A new approach to Naples parking functions through complete parking preferences, and its enumerative consequences

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Abstract

Naples parking functions were introduced as a generalization of classical parking functions, in which cars are allowed to park backwards, by checking up to a fixed number of previous spots, before proceeding forward as usual.

In this work we introduce the notion of a complete parking preference, through which we are able to give some information on the combinatorics of Naples parking functions. Roughly speaking, a *complete parking preference* is a parking preference such that, for any index j, there are more cars with preference at least j than spots available from j onward. We provide a characterization of Naples parking functions in terms of certain complete subsequences of them. As a consequence of this result we derive a characterization of permutation-invariant Naples parking functions which turns out to be equivalent to the one given by (Carvalho et al., 2021), but using a totally different approach (and language).

Concerning enumeration, we propose an effective approach to enumerate permutation-invariant Naples parking functions and complete Naples parking functions which is based on some natural combinatorial decompositions. We thus obtain formulas depending on some (generally simpler) quantities, which are of interest in their own right, and that can be described in a recursive fashion.

Mathematics Subject Classifications: 05A15, 05A99

1 Introduction

Parking functions are classical combinatorial objects, originally introduced by Konheim and Weiss [10] in a computer science context, but quickly gaining importance for themselves because of their appearance in numerous other contexts, such as hyperplane arrangements [14], representation theory [11] and combinatorial geometry [1]. More recently,

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there has been a renewed and growing interest in the combinatorics of parking functions, especially with regards to various generalizations that have been proposed. An exhaustive list of articles dealing with parking functions in the last decade is almost impossible to provide. We content ourselves to cite the survey [17] and the article [4], a lovely paper where a large number of possible variations on the theme of parking functions is proposed, with an engaging and appealing style particularly suitable for undergraduates and probably even high school students. All articles cited by [4], as well as all articles citing [4], would deserve to be included in the references of our paper.

One of the most studied variations of parking functions are the so-called *Naples parking functions*, in which cars are also allowed to check up to a certain number of previous spots before proceeding further. The enumerative combinatorics of Naples parking functions is not completely understood yet: in the literature there are recursive formulas for the number of k-Naples parking functions of length n [6] (where k denotes the number of previous spots that each car can check), but finding a closed form seems to be a rather difficult problem, as the only known explicit expression is rather involved [8]. In our work we propose a new approach to the enumerative combinatorics of Naples parking functions, which is based on the notion of a complete parking preference. Given a parking preference α of length n (see below for the formal definition of a parking preference), it may happen for a certain position j that the number of available spots from j onwards is strictly less than the number of cars whose preference is j or more. This of course prevents α from being a parking function, but not a Naples parking function. Such positions are therefore critical in studying Naples parking functions. A complete parking preference is precisely one in which *every* position (except the first, of course) is critical in the above sense. Using the notion of complete parking preference we are then able to provide characterizations of both generic and permutation-invariant Naples parking functions.

In order to illustrate the potential of our new approach, we then provide some new enumerative results on Naples parking functions. More specifically, we exploit our tools and our neat characterization results to enumerate permutation-invariant Naples parking functions and complete Naples parking functions in a recursive fashion.

The paper is organized as follows. After introducing and studying properties of the excess function of a parking preference (Section 2) and of complete parking preferences (Section 3), we describe our characterization results in Section 4. The second part of the paper deals with enumerative applications, and in particular Section 5 discusses the enumeration of permutation-invariant Naples parking functions, whereas Section 6 investigates the enumeration of complete Naples parking functions. More precisely, we describe an effective counting procedure to enumerate permutation-invariant Naples parking functions, thanks to which we are able to produce several terms of the associated counting sequence, whereas in the case of complete Naples parking functions we determine a recursion which, again, allows us to compute the desired values very efficiently. Most of the coefficients we find are not recorded in the Online Encyclopedia of Integer Sequences [13], with a few intriguing exceptions. For a couple of those we are able to prove the conjectured hit (we refer to Sections 5 and 6 for more precise details on the relevant entries in [13]). Finally, Section 7 is devoted to summarize our main results and state our projects for further work.

In closing this introduction, we collect the main definitions and notations that we need in our work.

As usual, given positive integers n, p, q, with $p \leqslant q$, we set $[p, q] = \{k \in \mathbb{N} \mid p \leqslant k \leqslant q\}$ and $[n] = [1, n]$. A *parking preference* $\alpha = (a_1, a_2, \ldots, a_n)$ of length n is a n-tuple of positive integers between 1 and n to be interpreted as follows: for an ordered list $c_1, c_2, \ldots, c_n$ of n cars attempting to park on n numbered spots in a street, a_i denotes the preferred spot for the i-th car c_i. The set of all parking preferences of length n will be denoted PP_n. The total number of cars having preference i will be denoted with $|\alpha|_i$. The *classical* parking rule requires that each car tries to park in its preferred spot and, if this is already occupied, then proceeds forward and parks in the first available spot, if any, otherwise it simply exits the street (thus failing to park). If all cars park successfully using the above rule, then we say that α is a *parking function* of length n, and we denote the set of all those with PF_n.

As an example, the parking preference $\alpha = (3, 1, 3, 5, 2, 4, 2) \in PP_7$ is a parking function, such that, for instance, $|\alpha|_3 = 2$ and $|\alpha|_6 = 0$. During the parking process, we have that car c_1 parks in spot 3, car c_2 parks in spot 1, car c_3 parks in spot 4 (because it finds spot 3 already occupied), then car c_4 parks in spot 5 and car c_5 parks in spot 2; regarding car c_6, which has preference 4, since both spot 4 and 5 are occupied, it parks in spot 6; finally car c_7, which has preference 2, parks in spot 7, which is the last remaining spot.

An elegant characterization of parking functions states that α is a parking function if and only if $\sum_{i=1}^{j} |\alpha|_i \geqslant j$, for all $1 \leqslant j \leqslant n$. The latter condition is equivalent to the fact that the nondecreasing rearrangement $(b_1, b_2, \ldots, b_n)$ of α is such that $b_j \leqslant j$, for all $1 \leqslant j \leqslant n$. As a consequence, we get that any rearrangement of a parking function is a parking function as well. This is usually expressed by saying that parking functions are *permutation-invariant*.

The classical parking rule can be modified into the k-Naples parking rule, thus introducing the notion of k-Naples parking function. Since this is the main object of our paper, we will record it with a formal definition.

Definition 1. The *k-Naples parking rule* is defined as follows: if a car drives up to its preferred spot and finds it occupied, it drives backwards and checks up to k spots prior to its preference, filling the first available one (if it exists); if none of such spots is free, then it proceeds forward looking for the first available spot, like in the classical parking rule. A *k-Naples parking function* is a parking preference such that all cars park successfully following the k-Naples parking rule, and the set of k-Naples parking functions of length n will be denoted $PF_{n,k}$.

We point out that, throughout the whole paper, we will commonly use the general term "Naples parking function" instead of "k-Naples parking function" whenever the role of k is irrelevant.

The notion of k-Naples parking function was originally introduced in [2, 6]. For example, the parking preference $(4, 1, 4, 4, 3)$ is a 1-Naples parking function, as illustrated in Figure 1. We remark that, unlike classical parking functions, a rearrangement of a k-Naples parking function is not a k-Naples parking function in general. For instance, $(3,3,2)$ is a 1-Naples parking function, whereas $(2,3,3)$ is not (see Figure 2). Another example is the 1-Naples parking function $(4,1,4,4,3)$ of Figure 1, since for example $(1,3,4,4,4)$ and $(3,4,4,4,1)$ are not 1-Naples parking functions.

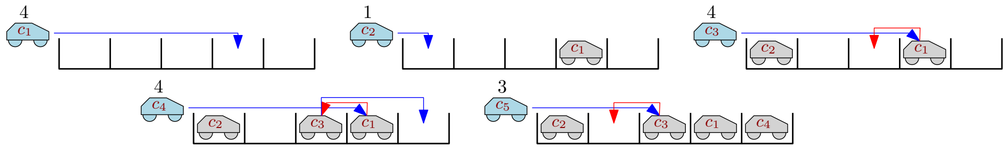

Figure 1: The 1-Naples parking function $(4,1,4,4,3)$.

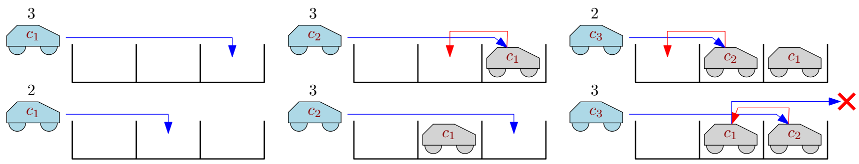

Figure 2: The parking preference $(3,3,2)$ is a 1-Naples parking function, whereas $(2,3,3)$ is not.

