The log-convexity of the n-th root sequence of the distinct partition function

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Abstract

Let $q(n)$ denote the total number of partitions of n into distinct parts and Δ be the difference operator with respect to n. We prove that for any real number α, there exists an integer $n(\alpha)$ such that the sequence $\{\sqrt[n]{q(n)/n^{\alpha}}\}_{n\geqslant n(\alpha)}$ is log-convex by obtaining a lower bound for $\triangle^2 \log \sqrt[n-1]{q(n-1)/(n-1)^{\alpha}}$, leading to a proof of the conjecture of Sun on the log-convexity of $\{\sqrt[n]{q(n)}\}_{n\geqslant 46}$. Moreover, we establish an inequality on the ratio $\sqrt[n-1]{q(n-1)}/\sqrt[n]{q(n)}$ by finding an upper bound of $\Delta^2 \log \sqrt[n-1]{q(n-1)}$. In the end, we prove the log-balancedness of the sequences $\{\sqrt[n]{q(n)}\}_{n\geqslant 46}$ and $\{\sqrt[n]{q(n)}/n\}_{n\geqslant 46}$, which can be extended to imply that the sequence $\{\sqrt[n]{q(n)}/n^{\alpha}\}_{n\geqslant N(\alpha)}$ is log-balanced for any nonnegative real number α.

Mathematics Subject Classifications: 05A20, 05A10

1 Introduction

The objective of this paper is to explore the log-convexity and its generalizations for the n-th root sequence of distinct partition function $q(n)$. Recall that a sequence $\{\alpha_n\}_{n\geqslant 0}$ of real numbers is said to be log-convex (resp. log-concave) if for $n \geqslant 1$, $\alpha_n^2 - \alpha_{n-1}\alpha_{n+1} \leqslant 0$ (resp. $\alpha_n^2 - \alpha_{n-1}\alpha_{n+1} \geqslant 0$).

Došlić [6] introduced that a log-convex sequence $\{\alpha_n\}_{n\geqslant 0}$ of real numbers is called log-balanced if $\{\alpha_n/n!\}_{n\geqslant 0}$ is log-concave. This implies that

$$\frac{\alpha_n}{n\alpha_{n-1}} \geqslant \frac{\alpha_{n+1}}{(n+1)\alpha_n}, n \geqslant 1.$$

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Motivated by Firoozbakht's conjecture [14] of the nth prime number, Sun [17] pioneered the investigation into the log-convexity properties related to the partition function $p(n)$, a series of researches that were subsequently resolved by Chen and Zheng [2]. Mukherjee [12] obtained the sequence $\left\{\sqrt[n]{\bar{p}(n)}\right\}_{n\geqslant 4}$ is log-convex. Xia and Zhang [19] proved $\left\{\sqrt[n]{a_n}\right\}_{n\geqslant 2}$ and $\left\{\sqrt[n]{b_n}\right\}_{n\geqslant 1}$ are log-concave conjectured by Sun [18] where

$$a_n := \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{4k^2-1}\binom{n-1}{k}^2\binom{n+k}{k}^2, \qquad b_n := \frac{1}{n^3}\sum_{k=0}^{n-1}(3k^2+3k+1)\binom{n-1}{k}^2\binom{n+k}{k}^2.$$

Hou and Li [9] established the criteria for the asymptotic ratio log-convexity of P-recursive sequences $\left\{\sqrt[n]{s_n}\right\}_{n\geqslant N}$, a P-recursive sequence [15] of order d satisfies a recurrence relation of the form

$$p_0(n)s_n + p_1(n)s_{n+1} + \cdots + p_d(n)s_{n+d} = 0.$$

Guo and Zhang [8] proved that the sequence $\{\sqrt[n]{spt(n)/n^{\alpha}}\}_{n\geqslant n(\alpha)}$ is log-convex for any real number α.

In [17], Sun also proposed the strict increasing conjecture regarding $q(n)$. He made the following conjecture:

Conjecture 1.1 ([17], Conjecture 2.12). The sequence $\{\sqrt[n+1]{q(n+1)}/\sqrt[n]{q(n)}\}_{n\geqslant 45}$ is strictly increasing.

The primary objective of this paper is to prove this conjecture. To align with the structure of this paper, we present its equivalent form, which is also one of our main results. Let $q(n)$ denote the number of partitions of n with distinct parts. It is known from Euler's partition theorem that $q(n)$ also counts the number of partitions of n with odd parts, see Andrews [1]. For example, there are eight partitions of 9 with distinct parts: $(9), (8,1), (7,2), (6,3), (6,2,1), (5,4), (5,3,1), (4,3,2)$.

Our study is based on asymptotic formula about $q(n)$. A considerable number of researches have been dedicated to the development of asymptotic formulas for $q(n)$ [5, 10, 11]. Here we mainly consider the most recent asymptotic formulation. Applying Chern's [3] asymptotic formula, Dong and Ji [[5], Theorem 1.2] obtained an effective bound for the error term of the correponding formula. For $v(n) \geqslant 21$, or equivalently, $n \geqslant 135$,

$$q(n) = \frac{\sqrt{2}\pi^2}{12v(n)}I_1(v(n)) + R(n), \qquad v(n) := \frac{\pi\sqrt{24n+1}}{6\sqrt{2}},$$

and

$$|R(n)| \leqslant \frac{\sqrt{3}\pi^{\frac{3}{2}}}{6v(n)^{\frac{1}{2}}}e^{\frac{v(n)}{3}},$$

where $I_1(s)$ is the first order modified Bessel function of first kind, defined as

$$I_1(s) = \frac{s}{\pi}\int_{-1}^{1}(1-t^2)^{\frac{1}{2}}e^{st}dt.$$

It is well-established that $q(n)$ possesses many arithmetic properties. Craig and Pun [4] showed that $q(n)$ satisfies the order d Turán inequalities for sufficiently large n by employing a result of Griffin, Ono, Rolen, and Zagier [[7], Theorem 3, Corollary 4]. Dong and Ji [5] proved that $\{q(n)\}_{n\geqslant 33}$ is log-concave and $\{q(n)\}_{n\geqslant 121}$ satisfies the higher order Turán inequality, conjectured by Craig and Pun [4]. Mukherjee, Zhang and Zhong [13] settled a conjecture of Dong and Ji on inequalities for $q(n)$ related to invariants of binary quartic forms. The properties and studies serve as a foundation for further exploration about $q(n)$. Our goal is to derive a generic inequality for $q(n)$ in the present context, stated in Theorem 1.2, which at once implies some special cases for $q(n)$, presented in Theorem 1.3-1.5.

Additionally, by definition, the log-balancedness implies the log-convexity [6]. It is also known that the quotient sequence of a log-balanced sequence does not grow too fast and it plays an important role in obtaining the growth rate of a sequence [16]. This is because the successive ratios of the quotient sequence are controlled by the factor $n/n+1$ [6], and consequently its growth rate is at most linear. This property is crucial for obtaining precise asymptotic estimates of the sequence and for analyzing higher-order differences. So it is very natural to consider the log-balancedness of the sequence $\left\{\sqrt[n]{q(n)}\right\}_{n\geqslant 46}$ with the help of theoretical foundations and the results obtained above, presented in Theorems1.7-1.9.

We hereby present the main results of this article as follows.

Theorem 1.2. *For any real number α,*

$$n(\alpha) = max\left\{\left[\left(\frac{800}{3\alpha}\right)\right] + 2, [(2\alpha+4)^4] + 2, 5505\right\}.$$

Then for $n \geqslant n(\alpha)$, the sequence $\{\sqrt[n]{q(n)/n^{\alpha}}\}$ is log-convex.

With the above theoretical foundations and computational results, it is imperative to deliberate upon two particular cases.

Theorem 1.3. *The sequence $\left\{\sqrt[n]{q(n)}\right\}_{n\geqslant 46}$ is log-convex.*

Theorem 1.4. *The sequence $\left\{\sqrt[n]{q(n)/n}\right\}_{n\geqslant 121}$ is log-convex.*

We also establish the following inequality on the ratio $\sqrt[n-1]{q(n-1)}/\sqrt[n]{q(n)}$ through the above calculations.

