Corners of self-conjugate $(t, tk \pm 1)$-core partitions

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Submitted: Dec 12, 2025; Accepted: Aug 8, 2026; Published: Sep 11, 2026

Abstract

A corner of a partition is a removable box in the Young diagram. Initiated by Huang and Wang, there has been research on the number of simultaneous core partitions with a fixed number of corners. In this paper, we give a bijection between the set of self-conjugate $(t, tk \pm 1)$-core partitions with a fixed number of corners and the set of $(\lfloor t/2 \rfloor + 1)$-tuples with certain conditions. As a result of this bijection, we count the number of such partitions.

Mathematics Subject Classifications: 05A17, 05A19

1 Introduction

A partition λ of a positive integer n is a sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell$ and $\lambda_1 + \cdots + \lambda_\ell = n$. The Young diagram of λ is a finite collection of boxes arranged in left-justified rows with λ_i boxes in row i. The box in row i and column j is said to be the box (i, j). For the Young diagram of λ, the hook length of the box (i, j) is the sum of the number of boxes on the right of the box (i, j), the number of boxes below it, and 1 for itself.

If a partition λ has no box of hook length divisible by t, we call it a t-core partition. Furthermore, a partition λ is a $(t_1, t_2, \ldots, t_p)$-core partition if λ is simultaneously a t_1-core, a t_2-core, …, and a t_p-core partition. For example, the partition $(5, 3, 2)$ is a 5-core partition. See Figure 1 for the Young diagram and the hook lengths of each box.

A corner of a partition λ is a removable box in the Young diagram of λ, which means that the removal of that box from the Young diagram of λ still keeps the diagram as

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7	6	4	2	1
4	3	1		
2	1			

Figure 1: The Young diagram of the 5-core partition $(5, 3, 2)$ and its hook lengths.

the Young diagram of a partition. Note that the number of corners of λ, the number of distinct parts of λ, and the number of boxes of hook length 1 are equal.

Huang and Wang [7] proved that, for positive integers t and m, the number of $(t, t+1)$-core partitions with m corners is the Narayana number $N(t, m+1) = \frac{1}{t}\binom{t}{m+1}\binom{t}{m}$ and the number of $(t, t+1, t+2)$-core partitions with m corners is $\binom{t}{2m}C_m$, where C_m is the mth Catalan number. As a generalization of Huang and Wang's result, Cho, Huh, and Sohn [4] gave an explicit formula for the number of $(t, t+1, \ldots, t+p)$-core partitions with m corners for positive integers t and $p \geqslant 2$. Recently, the authors of this paper [5] counted the number of $(t, tk \pm 1)$-core partitions with m corners, which gives another direction of generalizing Huang and Wang's result. In addition, there have been several studies on $(t, tk \pm 1)$-core partitions where parts of the partition have some restrictions. For example, see [8, 9, 10, 11, 12].

It is natural to ask whether we can have similar results for self-conjugate partitions. For a given partition λ, a partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_k)$ is called the conjugate of λ if λ'_j denotes the number of boxes in column j of the Young diagram of λ. We call a partition λ self-conjugate if $\lambda = \lambda'$. Cho and Hong [1] enumerated the number of self-conjugate $(t, t+1)$-core partitions with m corners. Cho and Huh [2] gave a lattice path interpretation of the set of self-conjugate $(t, t+1, \ldots, t+p)$-core partitions with m corners for $p \geqslant 2$, and counted the number of self-conjugate $(t, t+1, \ldots, t+p)$-core partitions with m corners for $p = 2, 3$.

In this paper, our goal is to give a nice interpretation of self-conjugate $(t, tk \pm 1)$-core partitions with m corners. An NE lattice path is a path consisting only of two steps $N = (0, 1)$ and $E = (1, 0)$. Ford, Mai, and Sze [6] showed that a self-conjugate (a, b)-core partition has an NE lattice path interpretation. In this paper, we extend the corresponding NE lattice path for a self-conjugate $(t, tk \pm 1)$-core partition to obtain a new NE lattice path $E^{x_1}NE^{x_2}N \cdots NE^{x_{t_1}}NE^{x_{t_1+1}}$, where $t_1 = \lfloor t/2 \rfloor$.

Throughout this paper, we denote by $\mathbb{N}$ the set of nonnegative integers. The following theorem is the main result of this paper.

Theorem 1.1. *Let t, k, and m be positive integers with $t \geqslant 2$ and $t_1 = \lfloor t/2 \rfloor$. We set*

$$\mathcal{A} := \mathcal{A}(t, k, m) = \left\{ (x_1, \ldots, x_{t_1+1}) \in \mathbb{N}^{t_1+1} \;\middle|\; \sum_{i=1}^{t_1+1} x_i = k(t_1+1), \sum_{x_i \geqslant k+1} (x_i - k) = \left\lceil \frac{m}{2} \right\rceil \right\}.$$

There is a bijection between the set of self-conjugate $(t, tk-1)$-core partitions with m

corners and the set $\mathcal{A}_{t,tk-1}(m)$*, where*

$$\mathcal{A}_{t,tk-1}(m) = \begin{cases} \mathcal{A} \cap \{(x_1,\ldots,x_{t_1+1}) \mid x_{t_1+1} \geqslant k+1\} & \text{if } 2 \nmid t \text{ and } 2 \nmid m, \\ \mathcal{A} \cap \{(x_1,\ldots,x_{t_1+1}) \mid \lceil \frac{k+1}{2} \rceil \leqslant x_{t_1+1} \leqslant k\} & \text{if } 2 \nmid t \text{ and } 2 \mid m, \\ \mathcal{A} \cap \{(x_1,\ldots,x_{t_1+1}) \mid x_1 \geqslant k+1, x_{t_1+1} \geqslant \frac{k+1}{2}\} & \text{if } 2 \mid t,\ 2 \nmid k, \text{ and } 2 \nmid m, \\ \mathcal{A} \cap \{(x_1,\ldots,x_{t_1+1}) \mid x_1 \geqslant \frac{k}{2}, x_{t_1+1} \geqslant k+1\} & \text{if } 2 \mid t,\ 2 \mid k, \text{ and } 2 \nmid m, \\ \mathcal{A} \cap \{(x_1,\ldots,x_{t_1+1}) \mid \frac{k+1}{2} \leqslant x_1 \leqslant k, x_{t_1+1} \geqslant \frac{k+1}{2}\} & \text{if } 2 \mid t,\ 2 \nmid k, \text{ and } 2 \mid m, \\ \mathcal{A} \cap \{(x_1,\ldots,x_{t_1+1}) \mid x_1 \geqslant \frac{k}{2}, \frac{k}{2}+1 \leqslant x_{t_1+1} \leqslant k\} & \text{if } 2 \mid t,\ 2 \mid k, \text{ and } 2 \mid m, \end{cases} \tag{1}$$

and the set of self-conjugate $(t, tk+1)$*-core partitions with* m *corners and the set* $\mathcal{A}_{t,tk+1}(m)$*, where*

$$\mathcal{A}_{t,tk+1}(m) = \begin{cases} \mathcal{A} \cap \{(x_1,\ldots,x_{t_1+1}) \mid \lfloor \frac{k}{2} \rfloor \leqslant x_{t_1+1} \leqslant k-1\} & \text{if } 2 \nmid t \text{ and } 2 \nmid m, \\ \mathcal{A} \cap \{(x_1,\ldots,x_{t_1+1}) \mid x_{t_1+1} \geqslant k\} & \text{if } 2 \nmid t \text{ and } 2 \mid m, \\ \mathcal{A} \cap \{(x_1,\ldots,x_{t_1+1}) \mid \frac{k-1}{2} \leqslant x_1 \leqslant k-1, x_{t_1+1} \geqslant \frac{k+1}{2}\} & \text{if } 2 \mid t,\ 2 \nmid k, \text{ and } 2 \nmid m, \\ \mathcal{A} \cap \{(x_1,\ldots,x_{t_1+1}) \mid x_1 \geqslant \frac{k}{2}, \frac{k}{2} \leqslant x_{t_1+1} \leqslant k-1\} & \text{if } 2 \mid t,\ 2 \mid k, \text{ and } 2 \nmid m, \\ \mathcal{A} \cap \{(x_1,\ldots,x_{t_1+1}) \mid x_1 \geqslant k, x_{t_1+1} \geqslant \frac{k+1}{2}\} & \text{if } 2 \mid t,\ 2 \nmid k, \text{ and } 2 \mid m, \\ \mathcal{A} \cap \{(x_1,\ldots,x_{t_1+1}) \mid x_1 \geqslant \frac{k}{2}, x_{t_1+1} \geqslant k\} & \text{if } 2 \mid t,\ 2 \mid k, \text{ and } 2 \mid m. \end{cases} \tag{2}$$

By Theorem 1.1, we enumerate the number of self-conjugate $(t, tk \pm 1)$-core partitions with m corners as follows. Since the empty partition is the only partition with no corners, we omit the case where $m = 0$. By convention, we denote $\binom{n}{k} = 0$ when $n < k$ or $k < 0$.

