w)$ as a w-up-down tableau T, we may rewrite (7.16) as

$$\begin{aligned} &e(\mathbf{x}_n) \det_{1\leqslant i,j\leqslant w} \left(f_{i+\chi(i>w-k)-j}(\mathbf{x}_n) - f_{i+\chi(i>w-k)+j-1}(\mathbf{x}_n)\right) \\ &\qquad = \sum_{(T,S)\in \mathrm{MUD}_n(w;(1^k,0^{w-k})\to(0^w))} \omega(T) \prod_{j\in S} x_j (-x_j)^{\chi(\ell(T_{2j-1})<w)}. \end{aligned} \tag{7.17}$$

We will further simplify the right-hand side of (7.17) by finding a sign-reversing involution. Let $(T, S) \in \mathrm{MUD}_n(w; (1^k, 0^{w-k}) \to (0^w))$. Find the smallest j satisfying one of the following conditions:



$j \in S,\ t \geqslant 1,\ u \notin \mathbf{P}$ $\qquad$ $j \notin S,\ t \geqslant 1$

Figure 3: Configurations for the involution in Lemma 7.10.

Case 1 $j \in S$ and $\ell(T_{2j-1}) < w$. In this case, let T' be the up-down tableau obtained from T by creating a new cell at the end of row one of T_{2j-1}, and let $S' = S \setminus \{j\}$. Then $2j-1$ is a length-peak of the resulting up-down tableau T' and $j \notin S'$.

Case 2 $j \notin S$ and $2j-1$ is a length-peak of T. In this case, let T' be the up-down tableau obtained from T by deleting the last row, which has one cell, from T_{2j-1}, and let $S' = S \cup \{j\}$. Then $\ell(T'_{2j-1}) < w$ and $j \in S'$.

See Figure 3. If there is no such j, define $T' = T$ and $S' = S$. Then the map $(T, S) \mapsto (T', S')$ is a sign-reversing involution on $\mathrm{MUD}_n(w; (1^k, 0^{w-k}) \to (0^w))$ with fixed point set $\mathrm{MUD}^o_n(w; (1^k, 0^{w-k}) \to (0^w))$. This shows the first identity, Equation (7.13).

The second identity, Equation (7.14), can be proved similarly, where the following equation plays the role of (7.17):

$$\begin{aligned}\overline{e}(\mathbf{x}_n) \det_{1\leqslant i,j\leqslant w} \left(f_{i+\chi(i>w-k)-j}(\mathbf{x}_n) + f_{i+\chi(i>w-k)+j-1}(\mathbf{x}_n)\right) \\ = \sum_{(T,S)\in\mathrm{MUD}_n(w;(1^k,0^{w-k})\to(0^w))} \omega(T) \prod_{j\in S}(-x_j)(-x_j)^{\chi(\ell(T_{2j-1})<w)}. \quad \square\end{aligned}$$

We are now in the position to state, and prove, our combinatorial interpretations of the sums of Schur functions over partitions with an odd bound on the number of columns and a given number of odd columns appearing in (5.1) and (5.2).

Theorem 7.11. For integers $0 \leqslant k \leqslant w$, we have

$$\sum_{\substack{\lambda:\lambda_1\leqslant 2w+1\\ c(\lambda)=k}} s_\lambda(\mathbf{x}_n) = \sum_{\substack{(T,S)\in\mathrm{MUD}^o_n(w;(1^k,0^{w-k})\to(0^w))\\ |S|\ \text{even}}} \omega(T)\omega(S), \tag{7.18}$$

$$\sum_{\substack{\lambda:\lambda_1\leqslant 2w+1\\ c(\lambda)=2w+1-k}} s_\lambda(\mathbf{x}_n) = \sum_{\substack{(T,S)\in\mathrm{MUD}^o_n(w;(1^k,0^{w-k})\to(0^w))\\ |S|\ \text{odd}}} \omega(T)\omega(S). \tag{7.19}$$

Proof. These identities are obtained by adding and subtracting the two equations (5.1) and (5.2), and applying the two equations in Lemma 7.10. ☐

7.4 Combinatorial interpretation for Theorem 1.3 (even bound case)

We finally turn our attention to the "even bound identities," that is, to (5.3) and (5.4).

Recall that $\widehat{\mathrm{UD}}_n(w;\mu\to\nu)$ is the set of $T\in\mathrm{UD}_n(w;\mu\to\nu)$ without any non-full length-peaks.