Theorem 1.5. *For $n \geqslant 12$, we have*

$$\frac{\sqrt[n]{q(n)}}{\sqrt[n+1]{q(n+1)}}\left(1+\frac{\sqrt{3}\pi}{4n^{5/2}}\right) > \frac{\sqrt[n-1]{q(n-1)}}{\sqrt[n]{q(n)}}. \tag{1.1}$$

By bounding $\triangle^2 \log \sqrt[n]{q(n)}$, we derive the following limit of $n^{5/2} \triangle^2 \log \sqrt[n]{q(n)}$:

Theorem 1.6.

$$\lim_{n\to+\infty} n^{5/2}\Delta^2 \log \sqrt[n]{q(n)} = \frac{\sqrt{3}\pi}{4}. \tag{1.2}$$

From the above relation 1.2, it can be seen that the coefficient $\frac{\sqrt{3}\pi}{4}$ in 1.1 is the best possible.

By use of the above theorems and the relation 1.1, it is straight to obtain the log-balancedness of the sequences $\left\{\sqrt[n]{q(n)}\right\}_{n\geqslant 46}$ and $\left\{\sqrt[n]{q(n)}/n\right\}_{n\geqslant 46}$.

Theorem 1.7. *The sequence* $\left\{\sqrt[n]{q(n)}\right\}_{n\geqslant 46}$ *is log-balanced.*

Theorem 1.8. *The sequence* $\left\{\sqrt[n]{q(n)}/n\right\}_{n\geqslant 46}$ *is log-balanced.*

Theorem 1.9. *The sequence* $\left\{\sqrt[n]{q(n)}/n^{\alpha}\right\}_{n\geqslant N(\alpha)}$ *is log-balanced for any nonnegative real number* α, *where* $N(\alpha) := \max\{4, 4\alpha\}$.

An outline of this paper is as follows. We devote Sec.2 to a discussion of the log-convexity of $r_\alpha(n) := \{\sqrt[n]{q(n)/n^{\alpha}}\}_{n\geqslant n(\alpha)}$ by proving the positivity of the lower bound for $\Delta^2 \log r_\alpha(n-1)$. In Sec.3, we establish an inequality on the ratio $\sqrt[n-1]{q(n-1)}/\sqrt[n]{q(n)}$ by finding an upper bound of $\Delta^2 \log \sqrt[n-1]{q(n-1)}$. In Sec. 4, we prove the log-balancedness of the sequences $\left\{\sqrt[n]{q(n)}\right\}_{n\geqslant 46}$ and $\left\{\sqrt[n]{q(n)}/n\right\}_{n\geqslant 46}$. Furthermore, we can conclude that the sequence $\left\{\sqrt[n]{q(n)}/n^{\alpha}\right\}_{n\geqslant N(\alpha)}$ is log-balanced for any nonnegative real number α.

2 The Log-convexity of sequences

In this section, we give a proof of Theorem 1.2 by proving the positivity of the lower bound for $\Delta^2 \log r_\alpha(n-1)$. Simultaneously, we state the log-convexity of $\{\sqrt[n]{q(n)}\}_{n\geqslant 46}$ and $\{\sqrt[n]{q(n)/n}\}_{n\geqslant 121}$, which completes the proofs of Theorem 1.3 and Theorem 1.4.

We shall provide a precise formulation regarding the expression of $q(n)$ at first. We can express $q(n)$ as

$$q(n) = T(n) + R(n),$$

where

$$T(n) := \frac{\sqrt{2}\pi^2}{12v(n)} I_1(v(n)), \tag{2.1}$$

$$v(n) := \frac{\pi\sqrt{24n+1}}{6\sqrt{2}},$$

$$|R(n)| \leqslant \frac{\sqrt{3}\pi^{\frac{3}{2}}}{6v(n)^{\frac{1}{2}}} e^{\frac{v(n)}{3}},$$

$$I_1(s) := \frac{s}{\pi}\int_{-1}^{1}(1-t^2)^{\frac{1}{2}} e^{st} dt.$$

Denote Δ be the difference operator respect to n, for $n \geqslant 2$ we have $\Delta a(n) := a(n) - a(n-1)$. The log-convexity of the sequence $\{r_\alpha(n)\}$ can be restated as the following relation for:

$$\log r_\alpha(n+1) + \log r_\alpha(n-1) - 2\log r_\alpha(n) > 0,$$

which is equivalent to,

$$\Delta^2 \log r_\alpha(n-1) > 0.$$

Based on the decomposition for $q(n)$, $\Delta^2 \log r_\alpha(n-1)$ can be equivalently expressed by the following relation for:

$$\Delta^2 \log r_\alpha(n-1) = \Delta^2 \frac{1}{n-1} \log q(n-1) - \alpha\Delta^2 \frac{1}{n-1} \log(n-1), \tag{2.2}$$

where

$$\Delta^2 \frac{1}{n-1} \log q(n-1) = \Delta^2 \frac{1}{n-1} \log T(n-1) + \Delta^2 \frac{1}{n-1} E(n-1), \tag{2.3}$$

$$E(n-1) = \log\left(1 + \frac{R(n-1)}{T(n-1)}\right). \tag{2.4}$$

The following lemma serves as a pivotal tool in the derivation of both the upper and lower bounds.

Lemma 2.1. [[2], Lemma 2.1] Suppose $f(x)$ has a continuous second derivative for $x \in [n-1, n+1]$. Then there exists $c \in [n-1, n+1]$ such that

$$\Delta^2 f(n-1) = f(n+1) + f(n-1) - 2f(n) = f''(c). \tag{2.5}$$

If $f(x)$ has an increasing second derivative, then

$$f''(n-1) < \Delta^2 f(n-1) < f''(n+1). \tag{2.6}$$

Conversely, if $f(x)$ has a decreasing second derivative, then

$$f''(n+1) < \Delta^2 f(n-1) < f''(n-1). \tag{2.7}$$

The following theorem establishes specific inequality related to the first order modified Bessel function $I_1(s)$ of the first kind.

Theorem 2.2. *For $s \geqslant 26$,*

$$\frac{e^s}{\sqrt{2\pi s}}(1-\frac{1}{2s}) \leqslant I_1(s) \leqslant \frac{e^s}{\sqrt{2\pi s}}(1-\frac{1}{4s}). \tag{2.8}$$

Proof. Dong and Ji[[5], Lemma 2.2] obtained an upper bound and a lower bound for $I_1(s)$. However, considering our practical situation, we need to establish new upper and lower bounds based on it. Based on their inequality, we provide the following proof.

For $s \geqslant 26$,

$$\begin{aligned}&\frac{1}{2s}-\left(\frac{3}{8s}+\frac{15}{128s^2}+\frac{105}{1024s^3}+\frac{4725}{32768s^4}+\frac{72765}{262144s^5}+\frac{31}{s^6}\right)\\&=\frac{32768s^5-30720s^4-26880s^3-37800s^2-72765s-8126464}{262144s^6}>0.\end{aligned}$$

In our equation, the condition for the inequality to hold is $n \geqslant 206$. So we have completed the proof of the lower bound. The upper bound can be completed by the method analogous to the above. □

The subsequent theorem employs lemma 2.1 to rigorously determine the upper and lower bounds for $\Delta^2 \frac{1}{n-1} \log T(n-1)$.