Theorem 1.2. *For positive integers* t, k, m *with* $t \geqslant 2$*, let* $t_1 = \lfloor t/2 \rfloor$*,* $m_1 = \lceil m/2 \rceil$*, and* $\alpha_{r,s} = t_1 + m_1 - s - (k+1)r$*. When* t *is odd, the number of self-conjugate* $(t, tk-1)$*-core partitions with* m *corners is*

$$\begin{cases} \sum_{s \geqslant 1} \binom{t_1}{s} \binom{m_1-1}{s-1} \sum_{r \geqslant 0} (-1)^r \binom{t_1-s}{r} \left[\binom{\alpha_{r,s}}{t_1-s} - \binom{\alpha_{r,s} - \lceil k/2 \rceil}{t_1-s} \right] & \text{if } 2 \mid m, \\ \sum_{s \geqslant 0} \binom{t_1}{s} \binom{m_1-1}{s} \sum_{r \geqslant 0} (-1)^r \binom{t_1-s}{r} \binom{\alpha_{r,s}-1}{t_1-1-s} & \text{if } 2 \nmid m. \end{cases}$$

When t *is even, the number of self-conjugate* $(t, tk-1)$*-core partitions with* m *corners is*

$$\begin{cases} \sum_{s \geqslant 1} \binom{t_1-1}{s-1} \binom{m_1-1}{s-1} \sum_{r \geqslant 0} (-1)^r \binom{t_1-s}{r} \left[\binom{\alpha_{r,s}}{t_1-s} - \binom{\alpha_{r,s} - \lceil k/2 \rceil}{t_1-s} \right] \\ + \sum_{s \geqslant 1} \binom{t_1-1}{s} \binom{m_1-1}{s-1} \sum_{r \geqslant 0} (-1)^r \binom{t_1-1-s}{r} \\ \times \left[\binom{\alpha_{r,s}}{t_1-s} - \binom{\alpha_{r,s} - \lfloor k/2 \rfloor - 1}{t_1-s} - \binom{\alpha_{r,s} - \lceil k/2 \rceil}{t_1-s} + \binom{\alpha_{r,s} - (k+1)}{t_1-s} \right] & \text{if } 2 \mid m, \\ \\ \sum_{s \geqslant 1} \binom{t_1-1}{s-1} \binom{m_1-1}{s} \sum_{r \geqslant 0} (-1)^r \binom{t_1-s}{r} \binom{\alpha_{r,s}-1}{t_1-1-s} \\ + \sum_{s \geqslant 0} \binom{t_1-1}{s} \binom{m_1-1}{s} \sum_{r \geqslant 0} (-1)^r \binom{t_1-1-s}{r} \left[\binom{\alpha_{r,s}-1}{t_1-1-s} - \binom{\alpha_{r,s} - \lfloor k/2 \rfloor - 2}{t_1-1-s} \right] & \text{if } 2 \nmid m. \end{cases}$$

Theorem 1.3. *For positive integers t, k, m with $t \geqslant 2$, let $t_1 = \lfloor t/2 \rfloor$, $m_1 = \lceil m/2 \rceil$, and $\beta_r = m_1 - kr - 1$. When t is odd, the number of self-conjugate $(t, tk+1)$-core partitions with m corners is*

$$\begin{cases} \sum\limits_{s \geqslant 0} \binom{t_1}{s}\binom{s+m_1}{s} \sum\limits_{r \geqslant 0} (-1)^r \binom{t_1-s}{r}\binom{\beta_r}{t_1-1-s} & \text{if} \quad 2 \mid m, \\ \sum\limits_{s \geqslant 1} \binom{t_1}{s}\binom{s+m_1-1}{s-1} \sum\limits_{r \geqslant 0} (-1)^r \binom{t_1-s}{r}\left[\binom{\beta_r}{t_1-s} - \binom{\beta_r - \lceil k/2 \rceil}{t_1-s}\right] & \text{if} \quad 2 \nmid m. \end{cases}$$

When t is even, the number of self-conjugate $(t, tk+1)$-core partitions with m corners is

$$\begin{cases} \sum\limits_{s \geqslant 1} \binom{t_1-1}{s-1}\binom{s+m_1}{s} \sum\limits_{r \geqslant 0} (-1)^r \binom{t_1-s}{r}\binom{\beta_r}{t_1-1-s} & \\ + \sum\limits_{s \geqslant 0} \binom{t_1-1}{s}\binom{s+m_1}{s} \sum\limits_{r \geqslant 0} (-1)^r \binom{t_1-1-s}{r}\left[\binom{\beta_r}{t_1-1-s} - \binom{\beta_r - \lfloor k/2 \rfloor}{t_1-1-s}\right] & \text{if} \quad 2 \mid m, \\ \sum\limits_{s \geqslant 1} \binom{t_1-1}{s-1}\binom{s+m_1-1}{s-1} \sum\limits_{r \geqslant 0} (-1)^r \binom{t_1-s}{r}\left[\binom{\beta_r}{t_1-s} - \binom{\beta_r - \lceil k/2 \rceil}{t_1-s}\right] & \\ + \sum\limits_{s \geqslant 1} \binom{t_1-1}{s}\binom{s+m_1-1}{s-1} \sum\limits_{r \geqslant 0} (-1)^r \binom{t_1-1-s}{r} & \\ \times \left[\binom{\beta_r}{t_1-s} - \binom{\beta_r - \lfloor k/2 \rfloor}{t_1-s} - \binom{\beta_r - \lceil k/2 \rceil}{t_1-s} + \binom{\beta_r - k}{t_1-s}\right] & \text{if} \quad 2 \nmid m. \end{cases}$$

The paper is organized as follows. In Section 2, we explain path interpretations of self-conjugate $(t, tk \pm 1)$-core partitions. The main results, which are Theorems 1.1, 1.2, and 1.3, are proved in Section 3. In Section 4, we give other expressions for the number of self-conjugate $(t, tk \pm 1)$-core partitions with m corners using generating functions. Furthermore, for small t, the number of self-conjugate $(t, tk \pm 1)$-core partitions with m corners admits relatively simple formulas, which we provide explicitly.

2 Path interpretations of self-conjugate $(t, tk \pm 1)$-core partitions with m corners

In this section, we give a lattice path interpretation of self-conjugate $(t, tk \pm 1)$-core partitions with m corners. We divide into two cases: self-conjugate $(t, tk-1)$-core partitions and self-conjugate $(t, tk+1)$-core partitions.

2.1 Path interpretations of self-conjugate $(t, tk-1)$-core partitions with m corners

In this subsection, we focus on the set of self-conjugate $(t, tk-1)$-core partitions with m corners. Cho, Huh, Nam, and Sohn [3, Proposition 17] found a formula for the number of self-conjugate t-core partitions with m corners, which is

$$\sum_{i=1}^{\min(m, \lfloor t/2 \rfloor)} \binom{\lfloor \frac{m-1}{2} \rfloor}{\lfloor \frac{i-1}{2} \rfloor}\binom{\lfloor \frac{m}{2} \rfloor}{\lfloor \frac{i}{2} \rfloor}\binom{\lfloor \frac{t}{2} \rfloor + m - i}{m}.$$

Using the fact that the maximum hook length of a t-core partition with m corners is $tm-1$, we obtain the number of self-conjugate $(t, tk-1)$-core partitions with a small number of corners.

Proposition 2.1. *Let t, k, and m be positive integers with $t \geqslant 2$, $k \geqslant 1$, and $m \leqslant k-1$. Then the number of self-conjugate $(t, tk-1)$-core partitions with m corners is*

$$\sum_{i=1}^{\min(m,\lfloor t/2 \rfloor)} \binom{\lfloor \frac{m-1}{2} \rfloor}{\lfloor \frac{i-1}{2} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor}{\lfloor \frac{i}{2} \rfloor} \binom{\lfloor \frac{t}{2} \rfloor + m - i}{m}.$$

For a partition λ, let $MD(\lambda)$ be the set of hook lengths in the main diagonal of λ. Note that if λ is self-conjugate, then $MD(\lambda)$ only contains distinct odd elements. We denote the number of corners of λ by $c(\lambda)$. For a self-conjugate partition λ, let $d(\lambda)$ be the number of $h \in MD(\lambda)$ such that $h-2 \notin MD(\lambda)$. It is known that $c(\lambda)$ and $d(\lambda)$ have the following relation.

Lemma 2.2 ([1, Lemma 2.2])**.** *Let λ be a self-conjugate partition. Then, the number of corners of λ is equal to*

$$c(\lambda) = \begin{cases} 2d(\lambda) & \text{if } 1 \notin MD(\lambda), \\ 2d(\lambda) - 1 & \text{if } 1 \in MD(\lambda). \end{cases}$$

In Figure 2, the self-conjugate partition $\lambda = (7, 4, 4, 3, 1, 1, 1)$ with $MD(\lambda) = \{13, 5, 3\}$ satisfies that $1 \notin MD(\lambda)$, $c(\lambda) = 4$, and $d(\lambda) = 2$, so $c(\lambda) = 2d(\lambda)$.

13	9	8	6	3	2	1
9	5	4	2			
8	4	3	1			
6	2	1				
3						
2						
1						

Figure 2: The Young diagram of $(7, 4, 4, 3, 1, 1, 1)$ with 4 corners and its hook lengths.

For relatively prime positive integers a and b, let $A(a, b)$ be an array of integers that fits in the rectangle whose vertices are at $(0, 0), (0, \lfloor \frac{a}{2} \rfloor), (\lfloor \frac{b}{2} \rfloor, 0)$, and $(\lfloor \frac{b}{2} \rfloor, \lfloor \frac{a}{2} \rfloor)$. The entries of the array $A(a, b)$ on the ith row and the jth column are

$$A_{i,j} = ab - a(2j-1) - b(2i-1)$$

for $1 \leqslant i \leqslant \lfloor \frac{a}{2} \rfloor$ and $1 \leqslant j \leqslant \lfloor \frac{b}{2} \rfloor$. Since a and b are relatively prime, the entries of $A(a, b)$ are all distinct. We write the entry $A_{i,j}$ inside the square whose vertices are $(j-1, \lfloor a/2 \rfloor - i)$, $(j, \lfloor a/2 \rfloor - i)$, $(j-1, \lfloor a/2 \rfloor - i + 1)$, and $(j, \lfloor a/2 \rfloor - i + 1)$.

Let $\mathcal{NE}(a,b)$ be the set of NE lattice paths from $(0,0)$ to (a,b) whose steps consist only of $N=(0,1)$ and $E=(1,0)$, and $\mathcal{SC}_{(a,b)}$ be the set of self-conjugate (a,b)-core partitions. Ford, Mai, and Sze [6] constructed a bijection $\phi_{a,b}: \mathcal{SC}_{(a,b)} \to \mathcal{NE}(\lfloor \frac{b}{2} \rfloor, \lfloor \frac{a}{2} \rfloor)$ by mapping a self-conjugate (a,b)-core partition λ to an NE lattice path $P=\phi_{a,b}(\lambda)$. From this construction, $MD(\lambda)$ consists of all positive integers below the path $P=\phi_{a,b}(\lambda)$ and the absolute values of all negative integers above P. A path $P \in \mathcal{NE}(\lfloor \frac{b}{2} \rfloor, \lfloor \frac{a}{2} \rfloor)$ can be expressed as $P=P_1P_2\cdots P_r$, where $P_i=N$ or E and $r=\lfloor \frac{a}{2} \rfloor+\lfloor \frac{b}{2} \rfloor$. Let $E_{k+1}(P)$ be the number of occurrences of $k+1$ consecutive E steps of a path P. Throughout this subsection, we consider the case where $a=t$ and $b=tk-1$.