As a final example, consider the parking preference $\zeta = (9,5,1,9,4,5,9,8,1,4) \in PP_{10}$. It can be easily checked that it is a 1-Naples parking function, so $\zeta \in PF_{10,1}$. This specific parking preference will play the role of a running example in the rest of the article, and will be used (together with other examples) to illustrate the main notions and results of our work. In particular, we will reserve the symbol ζ to denote it, so that the reader will easily recognize it throughout the paper.

2 The excess function of a parking preference

Let $\alpha = (a_1, a_2, \ldots, a_n) \in PP_n$. Then, for every $1 \leqslant j \leqslant n$, we set

$$u_\alpha(j) = \left(\sum_{i=j}^{n} |\alpha|_i \right) - (n-j+1). \tag{1}$$

In other words, $u_\alpha(j) \in \mathbb{Z}$ denotes the excess of cars having preference at least j with respect to the number of available spots from j onwards. The map u_α will be called the *excess function* of the parking preference α. It is clear that, if $\alpha \in PF_n$, then $u_\alpha(j) \leqslant 0$ for all j (and we will see in a moment that the reverse implication is also true). Thus those positions j for which $u_\alpha(j) > 0$ are somehow critical in trying to understand if the parking preference α is actually a Naples parking function. We set $U_\alpha = \{1 \leqslant j \leqslant n \,|\, u_\alpha(j) \geqslant 1\}$. In other words, U_α is the set of positions having positive excess function, and we remark that clearly $1 \notin U_\alpha$, for all parking preferences α.

Consider our running example ζ. Its excess function is $(0,-1,0,1,0,-1,0,1,1,-1)$, where the i-th entry is $u_\zeta(i)$. Therefore, the excess set of ζ is $U_\zeta = \{4\} \cup [8,9]$. As a further example, consider the parking preference $\alpha = (8,5,1,9,5,8,8,5,1) \in PP_9$. The excess function of α is $(0,-1,0,1,2,0,1,2,0)$. Therefore, the excess set of α is $U_\alpha = [4,5] \cup [7,8]$.

We would like to remark that our excess map is somewhat related to the notion of *defect* of a parking preference introduced in [3]. According to the cited paper, a *defective parking function* of *defect* k is a parking preference for which k cars fail to park. It is easy to realize that, using our terminology, the defect of a parking preference α of length n corresponds to the maximum value attained by the excess map u_α, that is $\max\{u_\alpha(j) \mid 1 \leqslant j \leqslant n\}$.

In the next couple of lemmas we record some basic properties of the excess function that are frequently used in all the proofs of the results presented in this paper.

Lemma 2. *Let $\alpha \in PP_n$. For all $1 \leqslant j, j_1, j_2 \leqslant n$, with $j_1 < j_2$, we have:*

(i) $u_\alpha(j) = j - 1 - \sum_{i=1}^{j-1} |\alpha|_i$;

(ii) $u_\alpha(j_2) - u_\alpha(j_1) = j_2 - j_1 - \sum_{i=j_1}^{j_2-1} |\alpha|_i$;

(iii) $u_\alpha(j) = u_\alpha(j+1) + |\alpha|_j - 1$, *when* $j < n$;

(iv) $u_\alpha(j) < j$.

Proof. Since of course $n = \sum_{i=1}^{j-1} |\alpha|_i + \sum_{i=j}^{n} |\alpha|_i$, using (1) we immediately get (i). Moreover, again using the definition of $u_\alpha(j)$ for $j = j_2, j_1$, we get (ii). Finally, we observe that (iii) is a special case of (ii) (when $j_1 = j$ and $j_2 = j+1$) and (iv) clearly follows from (i). $\square$

Lemma 3. *Let $\alpha \in PP_n$ and let $[p,q]$ be a maximal interval of consecutive positions contained in U_α. Then we have:*

(i) $u_\alpha(p) = 1$;

(ii) $u_\alpha(p-1) = 0$;

(iii) $|\alpha|_{p-1} = 0$;

(iv) $|\alpha|_q \geqslant 2$.

Proof. Recall that $1 \notin U_\alpha$, hence $p \geqslant 2$. Since $[p,q]$ is maximal, we have that $p-1 \notin U_\alpha$, and so $u_\alpha(p-1) \leqslant 0$ and $u_\alpha(p) \geqslant 1$. Using (iii) of Lemma 2 we thus get:

$$1 \leqslant u_\alpha(p) = u_\alpha(p-1) - |\alpha|_{p-1} + 1 \leqslant 1,$$

which gives (i). Moreover, we also have:

$$0 \geqslant u_\alpha(p-1) = u_\alpha(p) + |\alpha|_{p-1} - 1 = |\alpha|_{p-1} \geqslant 0,$$

which gives (ii) and (iii). In order to prove (iv), recall that $u_\alpha(q) \geqslant 1$. If $q = n$, then by definition of the excess function we have that $|\alpha|_n = u_\alpha(n) + 1 \geqslant 2$. If $q < n$, then clearly $u_\alpha(q+1) \leqslant 0$ and so $|\alpha|_q \geqslant |\alpha|_q + u_\alpha(q+1) = u_\alpha(q) + 1 \geqslant 2$, which gives (iv). $\square$

Using (i) of Lemma 2, we get that the inequalities $\sum_{i=1}^{j} |\alpha|_i \geqslant j$ are equivalent to the inequalities $u_\alpha(j+1) \leqslant 0$, at least for all $j = 1, \ldots, n-1$. Moreover, since we know that $u_\alpha(1) = 0$ and $\sum_{i=1}^{n} |\alpha|_i = n$, we have that α is a parking function if and only $u_\alpha(j) \leqslant 0$ for all $1 \leqslant j \leqslant n$. We record (and rephrase) this characterization of parking functions in the next proposition.

Proposition 4. *A parking preference α is a parking function if and only if $U_\alpha = \emptyset$.*

We may now wonder whether a similar result holds also for Naples parking functions. Since Naples parking functions are *not* in general permutation-invariant, a characterization in terms of the excess function only is not possible. However, we observe that the following necessary condition holds, which generalizes Proposition 4.

Proposition 5. *If $\alpha \in PF_{n,k}$, then $u_\alpha(j) \leqslant k$, for all $1 \leqslant j \leqslant n$.*

Proof. By contradiction, suppose that there exists $1 \leqslant j \leqslant n$ such that $u_\alpha(j) > k$. That is, the number of cars whose preference is at least j is strictly greater than the number of available spots from j onwards plus k. By (i) of Lemma 2 we know that $u_\alpha(i) \leqslant i-1$, for all $1 \leqslant i \leqslant n$, and so in particular we have $j \geqslant k+2$, i.e., there are at least $k+1$ parking spots before the j-th one. Since $\alpha \in PF_{n,k}$, all cars must be able to park, and our hypothesis implies that there is at least one car (among those with preference at least j) that needs to check at least up to $k+1$ spots before its preference in order to park. This is however impossible, since α is a k-Naples parking function. $\square$

Unfortunately, the above condition is not sufficient, as we have already mentioned that the parking preference $(2,3,3)$ is not a 1-Naples parking function (and $u_\alpha(j) = 1$, for $j = 2, 3$). Such a condition is however sufficient to show that there is at least one rearrangement of the parking preference that is a Naples parking function.

Proposition 6. *Suppose that $\alpha \in PP_n$ is such that $u_\alpha(j) \leqslant k$, for all $1 \leqslant j \leqslant n$. If α is nonincreasing, then $\alpha \in PF_{n,k}$.*

Proof. Suppose by contradiction that $\alpha \notin PF_{n,k}$, that is there is at least one index i such that car c_i is not able to park. This means that, when c_i tries to park, all the spots in $[a_i - k, n]$ are already occupied[1]. Since α is nonincreasing, all such spots have been occupied by cars with preference in $[a_i, n]$. As a consequence, the number of cars with preference in $[a_i, n]$ is at least $n - (a_i - k - 1) + 1 = n - a_i + k + 2$, hence

$$u_\alpha(a_i) = \sum_{j=a_i}^{n} |\alpha|_j - (n - a_i + 1) \geqslant (n - a_i + k + 2) - (n - a_i + 1) = k + 1,$$

which contradicts the hypothesis. $\square$

[1]Notice that, since there are as many cars as spots, then necessarily $a_i - k > 1$, otherwise all the n spots would be already occupied.

3 Complete parking preferences

We are now ready to introduce complete parking preferences. These are a special subclass of parking preferences that play a crucial role in our characterization of Naples parking functions.

Given $n \geqslant 2$, a parking preference $\alpha \in PP_n$ is said to be *complete* when $U_\alpha = [2, n]$. In other words, a complete parking preference is such that the set U_α is as large as possible (remember that $1 \notin U_\alpha$). As an example, the parking preference $(5, 3, 3, 5, 4) \in PP_5$ is complete, whereas $(5, 3, 3, 4, 4)$ is not (in the latter preference the excess at position 5 is zero). We will show in Corollary 25 that the total number of complete parking preferences of length n is $(n-1)^{n-1}$. Moreover, Theorem 30 will provide an efficient recursive formula to enumerate complete k-Naples parking functions.