Theorem 2.3. *Let*

$$B_1(n) = \frac{\sqrt{3}\pi}{4(n+1)^{5/2}} - \frac{2\log(\sqrt{2}\pi\sqrt{24n-23})}{(n-1)^3} - \frac{2\log(\frac{\pi\sqrt{6}(48n-46)^{1/4}}{6})}{(n-1)^3}, \tag{2.9}$$

$$B_2(n) = \frac{\sqrt{3}\pi}{4(n-1)^{5/2}} - \frac{2\log(\sqrt{2}\pi\sqrt{24n+25})}{(n+1)^3} - \frac{2\log(\frac{\pi\sqrt{6}(48n+50)^{1/4}}{6})}{(n+1)^3} + \frac{9}{(n-1)^3}, \tag{2.10}$$

we have

$$B_1(n) < \Delta^2 \frac{1}{n-1} \log T(n-1) < B_2(n).$$

Proof. By the definition of $T(n)$, we may write:

$$\frac{\log T(n)}{n} = \sum_{i=1}^{3} g_i(n),$$

where

$$g_1(n) = \frac{\log(\sqrt{2}\pi^2)}{n},\ g_2(n) = -\frac{\log\left(\sqrt{2}\pi\sqrt{24n+1}\right)}{n},\ g_3(n) = \frac{\log(I_1(v(n))}{n}.$$

Thus

$$\Delta^2 \frac{1}{n-1} \log T(n-1) = \sum_{i=1}^{3} \Delta^2 g_i(n-1).$$

Due to the complexity of $g_3(n)$, it is essential to study it in a more detailed discussion. Since $\log x$ is a monotonically increasing function when $x > 0$, by the use of 2.8, we can obtain that

$$\log(I_1(s)) \geqslant s + \log\left(1 - \frac{1}{2s}\right) - \log(\sqrt{2\pi s}).$$

So we divide $g_3(n)$ into the following three terms:

$$g_{31}(n) = \frac{\pi\sqrt{24n+1}}{6\sqrt{2}n},$$

$$g_{32}(n) = \frac{\log\left(1 - \frac{3\sqrt{2}}{\pi\sqrt{24n+1}}\right)}{n},$$

$$g_{33}(n) = -\frac{\log\left(\pi\sqrt{\frac{\sqrt{24n+1}}{3\sqrt{2}}}\right)}{n}.$$

Summarize what has been said above, we write:

$$\frac{\log T(n)}{n} = g_1(n) + g_2(n) + g_{31}(n) + g_{32}(n) + g_{33}(n).$$

It can be easily checked that for $n \geqslant 206$,

$$g_1^{'''}(n) = \frac{-6\log(\sqrt{2}\pi^2)}{n^4} < 0,$$

$$\begin{aligned} g_2^{'''}(n) &= -\frac{13824}{n(24n+1)^3} - \frac{864}{n^2(24n+1)^2} - \frac{72}{n^3(24n+1)} + \frac{6\log(\pi\sqrt{2}\sqrt{24n+1})}{n^4} \\ &= \frac{6\Big[-12n(1+60n+1056n^2) + (24n+1)^3\log(\sqrt{2}\pi\sqrt{24n+1})\Big]}{n^4(24n+1)^3} > 0, \end{aligned}$$

$$\begin{aligned} g_{31}^{'''}(n) &= \frac{432\sqrt{2}\pi}{n(24n+1)^{5/2}} + \frac{36\sqrt{2}\pi}{n^2(24n+1)^{3/2}} + \frac{6\sqrt{2}\pi}{n^3\sqrt{24n+1}} - \frac{\sqrt{2}\pi\sqrt{24n+1}}{2n^4} \\ &= -\frac{(1+60n+1080n^2+4320n^3)\pi}{\sqrt{2}n^4(1+24n)^{5/2}} < 0, \end{aligned}$$

$$g_{32}^{'''}(n) = \frac{77760\sqrt{2}}{\pi(24n+1)^{7/2}(1-\frac{3\sqrt{2}}{\pi\sqrt{24n+1}})n} + \frac{279936}{\pi^2(24n+1)^4(1-\frac{3\sqrt{2}}{\pi\sqrt{24n+1}})^2 n}$$
$$+ \frac{3888\sqrt{2}}{\pi(24n+1)^{5/2}(1-\frac{3\sqrt{2}}{\pi\sqrt{24n+1}})n^2} + \frac{186624\sqrt{2}}{\pi^3(24n+1)^{9/2}(1-\frac{3\sqrt{2}}{\pi\sqrt{24n+1}})^3 n}$$
$$+ \frac{7776}{\pi^2(24n+1)^3(1-\frac{3\sqrt{2}}{\pi\sqrt{24n+1}})^2 n^2} + \frac{216\sqrt{2}}{\pi(24n+1)^{3/2}(1-\frac{3\sqrt{2}}{\pi\sqrt{24n+1}})n^3}$$
$$- \frac{6\log\left(1-\frac{3\sqrt{2}}{\pi\sqrt{24n+1}}\right)}{n^4},$$

using the fact $6\log(1-\frac{3\sqrt{2}}{\pi\sqrt{24n+1}}) < 0$, we can conclude that $g_{32}^{'''}(n) > 0$,

$$g_{33}^{'''}(n) = -\frac{55296}{(48n+2)^3 n} - \frac{1728}{(48n+2)^2 n^2} - \frac{72}{(48n+2)n^3} + \frac{6\log\left(\frac{\pi\sqrt{6}(48n+2)^{1/4}}{6}\right)}{n^4} > 0.$$

Using Lemma 2.1, we can get a lower bound and an upper bound for $\Delta^2 g_i(n-1)$ in terms of $g_i^{''}(n-1)$ and $g_i^{''}(n+1)$:

$$g_1^{''}(n+1) < \Delta^2 g_1(n-1) < g_1^{''}(n-1),$$
$$g_2^{''}(n-1) < \Delta^2 g_2(n-1) < g_2^{''}(n+1),$$
$$g_{31}^{''}(n+1) < \Delta^2 g_{31}(n-1) < g_{31}^{''}(n-1),$$
$$g_{32}^{''}(n-1) < \Delta^2 g_{32}(n-1) < g_{32}^{''}(n+1),$$
$$g_{33}^{''}(n-1) < \Delta^2 g_{33}(n-1) < g_{33}^{''}(n+1).$$

Using the above inequalities, we find that

$$\Delta^2 \frac{1}{n-1}\log T(n-1) > g_1^{''}(n+1) + g_2^{''}(n-1) + g_{31}^{''}(n+1) + g_{32}^{''}(n-1) + g_{33}^{''}(n-1) \quad (2.11)$$

and

$$\Delta^2 \frac{1}{n-1}\log T(n-1) < g_1^{''}(n-1) + g_2^{''}(n+1) + g_{31}^{''}(n-1) + g_{32}^{''}(n+1) + g_{33}^{''}(n+1), \quad (2.12)$$

where

$$g_1^{''}(n) = \frac{2\log\left(\sqrt{2}\pi^2\right)}{n^3}, \quad (2.13)$$

$$g_2^{''}(n) = \frac{288}{n(24n+1)^2} + \frac{24}{(24n+1)n^2} - \frac{2\log\left(\sqrt{2}\pi\sqrt{24n+1}\right)}{n^3}, \quad (2.14)$$

$$g_{31}^{''}(n) = -\frac{12\pi\sqrt{2}}{n(24n+1)^{3/2}} - \frac{2\pi\sqrt{2}}{n^2\sqrt{24n+1}} + \frac{\pi\sqrt{24n+1}\sqrt{2}}{6n^3}, \quad (2.15)$$

$$g_{32}^{''}(n) = -\frac{1296\sqrt{2}}{\pi n(24n+1)^{5/2}(1-\frac{3\sqrt{2}}{\pi\sqrt{24n+1}})} - \frac{2592}{\pi^2 n(24n+1)^3(1-\frac{3\sqrt{2}}{\pi\sqrt{24n+1}})^2}$$
$$- \frac{72\sqrt{2}}{\pi n^2(24n+1)^{3/2}(1-\frac{3\sqrt{2}}{\pi\sqrt{24n+1}})} + \frac{2\log\left(1-\frac{3\sqrt{2}}{\pi\sqrt{24n+1}}\right)}{n^3}, \tag{2.16}$$

$$g_{33}^{''}(n) = \frac{576}{(48n+2)^2 n} + \frac{24}{(48n+2)n^2} - \frac{2\log\left(\frac{\pi\sqrt{6}(48n+2)^{1/4}}{6}\right)}{n^3}. \tag{2.17}$$

Utilizing the results established above, we get a lower bound for $\Delta^2 \frac{1}{n-1}\log T(n-1)$ at first.