For a given self-conjugate $(t,tk-1)$-core partition λ, we extend the path $\phi_{t,tk-1}(\lambda)$ to an NE lattice path $\phi^-(\lambda)$ as follows: let $\phi^-(\lambda)=E^x\phi_{t,tk-1}(\lambda)E^y$, where

$$(x,y)=\begin{cases}(\lceil \frac{k}{2} \rceil, \lceil \frac{k+1}{2} \rceil) & \text{if } 2 \mid t,\\ (0, \lceil \frac{k+1}{2} \rceil) & \text{if } 2 \nmid t.\end{cases} \tag{3}$$

We write $t_1=\lfloor t/2 \rfloor$. Note that the extended path $\phi^-(\lambda)$ has t_1 N steps and $k(t_1+1)$ E steps by the construction. We have the following relation between a self-conjugate $(t,tk-1)$ core partition λ and the corresponding path $\phi^-(\lambda)$.

Lemma 2.3. *Let t and k be positive integers such that $t \geqslant 2$ and λ be a self-conjugate $(t,tk-1)$-core partition. Then, $d(\lambda)=E_{k+1}(\phi^-(\lambda))$.*

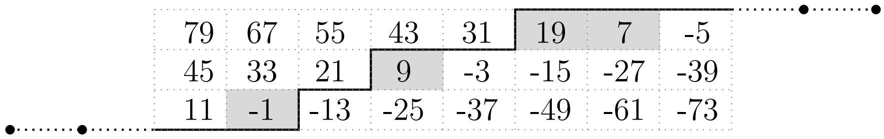
Proof. Recall that the entry $A_{i,j}$ in $A(t,tk-1)$ is located inside the square whose vertices are $(j-1,t_1-i),(j,t_1-i),(j-1,t_1-i+1),(j,t_1-i+1)$. For $h \in MD(\lambda)$, we consider two cases where $A_{i,j}=h$ or $A_{i,j}=-h$.

First, let $A_{i,j}=h$ for some $1 \leqslant i \leqslant t_1$, $1 \leqslant j \leqslant \lfloor (tk-1)/2 \rfloor$. When $i>1$, we have $A_{i-1,j+k}=h-2$. Hence, each $h \in MD(\lambda)$ with $h-2 \notin MD(\lambda)$ corresponds to $k+1$ consecutive E steps from $(j-1,t_1-i+1)$ to $(j+k,t_1-i+1)$. When $i=1$, it is clear that $h-2 \notin MD(\lambda)$. Let j_1 be the smallest number such that $A_{1,j_1} \in MD(\lambda)$. Therefore, the last $\lfloor \frac{tk-1}{2} \rfloor + \lceil \frac{k+1}{2} \rceil - j_1 + 1$ steps of $\phi^-(\lambda)$ are all E. Since $A_{1,j}=t(tk-1)-t(2j-1)-(tk-1)=t^2k-(2j+k)t+1>0$ for $j \leqslant \lfloor \frac{(t-1)k}{2} \rfloor$, there are $\lfloor \frac{(t-1)k}{2} \rfloor - j_1 + 1$ positive entries in the first row of $A(t,tk-1)$ located below $\phi^-(\lambda)$. Since $\lfloor \frac{tk-1}{2} \rfloor + \lceil \frac{k+1}{2} \rceil = \lfloor \frac{(t-1)k}{2} \rfloor + k$, we can check that $d(\lambda)=E_{k+1}(\phi^-(\lambda))$.

We now suppose that $A_{i,j}=-h$ for some $1 \leqslant i \leqslant t_1$, $1 \leqslant j \leqslant \lfloor (tk-1)/2 \rfloor$. Similarly to the first case, we have $A_{i+1,j-k}=-h+2$ when $i<t_1$. Hence, $h \in MD(\lambda)$ and $h-2 \notin MD(\lambda)$ if and only if there exist $k+1$ consecutive E steps from $(j-k-1,t_1-i)$ to (j,t_1-i). When $i=t_1$, we denote by j_2 the largest number satisfying $-A_{t_1,j_2} \in MD(\lambda)$. In this case, we divide into two cases by the parity of t. When t is even, the first $j_2+\lceil \frac{k}{2} \rceil$ steps of $\phi^-(\lambda)$ are all E. From the result that $A_{t_1,j}=t(tk-1)-t(2j-1)-(tk-1)(2t_1-1)=t(k-2j+1)-1<0$ when $j \geqslant \lceil \frac{k+1}{2} \rceil$, there are $j_2-\lceil \frac{k+1}{2} \rceil+1$ negative entries in the last row of $A(t,tk-1)$ located above $\phi^-(\lambda)$. Since $\lceil \frac{k}{2} \rceil + \lceil \frac{k+1}{2} \rceil = k+1$, we complete the proof. Since the case where t is odd is similar to the even case, we omit the details. □

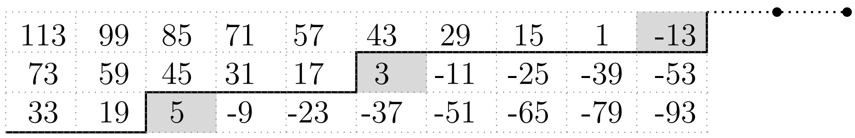
Example 2.4. Let λ be the self-conjugate $(6,17)$-core partition and μ be the self-conjugate $(7,20)$-core partition with $MD(\lambda)=\{19,9,7,1\}$ and $MD(\mu)=\{13,5,3\}$.

By definition, we have $d(\lambda) = 3$ and $d(\mu) = 2$. We have $k = 3$ for both partitions, so the extended paths are $\phi^-(\lambda) = E^4NE^1NE^2NE^5$ and $\phi^-(\mu) = E^2NE^3NE^5NE^2$ (see Figures 3 and 4). By Lemma 2.3, we also obtain that $d(\lambda) = E_4(\phi^-(\lambda)) = 3$ and $d(\mu) = E_4(\phi^-(\mu)) = 2$.



$$\phi^-(\lambda) = E^2\phi_{6,17}(\lambda)E^2 = EEEENENEENEEEEE = E^4NE^1NE^2NE^5$$

Figure 3: The extended path $\phi^-(\lambda)$ corresponding to the self-conjugate $(6, 17)$-core partition λ with $MD(\lambda) = \{19, 9, 7, 1\}$.



$$\phi^-(\mu) = \phi_{7,20}(\mu)E^2 = EENEEENEEEEENEE = E^2NE^3NE^5NE^2$$

Figure 4: The extended path $\phi^-(\mu)$ corresponding to the self-conjugate $(7, 20)$-core partition μ with $MD(\mu) = \{13, 5, 3\}$.

By Lemma 2.2, the parity of $c(\lambda)$ depends on whether $1 \in MD(\lambda)$ or not. By the construction of $A_{i,j}$, the array $A(t, tk-1)$ cannot have both 1 and -1 as its entry. Moreover, we characterize $\phi^-(\lambda)$ under the condition that $1 \in MD(\lambda)$.

Lemma 2.5. *For positive integers t, k with $t \geqslant 2$, let λ be a self-conjugate $(t, tk-1)$-core partition.*

1. *When t is even and k is odd, $\phi^-(\lambda)$ starts with at least $k+1$ consecutive E steps if and only if $1 \in MD(\lambda)$.*

2. *When t is odd or k is even, $\phi^-(\lambda)$ ends with at least $k+1$ consecutive E steps if and only if $1 \in MD(\lambda)$.*

Proof. (1) When t is even and k is odd, we have

$$A_{t/2,(k+1)/2} = t(tk-1) - tk - (tk-1)(t-1) = -1.$$

By the equation (3), $\phi^-(\lambda)$ starts with at least $\lceil \frac{k}{2} \rceil + \frac{k+1}{2} = k+1$ consecutive E steps if $1 \in MD(\lambda)$ and $\phi^-(\lambda)$ starts with at most k consecutive E steps otherwise.

(2) When t is odd or k is even, we have

$$A_{1,k(t-1)/2} = t(tk-1) - t\{k(t-1)-1\} - (tk-1) = 1.$$

Again, by the equation (3), $\phi^-(\lambda)$ ends with at least $\lceil \frac{k+1}{2} \rceil + \lfloor \frac{tk-1}{2} \rfloor - \frac{k(t-1)}{2} + 1 = k+1$ consecutive E steps if $1 \in MD(\lambda)$ and $\phi^-(\lambda)$ ends with at most k consecutive E steps otherwise. $\square$

Let t, k, and m be positive integers such that $t \geqslant 2$. For a self-conjugate $(t, tk-1)$-core partition λ, we can express $\phi^-(\lambda)$ as

$$\phi^-(\lambda) = E^{x_1} N E^{x_2} N \cdots N E^{x_{t_1}} N E^{x_{t_1+1}},$$

where $t_1 = \lfloor t/2 \rfloor$, $x_i \in \mathbb{N}$ for all i, and $x_1 + x_2 + \cdots + x_{t_1+1} = k(t_1+1)$. Recall that $x_i \in \mathbb{N}$ for all i. Now we regard λ as a self-conjugate $(t, tk-1)$-core partition with m corners, which means that $c(\lambda) = m$ and $d(\lambda) = m_1 := \lceil m/2 \rceil$ by Lemma 2.2. From Lemma 2.3, we have

$$d(\lambda) = E_{k+1}(\phi^-(\lambda)) = \sum_{x_i \geqslant k+1} (x_i - k) = \sum_{x_i \leqslant k} (k - x_i), \tag{4}$$

where the last equality comes from the fact that $\sum\limits_{i=1}^{t_1+1} x_i = k(t_1+1)$.

2.2 Path interpretations of self-conjugate $(t, tk+1)$-core partitions with m corners

Similar to Proposition 2.1, we obtain the number of self-conjugate $(t, tk+1)$-core partitions with a small number of corners.

Proposition 2.6. *Let t, k, and m be positive integers with $t \geqslant 2$, $k \geqslant 1$, and $m \leqslant k$. Then the number of self-conjugate $(t, tk+1)$-core partitions with m corners is*

$$\sum_{i=1}^{\min(m, \lfloor t/2 \rfloor)} \binom{\lfloor \frac{m-1}{2} \rfloor}{\lfloor \frac{i-1}{2} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor}{\lfloor \frac{i}{2} \rfloor} \binom{\lfloor \frac{t}{2} \rfloor + m - i}{m}.$$

Recall that $d(\lambda)$ is equal to the number of $h \in MD(\lambda)$ such that $h - 2 \notin MD(\lambda)$. We give another interpretation of $d(\lambda)$.