In order to collect some useful properties of complete parking preferences and functions, we need to introduce one more piece of notation. For a parking preference $\alpha = (a_1, a_2, \dots, a_n) \in PP_n$, and for a given integer $k \geqslant 0$, we consider the *outcome map* $\psi_k^{(\alpha)}$, which is defined by setting $\psi_k^{(\alpha)}(i)$ to be the parking spot occupied by car c_i following the k-Naples parking rule, with the convention that $\psi_k^{(\alpha)}(i) = \infty$ if car c_i is unable to park. This notion extends to arbitrary parking preferences the definition introduced in [8] for both classical and Naples parking functions. Referring again to Figure 1, the outcome map of the 1-Naples parking function $(4, 1, 4, 4, 3)$ is $(4, 1, 3, 5, 2)$. Considering our running example $\zeta = (9, 5, 1, 9, 4, 5, 9, 8, 1, 4)$, the outcome map $\psi_1^{(\zeta)}$ of ζ as a 1-Naples parking function is $(9, 5, 1, 8, 4, 6, 10, 7, 2, 3)$. Notice however that, if we consider ζ as a 2-Naples parking function, then its outcome map $\psi_2^{(\zeta)}$ is $(9, 5, 1, 8, 4, 3, 7, 6, 2, 10)$. This illustrates the intuitive fact that the same preference may have different outcome maps depending on the parking rule followed by the cars.

Our first goal is to provide a characterization of complete parking preferences that are also k-Naples parking functions. To this aim, the following lemma will be useful. It shows an interesting property of cars parking by driving forward, namely they create a sort of splitting of the set of spots.

Lemma 7. *Let $\alpha = (a_1, a_2, \dots, a_n) \in PP_n$ and assume that all cars follow the k-Naples parking rule (for some fixed $k \geqslant 1$). Let $i, j \leqslant n$ be such that $\psi_k^{(\alpha)}(i) = j$ (i.e., car c_i parks in spot j) and $a_i < j$ (i.e., car c_i parks by driving forward). Then there are no cars with preference in $[j, n]$ that park in any spot in $[1, j]$. Moreover, if $\alpha \in PF_{n,k}$, then $u_\alpha(j) \leqslant -1$.*

Proof. Let $h \leqslant n$ be such that $a_h \in [j, n]$. If $h < i$, then spot j is still unoccupied, so car c_h cannot park before position j (and of course it cannot park in position j, since we are assuming that car c_i will park in that position). Now suppose that $h > i$. Since car c_i has parked by driving forward, it has checked all the spots in the interval $[a_i - k, j]$, finding them already occupied. This implies that car c_h cannot find any reachable free spots before position j (since $a_h \geqslant j$ and c_h can check up to k spot before its preferred one). Finally, if $\alpha \in PF_{n,k}$, then all cars are able to park, and in particular all spots in

$[1, j]$ are occupied by cars having preferences in $[1, j-1]$. Therefore there are at least j cars having preferences in $[1, j-1]$ and so (using (i) of Lemma 2)

$$u_\alpha(j) = j - 1 - \sum_{i=1}^{j-1} |\alpha|_j \leqslant -1,$$

as desired. ☐

We now prove our characterization, which essentially states that complete Naples parking functions are those in which *every* car parks by driving (weakly) backwards.

Proposition 8. *Let $\alpha = (a_1, a_2, \ldots, a_n) \in PP_n$ be complete, and assume that all cars follow the k-Naples parking rule. The following are equivalent:*

(i) $\alpha \in PF_{n,k}$;

(ii) for all $1 \leqslant j \leqslant n$, *spot* j *is occupied by a car having preference at least* j;

(iii) for all $1 \leqslant j \leqslant n$, $\psi_k^{(\alpha)}(j) \leqslant a_j$.

Proof. Suppose that $\alpha \in PF_{n,k}$. If there is some $j \leqslant n$ such that spot j is occupied by a car whose preference is strictly less than j, then Lemma 7 tells us that $u_\alpha(j) \leqslant -1$, contradicting the fact that α is complete. This proves that (i) implies (ii). The fact that (ii) implies (iii) is obvious (notice, in particular, that by (ii) all spots are occupied, which excludes the possibility that $\psi_k^{(\alpha)}(j) = \infty$ for any j). Finally, if (iii) holds, then necessarily $\psi_k^{(\alpha)}(j) < \infty$ for all $1 \leqslant j \leqslant n$, which means that all cars are able to park, i.e., $\alpha \in PF_{n,k}$. ☐

We are also able to refine the above characterization by providing more precise quantitative results about it.

Proposition 9. *Let $\alpha \in PP_n$ be complete, and assume that all cars follow the k-Naples parking rule. Then, for all $1 \leqslant j \leqslant n$,*

$$|\{1 \leqslant i \leqslant n \,|\, a_i \geqslant j, \psi_k^{(\alpha)}(i) < j\}| \leqslant u_\alpha(j). \tag{2}$$

In other words, the number of cars having preference at least j which park in a spot strictly less than j is at most $u_\alpha(j)$. Moreover, if $\alpha \in PF_{n,k}$, then in (2) the equality holds.

Proof. We start by observing that, if j is a spot which remains unoccupied, then the left hand side of (2) is 0, and the thesis for j immediately follows because α is complete. Therefore in the rest of the proof we assume that spot j is occupied by some car.

Suppose first that $j = n$. If spot n is occupied by a car whose preference is strictly less than n, then Lemma 7 implies that there are no cars with preference n which are able to park, hence in particular the left hand side of (2) is 0 and the thesis follows for n. On the other hand, if spot n is occupied by a car having preference n, then by definition of the excess function there are exactly $u_\alpha(n)$ more cars having preference n, i.e. $u_\alpha(n)$ is an upper bound for the number of cars having preference n which are able to park somewhere strictly before n.

Now let $j < n$ and suppose by induction that (2) holds for $j+1$. If spot j is occupied by a car whose preference is strictly less than j, we can again invoke Lemma 7 to get the thesis for j. Suppose instead that spot j is occupied by a car having preference at least j. In this case we have:

$$\begin{aligned}|\{i \leqslant n \mid a_i \geqslant j, \psi_k^{(\alpha)}(i) < j\}| &= |\{i \leqslant n \mid a_i = j, \psi_k^{(\alpha)}(i) < j\}| + |\{i \leqslant n \mid a_i \geqslant j+1, \psi_k^{(\alpha)}(i) < j\}| \\ &\leqslant |\alpha|_j + |\{i \leqslant n \mid a_i \geqslant j+1, \psi_k^{(\alpha)}(i) < j+1\}| \\ &\leqslant |\alpha|_j + u_\alpha(j+1),\end{aligned}$$

where the last inequality follows from the inductive hypothesis. Since we are assuming that spot j is occupied by a car c having preference at least j, there are two possible options: either c has preference j, in which case $|\{i \leqslant n \mid a_i = j, \psi_k^{(\alpha)}(i) < j\}| \leqslant |\alpha|_j - 1$, or c has preference strictly larger than j, in which case $|\{i \leqslant n \mid a_i \geqslant j+1, \psi_k^{(\alpha)}(i) < j\}| < |\{i \leqslant n \mid a_i \geqslant j+1, \psi_k^{(\alpha)}(i) < j+1\}|$, and so $|\{i \leqslant n \mid a_i \geqslant j+1, \psi_k^{(\alpha)}(i) < j\}| \leqslant u_\alpha(j+1)-1$. We then have (also using (iii) of Lemma 2)

$$|\{i \leqslant n \mid a_i \geqslant j, \psi_k^{(\alpha)}(i) < j\}| \leqslant |\alpha|_j + u_\alpha(j+1) - 1 = u_\alpha(j),$$

as desired.

To conclude, we observe that, if $\beta \in PF_{n,k}$, then all cars are able to park, and so, by definition of the excess function, $u_\beta(j)$ is a lower bound for the number of cars having preference at least j and parking strictly before spot j. So, for a generic k-Naples parking function, the reverse of inequality (2) holds, hence for a complete k-Naples parking function the equality holds. □

Consider again the complete parking preference $\alpha = (5,3,3,5,4)$, having excess function $(0,1,2,1,1)$. It is easy to check that $\alpha \in PF_{5,3}$. Looking at how the cars park, we observe that cars c_1 and c_2 park in their preferred spots, whereas cars c_3 and c_4 take one backward step each, and car c_5 takes three backward steps. This is in accordance with Proposition 8, which says that each car parks either in its preferred spot or needs to drive backwards to find its spot. Moreover, according to Proposition 9, $u_\alpha(i)$ gives precisely the number of cars having preference at least i and parking strictly before spot i.

It is worth observing that, if $\alpha \in PP_n$ is not complete, then inequality (2) in general does not hold. We can illustrate this fact exploiting our running example $\zeta = (9,5,1,9,4,5,9,8,1,4)$, interpreted as a 2-Naples parking function. Indeed, ζ is not complete. Moreover $u_\zeta(9) = 1$, but there are two cars having preference 9 which park strictly before spot 9 (namely c_4 and c_7). Similarly, we have $u_\zeta(5) = 0$, however car c_5 has preference 5 and parks strictly before spot 5.