Definition 7.12. For $T\in\mathrm{UD}_n(w;\mu\to\nu)$, let

$$E_w(T)=\{j\in[n]:\ell(T_{2j-2})<\ell(T_{2j-1})=\ell(T_{2j})=w\}.$$

We define

$$\mathrm{MUD}_n^*(w;\mu\to\nu)=\{(T,S):T\in\widehat{\mathrm{UD}}_n(w;\mu\to\nu)\text{ and }S\subseteq E_w(T)\},\tag{7.20}$$

$$\mathrm{MUD}_n^<(w;\mu\to\nu)=\{(T,S):T\in\mathrm{UD}_n(w;\mu\to\nu),\ \ell(T_j)<w\text{ for }0\leqslant j\leqslant 2n,\ S\subseteq[n]\}.\tag{7.21}$$

The following lemma concerns the right-hand side in (5.3) and (5.4). Here, and also later, we will use the convention that $\det A=1$ if A is an empty matrix.

Lemma 7.13. For $0\leqslant k\leqslant w$, we have

$$\begin{aligned}&\frac{1}{2}\det_{1\leqslant i,j\leqslant w}\big(f_{i+\chi(i>w-k)-j}(\mathbf{x}_n)+f_{i+\chi(i>w-k)+j-2}(\mathbf{x}_n)\big)\\&=\sum_{(T,S)\in\mathrm{MUD}_n^*(w;(1^k,0^{w-k})\to(0^w))}\omega(T).\end{aligned}\tag{7.22}$$

For $0\leqslant k\leqslant w-1$, we have

$$\begin{aligned}&e(\mathbf{x}_n)\overline{e}(\mathbf{x}_n)\det_{2\leqslant i,j\leqslant w}\big(f_{i+\chi(i>w-k)-j}(\mathbf{x}_n)-f_{i+\chi(i>w-k)+j-2}(\mathbf{x}_n)\big)\\&=\sum_{(T,S)\in\mathrm{MUD}_n^<(w;(1^k,0^{w-k})\to(0^w))}\omega(T)\prod_{j\in S}(-x_j^2).\end{aligned}\tag{7.23}$$

Proof. For the first identity, by (7.5), we have

$$\frac{1}{2}\det_{1\leqslant i,j\leqslant w}\big(f_{i+\chi(i>w-k)-j}(\mathbf{x}_n)+f_{i+\chi(i>w-k)+j-2}(\mathbf{x}_n)\big)=\sum_{\mathbf{Q}\in X}\mathrm{sgn}(\mathbf{Q})\prod_{i=1}^{w}\omega(P_i),\tag{7.24}$$

where X is the set of tuples $\mathbf{Q}=(Q_1,\dots,Q_w)$ of even-marked lattice paths

$$Q_i=(P_i,M_i)\in L_1^e((i+\chi(i>w-k),0)\to(\sigma(i),2n)),$$

for some permutation σ on $[w]$, and $\mathrm{sgn}(\mathbf{Q})=\mathrm{sgn}(\sigma)$. Note that the factor $1/2$ accounts for the fact that, if $j=1$, then $f_{i+\chi(i>w-k)-j}(\mathbf{x}_n)=f_{i+\chi(i>w-k)+j-2}(\mathbf{x}_n)$.

We apply the Lindström–Gessel–Viennot lemma [18, Lemma 1] to (7.24) as in the proof of (7.13). However, in the present case, we must be careful because certain configurations cannot be canceled. To be specific, suppose that $\mathbf{Q}$ contains two even-marked paths $Q_r=(P_r,M_r)$ and $Q_s=(P_s,M_s)$ such that

- P_r passes the points $(1,2j-2),(2,2j-1),(2,2j)$, and $j \in M_r$,
- P_s passes the points $(2,2j-2),(2,2j-1),(1,2j)$, which forces that $j \notin M_s$.

Let P_r' and P_r'' be the subpaths of P_r before and after the point $u=(2,2j-1)$, respectively. Define P_s' and P_s'' similarly. Let $\widetilde{P}_r = P_r'P_s''$ and $\widetilde{P}_s = P_s'P_r''$ as shown in Figure 4. Since $(1,2j-2)$ is no longer an even-branch point for $\widetilde{P}_r$ or $\widetilde{P}_s$, we cannot cancel such $\mathbf{Q}$ by modifying the paths at u. Still, except for such an intersection point, we can apply the usual "tail exchange" involution. Thus, we have

$$\frac{1}{2}\det_{1\leqslant i,j\leqslant w}\left(f_{i+\chi(i>w-k)-j}(\mathbf{x}_n)+f_{i+\chi(i>w-k)+j-2}(\mathbf{x}_n)\right)=\sum_{\mathbf{Q}\in Y}\operatorname{sgn}(\mathbf{Q})\prod_{i=1}^{w}\omega(P_i), \tag{7.25}$$

where Y is the set of $\mathbf{Q}\in X$ with $Q_i=(P_i,M_i)$ satisfying the following conditions:

- P_i is a path from $(i+\chi(i>w-k),0)$ to $(i,2n)$ for $i=3,\dots,w$,
- if there is an intersection point u, then $u=(2,2j-1)$ for some $j\in M_1\cup M_2$, $u\notin P_i$ for $i\geqslant 3$, and P_1 visits $(1,2j-2),u,(1,2j)$, and P_2 visits $(2,2j-2),u,(2,2j)$, or vice versa.