Lemma 2.7. *For a self-conjugate partition λ, $d(\lambda)$ is equal to the number of $h \in MD(\lambda)$ such that $h + 2 \notin MD(\lambda)$.*

Proof. Let λ be a self-conjugate partition with $MD(\lambda) = \{\delta_1, \delta_2, \ldots, \delta_d\}$, where $\delta_1 > \delta_2 > \cdots > \delta_d$. We define two sets H and H' by

$$H = \{\delta_i \in MD(\lambda) \mid \delta_i - 2 \notin MD(\lambda)\} \quad \text{and} \quad H' = \{\delta_i \in MD(\lambda) \mid \delta_i + 2 \notin MD(\lambda)\}.$$

First, we have $\delta_d \in H$ and $\delta_1 \in H'$ by the construction of $MD(\lambda)$. For each $i = 1, 2, \ldots, d-1$, if $\delta_i - \delta_{i+1} > 2$, then $\delta_i \in H$ if and only if $\delta_{i+1} \in H'$. Therefore, we get $|H| = |H'|$, and by definition of $d(\lambda)$, we have $d(\lambda) = |H'|$. $\square$

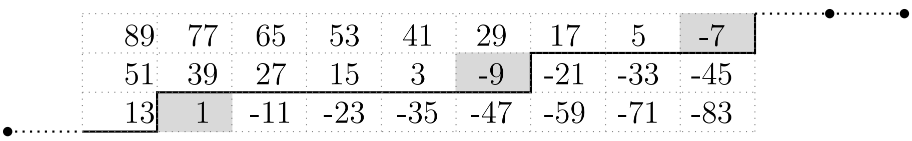
For a self-conjugate $(t, tk+1)$-core partition λ, we denote by $\phi_{t,tk+1}(\lambda)$ the corresponding NE lattice path of λ. We define the extended path $\phi^+(\lambda)$ by $\phi^+(\lambda) = E^x\phi_{t,tk+1}(\lambda)E^y$, where

$$(x, y) = \begin{cases} (\lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil) & \text{if } 2 \mid t, \\ (0, \lfloor \frac{k}{2} \rfloor) & \text{if } 2 \nmid t. \end{cases} \tag{5}$$

Note that the extended path $\phi^+(\lambda)$ has t_1 N steps and $k(t_1+1)$ E steps by the construction of ϕ^+. Similarly to Lemma 2.3, we give the following relation between λ and $\phi^+(\lambda)$. Using Lemma 2.7, Lemma 2.8 is proved analogously to the proof of Lemma 2.3, so we omit the proof.

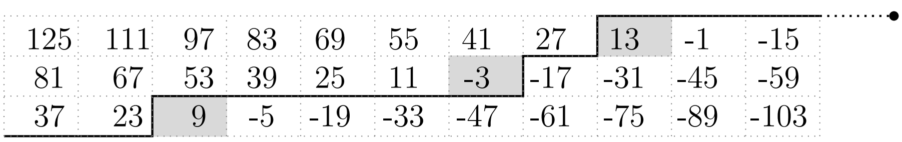
Lemma 2.8. *Let t and k be positive integers satisfying $t \geqslant 2$ and λ be a self-conjugate $(t, tk+1)$-core partition. Then, $d(\lambda) = E_{k+1}(\phi^+(\lambda))$.*

Example 2.9. Let λ be the self-conjugate $(6, 19)$-core partition with $MD(\lambda) = \{9, 7, 1\}$ and μ be the self-conjugate $(7, 22)$-core partition with $MD(\mu) = \{13, 9, 3\}$. Note that $d(\lambda) = 2$ and $d(\mu) = 3$. We have $k = 3$ for both λ and μ, so the extended paths are $\phi^+(\lambda) = E\phi_{6,19}(\lambda)E^2$ and $\phi^+(\mu) = \phi_{7,22}(\mu)E$ (see Figures 5 and 6). Since $E_4(\phi^+(\lambda)) = 2$ and $E_4(\phi^+(\mu)) = 3$, we can see that both λ and μ satisfy Lemma 2.8.



$$\phi^+(\lambda) = E\phi_{6,19}(\lambda)E^2 = EENEEEEENEEENEE = E^2NE^5NE^3NE^2$$

Figure 5: The extended path $\phi^+(\lambda)$ corresponding to the self-conjugate $(6, 19)$-core partition λ with $MD(\lambda) = \{9, 7, 1\}$.



$$\phi^+(\mu) = \phi_{7,22}(\mu)E = EENEEEEENENEEEE = E^2NE^5NENE^4$$

Figure 6: The extended path $\phi^+(\mu)$ corresponding to the self-conjugate $(7, 22)$-core partition μ with $MD(\mu) = \{13, 9, 3\}$.

Similar to Lemma 2.5, we characterize $\phi^+(\lambda)$ according to whether $1 \in MD(\lambda)$ or not.

Lemma 2.10. *For positive integers t and k with $t \geqslant 2$, let λ be a self-conjugate $(t, tk+1)$-core partition.*

1. *When t is even and k is odd, $\phi^+(\lambda)$ starts with at least k consecutive E steps if and only if $1 \notin MD(\lambda)$.*
2. *When t is odd or k is even, $\phi^+(\lambda)$ ends with at least k consecutive E steps if and only if $1 \notin MD(\lambda)$.*

Proof. (1) Assume that t is even and k is odd. We have

$$A_{t/2,(k+1)/2} = t(tk+1) - tk - (tk+1)(t-1) = 1.$$

By the equation (5), $\phi^+(\lambda)$ starts with at least $\frac{k-1}{2} + \frac{k+1}{2} = k$ consecutive E steps if $1 \notin MD(\lambda)$ and $\phi^+(\lambda)$ starts with at most $k-1$ consecutive E steps otherwise.

(2) First, we consider t is odd. Then, we get

$$A_{1,k(t-1)/2+1} = t(tk+1) - t\{k(t-1)+1\} - (tk+1) = -1.$$

By the equation (5), $\phi^+(\lambda)$ ends with at least $(\lfloor \frac{tk+1}{2} \rfloor - \frac{k(t-1)}{2}) + \lfloor \frac{k}{2} \rfloor = k$ consecutive E steps if $1 \notin MD(\lambda)$ and $\phi^+(\lambda)$ ends with at most $k-1$ consecutive E steps otherwise. Similarly, we get the result when k is even. $\square$

Let t, k, and m be positive integers such that $t \geqslant 2$. For a self-conjugate $(t, tk+1)$-core partition λ, we can express $\phi^+(\lambda)$ as

$$\phi^+(\lambda) = E^{x_1} N E^{x_2} N \cdots N E^{x_{t_1}} N E^{x_{t_1+1}},$$

where $t_1 = \lfloor t/2 \rfloor$, $x_i \in \mathbb{N}$ for all i, and $x_1 + x_2 + \cdots + x_{t_1+1} = k(t_1+1)$. Now, we consider λ as a self-conjugate $(t, tk+1)$-core partition with m corners, that is $c(\lambda) = m$ and $d(\lambda) = m_1 := \lceil m/2 \rceil$ by Lemma 2.2. From Lemma 2.8, we have

$$d(\lambda) = E_{k+1}(\phi^+(\lambda)) = \sum_{x_i \geqslant k+1} (x_i - k) = \sum_{x_i \leqslant k} (k - x_i). \tag{6}$$

3 Proofs of Theorems 1.1, 1.2, and 1.3

In this section, we give a proof of our main results.

Proof of Theorem 1.1. We first focus on self-conjugate $(t, tk-1)$-core partitions. We show that there is an injection from the set of self-conjugate $(t, tk-1)$-core partitions with m corners to the set $\mathcal{A}_{t,tk-1}(m)$. For a self-conjugate $(t, tk-1)$-core partition λ with m corners, we have the corresponding extended path

$$\phi^-(\lambda) = E^{x_1} N E^{x_2} N \cdots N E^{x_{t_1}} N E^{x_{t_1+1}},$$

which satisfies that $(x_1, x_2, \ldots, x_{t_1+1}) \in \mathcal{A}$.

First, we consider when t is odd. By Lemmas 2.2 and 2.5 (2), and the equation (3), we have $x_{t_1+1} \geqslant k+1$ if m is odd and $\lceil \frac{k+1}{2} \rceil \leqslant x_{t_1+1} \leqslant k$ otherwise. Hence, we get $(x_1, x_2, \ldots, x_{t_1+1}) \in \mathcal{A}_{t,tk-1}(m)$. Next, suppose that t and k are both even. By Lemmas

2.2 and 2.5 (2), and the equation (3), we get $x_1 \geqslant \frac{k}{2}$, and $x_{t_1+1} \geqslant k+1$ if m is odd and $\frac{k}{2}+1 \leqslant x_{t_1+1} \leqslant k$ otherwise. Therefore, we obtain $(x_1, x_2, \dots, x_{t_1+1}) \in \mathcal{A}_{t,tk-1}(m)$. The last case is when t is even and k is odd. By Lemmas 2.2 and 2.5 (1), and the equation (3), we get $x_{t_1+1} \geqslant \frac{k+1}{2}$, and $x_1 \geqslant k+1$ if m is odd and $\frac{k+1}{2} \leqslant x_1 \leqslant k$ otherwise. Hence, we have $(x_1, x_2, \dots, x_{t_1+1}) \in \mathcal{A}_{t,tk-1}(m)$.

For surjectivity, let $(x_1, x_2, \dots, x_{t_1+1}) \in \mathcal{A}_{t,tk-1}(m)$. There is a self-conjugate $(t, tk-1)$-core partition λ such that the extended path of λ is given by

$$\phi^-(\lambda) = E^{x_1} N E^{x_2} N \cdots N E^{x_{t_1+1}}.$$

From Lemmas 2.2 and 2.5, we have the following:

1. Suppose that t is even and k is odd. Then, $x_1 \geqslant k+1$ if m is odd and $x_1 \leqslant k$ otherwise.

2. Suppose that t is odd or k is even. Then, $x_{t_1+1} \geqslant k+1$ if m is odd and $x_{t_1+1} \leqslant k$ otherwise.