For $n \geqslant 206$, we can obtain that

$$\begin{aligned} g_{31}^{''}(n+1) &= -\frac{12\pi\sqrt{2}}{(24n+25)^{3/2}(n+1)} - \frac{2\pi\sqrt{2}}{\sqrt{24n+25}(n+1)^2} + \frac{\pi\sqrt{24n+25}\sqrt{2}}{6(n+1)^3} \\ &> -\frac{12\pi\sqrt{2}}{(24n+24)^{3/2}(n+1)} - \frac{2\pi\sqrt{2}}{\sqrt{24n+24}(n+1)^2} + \frac{\pi\sqrt{24n+24}\sqrt{2}}{6(n+1)^3} \\ &> \frac{\sqrt{3}\pi}{4(n+1)^{5/2}}. \end{aligned} \tag{2.18}$$

This lower bound provides a sharper estimate and enabling us to establish a tighter error term in the asymptotic expansion of $g_{31}^{''}(n+1)$. This plays an important role in the proof of Theorem 2.3.

$$\begin{aligned} g_{32}^{''}(n-1) = &-\frac{1296\sqrt{2}}{\pi(24n-23)^{5/2}(1-\frac{3\sqrt{2}}{\pi\sqrt{24n-23}})(n-1)} \\ &- \frac{2592}{\pi^2(24n-23)^3(1-\frac{3\sqrt{2}}{\pi\sqrt{24n-23}})^2(n-1)} \\ &- \frac{72\sqrt{2}}{\pi(24n-23)^{3/2}(1-\frac{3\sqrt{2}}{\pi\sqrt{24n-23}})(n-1)^2} + \frac{2\log\left(1-\frac{3\sqrt{2}}{\pi\sqrt{24n-23}}\right)}{(n-1)^3} \\ &> -\frac{1}{(n-1)^3}, \end{aligned} \tag{2.19}$$

$$
\begin{aligned}
&g_2''(n-1)+g_{33}''(n-1)\\
=&\frac{288}{(24n-23)^2(n-1)}+\frac{24}{(24n-23)(n-1)^2}-\frac{2\log\left(\sqrt{2}\pi\sqrt{24n-23}\right)}{(n-1)^3}\\
&+\frac{576}{(48n-46)^2(n-1)}+\frac{24}{(48n-46)(n-1)^2}-\frac{2\log\left(\frac{\pi\sqrt{6}(48n-46)^{1/4}}{6}\right)}{(n-1)^3}\\
&>\frac{288}{(25n-25)^2(n-1)}+\frac{24}{(25n-25)(n-1)^2}-\frac{2\log\left(\sqrt{2}\pi\sqrt{24n-23}\right)}{(n-1)^3}\\
&+\frac{576}{(50n-50)^2(n-1)}+\frac{24}{(50n-50)(n-1)^2}-\frac{2\log\left(\frac{\pi\sqrt{6}(48n-46)^{1/4}}{6}\right)}{(n-1)^3}\\
&>\frac{2}{(n-1)^3}-\frac{2\log(\sqrt{2}\pi\sqrt{24n-23})}{(n-1)^3}-\frac{2\log\left(\frac{\pi\sqrt{6}(48n-46)^{1/4}}{6}\right)}{(n-1)^3}.
\end{aligned}
\tag{2.20}
$$

Now, it is clear from 2.13, 2.18, 2.19 and 2.20:

$$
g_1''(n+1)+g_2''(n-1)+g_3''(n+1)-B_1(n)>\frac{2\log(\sqrt{2}\pi^2)}{(n+1)^3}-\frac{1}{(n-1)^3}+\frac{2}{(n-1)^3}. \tag{2.21}
$$

Let $C(n)$ be the right hand side of the above inequality. Then $B_1(n)<\Delta^2\frac{1}{n-1}\log T(n-1)$ will be proved if we can obtain $C(n)>0$.

It is straightforward to show that $C(n)>0$. We can draw the conclusion that

$$
\Delta^2\frac{1}{n-1}\log T(n-1)>B_1(n).
$$

We now return to ascertain the upper bound for $\Delta^2\frac{1}{n-1}\log T(n-1)$, this result can be completed by the method analogous to that used above.

This part about the Bessel integral needs to be reconsidered when we solve for the upper bound. By the use of 2.8, we can conclude that

$$
\log(I_1(s))\leqslant s+\log\left(1-\frac{1}{4s}\right)-\log(\sqrt{2\pi s}).
$$

So we also divide $g_3(n)$ into the following three terms:

$$
g_1(n)=\frac{\log(\sqrt{2}\pi^2)}{n},\, g_2(n)=-\frac{\log\left(\sqrt{2}\pi\sqrt{24n+1}\right)}{n},
$$

$$
g_{31}^*(n)=\frac{\pi\sqrt{24n+1}}{6\sqrt{2}n},\ g_{32}^*(n)=\frac{\log\left(1-\frac{3\sqrt{2}}{2\pi\sqrt{24n+1}}\right)}{n},\ g_{33}^*(n)=-\frac{\log\left(\pi\sqrt{\frac{\sqrt{24n+1}}{3\sqrt{2}}}\right)}{n}.
$$

Thus

$$\Delta^2 \frac{1}{n-1} \log T(n-1) < g_1^{''}(n-1) + g_2^{''}(n+1) + g_{33}^{''*}(n-1) + g_{32}^{''*}(n+1) + g_{33}^{''*}(n+1). \quad (2.22)$$

We can easily get $g_{32}^{''*}(n+1) < 0$ for $n \geqslant 206$ imitating the above method. The analyses of the remaining items are presented below.

For $n \geqslant 206$, we can see that

$$\begin{aligned} g_{31}^{''*}(n-1) &= -\frac{12\pi\sqrt{2}}{(24n-23)^{3/2}(n-1)} - \frac{2\pi\sqrt{2}}{\sqrt{24n-23}(n-1)^2} + \frac{\pi\sqrt{24n-23}\sqrt{2}}{6(n-1)^3} \\ &< \frac{\sqrt{3}\pi}{4(n-1)^{5/2}} + \frac{\pi\sqrt{2}}{6(n-1)^3}. \end{aligned} \quad (2.23)$$

One can readily check that for $n \geqslant 206, -\frac{12\pi\sqrt{2}}{(24n-23)^{3/2}(n-1)} - \frac{2\pi\sqrt{2}}{\sqrt{24n-23}(n-1)^2} + \frac{\pi\sqrt{24n-23}\sqrt{2}}{6(n-1)^3} - \frac{\sqrt{3}\pi}{4(n-1)^{5/2}} - \frac{\pi\sqrt{2}}{6(n-1)^3} < 0$ holds. We can obtain that

$$\begin{aligned} & g_2''(n+1) + g_{33}^{''*}(n+1) \\ & < \frac{288}{(24n+24)^2(n+1)} + \frac{24}{(24n+24)(n+1)^2} - \frac{2\log\left(\sqrt{2}\pi\sqrt{24n+25}\right)}{(n+1)^3} \\ & + \frac{576}{(48n+48)^2(n+1)} + \frac{24}{(48n+48)(n+1)^2} - \frac{2\log\left(\frac{\pi\sqrt{6}(48n+50)^{1/4}}{6}\right)}{(n+1)^3} \\ & = \frac{9}{4(n+1)^3} - \frac{2\log\left(\sqrt{2}\pi\sqrt{24n+25}\right)}{(n+1)^3} - \frac{2\log\left(\frac{\pi\sqrt{6}(48n+50)^{1/4}}{6}\right)}{(n+1)^3}, \end{aligned} \quad (2.24)$$

$$\begin{aligned} & g_1''(n-1) + g_2''(n+1) + g_{31}^{''*}(n-1) + g_{32}^{''*}(n+1) + g_{33}^{''*}(n+1) \\ & < \frac{2\log(\sqrt{2}\pi^2)}{(n-1)^3} + \frac{\sqrt{3}\pi}{4(n-1)^{5/2}} + \frac{\pi\sqrt{2}}{6(n-1)^3} \\ & + \frac{9}{4(n+1)^3} - \frac{2\log(\sqrt{2}\pi\sqrt{24n+25})}{(n+1)^3} - \frac{2\log\left(\frac{\pi\sqrt{6}(48n+50)^{1/4}}{6}\right)}{(n+1)^3} \\ & < \frac{\sqrt{3}\pi}{4(n-1)^{5/2}} - \frac{2\log(\sqrt{2}\pi\sqrt{24n+25})}{(n+1)^3} - \frac{2\log\left(\frac{\pi\sqrt{6}(48n+50)^{1/4}}{6}\right)}{(n+1)^3} + \frac{9}{(n-1)^3}. \end{aligned} \quad (2.25)$$

This completes the proof. □

The following theorem gives an effective upper bound for the $|\Delta^2 \frac{1}{n-1} E(n-1)|$.