Observe that, if $\mathbf{Q}\in Y$, then $M_i=\emptyset$ for all $i\geqslant 3$, and $P_2,P_3,\dots,P_w$ are nonintersecting.

We will further simplify the right-hand side of (7.25) by finding a sign-reversing involution. Let $\mathbf{Q}\in Y$ with $Q_i=(P_i,M_i)$, and let $\mathbf{P}=(P_1,P_2,\dots,P_w)$. Find the smallest j satisfying one of the following conditions:

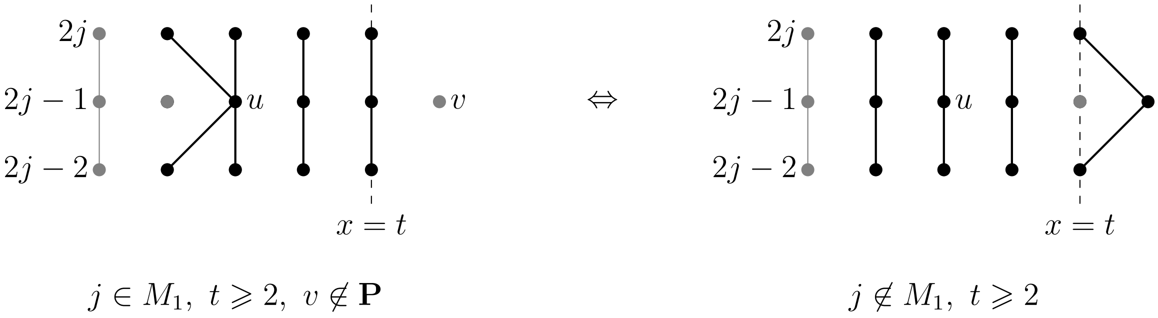
CASE 1 $j\in M_1$ and $u=(2,2j-1)\in P_1\cap P_2$. In this case, let $v=(t+1,2j-1)$, where $t\geqslant 2$ is the smallest integer such that $(t+1,2j-1)$ is not contained in any path in $\mathbf{P}$. Let $\mathbf{Q}'$ be the family obtained from $\mathbf{Q}$ by removing the integer j from M_1, exchanging all points (x,y) of P_1 and P_2 with $y\geqslant 2j$, and replacing $u\in P_1$ and $(t,2j-1)\in P_t$ by $(1,2j-1)$ and v, respectively.

CASE 2 $j\notin M_1$, P_i visits $(i,2j-2),(i,2j-1),(i,2j)$ for $i=1,2,\dots,t-1$, and P_t visits $(t,2j-2)$, $(t+1,2j-1),(t,2j)$ for some $t\geqslant 2$. In this case, let $u=(2,2j-1)$ and $v=(t+1,2j-1)$. Let $\mathbf{Q}'$ be the family obtained from $\mathbf{Q}$ by adding the integer j to M_1, exchanging all points (x,y) of P_1 and P_2 with $y\geqslant 2j$, and replacing $(1,2j-1)\in P_1$ and $v\in P_t$ by u and $(t,2j-1)$, respectively.

See Figure 5. If there is no such j, define $\mathbf{Q}'=\mathbf{Q}$. Then the map $\mathbf{Q}\mapsto\mathbf{Q}'$ is a sign-reversing involution on Y. Note that, if $\mathbf{Q}\in Y$ is a fixed point, then $M_i=\emptyset$ for all $i\geqslant 2$. Hence, a fixed point $\mathbf{Q}\in Y$ has no intersection point and can be identified with a pair $(T,S)\in\mathrm{MUD}_n^*(w;(1^k,0^{w-k})\to(0^w))$, where $S=M_1$. This shows the first identity, Equation (7.22).

The second identity, Equation (7.23), is easily obtained by combining the application of the Lindström–Gessel–Viennot lemma [18, Lemma 1] to (7.4) with the fact that $e(\mathbf{x}_n)\overline{e}(\mathbf{x}_n)=\overline{e}(\mathbf{x}_n^2)$. ☐

Figure 4: Suppose that the black path $P_r = P'_rP''_r$ has a marking $j \in M_r$. It intersects with the red path $P_s = P'_sP''_s$ at u but this cannot be canceled because the configuration (on the right) obtained by exchanging the two subpaths after u is not a valid configuration due to the marking $j \in M_r$.



$j \in M_1,\ t \geqslant 2,\ v \notin \mathbf{P}$

$j \notin M_1,\ t \geqslant 2$

Figure 5: An illustration of the involution in Lemma 7.13.