By Lemma 2.3, we check that λ has m corners. In a similar manner, we can show that there is a bijection between the set of self-conjugate $(t, tk+1)$-core partitions with m corners and the set $\mathcal{A}_{t,tk+1}(m)$. Here, we use Lemmas 2.2 and 2.10, and the equation (5) for injectivity and Lemma 2.8 for surjectivity. $\square$

From now on, we enumerate the cardinality of $\mathcal{A}_{t,tk-1}(m)$ in order to obtain the number of self-conjugate $(t, tk-1)$-core partitions with m corners. We divide into two cases whether t is even or odd.

First, assume that t is even. For odd k, the elements of $\mathcal{A}_{t,tk-1}(m)$ are obtained from those for even k by reversing the entries of each tuple, i.e., the elements in the set $\mathcal{A}_{t,tk-1}(m)$ for odd k and in the set $\mathcal{A}_{t,tk-1}(m)$ for even k are symmetric.

Theorem 3.1. *For positive integers t, k, m with even $t \geqslant 2$, let $t_1 = t/2$, $m_1 = \lceil m/2 \rceil$, and $\alpha_{r,s} = t_1 + m_1 - s - (k+1)r$. For even m, we have*

$$\begin{aligned}
&|\mathcal{A}_{t,tk-1}(m)| = \\
&\quad \sum_{s \geqslant 1} \binom{t_1-1}{s-1}\binom{m_1-1}{s-1} \sum_{r \geqslant 0} (-1)^r \binom{t_1-s}{r} \left[\binom{\alpha_{r,s}}{t_1-s} - \binom{\alpha_{r,s} - \lfloor (k+1)/2 \rfloor}{t_1-s} \right] \\
&\quad + \sum_{s \geqslant 1} \binom{t_1-1}{s}\binom{m_1-1}{s-1} \sum_{r \geqslant 0} (-1)^r \binom{t_1-s-1}{r} \\
&\quad \times \left[\binom{\alpha_{r,s}}{t_1-s} - \binom{\alpha_{r,s} - \lfloor k/2 \rfloor - 1}{t_1-s} - \binom{\alpha_{r,s} - \lfloor (k+1)/2 \rfloor}{t_1-s} + \binom{\alpha_{r,s} - (k+1)}{t_1-s} \right].
\end{aligned}$$

For odd m, we have

$$\begin{aligned}
|\mathcal{A}_{t,tk-1}(m)| &= \sum_{s \geqslant 1} \binom{t_1-1}{s-1}\binom{m_1-1}{s} \sum_{r \geqslant 0} (-1)^r \binom{t_1-s}{r} \binom{\alpha_{r,s}-1}{t_1-s-1} \\
&+ \sum_{s \geqslant 0} \binom{t_1-1}{s}\binom{m_1-1}{s} \sum_{r \geqslant 0} (-1)^r \binom{t_1-s-1}{r} \left[\binom{\alpha_{r,s}-1}{t_1-s-1} - \binom{\alpha_{r,s} - \lfloor k/2 \rfloor - 2}{t_1-s-1} \right].
\end{aligned}$$

Proof. Due to the symmetry, we only consider the case when k is even. Let $(x_1,\dots,x_{t_1+1})\in\mathcal{A}_{t,tk-1}(m)$. Throughout the proof, we denote $y_i=x_i-k-1$ if $x_i\geqslant k+1$, $z_i=k-x_i$ if $x_i\leqslant k$, and $s=|\{1\leqslant i\leqslant t_1 \mid x_i\geqslant k+1\}|$.

Suppose that m is even so that $x_1\geqslant k/2$ and $k/2+1\leqslant x_{t_1+1}\leqslant k$. When $s=0$, we get $x_i\leqslant k$ for all $1\leqslant i\leqslant t_1+1$, which implies that $m=0$. Now, we assume that $s\geqslant 1$.

First we consider the case when $x_1\geqslant k+1$. Without loss of generality, suppose that $x_1,\dots,x_s\geqslant k+1$ and $x_{s+1},\dots,x_{t_1+1}\leqslant k$. Since

$$m_1=\sum_{i=1}^{s}(x_i-k)=\sum_{i=s+1}^{t_1+1}(k-x_i),$$

it is sufficient to evaluate the number of solutions of the following equations:

$$\sum_{i=1}^{s} y_i=m_1-s \text{ for } y_1,\dots,y_s\geqslant 0, \tag{7}$$

$$\sum_{i=s+1}^{t_1+1} z_i=m_1 \text{ for } 0\leqslant z_{s+1},\dots,z_{t_1}\leqslant k \text{ and } 0\leqslant z_{t_1+1}\leqslant \frac{k}{2}-1. \tag{8}$$

The number of solutions of equation (7) is $\left(\!\binom{s}{m_1-s}\!\right)=\binom{m_1-1}{s-1}$. By the inclusion-exclusion principle and the condition of z_{t_1+1}, the number of solutions of equation (8) is

$$\sum_{r\geqslant 0}(-1)^r\binom{t_1-s}{r}\left[\binom{t_1+m_1-s-r(k+1)}{t_1-s}-\binom{t_1+m_1-s-r(k+1)-k/2}{t_1-s}\right],$$

where r represents the number of z_i's such that $z_i>k$ in each solution. Since $x_1\geqslant k+1$ and $x_{t_1+1}\leqslant k$, by considering the selection of $s-1$ many $x_i\geqslant k+1$ among $x_2,\dots,x_{t_1}$, we multiply a factor $\binom{t_1-1}{s-1}$.

Next, we suppose that $k/2\leqslant x_1\leqslant k$. Without loss of generality, we assume that $x_2,x_3,\dots,x_{s+1}\geqslant k+1$. Then, we evaluate the number of solutions of the following equations:

$$\sum_{i=2}^{s+1} y_i=m_1-s \text{ for } y_2,\dots,y_{s+1}\geqslant 0, \tag{9}$$

$$z_1+\sum_{i=s+2}^{t_1+1} z_i=m_1 \text{ for } 0\leqslant z_1\leqslant \frac{k}{2}, 0\leqslant z_{s+2},\dots,z_{t_1}\leqslant k, \text{ and } 0\leqslant z_{t_1+1}\leqslant \frac{k}{2}-1. \tag{10}$$

It is clear that the number of solutions of equation (9) is $\binom{m_1-1}{s-1}$. By the inclusion-exclusion principle and the conditions of z_1 and z_{t_1+1}, the number of solutions of equation (10) is

$$\sum_{r\geqslant 0}(-1)^r\binom{t_1-s-1}{r}\left[\binom{t_1+m_1-s-r(k+1)}{t_1-s}-\binom{t_1+m_1-s-r(k+1)-k/2-1}{t_1-s}\right.$$

$$-\binom{t_1+m_1-s-r(k+1)-k/2}{t_1-s}+\binom{t_1+m_1-s-r(k+1)-k-1}{t_1-s}\Bigg],$$

where r represents the number of z_i's such that $z_i > k$ in each solution. Since there are s many $x_i \geqslant k+1$ among $x_2, \ldots, x_{t_1}$, we also multiply $\binom{t_1-1}{s}$. By adding the two summands, we get the result.

Suppose that m is odd, so that $x_1 \geqslant k/2$ and $x_{t_1+1} \geqslant k+1$. We first assume that $x_1 \geqslant k+1$. Without loss of generality, suppose that

$$x_1, \ldots, x_s \geqslant k+1 \quad \text{and} \quad x_{s+1}, \ldots, x_{t_1} \leqslant k.$$

Then we find the number of solutions of the following equations:

$$y_{t_1+1} + \sum_{i=1}^{s} y_i = m_1 - s - 1 \text{ for } y_1, \ldots, y_s, y_{t_1+1} \geqslant 0, \tag{11}$$

$$\sum_{i=s+1}^{t_1} z_i = m_1 \text{ for } 0 \leqslant z_{s+1}, \ldots, z_{t_1} \leqslant k. \tag{12}$$

Now we assume that $k/2 \leqslant x_1 \leqslant k$. Without loss of generality, suppose that

$$x_2, \ldots, x_{s+1} \geqslant k+1 \text{ and } x_{s+2}, \ldots, x_{t_1} \leqslant k.$$

Thus, we evaluate the number of solutions of the following equations:

$$y_{t_1+1} + \sum_{i=2}^{s+1} y_i = m_1 - s - 1 \text{ for } y_2, \ldots, y_{s+1}, y_{t_1+1} \geqslant 0, \tag{13}$$

$$z_1 + \sum_{i=s+2}^{t_1} z_i = m_1 \text{ for } 0 \leqslant z_1 \leqslant k/2 \text{ and } 0 \leqslant z_{s+2}, \ldots, z_{t_1} \leqslant k. \tag{14}$$

Similarly to the case where m is even, by counting the number of solutions of equations (11), (12), (13), and (14), we get the desired result. $\square$

We now consider the case when t is odd.

Theorem 3.2. *For positive integers t, k, m with odd $t \geqslant 3$, let $t_1 = (t-1)/2$, $m_1 = \lceil m/2 \rceil$, and $\alpha_{r,s} = t_1 + m_1 - s - (k+1)r$ for some nonnegative integers r and s. Then, we have*

$$|\mathcal{A}_{t,tk-1}(m)| = \begin{cases} \sum\limits_{s\geqslant 0} \binom{t_1}{s}\binom{m_1-1}{s} \sum\limits_{r\geqslant 0} (-1)^r \binom{t_1-s}{r}\binom{\alpha_{r,s}-1}{t_1-1-s} & \text{if } 2 \nmid m, \\ \sum\limits_{s\geqslant 1} \binom{t_1}{s}\binom{m_1-1}{s-1} \sum\limits_{r\geqslant 0} (-1)^r \binom{t_1-s}{r}\left(\binom{\alpha_{r,s}}{t_1-s} - \binom{\alpha_{r,s}-\lfloor (k+1)/2 \rfloor}{t_1-s}\right) & \text{if } 2 \mid m. \end{cases}$$

Proof. Let $(x_1, x_2, \ldots, x_{t_1+1}) \in \mathcal{A}_{t,tk-1}(m)$. First, we suppose that m is odd. Let $s = |\{1 \leqslant i \leqslant t_1 \mid x_i \geqslant k+1\}|$. Without loss of generality, suppose that $x_1, \ldots, x_s \geqslant k+1$ and

$x_{s+1},\ldots,x_{t_1} \leqslant k$. Let $y_i = x_i - k - 1$ for $i = 1,\ldots,s$ or t_1+1 and $z_i = k - x_i$ for $i = s+1,\ldots,t_1$. We evaluate the number of solutions of the following equations:

$$\sum_{i=1}^{s} y_i + y_{t_1+1} = m_1 - s - 1 \text{ for } y_1,\ldots,y_s,y_{t_1+1} \geqslant 0,$$
$$\sum_{i=s+1}^{t_1} z_i = m_1 \text{ for } 0 \leqslant z_{s+1},\ldots,z_{t_1} \leqslant k.$$

We obtain the number of solutions for each equation in a similar way to the proof of Theorem 3.1, which gives the desired result.