Theorem 2.4. *For $n \geqslant 206$,*

$$\left|\Delta^2 \frac{1}{n-1} E(n-1)\right| < 5e^{-\frac{\sqrt{2}\pi\sqrt{24n-23}}{18}}. \quad (2.26)$$

Proof. Setting $y(n) := \frac{R(n)}{T(n)}$, we find that for $n \geqslant 206$,

$$\begin{aligned}
&\Big|\Delta^2 \frac{1}{n-1} E(n-1)\Big| \\
&= \frac{1}{n+1}\log(1+y(n+1)) - \frac{2}{n}\log(1+y(n)) + \frac{1}{n-1}\log(1+y(n-1)) \\
&\leqslant \frac{1}{n+1}\Big|\log(1+y(n+1))\Big| + \frac{2}{n}\Big|\log(1+y(n))\Big| + \frac{1}{n-1}\Big|\log(1+y(n-1))\Big|,
\end{aligned} \tag{2.27}$$

Using the lower bound of $I_1(s)$, we have

$$\begin{aligned}
|y(n)| < \left|\frac{\sqrt{3}\pi^{\frac{3}{2}}e^{\frac{v(n)}{3}}}{6v(n)^{\frac{1}{2}}\frac{\sqrt{2}\pi^2}{12v(n)}I_1(v(n))}\right| &< \left|\frac{\sqrt{3}\pi^{\frac{3}{2}}e^{\frac{v(n)}{3}}}{6v(n)^{\frac{1}{2}}\frac{\sqrt{2}\pi^2}{12v(n)}\frac{e^{v(n)}}{\sqrt{2\pi v(n)}}(1-\frac{1}{2v(n)})}\right| \\
&= \frac{\sqrt{3}\pi^2(24n+1)}{18\mathrm{e}^{\frac{\pi\sqrt{24n+1}\sqrt{2}}{18}}\left(\frac{\pi\sqrt{24n+1}\sqrt{2}}{6}-1\right)}.
\end{aligned}$$

Note that for $n \geqslant 206$,

$$|y(n)| < \frac{\sqrt{3}\pi^2(24n+1)}{18\mathrm{e}^{\frac{\pi\sqrt{24n+1}\sqrt{2}}{18}}\left(\frac{\pi\sqrt{24n+1}\sqrt{2}}{6}-1\right)} < \frac{n}{e^{\frac{\sqrt{2}\pi\sqrt{24n+1}}{18}}}. \tag{2.28}$$

For $n \geqslant 206$,

$$|y(n)| < \frac{\sqrt{3}\pi^2(24n+1)}{18\mathrm{e}^{\frac{\pi\sqrt{24n+1}\sqrt{2}}{18}}\left(\frac{\pi\sqrt{24n+1}\sqrt{2}}{6}-1\right)} < \frac{1}{5}. \tag{2.29}$$

Combining 2.28 and 2.29, it is clear that for $n \geqslant 206$,

$$|y(n)| < \frac{\sqrt{3}\pi^2(24n+1)}{18\mathrm{e}^{\frac{\pi\sqrt{24n+1}\sqrt{2}}{18}}\left(\frac{\pi\sqrt{24n+1}\sqrt{2}}{6}-1\right)} < \frac{n}{e^{\frac{\sqrt{2}\pi\sqrt{24n+1}}{18}}} < \frac{1}{5}.$$

It is a well-established fact that $\log(1+x) < x$ for $0 < x < 1$ and $-\log(1+x) < -x/(1+x)$ for $-1 < x < 0$. Thus, for $|x| < 1$,

$$|\log(1+x)| \leqslant \frac{|x|}{1-|x|}. \tag{2.30}$$

By 2.29 and 2.30, we can conclude that for $n \geqslant 206$,

$$|\log(1+y(n))| \leqslant \frac{|y(n)|}{1-|y(n)|} \leqslant \frac{5}{4}\Big|y(n)\Big|. \tag{2.31}$$

Applying 2.31 to 2.27, we obtain that for $n \geqslant 206$,

$$\begin{aligned}
&\Big|\Delta^2 \frac{1}{n-1} E(n-1)\Big| \\
&\leqslant \frac{5}{4}\left(\frac{\Big|y(n+1)\Big|}{n+1} + \frac{2\Big|y(n)\Big|}{n} + \frac{\Big|y(n-1)\Big|}{n-1}\right).
\end{aligned} \tag{2.32}$$

Plugging 2.28 into 2.32, we infer that for $n \geqslant 206$,

$$\begin{aligned}
&\Big|\Delta^2 \frac{1}{n-1} E(n-1)\Big| \\
&< \frac{5}{4}\left(\frac{1}{n+1}\frac{n+1}{e^{\frac{\sqrt{2}\pi\sqrt{24n+25}}{18}}} + \frac{2}{n}\frac{n}{e^{\frac{\sqrt{2}\pi\sqrt{24n+1}}{18}}} + \frac{1}{n-1}\frac{n-1}{e^{\frac{\sqrt{2}\pi\sqrt{24n-23}}{18}}}\right) \\
&= \frac{5}{4}\left(e^{-\frac{\sqrt{2}\pi\sqrt{24n+25}}{18}} + 2e^{-\frac{\sqrt{2}\pi\sqrt{24n+1}}{18}} + e^{-\frac{\sqrt{2}\pi\sqrt{24n-23}}{18}}\right).
\end{aligned}$$

But $\{e^{-\frac{\sqrt{2}\pi\sqrt{24n+1}}{18}}\}_{n\geqslant 206}$ is a decreasing sequence. So we obtain the upper bound,

$$\Big|\Delta^2 \frac{1}{n-1} E(n-1)\Big| < 5e^{-\frac{\sqrt{2}\pi\sqrt{24n-23}}{18}}.$$

It completes the proof. $\square$

The following theorem gives the bounds for $-\Delta^2 \frac{1}{n-1}\log(n-1)$.

Theorem 2.5. *For $n \geqslant 7$,*

$$-\frac{2\log(n-1)}{(n-1)^3} + \frac{3}{(n-1)^3} < -\Delta^2 \frac{1}{n-1}\log(n-1) < -\frac{2\log(n+1)}{(n+1)^3} + \frac{3}{(n+1)^3}.$$

Proof. We observe that, for $n \geqslant 7$,

$$(-\frac{\log n}{n})''' = -\frac{11}{n^4} + \frac{6\log n}{n^4} > 0.$$

Setting $f(n) = -\frac{\log n}{n}$ and applying Lemma 2.1, we obtain for $n \geqslant 7$,

$$-\Delta^2 \frac{1}{n-1}\log(n-1) > (-\frac{\log(n-1)}{n-1})'',$$

$$-\Delta^2 \frac{1}{n-1}\log(n-1) < (-\frac{\log(n+1)}{n+1})''.$$

We finish the proof.

When α is a positive real number, we can obtain the inequality for $n \geqslant 7$,

$$-\frac{2\alpha\log(n-1)}{(n-1)^3} + \frac{3\alpha}{(n-1)^3} < -\alpha\Delta^2 \frac{1}{n-1}\log(n-1) < -\frac{2\alpha\log(n+1)}{(n+1)^3} + \frac{3\alpha}{(n+1)^3}.$$

$\square$

With the aid of Theorems 2.3, 2.4 and 2.5, we are ready to prove the log-convexity of $\{r_{\alpha(n)}\}_{n\geqslant n(\alpha)}$.

Proof of Theorem 1.2. We now consider the case $\alpha \geqslant 0$.