Definition 7.14. Let $p(T)$ denote the number of length-peaks in T. We define

$$\begin{aligned}\mathrm{MUD}_n^{e,e}(w;\mu\to\nu) &= \mathrm{MUD}_n^{1}(w;\mu\to\nu)\cup\mathrm{MUD}_n^{0,e}(w;\mu\to\nu),\\ \mathrm{MUD}_n^{e,o}(w;\mu\to\nu) &= \mathrm{MUD}_n^{1}(w;\mu\to\nu)\cup\mathrm{MUD}_n^{0,o}(w;\mu\to\nu),\end{aligned}$$

where

$$\begin{aligned}\mathrm{MUD}_n^{1}(w;\mu\to\nu) &= \{(T,S): T\in\widehat{\mathrm{UD}}_n(w;\mu\to\nu), E_w(T)\neq\emptyset,\\ &\qquad\qquad \text{and } S\subseteq E_w(T)\setminus\{\min E_w(T)\}\},\\ \mathrm{MUD}_n^{0,e}(w;\mu\to\nu) &= \{(T,\emptyset): T\in\widehat{\mathrm{UD}}_n(w;\mu\to\nu), E_w(T)=\emptyset, \text{ and } p(T) \text{ is even}\},\\ \mathrm{MUD}_n^{0,o}(w;\mu\to\nu) &= \{(T,\emptyset): T\in\widehat{\mathrm{UD}}_n(w;\mu\to\nu), E_w(T)=\emptyset, \text{ and } p(T) \text{ is odd}\}.\end{aligned}$$

Note that, if (T,S) is an element in $\mathrm{MUD}_n^{0,e}(w;\mu\to\nu)$ or $\mathrm{MUD}_n^{0,o}(w;\mu\to\nu)$, then $S=\emptyset$.

We are ready to state our combinatorial interpretations of the sums of Schur functions over partitions with an even bound on the number of columns and a given number of odd columns appearing in (5.3) and (5.4).

Theorem 7.15. For $0\leqslant k\leqslant w-1$, we have

$$\sum_{\substack{\lambda:\lambda_1\leqslant 2w\\ c(\lambda)=k}} s_\lambda(\mathbf{x}_n) = \sum_{(T,S)\in\mathrm{MUD}_n^{e,e}(w;(1^k,0^{w-k})\to(0^w))}\omega(T), \tag{7.26}$$

$$\sum_{\substack{\lambda:\lambda_1\leqslant 2w\\ c(\lambda)=2w-k}} s_\lambda(\mathbf{x}_n) = \sum_{(T,S)\in\mathrm{MUD}_n^{e,o}(w;(1^k,0^{w-k})\to(0^w))}\omega(T). \tag{7.27}$$

For the case $k=w$, we have

$$\sum_{\substack{\lambda:\lambda_1\leqslant 2w\\ c(\lambda)=w}} s_\lambda(\mathbf{x}_n) = \sum_{(T,S)\in\mathrm{MUD}_n^{*}(w;(1^w)\to(0^w))}\omega(T). \tag{7.28}$$

Proof. The third identity, Equation (7.28), follows immediately from (5.3) and (7.22). We will only prove the first identity, Equation (7.26), because (7.27) can be proved similarly.

From now on we assume that $0\leqslant k\leqslant w-1$ and let $\mu=(1^k,0^{w-k})$ and $\nu=(0^w)$. By adding the two equations (5.3) and (5.4) and subsequently applying (7.22) and (7.23), we arrive at

$$\sum_{\substack{\lambda:\lambda_1\leqslant 2w\\ c(\lambda)=k}} s_\lambda(\mathbf{x}_n) = \frac{1}{2}\left(\sum_{(T,S)\in\mathrm{MUD}_n^{*}(w;\mu\to\nu)}\omega(T) + \sum_{(T,S)\in\mathrm{MUD}_n^{<}(w;\mu\to\nu)}\omega(T)\prod_{j\in S}(-x_j^2)\right). \tag{7.29}$$

We divide $\mathrm{MUD}_n^{*}(w;\mu\to\nu)=A\cup B$ into two subsets, where

$$A=\{(T,S)\in\mathrm{MUD}_n^{*}(w;\mu\to\nu): E_w(T)\neq\emptyset\},$$

$$B = \{(T,S) \in \mathrm{MUD}_n^*(w;\mu \to \nu) : E_w(T) = \emptyset\}. \tag{7.30}$$

Then A is the disjoint union of the following two sets:

$$\begin{aligned}\mathrm{MUD}_n^1(w;\mu \to \nu) &= \{(T,S) \in \mathrm{MUD}_n^*(w;\mu \to \nu) : E_w(T) \neq \emptyset \text{ and } \min E_w(T) \notin S\},\\ A' &= \{(T,S) \in \mathrm{MUD}_n^*(w;\mu \to \nu) : E_w(T) \neq \emptyset \text{ and } \min E_w(T) \in S\}.\end{aligned}$$