When m is even, we consider the condition where $\lceil (k+1)/2 \rceil \leqslant x_{t_1+1} \leqslant k$. We omit the details since it is analogous to the case when m is odd. □

Proof of Theorem 1.2. It comes directly by combining Theorems 1.1, 3.1, and 3.2. □

In the proof of Theorem 3.3, we define new variables y_i and z_i depending on x_i and find an equation that each variable satisfies. The number of solutions of such equations gives an expression for $|A_{t,tk+1}(m)|$.

Theorem 3.3. *For positive integers t, k, m with even $t \geqslant 2$, let $t_1 = t/2$, $m_1 = \lceil m/2 \rceil$, and $\beta_r = m_1 - rk - 1$. For even m, we have*

$$\begin{aligned}|\mathcal{A}_{t,tk+1}(m)| &= \sum_{s\geqslant 1}\binom{t_1-1}{s-1}\binom{s+m_1}{s}\sum_{r\geqslant 0}(-1)^r\binom{t_1-s}{r}\binom{\beta_r}{t_1-1-s}\\ &+\sum_{s\geqslant 0}\binom{t_1-1}{s}\binom{s+m_1}{s}\sum_{r\geqslant 0}(-1)^r\binom{t_1-1-s}{r}\left[\binom{\beta_r}{t_1-1-s}-\binom{\beta_r-\lfloor\frac{k}{2}\rfloor}{t_1-1-s}\right].\end{aligned}$$

For odd m, we have

$$\begin{aligned}|\mathcal{A}_{t,tk+1}(m)| = &\sum_{s\geqslant 1}\binom{t_1-1}{s-1}\binom{s+m_1-1}{s-1}\sum_{r\geqslant 0}(-1)^r\binom{t_1-s}{r}\left[\binom{\beta_r}{t_1-s}-\binom{\beta_r-\lceil\frac{k}{2}\rceil}{t_1-s}\right]\\ &+\sum_{s\geqslant 1}\binom{t_1-1}{s}\binom{s+m_1-1}{s-1}\sum_{r\geqslant 0}(-1)^r\binom{t_1-1-s}{r}\\ &\times\left[\binom{\beta_r}{t_1-s}-\binom{\beta_r-\lfloor\frac{k}{2}\rfloor}{t_1-s}-\binom{\beta_r-\lceil\frac{k}{2}\rceil}{t_1-s}+\binom{\beta_r-k}{t_1-s}\right].\end{aligned}$$

Proof. The proof is similar to the proof of Theorem 3.1, so here we only show the case where both k and m are even and omit the rest of the cases.

For $(x_1,\ldots,x_{t_1+1}) \in \mathcal{A}_{t,tk+1}(m)$, let $y_i = x_i - k$ if $x_i \geqslant k$ and $z_i = k-1-x_i$ otherwise. Since m is even, we have $x_1 \geqslant k/2$ and $x_{t_1+1} \geqslant k$. Define $s = |\{1 \leqslant i \leqslant t_1 \mid x_i \geqslant k\}|$.

Suppose first that $x_1 \geqslant k$ so that $s \geqslant 1$. Without loss of generality, assume that $x_i \geqslant k$ for $1 \leqslant i \leqslant s$ or t_1+1, and $x_i \leqslant k-1$ otherwise. Since

$$m_1 = \sum_{i=1}^{s}(x_i - k) + (x_{t_1+1} - k) = \sum_{i=s+1}^{t_1}(k - x_i),$$

it suffices to enumerate the number of solutions of the following equations:

$$\sum_{i=1}^{s} y_i + y_{t_1+1} = m_1 \text{ for } y_1, \ldots, y_s, y_{t_1+1} \geqslant 0, \tag{15}$$

$$\sum_{i=s+1}^{t_1} z_i = m_1 - t_1 + s \text{ for } 0 \leqslant z_{s+1}, \ldots, z_{t_1} \leqslant k-1. \tag{16}$$

The number of solutions of equation (15) is $\left(\!\binom{s+1}{m_1}\!\right) = \binom{s+m_1}{s}$. By the inclusion-exclusion principle, the number of solutions of equation (16) is

$$\sum_{r \geqslant 0} (-1)^r \binom{t_1 - s}{r} \binom{m_1 - rk - 1}{t_1 - s - 1},$$

where r represents the number of z_i's such that $z_i \geqslant k$ in each solution. Since $x_1 \geqslant k$, by considering the choice of selecting $s-1$ many $x_i \geqslant k$ among $x_2, \ldots, x_{t_1}$, we multiply a factor $\binom{t_1-1}{s-1}$.

Next, assume that $k/2 \leqslant x_1 \leqslant k-1$. Without loss of generality, suppose that $x_2, x_3, \ldots, x_{s+1} \geqslant k$. Then, we evaluate the number of solutions of the following equations:

$$\sum_{i=2}^{s+1} y_i + y_{t_1+1} = m_1 \text{ for } y_2, \ldots, y_{s+1}, y_{t_1+1} \geqslant 0, \tag{17}$$

$$z_1 + \sum_{i=s+2}^{t_1} z_i = m_1 - t_1 + s \text{ for } 0 \leqslant z_1 \leqslant k/2 - 1 \text{ and } 0 \leqslant z_{s+2}, \ldots, z_{t_1} \leqslant k-1. \tag{18}$$

The number of the solutions of equation (17) is $\left(\!\binom{s+1}{m_1}\!\right) = \binom{s+m_1}{s}$. By the inclusion-exclusion principle and the condition of z_1, the number of solutions of equation (18) is

$$\sum_{r \geqslant 0} (-1)^r \binom{t_1 - 1 - s}{r} \left[\binom{m_1 - rk - 1}{t_1 - 1 - s} - \binom{m_1 - rk - k/2 - 1}{t_1 - 1 - s}\right],$$

where r represents the number of z_i's such that $z_i \geqslant k$ in each solution. Since there are s many $x_i \geqslant k$ among $x_2, \ldots, x_{t_1}$, we also multiply $\binom{t_1-1}{s}$. By adding the two summands, we get the desired result. $\square$

By an argument similar to the proof of Theorem 3.3, we obtain the following theorem.

Theorem 3.4. *For positive integers t, k, m with odd $t \geqslant 3$, let $t_1 = (t-1)/2$, $m_1 = \lceil m/2 \rceil$, and $\beta_r = m_1 - kr - 1$ for some nonnegative integers r and s. Then, we have*

$$|\mathcal{A}_{t,tk+1}(m)| = \begin{cases} \sum\limits_{s \geqslant 0} \binom{t_1}{s}\binom{s+m_1}{s} \sum\limits_{r\geqslant 0} (-1)^r \binom{t_1-s}{r}\binom{\beta_r}{t_1-1-s} & \text{if } \ 2 \mid m, \\ \sum\limits_{s\geqslant 1} \binom{t_1}{s}\binom{s+m_1-1}{s-1} \sum\limits_{r\geqslant 0}(-1)^r\binom{t_1-s}{r}\left(\binom{\beta_r}{t_1-s} - \binom{\beta_r - \lceil \frac{k}{2}\rceil}{t_1-s}\right) & \text{if } \ 2 \nmid m. \end{cases}$$

Proof of Theorem 1.3. It is directly derived from Theorems 1.1, 3.3, and 3.4. □

4 Additional results

In this section, we give additional results on self-conjugate $(t, tk \pm 1)$-core partitions with m corners.

4.1 Another expression for self-conjugate $(t, tk \pm 1)$-core partitions with m corners

We have another expression for the number of self-conjugate $(t, tk \pm 1)$-core partitions with m corners. For a formal power series $f(x, y)$, the term $[x^a y^b]f(x, y)$ refers to the coefficient of $x^a y^b$ in $f(x, y)$.

Recall that the elements in the set $\mathcal{A}_{t,tk\pm1}(m)$ for odd k and in the set $\mathcal{A}_{t,tk\pm1}(m)$ for even k are symmetric.

Theorem 4.1. *Let t, k, and m be positive integers with $t \geqslant 2$, and $t_1 = \lfloor t/2 \rfloor$. The number of self-conjugate $(t, tk-1)$-core partitions with m corners is*

$$\begin{cases} [z^{k(t_1+1)}q^m]\left(\sum\limits_{x=0}^{\infty} z^x q^{|x-k|}\right)^{t_1-1}\left(\sum\limits_{x=\lceil \frac{k}{2}\rceil}^{\infty} z^x q^{|x-k|}\right)\left(\sum\limits_{x=\lceil\frac{k+1}{2}\rceil}^{k} z^x q^{|x-k|}\right) & \text{if } \ 2\mid t, 2\mid m, \\ [z^{k(t_1+1)}q^{m+1}]\left(\sum\limits_{x=0}^{\infty} z^x q^{|x-k|}\right)^{t_1-1}\left(\sum\limits_{x=\lceil \frac{k}{2}\rceil}^{\infty} z^x q^{|x-k|}\right)\left(\sum\limits_{x=k+1}^{\infty} z^x q^{|x-k|}\right) & \text{if } \ 2\mid t, 2\nmid m, \\ [z^{k(t_1+1)}q^m]\left(\sum\limits_{x=0}^{\infty} z^x q^{|x-k|}\right)^{t_1}\left(\sum\limits_{x=\lceil\frac{k+1}{2}\rceil}^{k} z^x q^{|x-k|}\right) & \text{if } \ 2\nmid t, 2\mid m, \\ [z^{k(t_1+1)}q^{m+1}]\left(\sum\limits_{x=0}^{\infty} z^x q^{|x-k|}\right)^{t_1}\left(\sum\limits_{x=k+1}^{\infty} z^x q^{|x-k|}\right) & \text{if } \ 2\nmid t, 2\nmid m. \end{cases}$$

Proof. First, assume that t is even. Consider the term

$$\left(\sum_{x=0}^{\infty} z^x q^{|x-k|}\right)^{t_1-1} \quad \text{as} \quad \prod_{i=2}^{t_1}\left(\sum_{x_i=0}^{\infty} z^{x_i} q^{|x_i-k|}\right).$$

Due to the symmetry of k, we only consider the case when k is even. Then, the number of self-conjugate $(t, tk-1)$-core partitions with m corners is $|\mathcal{A}_{t,tk-1}(m)|$, where each element

$(x_1, x_2, \ldots, x_{t_1+1}) \in \mathcal{A}_{t,tk-1}(m)$ satisfies the conditions $\sum_{i=1}^{t_1+1} x_i = k(t_1+1)$, $x_1 \geqslant k/2$, $x_{t_1+1} \geqslant k/2+1$, and $\sum_{x_i \geqslant k+1} (x_i - k) = m_1$ with $m_1 = \lceil m/2 \rceil$. Since $\sum_{x_i \geqslant k+1} (x_i - k) = \sum_{x_i \leqslant k} (k - x_i)$, we have $\sum_{i=1}^{t_1+1} |x_i - k| = 2m_1$. The second term in each case is obtained by the condition $x_1 \geqslant k/2$. By Lemma 2.5, we have $x_{t_1+1} \geqslant k+1$ when m is odd and $k/2+1 \leqslant x_{t_1+1} \leqslant k$ when m is even, so we get the third term of each product.