4 A characterization of k-Naples parking functions through complete subsequences

The present section contains our main result, namely a characterization of Naples parking functions which makes use of the notion of complete parking preference that we have introduced and studied in the previous section. In order to prove such a characterization we need a preliminary result, which is of interest in itself. The following proposition gives a characterization of k-Naples parking functions in terms of the filling of certain critical spots which only depend on the sets U_α.

Proposition 10. *Given $\alpha \in PP_n$ and $k \geqslant 1$, we have that α is a k-Naples parking function if and only if, for every $p \in U_\alpha$ such that $p-1 \notin U_\alpha$, spot $p-1$ is occupied.*

Proof. Clearly, if α is a k-Naples parking function, then every spot is occupied by some car.

On the other hand, suppose that α is not a k-Naples parking function. Let h be the smallest spot which remains unoccupied. Then all spots in $[1, h-1]$ are occupied by cars having preferences in $[1, h-1]$, and there is no car having preference in $[1, h-1]$ which parks after spot h. We thus have (by (i) of Lemma 2) that $u_\alpha(h) = h - 1 - \sum_{i=1}^{h-1} |\alpha|_i = 0$, i.e., $h \notin U_\alpha$. Moreover, since h remains unoccupied, then $|\alpha|_h = 0$, which gives (using (iii) of Lemma 2) $u_\alpha(h+1) = u_\alpha(h) - |\alpha|_h + 1 = 1$, i.e., $h+1 \in U_\alpha$. Thus, for $p = h+1$, we have $p \in U_\alpha$, $p-1 \notin U_\alpha$, and spot $p-1$ is unoccupied. $\square$

Notice that, under the same assumptions on the spot p as in Proposition 10, if $\alpha \in PF_{n,k}$, then all spots are occupied, and (by Lemma 3) $u_\alpha(p-1) = 0$ and $|\alpha|_{p-1} = 0$, hence Lemma 7 implies that spot $p-1$ must be occupied by a car driving backwards (i.e., having preference at least p). Proposition 10 thus identifies a set of "critical spots" $\{i \in [n] \mid i \notin U_\alpha, i+1 \in U_\alpha\}$ that need to be occupied by cars driving backwards in order for the parking preference to be a Naples parking function. Some trivial ways to ensure this condition is satisfied are, for instance, choosing a sufficiently large k (as every parking preference is a $(n-1)$-Naples parking function), or sorting the cars in nonincreasing order of their preferences (as per Proposition 6).

From the point of view of Proposition 10, classical parking functions are characterized by the fact that the above set of critical spots is empty, which is consistent with the fact that $U_\alpha = \emptyset$.

We would like to point out that Proposition 10 also allows us to design an efficient procedure to determine whether a given parking preference is a k-Naples parking function. Indeed, given $\alpha \in PP_n$, just performing the k-Naples parking rules gives an algorithm whose complexity (in the worst case) is $\mathcal{O}(n^2)$ (when each car has to check all spots already occupied by the previous cars, which may occur, for instance, if α is constant). However, Proposition 10 tells us that it is enough to check that the critical spots are occupied by driving backwards. The set of critical spots of α can be computed in time $\mathcal{O}(n)$ (as this is the time needed to compute the excess set U_α). Moreover, we can design an algorithm that, for each car c_i, determines the closest reachable spot on its left not already occupied by some previous car by driving backwards. Since we want to know if α is a k-Naples parking function, this can be done in time $\mathcal{O}(n)$ (as each car is allowed to check at most k spots by driving backwards). Denoting by B the set of spots determined by the above algorithm, it will be enough to check that each critical spot is contained in B to conclude that α is a k-Naples parking function. The total time complexity of the whole procedure is thus $\mathcal{O}(n)$ (or $\mathcal{O}(kn)$, in case k is used as a parameter). A pseudocode describing the procedure outlined above is reported in Algorithm 1.

The last thing we need before stating our characterization of Naples parking functions is a couple of technical definitions, followed by a technical (but very useful) lemma.

Given $\alpha = (a_1, a_2, \dots, a_n) \in PP_n$ and $w \in \mathbb{Z}$, the *w-shift* of α is the n-tuple $\tau_w(\alpha) = (a_1 - w, a_2 - w, \dots, a_n - w)$. This is particularly meaningful when w is nonnegative and $w \leqslant \min(a_1, a_2, \dots, a_n)$. Moreover, it will be useful to extend the above notation to sets: given $W \subseteq \mathbb{Z}$, we set $\tau_w(W) = \{i \in \mathbb{Z} \mid i + w \in W\}$. Finally, if $J = \{j_1, j_2, \dots, j_h\} \subseteq [1, n]$, with $j_1 < j_2 < \cdots < j_h$, the *restriction* of α to J is the h-tuple $\alpha_{|_J} = (a_{j_1}, a_{j_2}, \dots, a_{j_h})$.

Now let $\alpha \in PP_n$ and suppose that $1 \leqslant j \leqslant n$ is such that $u_\alpha(j) = 0$. Set $J = \{1 \leqslant i \leqslant n \mid a_i \geqslant j\}$ and $J^c = [1, n] \setminus J$. Then it follows immediately from the definition of the excess function that $|J| = n - j + 1$, and so $\alpha_{|_{J^c}} \in PP_{j-1}$ and $\tau_{j-1}(\alpha_{|_J}) \in PP_{n-j+1}$. Moreover, for all i for which it makes sense, an easy computation shows that $u_{\alpha_{|_{J^c}}}(i) = u_\alpha(i)$ (and so $U_{\alpha_{|_{J^c}}} = U_\alpha \cap [1, j-1]$) and $u_{\tau_{j-1}(\alpha_{|_J})}(i) = u_\alpha(i + j - 1)$ (and so $U_{\tau_{j-1}(\alpha_{|_J})} = \tau_{j-1}(U_\alpha) \cap [1, n-j+1] = \tau_{j-1}(U_\alpha \cap [j, n])$). Therefore, roughly speaking, every parking preference α can be decomposed into two smaller parking preferences in correspondence of every position j for which $u_\alpha(j) = 0$, whose corresponding excess functions are essentially the same as the excess function of α (up to a suitable shift when needed).

To illustrate, we consider again our running example $\zeta = (9, 5, 1, 9, 4, 5, 9, 8, 1, 4)$, for which we have $u_\zeta(5) = 0$. With the above notations, setting $j = 5$, we have $J = \{1, 2, 4, 6, 7, 8\}$ and $J^c = \{3, 5, 9, 10\}$, hence $\zeta_{|_{J^c}} = (1, 4, 1, 4)$ and $\tau_4(\zeta_{|_J}) = (5, 1, 5, 1, 5, 4)$. A direct computation shows that indeed the excess function of $\zeta_{|_{J^c}}$ coincides with the excess function of ζ restricted to $[1, 4]$, whereas the excess function of $\tau_4(\zeta_{|_J})$ coincides with the excess function of ζ restricted to $[5, 10]$.

Lemma 11. *Let $\alpha \in PF_{n,k}$ and suppose that $1 \leqslant j \leqslant n$ is such that $u_\alpha(j) = 0$. Then (with the above notations) $\tau_{j-1}(\alpha_{|_J}) \in PF_{n-j+1,k}$.*

Proof. When $j = 1$ the lemma is immediate, so we can suppose that $j \geqslant 2$. Suppose by contradiction that $\tau_{j-1}(\alpha_{|_J})$ is *not* a k-Naples parking function. Denote with h the minimum spot which remains empty in the restricted problem (where of course $1 \leqslant h \leqslant n - j + 1$). We observe that, in the restricted problem, all cars having preferences in $[1, h-1]$ are able to park, and they necessarily park in the spots between 1 and $h - 1$, which are all filled. Thus the number of cars having preferences between 1 and $h - 1$ is precisely $h - 1$, and consequently the number of cars having preference at least h coincides with the number of available spots from h to $n - j + 1$. In other words, we have that $u_{\tau_{j-1}(\alpha_{|_J})}(h) = u_\alpha(h + j - 1) = 0$. Now, since $\alpha \in PF_{n,k}$, in the original problem spot $h + j - 1$ is occupied by some car. We observe that it cannot be occupied by a car having preference strictly less than $h + j - 1$, otherwise, by Lemma 7, we would have