Since $(T,S) \mapsto (T, S \cup \{\min E_w(T)\})$ is a weight-preserving bijection from $\mathrm{MUD}_n^1(w;\mu \to \nu)$ to A', we obtain

$$\frac{1}{2}\sum_{(T,S)\in A} \omega(T) = \sum_{(T,S)\in \mathrm{MUD}_n^1(w;\mu\to\nu)} \omega(T). \tag{7.31}$$

Now we claim that

$$\frac{1}{2}\left(\sum_{(T,S)\in B} \omega(T) + \sum_{(T,S)\in \mathrm{MUD}_n^<(w;\mu\to\nu)} \omega(T)\prod_{j\in S}(-x_j^2)\right) = \sum_{(T,S)\in \mathrm{MUD}_n^{0,e}(w;\mu\to\nu)} \omega(T). \tag{7.32}$$

Note that the first identity, Equation (7.26), follows from (7.29), (7.31), and (7.32). Thus it remains to prove the above claim.

By (7.20) and (7.30), we have $S = \emptyset$ for every $(T,S) \in B$. Thus, we can rewrite B as

$$B = \{(T,\emptyset) : T \in \widehat{\mathrm{UD}}_n(w;\mu \to \nu), E_w(T) = \emptyset\}.$$

Let L be the left-hand side of (7.32). Then

$$L = \sum_{(T,\emptyset)\in X} \omega(T) + \frac{1}{2}\sum_{(T,\emptyset)\in Y} \omega(T) + \frac{1}{2}\sum_{(T,S)\in Z} \omega(T)\prod_{j\in S}(-x_j^2), \tag{7.33}$$

where

$$\begin{aligned} X &= B \cap \mathrm{MUD}_n^<(w;\mu \to \nu),\\ Y &= B \setminus \mathrm{MUD}_n^<(w;\mu \to \nu),\\ Z &= \mathrm{MUD}_n^<(w;\mu \to \nu) \setminus B.\end{aligned}$$

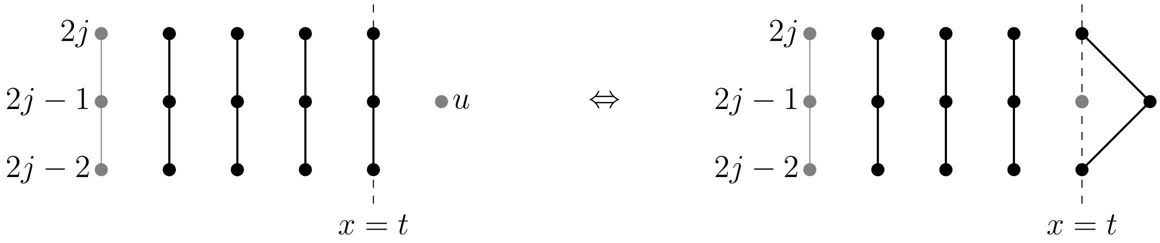
We need to find equivalent descriptions for the sets X, Y, and Z. Suppose $(T,\emptyset) \in X$. Since $(T,\emptyset) \in B$, we have $E_w(T) = \emptyset$, and T has no non-full length-peaks. Moreover, since $(T,\emptyset) \in \mathrm{MUD}_n^<(w;\mu \to \nu)$, we have $\ell(T_j) < w$ for all j, which implies that T has no full length-peaks. Conversely, any pair $(T,\emptyset)$ satisfying these properties is an element of X because $E_w(T) = \emptyset$ and $p(T) = 0$ imply that $\ell(T_j) < w$ for all j. This shows that

$$X = \{(T,\emptyset) : T \in \widehat{\mathrm{UD}}_n(w;\mu \to \nu), E_w(T) = \emptyset, p(T) = 0\}. \tag{7.34}$$

By similar arguments, we have

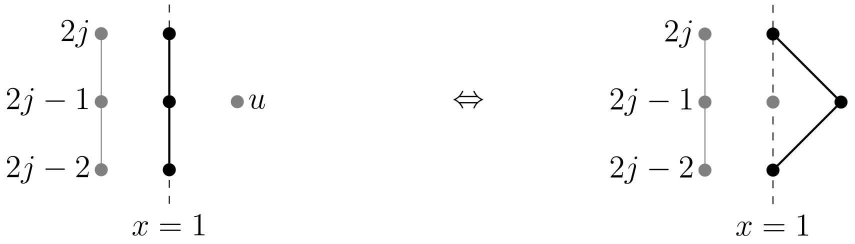
$$Y = \{(T,\emptyset) : T \in \widehat{\mathrm{UD}}_n(w;\mu \to \nu), E_w(T) = \emptyset, p(T) \neq 0\}, \tag{7.35}$$
$$Z = \{(T,S) \in \mathrm{MUD}_n^<(w;\mu \to \nu) : p(T) \neq 0 \text{ or } S \neq \emptyset\}. \tag{7.36}$$

Now we simplify the sum over Z in (7.33) by finding a sign-reversing involution on Z. Suppose $(T,S) \in Z$. Let j be the smallest integer satisfying one of the following conditions:



$j \in M,\ t \geqslant 2,\ u \notin \mathbf{P}$ $\quad$ $j \notin M,\ t \geqslant 2$

Figure 6: Configurations for the involution ϕ.