The proof where t is odd is similar to the proof where t is even, so we omit the details. $\square$

Similarly, we have another expression for the number of self-conjugate $(t, tk+1)$-core partitions with m corners in terms of generating functions.

Theorem 4.2. *Let t, k, and m be positive integers with $t \geqslant 2$, and $t_1 = \lfloor t/2 \rfloor$. The number of self-conjugate $(t, tk+1)$-core partitions with m corners is*

$$\begin{cases} [z^{k(t_1+1)}q^m] \left(\sum_{x=0}^{\infty} z^x q^{|x-k|}\right)^{t_1-1} \left(\sum_{x=\lceil \frac{k}{2} \rceil}^{\infty} z^x q^{|x-k|}\right) \left(\sum_{x=k}^{\infty} z^x q^{|x-k|}\right) & \text{if } 2 \mid t, 2 \mid m, \\ [z^{k(t_1+1)}q^{m+1}] \left(\sum_{x=0}^{\infty} z^x q^{|x-k|}\right)^{t_1-1} \left(\sum_{x=\lceil \frac{k}{2} \rceil}^{\infty} z^x q^{|x-k|}\right) \left(\sum_{x=\lfloor \frac{k}{2} \rfloor}^{k-1} z^x q^{|x-k|}\right) & \text{if } 2 \mid t, 2 \nmid m, \\ [z^{k(t_1+1)}q^m] \left(\sum_{x=0}^{\infty} z^x q^{|x-k|}\right)^{t_1} \left(\sum_{x=k}^{\infty} z^x q^{|x-k|}\right) & \text{if } 2 \nmid t, 2 \mid m, \\ [z^{k(t_1+1)}q^{m+1}] \left(\sum_{x=0}^{\infty} z^x q^{|x-k|}\right)^{t_1} \left(\sum_{x=\lfloor \frac{k}{2} \rfloor}^{k-1} z^x q^{|x-k|}\right) & \text{if } 2 \nmid t, 2 \nmid m. \end{cases}$$

Proof. We first prove the case where t is even. Due to the symmetry of k, we consider the case where k is even. By the definition of $\mathcal{A}_{t,tk+1}(m)$, we have $\sum_{i=1}^{t_1+1} x_i = k(t_1+1)$, $x_1 \geqslant k/2$, $x_{t_1+1} \geqslant k/2$, and $\sum_{x_i \geqslant k+1} (x_i - k) = m_1$ for $(x_1, \ldots, x_{t_1+1}) \in \mathcal{A}_{t,tk+1}(m)$, where $m_1 = \lceil m/2 \rceil$. Since $\sum_{x_i \geqslant k} (x_i - k) = \sum_{x_i < k} (k - x_i) = m_1$, we get $\sum_{i=1}^{t_1+1} |x_i - k| = 2m_1$. The second term in each product is obtained by the inequality $x_1 \geqslant k/2$. By Lemma 2.10, we have $x_{t_1+1} \geqslant k$ when m is even and $k/2 \leqslant x_{t_1+1} \leqslant k-1$ when m is odd, so we obtain the third term of each product. Since the proof where t is odd is similar, we omit the details. $\square$

4.2 The case when t is small

We have nice results for the cases where $t = 2, 3, 4, 5$. Although the main result contains double sums, we obtain simple results for each case without any summation terms.

Corollary 4.3. *Let k and m be positive integers. Then, the number of self-conjugate $(2, 2k-1)$-core partitions with m corners is*

$$\begin{cases} 1 & \text{if } 1 \leqslant m \leqslant k-1, \\ 0 & \text{if } m \geqslant k. \end{cases}$$

Proof. Note that each nonempty 2-core partition is always the staircase partition $(j, j-1, \dots, 2, 1)$ for a positive integer j and has j corners. Thus, all possible $(2, 2k-1)$-core partitions are of the form $(j, j-1, \dots, 1)$ for $j = 1, 2, \dots, k-1$, so we obtain the desired result. □

Corollary 4.4. *Let k and m be positive integers. When m is even, so that $m = 2m_1$ for some positive integer m_1, the number of self-conjugate $(3, 3k-1)$-core partitions with m corners is*

$$\begin{cases} 1 & \text{if } 1 \leqslant m_1 \leqslant \lfloor \frac{k-1}{2} \rfloor, \\ 0 & \text{if } m_1 \geqslant \lfloor \frac{k-1}{2} \rfloor + 1. \end{cases}$$

When m is odd, so that $m = 2m_1 - 1$ for some positive integer m_1, the number of self-conjugate $(3, 3k-1)$-core partitions with m corners is

$$\begin{cases} 1 & \text{if } 1 \leqslant m_1 \leqslant k, \\ 0 & \text{if } m_1 \geqslant k+1. \end{cases}$$

Proof. It follows from Theorem 1.1 that there is a bijection between the set of self-conjugate $(3, 3k-1)$-core partitions with m corners and the set $\mathcal{A}_{3,3k-1}(m)$. Let $(x_1, x_2) \in \mathcal{A}_{3,3k-1}(m)$.

Suppose that m is even. By the definition of $\mathcal{A}_{3,3k-1}(m)$, we have the condition $\lceil (k+1)/2 \rceil \leqslant x_2 \leqslant k$. When $\lceil k/2 \rceil \leqslant x_1 \leqslant k$, both x_1 and x_2 must be k because $x_1 + x_2 = 2k$, so that $m = 0$. When $x_1 \geqslant k+1$, we have $x_1 - k = k - x_2 = m_1$ and $x_1 \leqslant 2k - \lceil (k+1)/2 \rceil = k + \lfloor (k-1)/2 \rfloor$. Therefore, the equation $x_1 - k = k - x_2 = m_1$ has exactly one solution for each $1 \leqslant m_1 \leqslant \lfloor (k-1)/2 \rfloor$, and has no solution otherwise. Suppose that m is odd. In this case, we have $x_2 \geqslant k+1$ so that $x_1 \leqslant k-1$ since $x_1 + x_2 = 2k$. Therefore, the equation $k - x_1 = x_2 - k = m_1$ has the unique solution for each $1 \leqslant m_1 \leqslant k$, and has no solution otherwise. □

The following result gives the number of such partitions when $t = 4$.

Corollary 4.5. *Let k and m be positive integers. When m is even, so that $m = 2m_1$ for some positive integer m_1, the number of self-conjugate $(4, 4k-1)$-core partitions with m corners is*

$$\begin{cases} 3m_1 + 1 & \text{if } 1 \leqslant m_1 \leqslant \lfloor \frac{k-1}{2} \rfloor, \\ 2k - \lfloor \frac{k}{2} \rfloor - m_1 & \text{if } \lceil \frac{k}{2} \rceil \leqslant m_1 \leqslant k + \lfloor \frac{k-1}{2} \rfloor, \\ 0 & \text{if } m_1 \geqslant k + \lceil \frac{k}{2} \rceil. \end{cases}$$

When m is odd, so that $m = 2m_1 - 1$ for some positive integer m_1, the number of self-conjugate $(4, 4k-1)$-core partitions with m corners is

$$\begin{cases} 3m_1 - 1 & \text{if } 1 \leqslant m_1 \leqslant \lfloor \frac{k}{2} \rfloor, \\ \lfloor \frac{k}{2} \rfloor + m_1 & \text{if } \lceil \frac{k+1}{2} \rceil \leqslant m_1 \leqslant k, \\ k + \lceil \frac{k+1}{2} \rceil - m_1 & \text{if } k+1 \leqslant m_1 \leqslant k + \lfloor \frac{k}{2} \rfloor, \\ 0 & \text{if } m_1 \geqslant k + \lceil \frac{k+1}{2} \rceil. \end{cases}$$

Proof. We show the case where both k and m are even and the rest of the cases can be proved in a similar way, so we omit the details. Let $(x_1, x_2, x_3) \in \mathcal{A}_{4,4k-1}(m)$. Then, by the definition of $\mathcal{A}_{4,4k-1}(m)$, we have

$$m_1 = \sum_{x_i \geqslant k+1} (x_i - k) = \sum_{x_i \leqslant k} (k - x_i), \quad x_1 \geqslant \frac{k}{2}, \quad \frac{k}{2} + 1 \leqslant x_3 \leqslant k.$$

Now, we consider the following three cases:

$$(1)\ x_1 \leqslant k, \quad (2)\ x_1 \geqslant k+1,\ x_2 \leqslant k, \quad (3)\ x_1 \geqslant k+1,\ x_2 \geqslant k+1.$$

Case (1): We obtain that $x_2 \geqslant k+1$ since $x_1, x_3 \leqslant k$ and $x_1 + x_2 + x_3 = 3k$. We only need to find the number of solutions of

$$x_2 - k = 2k - x_1 - x_3 = m_1,\ \frac{k}{2} \leqslant x_1 \leqslant k,\ \frac{k}{2} + 1 \leqslant x_3 \leqslant k. \tag{19}$$

When $1 \leqslant m_1 \leqslant k/2 - 1$, let $x_1' = k - x_1$ and $x_3' = k - x_3$. Then we have $0 \leqslant x_1' \leqslant k/2$, $0 \leqslant x_3' \leqslant k/2 - 1$, and $x_1' + x_3' = m_1$, so the number of solutions of equation (19) is $m_1 + 1$.