Algorithm 1: Determining if $\alpha \in PF_{n,k}$

Data: $\alpha = (a_1, \ldots, a_n)$; $0 \leqslant k \leqslant n$
Result: TRUE (if success); FALSE (if failure)
$(|\alpha|_1, \ldots, |\alpha|_n) \leftarrow (0, \ldots, 0)$;
for $i \in [n]$ **do**
  $|\alpha|_{a_i} \leftarrow |\alpha|_{a_i} + 1$; `/* compute` $(|\alpha|_1, \ldots, |\alpha|_n)$ `*/`
end
Let $U_\alpha = (u_\alpha(1), \ldots, u_\alpha(n))$;
$u_\alpha(n) \leftarrow |\alpha|_n - 1$;
for $j = n-1, \ldots, 1$ **do**
  $u_\alpha(j) \leftarrow u_\alpha(j+1) + |\alpha|_j - 1$; `/* compute` U_α `*/`
end
Let $K = \{k_1, \ldots, k_n\}$; `/* Critical spots */`
$k_n \leftarrow$ not critical;
for $i \in [n-1]$ **do**
  if $u_\alpha(i) == 0$ *and* $u_\alpha(i+1) \geqslant 1$ **then**
    $k_i \leftarrow$ critical;
  else
    $k_i \leftarrow$ not critical;
  end
end
Let Spots $= (s_1, \ldots, s_n)$; Spots $\leftarrow$ (*empty*, …, *empty*);
for $i \in [n]$ **do**
  success $\leftarrow$ $false$;
  $b \leftarrow 0$; `/* backward steps made */`
  while *success* $\neq$ *true and* $b \leqslant k$ *and* $a_i - b \geqslant 1$ **do**
    if $s_{a_i - b} ==$ *empty* **then**
      $s_{a_i - b} \leftarrow full$;
      success $\leftarrow$ $true$;
    else
      $b \leftarrow b + 1$;
    end
  end
end
for ***each*** i *such that* k_i *is critical* **do**
  if $s_i ==$ *empty* **then**
    return FALSE ; `/* parking fails precisely when critical spots are not filled by moving backwards */`
  end
end
return TRUE;

$u_\alpha(h+j-1) \leqslant -1$, against what we have proved above. Therefore spot $h+j-1$ is occupied by some car driving backward, i.e., having preference strictly greater than $h+j-1$ (notice that there are no cars with preference $h+j-1$, since we are supposing that in the restricted problem there are no cars having preference h, as that spot is empty). However, before spot $h+j-1$ is occupied in the original problem, the behavior of the cars having preference strictly greater than $h+j-1$ is the same in the original and in the restricted problem. This implies that the car which occupies spot $h+j-1$ in the original problem will do the same in the restricted problem, against the hypothesis. $\square$

It is worth observing that Lemma 11 does *not* imply that, if $u_\alpha(j) = 0$, then all cars having preference in $[j, n]$ necessarily park in positions belonging to $[j, n]$. Consider, for instance, the parking preference $\alpha = (4, 4, 3, 2, 3) \in PF_{5,1}$. We have that $u_\alpha(4) = 0$, and the set of cars having preference at least 4 is $\{c_1, c_2\}$. Clearly $\tau_3(\alpha_{|_{\{1,2\}}}) = (1, 1)$ is a 1-Naples parking function, in accordance with Lemma 11. However, during the parking process for α, car c_2 parks in spot $3 \notin \{4, 5\}$.

We also remark that a similar result does not hold in general for the parking preference $\alpha_{|_{J^c}}$. The same preference $\alpha = (4, 4, 3, 2, 3)$ considered above serves as a counterexample. Indeed, recalling that $u_\alpha(4) = 0$, we have that $\alpha_{|_{\{3,4,5\}}} = (3, 2, 3) \notin PF_{3,1}$.

The next theorem contains the announced characterization of Naples parking functions in terms of complete parking subsequences. Recall that Proposition 10 states that a necessary and sufficient condition for α to be a k-Naples parking function is that, for every maximal interval $[p, q] \subseteq U_\alpha$, spot $p-1$ is occupied. Theorem 12 improves upon that, by showing that the only way for that spot to be correctly filled (i.e., by driving backwards) is to have a subsequence with the structure of a complete k-Naples parking function, with the proper shift.

Theorem 12. *Let $\alpha \in PP_n$. We have that $\alpha \in PF_{n,k}$ if and only if, for every maximal interval $[p, q] \subseteq U_\alpha$, there exists $J \subseteq [n]$ such that $a_j \in [p, p-2+|J|]$ (for all $j \in J$) and $\tau_{p-2}(\alpha_{|_J}) \in PF_{|J|,k}$ is a complete k-Naples parking function.*

Proof. We start by supposing that $\alpha \in PP_n$ satisfies the condition in the statement of the theorem. Let $[p, q] \subseteq U_\alpha$ be a maximal interval, and suppose that there exists a subset $J = \{j_1, j_2, \ldots, j_{|J|}\}$ of $[n]$ as above. In the hope of improving the clarity of our argument, we introduce some dedicated notations for the restricted preference. First of all, we denote with $\tilde{\psi}_k$ the outcome map of $\tau_{p-2}(\alpha_{|_J})$. Moreover, we denote with $\tilde{c}_i = c_{j_i}$ the i-th car in the restricted problem, and with $\tilde{a}_i = a_{j_i} - (p-2)$ the corresponding preference. Therefore, to be even more specific, the notation $\tilde{\psi}_k(i)$ denotes the position occupied by car $\tilde{c}_i = c_{j_i}$ in the restricted problem $\tau_{p-2}(\alpha_{|_J})$. Since by hypothesis $\tau_{p-2}(\alpha_{|_J}) \in PF_{|J|,k}$, we know that, in the restricted problem, all spots in $[1, |J|]$ are occupied. We now want to show (by induction) that, in the original problem, the corresponding spots in $[p-1, |J|+(p-2)]$ are also occupied. More formally, we want to show that, for all $i \in [1, |J|]$, all spots of the form $\tilde{\psi}_k(i) + (p-2)$ are occupied in the original problem.

We start by considering $i = 1$. Since car $\tilde{c}_1$ is the first car in the restricted problem, $\tilde{\psi}_k(1) = \tilde{a}_1 = a_{j_1} - (p-2)$. As a consequence, in the original problem either $\psi_k^{(\alpha)}(j_1) = a_{j_1}$

or car c_{j_1} finds spot a_{j_1} already occupied. In either case, after car c_{j_1}'s turn, spot $a_{j_1} = \tilde{\psi}_k(1)+(p-2)$ is occupied in the original problem. Now suppose that, for all $i \in [1, h-1]$, after car c_{j_i}'s turn, spot $\tilde{\psi}_k(i) + (p-2)$ is occupied in the original problem. If $\tilde{\psi}_k(h) = \tilde{a}_h = a_{j_h} - (p-2)$, then we can argue as in the base case above. Otherwise, since $\tau_{p-2}(\alpha_{|J})$ is complete, by Proposition 8 we have that $\tilde{\psi}_k(h) = \tilde{a}_h - m$, for some $m \leqslant k$. This implies that, in the restricted problem, all spots in the interval $[\tilde{a}_h - (m-1), \tilde{a}_h]$ are of the form $\tilde{\psi}_k(i)$, for some $i < h$. Thus, using the induction hypothesis, we get that in the original problem, when car c_h tries to park, all spots in the interval $[a_{j_h} - (m-1), a_{j_h}]$ are already occupied. This means that car c_h reaches spot $a_{j_h} - m = \tilde{a}_h - m + (p-2) = \tilde{\psi}_k(h) + (p-2)$, which is then certainly occupied after c_h's turn (either by c_h itself or by some previous car).

We have thus shown that, in the original problem, all spots in the interval $[p-1, |J| + (p-2)]$ are occupied, so in particular spot $p-1$ is occupied. Repeating the same argument for all maximal intervals $[p, q] \subseteq U_\alpha$, using Proposition 10, we can conclude that $\alpha \in PF_{n,k}$.

Now suppose that $\alpha \in PF_{n,k}$ and let $[p, q] \subseteq U_\alpha$ be a maximal interval. We start by considering the special case $p = 2$. Set $M = \min\{m \leqslant n \mid [1, m] \subseteq \psi_k^{(\alpha)}(\{i \mid a_i \in [1, m]\})\}$. In other words, M is the minimum of the set of all indices m such that all spots in $[1, m]$ are occupied by cars having preferences in $[1, m]$. Observe that such a set is nonempty, since $\alpha \in PF_{n,k}$, and so n belongs to the set, hence M is well defined. Moreover, using (iii) of Lemma 3, we get that $|\alpha|_1 = 0$, and so $M > 1$. Let $J = \{j_1, j_2, \ldots, j_M\}$ be the set of the indices of the cars parking in the spots of the interval $[1, M]$, with $j_1 < j_2 < \cdots < j_M$. Our goal is to show that the restriction $\alpha_{|J}$ is a complete k-Naples parking function.

First of all, it is obvious that $\alpha_{|J} \in PF_{M,k}$. In order to prove that $\alpha_{|J}$ is complete, we now proceed by "reverse induction" to show that, for all $h \in [2, M]$, $u_{\alpha_{|J}}(h) \geqslant 1$. For the case $h = M$, we observe that we cannot have $|\alpha_{|J}|_M = 0$, since otherwise we would have in particular that all spots in $[1, M-1]$ would be occupied by cars having preference in $[1, M-1]$, which contradicts the minimality of M. Similarly, we cannot have $|\alpha_{|J}|_M = 1$. In fact, assuming that c_j is the only car having preference M, spot M is occupied either by a car having preference $< M$, or by c_j; in the former case Lemma 7 would imply that c_j is not able to park (which is not the case, since $\alpha \in PF_{n,k}$), whereas in the latter case we would still get that all spots in $[1, M-1]$ would be occupied by cars having preference in $[1, M-1]$. We can thus conclude that $|\alpha_{|J}|_M \geqslant 2$, hence $u_{\alpha_{|J}}(M) \geqslant 1$.