$j \in M,\ u \notin \mathbf{P}$ $\quad$ $j \in M$

Figure 7: Configurations for the bijection ψ.

- $j \in S$ and $\ell(T_{2j-1}) < w-1$;
- $j \notin S$ and $2j-1$ is a length-peak of T.

If there is such j, we define $\phi(T,S) = (T',S')$ as shown in Figure 6. Otherwise, let $\phi(T,S) = (T,S)$.

Then ϕ is a sign-reversing involution on Z whose fixed point set Z_0 is the set of pairs $(T,S) \in \mathrm{MUD}_n^{<}(w;\mu \to \nu)$ satisfying the following conditions:

- $S \neq \emptyset$;
- if $j \in S$, then $\ell(T_{2j-1}) = w-1$;
- if $j \notin S$, then $2j-1$ is not a length-peak of T.

Now, for $(T,S) \in Z_0$, let $\psi(T,S) = (T',\emptyset)$, where T' is the w-up-down tableau obtained from T by applying the operation in Figure 7 for each $j \in S$.

It is easy to see that ψ is a bijection from Z_0 to Y such that, if $\psi(T,S) = (T',\emptyset)$, then $\omega(T)\prod_{j\in S}(-x_j^2) = \omega(T')(-1)^{p(T')}$. This shows that

$$\frac{1}{2}\sum_{(T,S)\in Z}\omega(T)\prod_{j\in S}(-x_j^2) = \frac{1}{2}\sum_{(T,S)\in Z_0}\omega(T)\prod_{j\in S}(-x_j^2) = \frac{1}{2}\sum_{(T,\emptyset)\in Y}\omega(T)(-1)^{p(T)}.$$

Thus (7.33) can be rewritten as

$$L = \sum_{(T,\emptyset)\in X} \omega(T) + \sum_{(T,\emptyset)\in Y_1} \omega(T),$$

where Y_1 is the set of $(T,\emptyset)$ such that $T \in \widehat{\mathrm{UD}}_n(w;\mu \to \nu)$, $E_w(T) = \emptyset$, and $p(T)$ is a nonzero even integer. Then, by (7.34), $X \cup Y_1$ is the set of $(T,\emptyset)$ such that $T \in \widehat{\mathrm{UD}}_n(w;\mu \to \nu)$, $E_w(T) = \emptyset$, and $p(T)$ is even, which is exactly the set $\mathrm{MUD}_n^{0,e}(w;\mu \to \nu)$. Therefore

$$L = \sum_{(T,S)\in \mathrm{MUD}_n^{0,e}(w;\mu\to\nu)} \omega(T),$$

and we obtain the claim (7.32). This completes the proof of (7.26). □

8 Standard Young tableaux and lattice walks

In this section, we show that the number of standard Young tableaux of bounded width with given number of odd columns is equal to the number of certain lattice walks.

We begin by introducing notation for the relevant sets of standard Young tableaux and of vacillating tableaux.

Definition 8.1. Let $\mathrm{SYT}_{n,w}[k]$ denote the set of standard Young tableaux in $\mathrm{SYT}_{n,w}$ having exactly k columns of odd length.

The following definition introduces the lattice paths that will feature in our results. We rather prefer to use the language of *vacillating tableaux*, although everything could equivalently be formulated in terms of lattice paths.

Definition 8.2. A *w-vacillating tableau* of length n is a sequence $T = (T_0, T_1, \dots, T_n)$ of partitions with $\ell(T_i) \leqslant w$, where the partitions T_{i-1} and T_i differ by at most one cell for $i = 1, 2, \dots, n$. We denote by $\mathrm{VT}_n(w;\mu \to \nu)$ the set of w-vacillating tableaux $(T_0, T_1, \dots, T_n)$ satisfying $T_0 = \mu$ and $T_n = \nu$. We define $\mathrm{VT}_n^{>}(w;\mu \to \nu)$ to be the subset of w-vacillating tableaux in $\mathrm{VT}_n(w;\mu \to \nu)$ with the property that equality of T_{i-1} and T_i can only occur when $\ell(T_{i-1}) = \ell(T_i) = w$.

Suppose that $T = (T_0, T_1, \dots, T_n) \in \mathrm{VT}_n(w;\mu \to \nu)$. Note that each partition T_i can be considered as a w-tuple of non-increasing integers. Hence, by definition, T can be identified with a walk of length n from μ to ν in the region $\{(x_1, \dots, x_w) : x_1 \geqslant \dots \geqslant x_w \geqslant 0\}$ using steps in $\{\mathbf{0}, \pm\epsilon_1, \dots, \pm\epsilon_w\}$, where ϵ_i is the i-th standard basis vector.