Using the fact that for $x > 5504$, $\log x < x^{1/4}$, we deduce that for $n \geqslant 5505$,

$$\frac{2\log(\sqrt{2}\pi\sqrt{24n-23})}{(n-1)^3} < \frac{2\log(n-1)}{(n-1)^3} < \frac{2\sqrt[4]{n-1}}{(n-1)^3} = \frac{2}{(n-1)^{11/4}}, \tag{2.33}$$

$$\frac{2\log\left(\frac{\pi\sqrt{6}(48n-46)^{1/4}}{6}\right)}{(n-1)^3} < \frac{2\log(n-1)}{(n-1)^3} < \frac{2\sqrt[4]{n-1}}{(n-1)^3} = \frac{2}{(n-1)^{11/4}}, \tag{2.34}$$

$$\frac{2\log(n-1)}{(n-1)^3} < \frac{2(n-1)^{1/4}}{(n-1)^3} = \frac{2}{(n-1)^{11/4}}. \tag{2.35}$$

Since $e^x > x^8/720$ for $x > 15$, we see that for $n \geqslant 5505$,

$$e^{-\frac{\sqrt{2}\pi\sqrt{24n-23}}{18}} < e^{-\frac{\sqrt{2}\pi\sqrt{24n-24}}{18}} < \frac{720}{\left(\frac{\sqrt{2}\pi\sqrt{24n-24}}{18}\right)^8} < \frac{160}{(n-1)^4}. \tag{2.36}$$

We can conclude that

$$\begin{aligned}
&\Delta^2 \log r_\alpha(n-1)\\
&= \Delta^2 \frac{1}{n-1}\log q(n-1) - \alpha\Delta^2\frac{1}{n-1}\log(n-1)\\
&> \frac{\sqrt{3}\pi}{4(n+1)^{5/2}} - \frac{2\log\left(\sqrt{2}\pi\sqrt{24n-23}\right)}{(n-1)^3} - \frac{2\log\left(\frac{\pi\sqrt{6}(48n-46)^{1/4}}{6}\right)}{(n-1)^3}\\
&- 5e^{-\frac{\sqrt{2}\pi\sqrt{24n-23}}{18}} - \frac{2\alpha\log(n-1)}{(n-1)^3} + \frac{3\alpha}{(n-1)^3}.
\end{aligned} \tag{2.37}$$

By the inequality 2.36 and direct computation, it is evident that for $n \geqslant max\{[\frac{800}{3\alpha}] + 2, 5505\}$,

$$\frac{3\alpha}{(n-1)^3} - 5e^{-\frac{\sqrt{2}\pi\sqrt{24n-23}}{18}} > \frac{3\alpha}{(n-1)^3} - \frac{800}{(n-1)^4} > 0. \tag{2.38}$$

Combining 2.33 and 2.34, we can obtain that for $n \geqslant max\{[(2\alpha+4)^4] + 2, 5505\}$,

$$\begin{aligned}
&-\frac{2\log\left(\sqrt{2}\pi\sqrt{24n-23}\right)}{(n-1)^3} - \frac{2\log\left(\frac{\pi\sqrt{6}(48n-46)^{1/4}}{6}\right)}{(n-1)^3} - \frac{2\alpha\log(n-1)}{(n-1)^3}\\
&\qquad > -\frac{4}{(n-1)^{11/4}} - \frac{2\alpha}{(n-1)^{11/4}} > -\frac{1}{(n-1)^{5/2}}.
\end{aligned} \tag{2.39}$$

Let

$$n(\alpha) = max\left\{\left[\left(\frac{800}{3\alpha}\right)\right] + 2, [(2\alpha+4)^4] + 2, 5505\right\}.$$

It is immediately apparent that for $n \geqslant n(\alpha)$ by 2.37, 2.38 and 2.39 when $\alpha \geqslant 0$,

$$\Delta^2 \log r_\alpha(n-1) > \frac{\sqrt{3}\pi}{4(n+1)^{5/2}} - \frac{1}{(n-1)^{5/2}} > 0. \tag{2.40}$$

With the above theoretical foundations and computational results, it is imperative to deliberate upon two particular cases, which are poised to substantiate the proofs of Theorem 1.3 and Theorem 1.4. □

Proof of Theorem 1.3. For α=0, we have already shown that for $n \geqslant 5505$,

$$\begin{aligned}
&\Delta^2 \frac{1}{n-1} \log q(n-1) \\
&> \frac{\sqrt{3}\pi}{4(n+1)^{5/2}} - \frac{2\log\left(\sqrt{2}\pi\sqrt{24n-23}\right)}{(n-1)^3} - \frac{2\log\left(\frac{\pi\sqrt{6}(48n-46)^{1/4}}{6}\right)}{(n-1)^3} - 5e^{-\frac{\sqrt{2}\pi\sqrt{24n-23}}{18}} \\
&> \frac{\sqrt{3}\pi}{4(n+1)^{5/2}} - \frac{2}{(n-1)^{11/4}} - \frac{2}{(n-1)^{11/4}} - \frac{800}{(n-1)^4}.
\end{aligned}$$

It can be easily obtained that $\Delta^2 \frac{1}{n-1} \log q(n-1) > 0$ for $n \geqslant 5505$, verifying that $\Delta^2 \frac{1}{n-1} \log q(n-1) > 0$ for $46 \leqslant n \leqslant 5504$ completes the proof. □

Proof of Theorem 1.4: For α=1, we have already shown that holds for $n \geqslant 5505$.

Combining 2.33, 2.34, 2.35 and 2.36, we find that for $n \geqslant 5505$,

$$\begin{aligned}
&\Delta^2 \frac{1}{n-1} \log(q(n-1)/n-1) \\
&= \Delta^2 \frac{1}{n-1} \log q(n-1) - \Delta^2 \frac{1}{n-1} \log(n-1) \\
&> \frac{\sqrt{3}\pi}{4(n+1)^{5/2}} - \frac{2}{(n-1)^{11/4}} - \frac{2}{(n-1)^{11/4}} - \frac{800}{(n-1)^4} - \frac{2}{(n-1)^{11/4}} + \frac{3}{(n-1)^3} \\
&= \frac{\sqrt{3}\pi}{4(n+1)^{5/2}} - \frac{6}{(n-1)^{11/4}} - \frac{800}{(n-1)^4} + \frac{3}{(n-1)^3}.
\end{aligned}$$

It can be easily obtained that $\Delta^2 \frac{1}{n-1} \log(q(n-1)/n-1) > 0$ for $n \geqslant 5505$, verifying that $\Delta^2 \frac{1}{n-1} \log(q(n-1)/n-1) > 0$ for $121 \leqslant n \leqslant 5504$ completes the proof. □

When $\alpha < 0$, it is clear that $\frac{1}{\sqrt[n]{n^\alpha}}$ is log-convex for $n \geqslant 1$ and $\{\sqrt[n]{q(n)}\}$ is log-convex for $n \geqslant 46$ based on Theorem 1.3. It is widely acknowledged that the product of two log-convex sequences is still log-convex. With the aid of 2.40, we can conclude that the sequence $\{r_\alpha(n)\}_{n \geqslant n(\alpha)}$ is log-convex for any real number α. It completes the proof of Theorem 1.2. □

3 An inequality on the ratio $\frac{\sqrt[n-1]{q(n-1)}}{\sqrt[n]{q(n)}}$

In this section, with the aid of Theorems 2.3 and 2.4, we aim to ascertain the limit of $n^{5/2}\Delta^2 \log \sqrt[n]{q(n)}$. Subsequently, we give an upper bound for $\Delta^2 \log \sqrt[n-1]{q(n-1)}$. This derivation yields the ratio inequality 1.1. We first prove the Theorem 1.6.