Now consider $h \in [2, M-1]$ and suppose that $u_{\alpha_{|J}}(i) \geqslant 1$ for all $i \geqslant h+1$. Our goal is to prove that $u_{\alpha_{|J}}(h) \geqslant 1$. By (iii) of Lemma 2 we have that $u_{\alpha_{|J}}(h) = u_{\alpha_{|J}}(h+1) + |\alpha_{|J}|_h - 1$. If $|\alpha_{|J}|_h \geqslant 1$, then clearly $u_{\alpha_{|J}}(h) \geqslant u_{\alpha_{|J}}(h+1) \geqslant 1$. So suppose that $|\alpha_{|J}|_h = 0$ and suppose also, by contradiction, that $u_{\alpha_{|J}}(h) \leqslant 0$ (which means that, concerning the parking preference $\alpha_{|J}$, there are *at least* $h-1$ cars having preference $\leqslant h-1$). Again by the minimality of M, there must be at least one car having preference in $[1, h-1]$ which parks in a spot $i \geqslant h$. If $i \geqslant h+1$, then by Lemma 7 $u_{\alpha_{|J}}(i) \leqslant -1$, which contradicts our induction hypothesis. If instead $i = h$, then again by Lemma 7 we have that $u_{\alpha_{|J}}(h) \leqslant -1$, and so $u_{\alpha_{|J}}(h+1) = u_{\alpha_{|J}}(h) - |\alpha_{|J}|_h + 1 \leqslant 0$, which again is in contrast with the induction hypothesis.

To conclude the proof we now turn to the case of a general p. Given a maximal interval $[p, q] \subseteq U_\alpha$, set $\hat{J} = \{i \leqslant n \mid a_i \geqslant p - 1\}$. By Lemma 11, and recalling that $u_\alpha(p-1) = 0$ (by (ii) of Lemma 3), we have that $\tau_{p-2}(\alpha_{|\hat{J}}) \in PF_{n-(p-2),k}$ and $[2, q-(p-2)]$ is a maximal interval of $U_{\tau_{p-2}(\alpha_{|\hat{J}})}$. Therefore we can exploit what we have done in the case $p = 2$ to get that there exists a nonempty set $J \subseteq \hat{J}$ such that $\tau_{p-2}(\alpha_{|J}) \in PF_{|J|,k}$ is a complete k-Naples parking function. Moreover, since the preferences of all cars in the restricted problem clearly belong to the interval $[2, |J|]$, we also have that $a_j \in [p, |J| + p - 2]$ for all $j \in J$, which concludes the proof. $\square$

Corollary 13. *Suppose that $\alpha \in PF_{n,k}$. Then, for any maximal interval $[p, q]$ of U_α, a set $J \subseteq [n]$ satisfying the conditions of Theorem 12 is such that $|J| \geqslant q - p + 2$.*

Proof. Suppose first that $p = 2$. In the proof of the above theorem we found a set J satisfying the required conditions such that $|J| = M$, where M is the minimum of the set of all indices m such that all spots in $[1, m]$ are occupied by cars having preference in $[1, m]$. We observe that $M + 1 \notin U_\alpha$, since, by (i) of Lemma 2,

$$u_\alpha(M+1) = M - \sum_{i=1}^{M} |\alpha|_i \leqslant 0.$$

As a consequence, referring again to the notations of the above theorem, $|J| \geqslant q$, which is our thesis when $p = 2$.

The general case ($p > 2$) follows from the case $p = 2$ and the last part of the proof of the previous theorem, and is thus left to the reader. $\square$

To illustrate Theorem 12, consider our running example $\zeta = (9, 5, 1, 9, 4, 5, 9, 8, 1, 4) \in PF_{10,1}$, for which we recall that $U_\zeta = \{4\} \cup [8, 9]$. Set $J_1 = \{5, 10\}$, and observe that, for all $j \in J_1$, $a_j = 4$. Now we have that $\tau_2(\zeta_{|J_1}) = (2, 2)$, which is a complete 1-Naples parking function. Similarly, setting $J_2 = \{1, 4, 8\}$, we see that, for all $j \in J_2$, $a_j \in [8, 9]$. In this case we have that $\tau_6(\zeta_{|J_2}) = (3, 3, 2)$, and it is easy to check that this is a complete 1-Naples parking function.

We also observe that, for a given maximal interval of U_α, the set J of Theorem 12 is not necessarily unique. For instance, in the above example we could choose the set $J_1 = \{2, 6, 10\}$ for the maximal interval $\{4\}$, or (if we interpret ζ as a 2-Naples parking function) we could choose $J_2 = \{1, 4, 7\}$ for the maximal interval $[8, 9]$.

A remarkable corollary of the above theorem is a neat characterization of permutation-invariant Naples parking functions in terms of the cardinalities of the maximal intervals of U_α. The following lemma will be useful in the proof of such a corollary.

Lemma 14. *Let $\alpha \in PF_{n,k}$ be a complete k-Naples parking function. Then $\psi_k^{(\alpha)}(n) = 1$ and $a_n \leqslant k + 1$. In words, this means that the last car c_n has preference at most $k + 1$ and parks in spot 1.*

Proof. Let $\psi_k^{(\alpha)}(n) = h$. If $h > 1$, then the set of spots $[1, h-1]$ is nonempty, and all such spots are occupied by cars whose preferences belongs to $[1, h-1]$ (since a car with preference at least h would check spot h before checking previous spots). Therefore the number of cars having preference in $[1, h-1]$ is at least $h-1$, hence $u_\alpha(h) = h - 1 - \sum_{i=1}^{h-1} |\alpha|_i \leqslant 0$, against the fact that α is complete. Thus $h = 1$, as desired. Moreover we get that $1 = \psi_k^{(\alpha)}(n) \geqslant a_n - k$, i.e. $a_n \leqslant k+1$. $\square$

Corollary 15. *Let $\alpha = (a_1, a_2, \ldots, a_n) \in PP_n$ and $k \geqslant 1$. The following are equivalent:*

(i) for every maximal interval $[p, q] \subseteq U_\alpha$, $|[p, q]| \leqslant k$;

(ii) for all permutations σ of size n, the n-tuple $\sigma(\alpha) = (a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)})$ is a k-Naples parking function of length n. In particular, $\alpha \in PF_{n,k}$.

Proof. We start by supposing that condition (i) holds, and let $[p, q]$ be a maximal interval of U_α. Then $u_\alpha(q+1) \leqslant 0$, and so by (iii) of Lemma 2 we have that $|\alpha|_q = u_\alpha(q) - u_\alpha(q+1) + 1 \geqslant u_\alpha(q) + 1$. Now choose arbitrarily $u_\alpha(q) + 1$ cars among those having preference q and let H be the set of their indices. Set $J = \{i \leqslant n \mid a_i \in [p, q-1]\} \cup H$. We want to show that J satisfies the properties stated in Theorem 12. First of all we observe that, by (ii) of Lemma 2, we have

$$\sum_{i=p}^{q-1} |\alpha|_i + u_\alpha(q) - u_\alpha(p) = q - p.$$

Moreover, by (i) of Lemma 3, we also have $u_\alpha(p) = 1$, and so

$$|J| = \sum_{i=p}^{q-1} |\alpha|_i + u_\alpha(q) + 1 = (q-p) + u_\alpha(p) + 1 = q - p + 2.$$

Now consider the parking preference $\tau_{p-2}(\alpha_{|J})$. We show by "reverse induction" that, for all $j \in [1, q-p+2]$, $u_{\tau_{p-2}(\alpha_{|J})}(j) = u_\alpha(j + (p-2))$. Indeed we have:

$$\begin{aligned} u_{\tau_{p-2}(\alpha_{|J})}(q-p+2) &= |\tau_{p-2}(\alpha_{|J})|_{q-p+2} - 1 \\ &= |\alpha_{|J}|_q - 1 = |H| - 1 = u_\alpha(q) \geqslant 1. \end{aligned}$$

Moreover, supposing that $u_{\tau_{p-2}(\alpha_{|J})}(i) = u_\alpha(i + (p-2))$, for all $i \in [j+1, q-p+2]$, we have that

$$\begin{aligned} u_{\tau_{p-2}(\alpha_{|J})}(j) &= u_{\tau_{p-2}(\alpha_{|J})}(j+1) + |\tau_{p-2}(\alpha_{|J})|_{j+1} - 1 \\ &= u_\alpha(j+1+(p-2)) + |\alpha_{|J}|_{j+1+(p-2)} - 1 \\ &= u_\alpha(j+1+(p-2)) + |\alpha|_{j+1+(p-2)} - 1 \\ &= u_\alpha(j + (p-2)), \end{aligned}$$

which concludes the induction argument.