Our first result concerns standard Young tableaux with an odd bound on the width and a given number of odd-length columns. Note that if $n \not\equiv k \mod 2$, then $|\mathrm{SYT}_{n,2w+1}[k]| = 0$. By taking the coefficients of $x_1 x_2 \cdots x_n$ on both sides of (7.18) and (7.19), we obtain the following theorem, which was first proved by the fourth author in [21, Corollary 5.5(3)] by considering the n-th tensor power of the natural representation of the Lie algebra $\mathfrak{so}_{2w+1}$.

Theorem 8.3. For nonnegative integers n, w and k such that $0 \leqslant k \leqslant 2w+1$ and $n \equiv k \mod 2$, we have

$$|\operatorname{SYT}_{n,2w+1}[k]| = |\operatorname{VT}_n^{>}(w; (1^t, 0^{w-t}) \to (0^w))|,$$

where $t = k$ if $k \leqslant w$ and $t = 2w+1-k$ if $k > w$.

Proof. First, suppose $0 \leqslant k \leqslant w$. Extracting the coefficient of $x_1 \cdots x_n$ on both sides of (7.18), we obtain that $|\operatorname{SYT}_{n,2w+1}[k]|$ is equal to the number of pairs $(T, S) \in \operatorname{MUD}_n^o(w; (1^k, 0^{w-k}) \to (0^w))$ such that $\omega(T)\omega(S) = x_1 x_2 \cdots x_n$ and $|S|$ is even. By Definition 7.9, this is equal to the number of the pairs (T, S) of a w-up-down tableau $T = (T_0, T_1, \dots, T_{2n})$ and a set $S \subseteq [n]$ such that, for each $j \in [n]$, one of the following two conditions holds:

- $j \notin S$, T_{2j-2} and T_{2j} differ by one cell, and T_{2j-1} is the larger partition between T_{2j-2} and T_{2j};
- $j \in S$, $T_{2j-2} = T_{2j-1} = T_{2j}$, and $\ell(T_{2j-1}) = w$.

Since the odd-indexed partitions T_{2j-1} are redundant and $j \in S$ if and only if $T_{2j-2} = T_{2j}$, we can identify these pairs (T, S) with the walks $(T_0, T_2, \dots, T_{2n}) \in \operatorname{VT}_n(w; (1^k, 0^{w-k}) \to (0^w))$ with the property that a zero step can only occur when $x_w > 0$. This shows that $|\operatorname{SYT}_{n,2w+1}[k]| = |\operatorname{VT}_n^{>}(w; (1^k, 0^{w-k}) \to (0^w)|$. The other case $w+1 \leqslant k \leqslant 2w+1$ can be handled similarly using (7.19). $\square$

Remark 8.4. It is easy to see that $|\operatorname{VT}_n^{>}(1; \mathbf{0} \to \mathbf{0})|$ is equal to the number of Motzkin paths of length n having no horizontal step on the x-axis, which is called *Riordan number*. The Riordan number also counts the number of noncrossing (respectively nonnesting) partitions of $[n]$ without singletons; see A005043. In general, $|\operatorname{VT}_n^{>}(w; \mathbf{0} \to \mathbf{0})|$ equals the multiplicity of the trivial representation in the n-th tensor power of the natural representation of $\mathfrak{so}_{2w+1}$. See [21].

Now we turn to numbers of standard Young tableaux with an even bound on the width and a given number of odd-length columns. On the other side of the identities to come we need to consider *marked* vacillating tableaux.

Definition 8.5. Let $\operatorname{MVT}_n(w; \mu \to \nu)$ denote the set of pairs (T, S) of $T \in \operatorname{VT}_n(w; \mu \to \nu)$ and $S \subseteq [n]$. We call each element $(T, S) \in \operatorname{MVT}_n(w; \mu \to \nu)$ a *marked vacillating tableau.*

Definition 8.6. Let $\operatorname{MVT}_n^*(w; \mu \to \nu)$ denote the set of elements $(T, S) \in \operatorname{MVT}_n(w; \mu \to \nu)$ such that T has no zero step and, if $j \in S$, then the j-th step of T is ϵ_w whose starting point is on the hyperplane $x_w = 0$.

Let $\operatorname{MVT}_n^0(w; \mu \to \nu)$ denote the set of elements $(T, S) \in \operatorname{MVT}_n^*(w; \mu \to \nu)$ satisfying one of the following conditions:

- T has no ϵ_w steps, or

- T has at least one ϵ_w step and $j \notin S$, where the j-th step of T is the first ϵ_w step in T.

Let $\mathrm{MVT}_n^1(w;\mu \to \nu)$ denote the set of elements $(T,S) \in \mathrm{MVT}_n^*(w;\mu \to \nu)$ such that T has at least one ϵ_w step and $j \in S$, where the j-th step of T is the first ϵ_w step in T.