Proof of Theorem 1.6. Using the formula for $q(n)$, we find that

$$\log \sqrt[n]{q(n)} = \frac{1}{n}\log T(n) + \frac{1}{n}E(n),$$

where $T(n)$ and $E(n)$ are given by 2.1 and 2.4, we get

$$\Delta^2 \log \sqrt[n-1]{q(n-1)} = \Delta^2 \frac{1}{n-1}\log T(n-1) + \Delta^2 \frac{1}{n-1}E(n-1). \tag{3.1}$$

Applying Theorem 2.3, we find that

$$\lim_{n\to+\infty}(n-1)^{5/2}\Delta^2 \frac{1}{n-1}\log T(n-1) = \beta. \tag{3.2}$$

From Theorem 2.4, it follows that

$$\lim_{n\to+\infty}(n-1)^{5/2}\Delta^2 \frac{1}{n-1}E(n-1) = 0. \tag{3.3}$$

Using 3.1, 3.2 and 3.3, we deduce that

$$\lim_{n\to+\infty} n^{5/2}\Delta^2 \log \sqrt[n]{q(n)} = \beta.$$

In view of proving Theorem 1.5, we need the following upper bound for $\Delta^2 \log \sqrt[n-1]{q(n-1)}$. □

Theorem 3.1. *For $n \geqslant 12$,*

$$\Delta^2 \log \sqrt[n-1]{q(n-1)} < \frac{\sqrt{3}\pi}{4n^{5/2}+\sqrt{3}\pi}. \tag{3.4}$$

Proof. By the upper bound of $\Delta^2 \frac{1}{n-1}\log T(n-1)$ given in Theorem 2.3, the upper bound of $|\Delta^2 \frac{1}{n-1}E(n-1)|$ given in Theorem 2.4, and the relation 1.2, we obtain the following upper bound of $\Delta^2 \log \sqrt[n-1]{q(n-1)}$ for $n \geqslant 206$:

$$\begin{aligned}\Delta^2 \log \sqrt[n-1]{q(n-1)} <& \frac{\sqrt{3}\pi}{4(n-1)^{5/2}} - \frac{2\log(\sqrt{2}\pi\sqrt{24n+25})}{(n+1)^3} - \frac{2\log\left(\frac{\pi\sqrt{6}(48n+50)^{1/4}}{6}\right)}{(n+1)^3}\\ &+ \frac{9}{(n-1)^3} + 5e^{-\frac{\sqrt{2}\pi\sqrt{24n-23}}{18}}.\end{aligned}$$

To prove 3.4, we take into account that for $n \geqslant 802$,

$$\frac{\sqrt{3}\pi}{4(n-1)^{5/2}} - \frac{2\log(\sqrt{2}\pi\sqrt{24n+25})}{(n+1)^3} - \frac{2\log\left(\frac{\pi\sqrt{6}(48n+50)^{1/4}}{6}\right)}{(n+1)^3} + \frac{9}{(n-1)^3} + 5e^{-\frac{\sqrt{2}\pi\sqrt{24n-23}}{18}} < \frac{\sqrt{3}\pi}{4n^{5/2}+\sqrt{3}\pi}. \tag{3.5}$$

At first, we show that for $n \geqslant 206$,

$$\frac{\sqrt{3}\pi}{4(n-1)^{5/2}} - \frac{\sqrt{3}\pi}{4n^{5/2}+\sqrt{3}\pi} < \frac{1}{(n-1)^3}. \tag{3.6}$$

Let's give it a proof.

Symbolized as $\beta = \frac{\sqrt{3}\pi}{4}$, we have

$$\frac{\sqrt{3}\pi}{4(n-1)^{5/2}} - \frac{\sqrt{3}\pi}{4n^{5/2}+\sqrt{3}\pi} = \frac{\beta}{(n-1)^{5/2}} - \frac{\beta}{(n^{5/2}+\beta)} = \frac{\beta^2+\beta[n^{5/2}-(n-1)^{5/2}]}{(n-1)^{5/2}(n^{5/2}+\beta)}. \tag{3.7}$$

Using the fact $n^{5/2}+\beta > (n-1)^{5/2}$, we find that for $n \geqslant 206$,

$$\frac{\sqrt{3}\pi}{4(n-1)^{5/2}} - \frac{\sqrt{3}\pi}{4n^{5/2}+\sqrt{3}\pi} < \frac{\beta^2+\beta[n^{5/2}-(n-1)^{5/2}]}{(n-1)^5}. \tag{3.8}$$

To prove 3.6, we proceed to show that the right hand side of 3.8 is bounded by $\frac{1}{(n-1)^3}$. According to the following asymptotic expansion:

$$n^{5/2}-(n-1)^{5/2} = \frac{5}{2}n^{3/2} - \frac{15}{8}n^{1/2} + \frac{5}{16}n^{-1/2} - \theta_n,$$

where $0 < \theta_n < \frac{5}{128}n^{-3/2}$. Consequently,

$$n^{5/2}-(n-1)^{5/2} < \frac{5}{2}n^{3/2}. \tag{3.9}$$

Applying 3.9 to 3.8, we can conclude that

$$\begin{aligned}\frac{\sqrt{3}\pi}{4(n-1)^{5/2}} - \frac{\sqrt{3}\pi}{4n^{5/2}+\sqrt{3}\pi} &< \frac{\beta^2+\beta[n^{5/2}-(n-1)^{5/2}]}{(n-1)^5} \\ &< \frac{\beta^2}{(n-1)^5} + \frac{5\beta n^{3/2}}{2(n-1)^5}.\end{aligned} \tag{3.10}$$

Using the fact that $\beta < 2$, we see that

$$\frac{\sqrt{3}\pi}{4(n-1)^{5/2}} - \frac{\sqrt{3}\pi}{4n^{5/2}+\sqrt{3}\pi} < \frac{4}{(n-1)^5} + \frac{5n^{3/2}}{(n-1)^5} < \frac{1}{(n-1)^3}. \tag{3.11}$$

This completes 3.6.

To prove the claim 3.5, it is enough to show that for $n \geqslant 802$,

$$\frac{2\log(\sqrt{2}\pi\sqrt{24n+25})}{(n+1)^3}+\frac{2\log\left(\frac{\pi\sqrt{6}(48n+50)^{1/4}}{6}\right)}{(n+1)^3}-\frac{9}{(n-1)^3}-5e^{-\frac{\sqrt{2}\pi\sqrt{24n-23}}{18}}>\frac{1}{(n-1)^3}. \tag{3.12}$$

From 2.36, it can be seen that for $n \geqslant 802$,

$$5\sqrt{n-1}e^{-\frac{\pi\sqrt{24n-25}}{12}}<\frac{800}{(n-1)^4}<\frac{1}{(n-1)^3}. \tag{3.13}$$

Since $2\log(\sqrt{2}\pi\sqrt{24n+25}) > 12$ and $2\log\left(\frac{\pi\sqrt{6}(48n+50)^{1/4}}{6}\right) > 5$ for $n \geqslant 802$, it follows from 3.13 that for $n \geqslant 802$,

$$\begin{aligned}\frac{2\log(\sqrt{2}\pi\sqrt{24n+25})}{(n+1)^3}&+\frac{2\log\left(\frac{\pi\sqrt{6}(48n+50)^{1/4}}{6}\right)}{(n+1)^3}-\frac{9}{(n-1)^3}-5\sqrt{n-1}e^{-\frac{\pi\sqrt{24n-25}}{12}}\\&>\frac{17}{(n+1)^3}-\frac{10}{(n-1)^3}.\end{aligned} \tag{3.14}$$

The inequality 3.12 will be proved if we can show $\frac{17}{(n+1)^3}-\frac{10}{(n-1)^3}>\frac{1}{(n-1)^3}$ for $n \geqslant 802$, it follows that

$$\frac{17}{(n+1)^3}-\frac{11}{(n-1)^3}=\frac{6n^3-84n^2+18n-28}{(n-1)^3(n+1)^3}>0.$$

So we obtain 3.12. Combining 3.12 and 3.6, we arrive at 3.5. For $12 \leqslant n \leqslant 801$, the inequality can be readily verifiable. This completes the proof. $\square$

We are now in a position to complete the proof of Theorem 1.5.

Proof of Theorem 1.5. It is known that for $x > 0$,

$$\frac{x}{1+x}<\log(1+x),$$

so that for $n \geqslant 1$,

$$\frac{\sqrt{3}\pi}{4n^{5/2}+\sqrt{3}\pi}<\log\left(1+\frac{\sqrt{3}\pi}{4n^{5/2}}\right).$$

In light of the above relation, Theorem 3.1 implies that for $n \geqslant 12$,

$$\Delta^2\log\sqrt[n-1]{q(n-1)}<\log\left(1+\frac{\sqrt{3}\pi}{4n^{5/2}}\right),$$

which is equivalent to,

$$\sqrt[n+1]{q(n+1)}\sqrt[n-1]{q(n-1)}<\left(1+\frac{\sqrt{3}\pi}{4n^{5/2}}\right)\left(\sqrt[n]{q(n)}\right)^2,$$

as required.