Since $u_\alpha(i) \geqslant 1$ for all $i \in [p, q]$, we thus have that $u_{\tau_{p-2}(\alpha_{|J})}(j) \geqslant 1$ for all $j \in [2, q-p+2]$, i.e. $\tau_{p-2}(\alpha_{|J})$ is complete. Moreover, since $|J| = q-p+2 = |[p, q]|+1 \leqslant k+1$, it follows that $\tau_{p-2}(\alpha_{|_J})$ is a k-Naples parking function. We can thus invoke Theorem 12 to conclude that $\alpha \in PF_{n,k}$. Finally, we notice that the definition of J does not depend on the specific order of the cars. In other words, for any permutation $\sigma \in S_n$, we can construct exactly the same set J also for $\sigma(\alpha) = (a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)})$, so $\sigma(\alpha)$ is also a k-Naples parking function.

Conversely, suppose that there exists a maximal interval $[p, q]$ of U_α such that $|[p, q]| = q - p + 1 > k$. Since to disprove condition (ii) we just need to show that there is at least one rearrangement of α which is not a k-Naples parking function, we can assume without loss of generality that α is weakly increasing. Suppose by contradiction that $\alpha \in PF_{n,k}$. By Theorem 12 we thus have that there exists some J such that $\tau_{p-2}(\alpha_{|_J})$ is a complete k-Naples parking function. Clearly also $\tau_{p-2}(\alpha_{|_J})$ is weakly increasing, and it is immediate to observe that the preference of the last car is equal to $|J|$ (indeed there must be at least one car having preference $|J|$, because $\tau_{p-2}(\alpha_{|_J})$ is complete). However, by Lemma 14, we should have $|J| \leqslant k + 1$, whereas, by Corollary 13, we have $|J| \geqslant q - p + 2 > k + 1$, hence the contradiction. □

Recall that our running example $\zeta = (9, 5, 1, 9, 4, 5, 9, 8, 1, 4)$ has excess set $U_\zeta = \{4\} \cup [8, 9]$. In particular, the largest interval of U_ζ has size 2. Thus, Corollary 15 gives us that, even though $\zeta \in PF_{10,1}$, it is not permutation-invariant (indeed, it is easy to check that the increasing rearrangement $(1, 1, 4, 4, 5, 5, 8, 9, 9, 9)$ is not a 1-Naples parking function). On the other hand, by the same corollary, ζ is permutation-invariant as a k-Naples parking function for all $k \geqslant 2$.

An equivalent characterization of permutation-invariant Naples parking functions is given in [5], where the authors prove that a k-Naples parking function is permutation-invariant if and only if its ascending rearrangement is a k-Naples parking function. It is then shown by the authors of [5] that ascending parking preferences are related to Grand-Dyck paths[2], and in particular an ascending parking preference α is a k-Naples parking function if and only if in the corresponding Grand-Dyck path the length of every factor lying entirely below the x-axis (except at most for the starting and ending points) is at most $2k$. This is in fact equivalent to our characterization in terms of the cardinalities of the maximal intervals of U_α, which has the advantage of being intrinsic to the parking preference α and does not involve other combinatorial structures. Moreover, the characterization given in Corollary 15 does not require to explicitly perform any parking process, but it just requires to check some counting property directly on the parking preference.

5 On the enumeration of permutation-invariant Naples parking functions

In the present section, as well as in the following one, we exploit the tools developed above to get some information concerning the enumeration of certain classes of Naples parking

[2] A *Grand-Dyck path* is a lattice path starting at the origin of a fixed cartesian coordinate system, ending on the x-axis and using only two kinds of steps, namely $U = (1, 1)$ and $D = (1, -1)$.

functions. Specifically, this section deals with permutation-invariant Naples parking functions, whereas the next one concerns complete Naples parking functions.

We start with some preliminary observations.

Lemma 16. *Let $\alpha_1 \in PP_{n_1}$ and $\alpha_2 \in PP_{n_2}$ be two parking preferences, and consider a subset $J \subseteq [1, n_1+n_2]$ such that $|J| = n_1$. Let $\alpha \in PP_{n_1+n_2}$ be the parking preference such that $\alpha_{|J} = \alpha_1$, and $\tau_{n_1}(\alpha_{|J^c}) = \alpha_2$. Then*

- *$u_\alpha(j) = u_{\alpha_1}(j)$, for all $j \in [1, n_1]$;*
- *$u_\alpha(j) = u_{\alpha_2}(j - n_1)$, for all $j \in [n_1+1, n_1+n_2]$.*

As a consequence, $U_\alpha = U_{\alpha_1} \cup \tau_{-n_1}(U_{\alpha_2})$.

Proof. For all $i \in [n_1+1, n_1+n_2]$, we have $|\alpha|_i = |\alpha_2|_{i-n_1}$, thus

$$u_\alpha(j) = \sum_{i=j}^{n_1+n_2} |\alpha|_i - (n_1+n_2-j+1) = \sum_{i=j-n_1}^{n_2} |\alpha_2|_i - (n_2-(j-n_1)+1) = u_{\alpha_2}(j-n_1),$$

for all $j \in [n_1+1, n_1+n_2]$. In particular, $u_\alpha(n_1+1) = u_{\alpha_2}(1) = 0$. Furthermore, $|\alpha|_i = |\alpha_1|_i$ for all $i \in [1, n_1]$. Thus by (ii) of Lemma 2

$$u_\alpha(j) = \sum_{i=j}^{(n_1+1)-1} |\alpha|_i - (n_1+1) + j + u_\alpha(n_1+1) = \sum_{i=j}^{n_1} |\alpha_1|_i - (n_1-j+1) + 0 = u_{\alpha_1}(j),$$

for all $j \in [1, n_1]$.

From the above it also follows that $U_\alpha = U_{\alpha_1} \cup \tau_{-n_1}(U_{\alpha_2})$. □

Roughly speaking, this means that the sets U_{α_i} behave well with respect to the above described construction. In particular, we have the following remarkable corollary.

Corollary 17. *If α_1 and α_2 are both permutation-invariant k-Naples parking functions, then the parking function α constructed starting from α_1 and α_2 as in Lemma 16 is also a permutation-invariant k-Naples parking function.*

Proof. Indeed, by Corollary 15, the only relevant thing to check is the size of the maximal intervals of U_α, so the conclusion immediately follows from Lemma 16. □

In light of these considerations, our goal throughout this section will be to decompose permutation-invariant Naples parking functions into smaller preferences (in a recursive fashion), such that the excess set of each of them consists of a single interval.

Denote with $T_{n,k}$ the number of permutation-invariant k-Naples parking functions of length n and with $t_{n,k}$ the number of those for which $2 \in U_\alpha$. Our first goal is to express $T_{n,k}$ in terms of the simpler coefficients $t_{n,k}$. The idea behind the proof of the next proposition is that every permutation-invariant k-Naples parking function can be decomposed into two smaller parking preferences, one being a parking function and the other being a permutation-invariant k-Naples parking function whose excess set contains 2.

Proposition 18. *For all $n, k \geqslant 0$, and setting $t_{0,k} = 1$, we have*

$$T_{n,k} = \sum_{j=0}^{n} \binom{n}{j} (j+1)^{j-1} t_{n-j,k}. \tag{3}$$

Proof. We start by observing that, using Proposition 5, it is easy to realize that every parking function is a k-Naples parking function (for any k), and so in particular permutation-invariant. Therefore we will focus on Naples parking functions that are *not* parking functions. So let $\alpha = (a_1, a_2, \ldots, a_n) \in PF_{n,k} \setminus PF_n$ be permutation-invariant, then $U_\alpha \neq \emptyset$, and we set $h = \min U_\alpha \geqslant 2$ and $H = \{i \leqslant n \mid a_i \geqslant h-1\}$. Lemma 3 implies that $|H| \geqslant 2$ and that $u_\alpha(h-1) = 0$.

Thanks to the properties of shifts and restrictions of parking preferences stated before Lemma 11, we have that $|H^c| = h-2$, $\alpha_{|H^c} \in PP_{h-2}$ and $U_{\alpha_{|H^c}} = U_\alpha \cap [1, h-2] = \emptyset$, hence $\alpha_{|H^c} \in PF_{h-2}$. Moreover we also have that $\tau_{h-2}(\alpha_{|H}) \in PP_{n-h+2}$ and $U_{\tau_{h-2}(\alpha_{|H})} = \tau_{h-2}(U_\alpha \cap [h-1, n]) = \tau_{h-2}(U_\alpha)$. Since α is k-Naples permutation-invariant, all maximal intervals of U_α have size at most k (by Corollary 15). The same holds for $U_{\tau_{h-2}(\alpha_{|H})}$ and so $\tau_{h-2}(\alpha_{|H}) \in PF_{n-h+2,k}$ (by Lemma 11) and is permutation-invariant (again by Corollary 15). Finally, we also have that $u_{\tau_{h-2}(\alpha_{|H})}(2) = u_\alpha(h) = 1$ (by Lemma 3), thus $2 \in U_{\tau_{h-2}(\alpha_{|H})}$.