If $k/2 \leqslant m_1 \leqslant k - 1$, let $x_1'' = x_1 - k/2$ and $x_3'' = x_3 - k/2 - 1$. Then we have $0 \leqslant x_1'' \leqslant k/2$, $0 \leqslant x_3'' \leqslant k/2 - 1$, and $x_1'' + x_3'' = k - 1 - m_1$, so the number of solutions of equation (19) is $k - m_1$.

If $m_1 \geqslant k$, the equation (19) has no solution since $2k - m_1 = x_1 + x_3 \geqslant k + 1$.

Case (2): In this case, we find the number of solutions of

$$x_1 - k = 2k - x_2 - x_3 = m_1,\ x_1 \geqslant k+1,\ x_2 \leqslant k,\ \frac{k}{2} + 1 \leqslant x_3 \leqslant k. \tag{20}$$

Similarly to Case (1), the number of solutions of equation (20) is

$$\begin{cases} m_1 + 1 & \text{if } 1 \leqslant m_1 \leqslant \frac{k}{2} - 1, \\ \frac{k}{2} & \text{if } \frac{k}{2} \leqslant m_1 \leqslant k - 1, \\ \frac{3}{2}k - m_1 & \text{if } k \leqslant m_1 \leqslant \frac{3}{2}k - 1, \\ 0 & \text{if } m_1 \geqslant \frac{3}{2}k. \end{cases}$$

Case (3): We find the number of solutions of

$$x_1 + x_2 - 2k = k - x_3 = m_1,\ x_1 \geqslant k+1,\ x_2 \geqslant k+1,\ \frac{k}{2} + 1 \leqslant x_3 \leqslant k. \tag{21}$$

Using a similar argument to Case (1), the number of solutions of equation (21) is

$$\begin{cases} m_1 - 1 & \text{if } 1 \leqslant m_1 \leqslant \frac{k}{2} - 1, \\ 0 & \text{if } m_1 \geqslant \frac{k}{2}. \end{cases}$$

Combining all the cases, we have the result. □

The case where $t = 5$ is similar to the case where $t = 4$, so we omit the proof.

Corollary 4.6. *Let k and m be positive integers. When m is even, so that $m = 2m_1$ for some positive integer m_1, the number of self-conjugate $(5, 5k-1)$-core partitions with m corners is*

$$\begin{cases} 3m_1 + 1 & \textit{if } 1 \leqslant m_1 \leqslant \lfloor \frac{k-1}{2} \rfloor, \\ 2\lfloor \frac{k+1}{2} \rfloor & \textit{if } \lceil \frac{k}{2} \rceil \leqslant m_1 \leqslant k, \\ 2k + 2\lfloor \frac{k+1}{2} \rfloor - 2m_1 & \textit{if } k+1 \leqslant m_1 \leqslant k + \lfloor \frac{k-1}{2} \rfloor, \\ 0 & \textit{if } m_1 \geqslant k + \lceil \frac{k}{2} \rceil. \end{cases}$$

When m is odd, so that $m = 2m_1 - 1$ for some positive integer m_1, the number of self-conjugate $(5, 5k-1)$-core partitions with m corners is

$$\begin{cases} 3m_1 - 1 & \textit{if } 1 \leqslant m_1 \leqslant k, \\ 2k - m_1 + 1 & \textit{if } k+1 \leqslant m_1 \leqslant 2k, \\ 0 & \textit{if } m_1 \geqslant 2k+1. \end{cases}$$

A proof for the number of $(t, tk+1)$-core partitions with m corners when $t = 2, 3, 4$, or 5 is similar to a proof of Corollaries 4.3, 4.4, 4.5, and 4.6, respectively, so we obtain the following corollaries.

Corollary 4.7. *Let k and m be positive integers. Then, the number of self-conjugate $(2, 2k+1)$-core partitions with m corners is*

$$\begin{cases} 1 & \textit{if } 1 \leqslant m \leqslant k, \\ 0 & \textit{if } m \geqslant k+1. \end{cases}$$

Corollary 4.8. *Let k and m be positive integers. When m is even, so that $m = 2m_1$ for some positive integer m_1, the number of self-conjugate $(3, 3k+1)$-core partitions with m corners is*

$$\begin{cases} 1 & \textit{if } 1 \leqslant m_1 \leqslant k, \\ 0 & \textit{if } m_1 \geqslant k+1. \end{cases}$$

When m is odd, so that $m = 2m_1 - 1$ for some positive integer m_1, the number of self-conjugate $(3, 3k+1)$-core partitions with m corners is

$$\begin{cases} 1 & \textit{if } 1 \leqslant m_1 \leqslant \lceil \frac{k}{2} \rceil, \\ 0 & \textit{if } m_1 \geqslant \lceil \frac{k}{2} \rceil + 1. \end{cases}$$

Corollary 4.9. *Let k and m be positive integers. When m is even, so that $m = 2m_1$ for some positive integer m_1, the number of self-conjugate $(4, 4k+1)$-core partitions with m corners is*

$$\begin{cases} 3m_1 + 1 & \text{if } 1 \leqslant m_1 \leqslant \lfloor \frac{k}{2} \rfloor, \\ \lfloor \frac{k}{2} \rfloor + m_1 + 1 & \text{if } \lceil \frac{k+1}{2} \rceil \leqslant m_1 \leqslant k, \\ k + \lfloor \frac{k}{2} \rfloor + 1 - m_1 & \text{if } k+1 \leqslant m_1 \leqslant k + \lfloor \frac{k}{2} \rfloor, \\ 0 & \text{if } m_1 \geqslant k + \lceil \frac{k+1}{2} \rceil. \end{cases}$$

When m is odd, so that $m = 2m_1 - 1$ for some positive integer m_1, the number of self-conjugate $(4, 4k+1)$-core partitions with m corners is

$$\begin{cases} 3m_1 - 1 & \text{if } 1 \leqslant m_1 \leqslant \lceil \frac{k}{2} \rceil, \\ k + \lceil \frac{k}{2} \rceil - m_1 + 1 & \text{if } \lceil \frac{k}{2} \rceil + 1 \leqslant m_1 \leqslant k + \lceil \frac{k}{2} \rceil, \\ 0 & \text{if } m_1 \geqslant k + \lceil \frac{k}{2} \rceil + 1. \end{cases}$$

Corollary 4.10. *Let k and m be positive integers. When m is even, so that $m = 2m_1$ for some positive integer m_1, the number of self-conjugate $(5, 5k+1)$-core partitions with m corners is*

$$\begin{cases} 3m_1 + 1 & \text{if } 1 \leqslant m_1 \leqslant k, \\ 2k - m_1 + 1 & \text{if } k+1 \leqslant m_1 \leqslant 2k, \\ 0 & \text{if } m_1 \geqslant 2k + 1. \end{cases}$$

When m is odd, so that $m = 2m_1 - 1$ for some positive integer m_1, the number of self-conjugate $(5, 5k+1)$-core partitions with m corners is

$$\begin{cases} 3m_1 - 1 & \text{if } 1 \leqslant m_1 \leqslant \lceil \frac{k}{2} \rceil, \\ 2\lceil \frac{k}{2} \rceil & \text{if } \lceil \frac{k}{2} \rceil + 1 \leqslant m_1 \leqslant k, \\ 2k + 2\lceil \frac{k}{2} \rceil - 2m_1 + 2 & \text{if } k+1 \leqslant m_1 \leqslant k + \lceil \frac{k}{2} \rceil, \\ 0 & \text{if } m_1 \geqslant k + \lceil \frac{k}{2} \rceil + 1. \end{cases}$$

We remark that the case $k = 1$ of Theorem 1.3 provides an alternative proof of [1, Theorem 1.4], which is stated and proved in Corollary 4.11.

Corollary 4.11. *[1, Theorem 1.4] For positive integers t and m, the number of self-conjugate $(t, t+1)$-core partitions with m corners is*

$$\binom{\lfloor \frac{t-1}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor} \binom{\lfloor \frac{t}{2} \rfloor}{\lfloor \frac{m+1}{2} \rfloor}.$$

Proof. First, we consider when t and m are even. Note that

$$\mathcal{A}_{t,t+1}(m) = \mathcal{A} \cap \{(x_1, x_2, \ldots, x_{t_1+1}) \mid x_1 \geqslant 1, x_{t_1+1} \geqslant 1\}.$$

Let $(x_1, x_2, \ldots, x_{t_1+1}) \in \mathcal{A}_{t,t+1}(m)$. Then, letting $m_1 = \lceil m/2 \rceil$, the number of self-conjugate $(t, t+1)$-core partitions with m corners is equal to the number of solutions of equations as follows:

$$\sum_{x_i \geqslant 1}(x_i - 1) = \sum_{x_i < 1}(1 - x_i) = m_1, \quad x_1 \geqslant 1, \quad x_{t_1+1} \geqslant 1.$$

We have $m_1 = |\{i \mid x_i = 0\}|$ from the above equations. Without loss of generality, let $x_2 = x_3 = \cdots = x_{m_1+1} = 0$. Then, we have $x_i \geqslant 1$ for $i = 1, m_1+2, \ldots, t_1+1$ and

$$(x_1 - 1) + (x_{m_1+2} - 1) + \cdots + (x_{t_1+1} - 1) = m_1. \tag{22}$$

The number of solutions of equation (22) is $\left(\!\!\binom{t_1-m_1+1}{m_1}\!\!\right) = \binom{t_1}{m_1}$. Since there are m_1 many $x_i = 0$ among $x_2, \ldots, x_{t_1}$, we also multiply $\binom{t_1-1}{m_1}$. Hence, we obtain that the number of self-conjugate $(t, t+1)$-core partitions with m corners is

$$\binom{t_1}{m_1}\binom{t_1-1}{m_1} = \binom{\frac{t}{2}}{\frac{m}{2}}\binom{\frac{t}{2}-1}{\frac{m}{2}}.$$

Similarly, we have that the result also holds where t or m is odd. $\square$

Acknowledgements

The authors thank the referees for the valuable and constructive comments. This work is supported by Korea Institute for Advanced Study (KIAS) grant funded by the Korea government. Hyunsoo Cho was supported by the NRF grant No. RS–2026–25481081. Kyeongjun Lee was supported by the NRF grant No. RS–2025–02262988. Ho-Hyeong Lee, Hayan Nam and Jaebum Sohn were supported by the NRF grant No. RS–2024-00347040.

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