The validity of $\beta = \frac{\sqrt{3}\pi}{4}$ has been demonstrated above. We will now show that $\beta = \frac{\sqrt{3}\pi}{4}$ is optimal, which is equivalent to showing that $\beta = \frac{\sqrt{3}\pi}{4}$ is the smallest possible number for the inequality in Theorem 1.5. Suppose that $0 < \gamma < \beta$. By Theorem 1.6, there exists an integer N such that for $n > N$,

$$n^{5/2}\Delta^2 \log \sqrt[n-1]{q(n-1)} > \gamma,$$

It follows that

$$\Delta^2 \log \sqrt[n-1]{q(n-1)} > \frac{\gamma}{n^{5/2}} > \log(1 + \frac{\gamma}{n^{5/2}}),$$

which implies that for $n > N$,

$$\frac{\sqrt[n]{q(n)}}{\sqrt[n+1]{q(n+1)}}(1 + \frac{\gamma}{n^{5/2}}) < \frac{\sqrt[n-1]{q(n-1)}}{\sqrt[n]{q(n)}}.$$

This contradicts the inequality relation we are trying to obtain. So we complete the proof of Theorem 1.5. $\square$

4 The log-balancedness of the sequence $\sqrt[n]{q(n)}$

In this section, with the help of theoretical foundations and the results obtained above, we first state the log-balancedness of the sequences $\{\sqrt[n]{q(n)}\}$ and $\left\{\sqrt[n]{q(n)}/n\right\}$. More generally, we consider that the sequence $\left\{\sqrt[n]{q(n)}/n^{\alpha}\right\}_{n \geqslant N_\alpha}$ is log-balanced for any nonnegative real number α. Now let us begin the proofs of log-balanced property.

Proof of Theorem 1.7. We have obtained that the sequence $\{\sqrt[n]{q(n)}\}$ is log-convex for $n \geqslant 46$ based on Theorem 1.3. In order to prove the log-balancedness of $\{\sqrt[n]{q(n)}\}_{n \geqslant 46}$, it is enough to prove that for $n \geqslant 46$,

$$\frac{\sqrt[n]{q(n)}}{n\sqrt[n-1]{q(n-1)}} \geqslant \frac{\sqrt[n+1]{q(n+1)}}{(n+1)\sqrt[n]{q(n)}},$$

which is equivalent to,

$$\frac{\sqrt[n]{q(n)}}{\sqrt[n+1]{q(n+1)}} \geqslant \frac{\sqrt[n-1]{q(n-1)}}{\sqrt[n]{q(n)}}\frac{n}{n+1}. \tag{4.1}$$

According to 1.1, we can conclude that

$$\frac{\sqrt[n]{q(n)}}{\sqrt[n+1]{q(n+1)}} \geqslant \frac{\sqrt[n-1]{q(n-1)}}{\sqrt[n]{q(n)}}\frac{1}{(1 + \frac{\sqrt{3}\pi}{4n^{5/2}})}. \tag{4.2}$$

By the use of 4.2, the inequality 4.1 will be proved if we can show that for $n \geqslant 46$,

$$\frac{1}{(1+\frac{\sqrt{3}\pi}{4n^{5/2}})} \geqslant \frac{n}{n+1}. \tag{4.3}$$

It can be easily checked that the inequality 4.3 is right, so we can draw the conclusion that the sequence $\{\sqrt[n]{q(n)}\}_{n\geqslant 46}$ is log-balanced. It completes the proof. □

Proof of Theorem 1.8. It is evident that the sequence $\{\frac{1}{n}\}_{n\geqslant 1}$ is log-convex. By the Theorem 1.3, we have seen that the sequence$\{\sqrt[n]{q(n)}\}_{n\geqslant 46}$ is log-convex. In order to prove the log-balancedness of the sequence $\{\sqrt[n]{q(n)}/n\}_{n\geqslant 46}$, it is enough to prove that for $n \geqslant 46$,

$$\frac{\frac{\sqrt[n]{q(n)}}{n}}{n\frac{\sqrt[n-1]{q(n-1)}}{n-1}} \geqslant \frac{\frac{\sqrt[n+1]{q(n+1)}}{n+1}}{(n+1)\frac{\sqrt[n]{q(n)}}{n}},$$

which is equivalent to,

$$\frac{\sqrt[n]{q(n)}}{\sqrt[n+1]{q(n+1)}} \geqslant \frac{\sqrt[n-1]{q(n-1)}}{\sqrt[n]{q(n)}}\frac{n^3}{(n-1)(n+1)^2}. \tag{4.4}$$

By the use of 4.2, the inequality 4.4 will be proved if we can show

$$\frac{1}{(1+\frac{\sqrt{3}\pi}{4n^{5/2}})} \geqslant \frac{n^3}{(n-1)(n+1)^2}. \tag{4.5}$$

It can be easily checked that the inequality 4.5 is proper, so we can draw the conclusion that the sequence $\{\sqrt[n]{q(n)}/n\}_{n\geqslant 46}$ is log-balanced. The proof is completed. □

Proof of Theorem 1.9. Similarly, using the method described above, we need to prove that this sequence is log-balanced, which is equivalent to prove that,

$$\frac{1}{1+\frac{\sqrt{3}\pi}{4n^{5/2}}} \geqslant \frac{n^{2\alpha+1}}{(n-1)^{\alpha}(n+1)^{\alpha+1}}. \tag{4.6}$$

We first focus on the right-hand side of the inequality,

$$\frac{n^{2\alpha+1}}{(n-1)^{\alpha}(n+1)^{\alpha+1}} = \frac{n^{2\alpha+1}}{(n^2-1)^{\alpha}(n+1)} = \frac{n}{n+1}\left(1-\frac{1}{n^2}\right)^{-\alpha} = \frac{1}{(1+\frac{1}{n})}\frac{1}{(1-\frac{1}{n^2})^{\alpha}}. \tag{4.7}$$

By substituting 4.7 into 4.6, we transform the problem into proving that

$$1+\frac{\sqrt{3}\pi}{4n^{5/2}} \leqslant \left(1+\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)^{\alpha}.$$

Since $\ln(1-x) \geqslant -2x$ when $0 \leqslant x \leqslant \frac{1}{2}$, we have

$$\ln\left(1-\frac{1}{n^2}\right) \geqslant -\frac{2}{n^2}, \ (n \geqslant 2).$$

Hence

$$\left(1-\frac{1}{n^2}\right)^{\alpha} = \exp\left(\alpha \ln\left(1-\frac{1}{n^2}\right)\right) \geqslant e^{-2\alpha/n^2}.$$

Using the elementary inequality $e^{-\mu} \geqslant 1-\mu$ for $\mu \geqslant 0$, we further get

$$e^{-2\alpha/n^2} \geqslant 1-\frac{2\alpha}{n^2}$$

and therefore

$$\left(1-\frac{1}{n^2}\right)^{\alpha} \geqslant \left(1-\frac{2\alpha}{n^2}\right).$$

It follows that

$$\left(1+\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)^{\alpha} \geqslant \left(1+\frac{1}{n}\right)\left(1-\frac{2\alpha}{n^2}\right).$$

Expanding the right-hand side gives

$$\left(1+\frac{1}{n}\right)\left(1-\frac{2\alpha}{n^2}\right) = 1+\frac{1}{n}-\frac{2\alpha}{n^2}-\frac{2\alpha}{n^3}.$$

If $n \geqslant 4\alpha$, then

$$\frac{2\alpha}{n^2} \leqslant \frac{1}{2n} \qquad \text{and} \qquad \frac{2\alpha}{n^3} \leqslant \frac{1}{2n^2},$$

so

$$1+\frac{1}{n}-\frac{2\alpha}{n^2}-\frac{2\alpha}{n^3} \geqslant 1+\frac{1}{n}-\frac{1}{2n}-\frac{1}{2n^2} = 1+\frac{1}{2n}-\frac{1}{2n^2}.$$

Moreover, for $n \geqslant 4$,

$$\frac{1}{2n}-\frac{1}{2n^2} = \frac{n-1}{2n^2} \geqslant \frac{1}{4n} \geqslant \frac{\sqrt{3}\pi}{4n^{5/2}}.$$

Therefore, whenever

$$n \geqslant N(\alpha) := \max\{4, 4\alpha\},$$

we have

$$1+\frac{\sqrt{3}\pi}{4n^{5/2}} \leqslant \left(1+\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)^{\alpha}.$$

This completes the proof of Theorem 1.9. □

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