Note that $\mathrm{MVT}_n^*(w;\mu \to \nu) = \mathrm{MVT}_n^0(w;\mu \to \nu) \sqcup \mathrm{MVT}_n^1(w;\mu \to \nu)$.

Here is our result on equality of numbers of standard Young tableaux with an even bound on the number of columns and a fixed number of odd-length columns and numbers of marked vacillating tableaux. Note that, if $n \not\equiv k \mod 2$, then $|\,\mathrm{SYT}_{n,2w}[k]| = 0$. The third equation of (1) and the equality in (2) below already appeared in [21, Corollary 5.5 (4)].

Theorem 8.7. Let n, w, and k be nonnegative integers such that $0 \leqslant k \leqslant w$.

1. For $0 \leqslant k \leqslant w-1$, we have

$$\begin{aligned} |\,\mathrm{SYT}_{n,2w}[k]| &= |\,\mathrm{MVT}_n^0(w;(1^k,0^{w-k}) \to (0^w))|, \\ |\,\mathrm{SYT}_{n,2w}[2w-k]| &= |\,\mathrm{MVT}_n^1(w;(1^k,0^{w-k}) \to (0^w))|, \\ |\,\mathrm{SYT}_{n,2w}[k]| + |\,\mathrm{SYT}_{n,2w}[2w-k]| &= |\,\mathrm{MVT}_n^*(w;(1^k,0^{w-k}) \to (0^w))|. \end{aligned}$$

2. For $k = w$, we have $|\,\mathrm{SYT}_{n,2w}[w]| = |\,\mathrm{MVT}_n^*(w;(1^w) \to (0^w))|$.

Proof. This can be proved in a similar manner as Theorem 8.3 by extracting the coefficients of $x_1x_2\cdots x_n$ in the equations in Theorem 7.15, except that in the present case the elements $j \in S$ do not affect the weights x_j. □

Remark 8.8. It is easy to see that $|\,\mathrm{MVT}_{2n}^*(1;\mathbf{0} \to \mathbf{0})|$ is equal to the number of Dyck paths of semilength n with the property that each up step starting on the x-axis is allowed to be marked. The number of these marked Dyck paths is given by the central binomial coefficients; see A000984. Furthermore, we have $|\,\mathrm{MVT}_{2n}^*(2;\mathbf{0} \to \mathbf{0})| = (\mathrm{Cat}(n))^2$, where $\mathrm{Cat}(n)$ is the n-th Catalan number; see A001246. This follows from the following three claims:

(1) $|\,\mathrm{MVT}_{2n}^*(w;\mathbf{0} \to \mathbf{0})|$ is the multiplicity of the trivial representation in the $(2n)$-th tensor power of the natural representation of $\mathfrak{so}_{2w}$.

(2) The Lie algebra $\mathfrak{so}_4$ is isomorphic to the direct sum $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, and the natural representation of $\mathfrak{so}_4$ corresponds to the outer tensor product of the natural representations of $\mathfrak{sl}_2$.

(3) The multiplicity of the trivial representation in the $(2n)$-th tensor power of the natural representation of $\mathfrak{sl}_2$ is equal to the Catalan number $\mathrm{Cat}(n)$.

9 Final questions

For $w \geqslant 1$, we have the following result; see [19, Example 2 on page 423] and also [11, Eq. (10.5)].

Proposition 9.1. The number $|\operatorname{SYT}_{n,w}[0]|$ is the expected value of $\operatorname{trace}(O)^n$, where O is a $w \times w$ orthogonal matrix randomly selected according to the Haar measure.

The result of Proposition 9.1 raises the following questions.

Question 9.2. Is there an interpretation for $|\operatorname{SYT}_{n,w}[k]|$ using orthogonal matrices for any k?

Further questions that our results in Section 7 (and in Section 8) naturally suggest are the following.

Question 9.3. Are there bijective proofs for the identities in Theorems 7.7, 7.11, and 7.15? Are there bijective proofs for the identities in Theorems 8.3 and 8.7?

There are strong indications that an approach using Fomin's growth diagrams (cf. [17]) should be successful. As for evidence, we point out that Theorem 5 in [17] gives a bijection between semistandard Young tableaux with an even bound on the number of *rows* and a fixed number of odd-length columns and certain down-up tableaux. We are convinced that this does produce a bijective proof of Goulden's identity (1.5), however after application of the symmetric function involution ω that maps elementary symmetric functions to complete homogeneous symmetric functions (which is in fact Goulden's original formulation of the identity). Furthermore, we believe that, by modifying the proof in [17] so that instead of the first and fourth variation of the Robinson–Schensted–Knuth algorithm we would use the second and third variation of that algorithm in terms of growth diagrams (we refer the reader to [17, proof of Theorem 5] and [16, Sec. 4]), one would obtain a bijective proof of Theorem 7.5.

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