Summing up, the above analysis shows that a permutation-invariant k-Naples parking function of length n which is not a parking function is uniquely determined by a set of positive integers $H \subseteq [n]$, with $|H| = n-h+2$, together with a parking function $\alpha_1 \in PF_{h-2}$ (on the cars associated with H^c) and a permutation-invariant k-Naples parking function $\alpha_2 \in PF_{n-h+2,k}$ such that $2 \in U_{\alpha_2}$. Since $2 \leqslant |H| \leqslant n$, we have that $2 \leqslant h \leqslant n$. Setting $j = h-2$, we thus get that the number of permutation-invariant k-Naples parking functions of length n which are not parking functions is given by $\sum_{j=0}^{n-2} \binom{n}{j} (j+1)^{j-1} t_{n-j,k}$. To conclude, we observe that, if we complete the previous summation with the summands corresponding to $j = n-1$ and $j = n$, then the former one vanishes (since clearly $t_{1,k} = 0$), whereas the latter one counts (standard) parking functions of length n: this gives precisely formula (3). $\square$

The second step of our approach consists of expressing the coefficients $t_{n,k}$ in terms of somehow simpler ones. Given $n \geqslant 0, k \geqslant 1$, we set $\Theta_{n,k}^{(=)} = |\{\alpha \in PP_n \mid U_\alpha = [2, k+1]\}|$ and $\Theta_{n,k}^{(\leqslant)} = \sum_{i=1}^{k} \Theta_{n,i}^{(=)}$. In other words, $\Theta_{n,k}^{(\leqslant)}$ is the number of parking preferences whose excess set is a single interval of size at most k having minimum 2. We observe that clearly $\Theta_{0,k}^{(=)} = \Theta_{1,k}^{(=)} = 0$ for all $k \geqslant 1$, and also that $\Theta_{n,k}^{(=)} = 0$ when $n \leqslant k$.

Notice that, as a consequence of Corollary 15, all parking preferences counted by $\Theta_{n,k}^{(\leqslant)}$ are (permutation-invariant) k-Naples parking functions (and the same is true for $\Theta_{n,i}^{(=)}$ when $i \leqslant k$). An argument analogous to that of the previous proposition provides the following recursive description of $t_{n,k}$.

Proposition 19. *For all $n \geqslant 0, k \geqslant 1$, we have*

$$t_{n,k} = \sum_{i=0}^{n} \binom{n}{i} \Theta_{i,k}^{(\leqslant)} t_{n-i,k}. \tag{4}$$

Proof. Let $\alpha \in PF_{n,k}$ be permutation-invariant, with $2 \in U_\alpha$. Suppose that $[2,j]$ is maximal in U_α, with $j \leqslant k+1$. Denote with h the minimum of $U_\alpha \setminus [2,j]$ (if it exists), and set $H = \{i \leqslant n \mid h-1 \leqslant a_i \leqslant n\}$. It is clear that $\alpha_{|_{H^c}}$ is a k-Naples parking function such that $U_{\alpha_{|_{H^c}}}$ consists of the single interval $[2,j]$, whereas $\tau_{h-2}(\alpha_{|_H})$ is a permutation-invariant k-Naples parking function such that $2 \in U_{\tau_{h-2}(\alpha_{|_H})}$. This gives exactly formula (4). $\square$

We remark that, since $\Theta_{0,k}^{(\leqslant)} = 0$, in the right hand side of (4) the summand corresponding to $i = 0$ vanishes, thus giving a properly defined recursion to determine $t_{n,k}$.

The proofs of Propositions 18 and 19 suggest how to decompose (in a unique way) a permutation-invariant Naples parking function into smaller (and simpler) ones, in a recursive fashion. We now illustrate this claim with an example.

Let $\alpha = (3,7,10,1,4,10,4,10,8,8,4) \in PP_{11}$. A direct computation shows that $U_\alpha = [3,4] \cup [7,8] \cup \{10\}$. Thanks to Corollary 15 we therefore have that α is a permutation-invariant 2-Naples parking function. Since $\min U_\alpha = 3$, following the proof of Proposition 18, we consider the set of indices $H = \{1,2,3,5,6,7,8,9,10,11\}$ whose associated preferences are $\geqslant 2$. We thus obtain that α can be decomposed into the two (smaller) 2-Naples parking functions $\alpha_{|_{H^c}} = (1)$ and $\tau_1(\alpha_{|_H}) = (2,6,9,3,9,3,9,7,7,3)$, where the latter is also permutation-invariant. Moreover, $U_{\tau_1(\alpha_{|_H})} = [2,3] \cup [6,7] \cup \{9\}$, and so $2 \in U_{\tau_1(\alpha_{|_H})}$. Now, setting $\beta = \tau_1(\alpha_{|_H})$, we can follow the proof of Proposition 19 to recursively decompose β into smaller permutation-invariant Naples parking functions whose excess sets are all of the form $[2,i]$, for suitable values of i. For the first step, define $H_1 = \{2,3,5,7,8,9\}$ (the set of indices of β whose associated preferences are $\geqslant 5$, where $5 = \min(U_\beta \setminus [2,3]) - 1$). We thus decompose β into $\beta_{|_{H_1^c}} = (2,3,3,3)$ and $\tau_4(\beta_{|_{H_1}}) = (2,5,5,5,3,3)$. Setting $\gamma = \tau_4(\beta_{|_{H_1}})$, we now have $U_\gamma = [2,3] \cup \{5\}$, and so $\min(U_\gamma \setminus [2,3]) = 5$. Setting $H_2 = \{2,3,4\}$ (the set of indices of γ whose associated preferences are $\geqslant 4$), we can now decompose γ into the smaller permutation-invariant 2-Naples parking functions $\gamma_{|_{H_2^c}} = (2,3,3)$ and $\tau_3(\gamma_{|_{H_2}}) = (2,2,2)$. Since the two latter parking preferences both have excess set consisting of a single interval with minimum 2, the decomposition is finished.

The above recursive process can be described in a single step by considering α and by partitioning its entries as follows:

$$\alpha = (3_1, 7_5, 10_8, 1_0, 4_1, 10_8, 4_1, 10_8, 8_5, 8_5, 4_1).$$

Collecting all entries having the same index and diminishing each entry by its index gives the four 2-Naples parking functions into which α is decomposed (where, in particular, the parking functions associated with the entries > 1 are permutation-invariant and their excess sets are constituted by a single interval having minimum 2).

Propositions 18 and 19 give a recipe to express the number of permutation-invariant Naples parking functions as a function of the $\Theta_{n,k}^{(\leqslant)}$'s only. From a purely combinatorial point of view, also referring to the above example, this can be expressed by saying that a permutation-invariant k-Naples parking function α of length n can be uniquely determined as follows. Given a composition $\beta = (b_0, b_1, \ldots, b_m)$ of n (i.e., $b_0 + b_1 + \cdots + b_m = n$), with $b_0 \geqslant 0$ and $b_i \geqslant 2$ for all $i \neq 0$, choose a set partition $\{B_0, B_1, \ldots, B_m\}$ of $\{1, 2, \ldots n\}$ such that $|B_i| = b_i$, for all i. Then assign:

- a parking function of size b_0 on B_0 (these will be those cars having preferences between 1 and b_0), and
- for each $i = 1, 2, \ldots, m$, a permutation-invariant k-Naples parking function α_i of size b_i with the property that, for all i, the excess set U_{α_i} consists of a single interval (of cardinality at most k) containing 2 (these will be those cars having preferences between $\sum_{j=0}^{i-1} b_j + 1$ and $\sum_{j=0}^{i} b_j$).

The above considerations can be translated into the following counting formula

$$T_{n,k} = \sum_m \sum_{\beta \in W_m} \binom{n}{b_0, b_1, \ldots, b_m} (b_0 + 1)^{b_0 - 1} \prod_{i=1}^{m} \Theta_{b_i,k}^{(\leqslant)}, \tag{5}$$

where $W_m = \{(b_0, b_1, \ldots, b_m) \,|\, b_0 + b_1 + \cdots + b_m = n, b_0 \geqslant 0, b_i \geqslant 2, i = 1, 2, \ldots, m\}$ and $\binom{n}{b_0, b_1, \ldots, b_m} = \frac{n!}{b_0! b_1! \cdots b_m!}$ denotes the usual multinomial coefficient.

As it is clear from formula (5), in order to effectively compute $T_{n,k}$ we need to find information on the coefficients $\Theta_{n,k}^{(\leqslant)}$. In the rest of the present section we will in fact provide a recursive recipe to compute the coefficients $\Theta_{n,k}^{(=)}$, by which we can in turn compute the $\Theta_{n,k}^{(\leqslant)}$'s. We start by showing an interesting symmetry property.

Lemma 20. *For any $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$, we have $\Theta_{n,k}^{(=)} = \Theta_{n,n-k}^{(=)}$.*

Proof. Given $\alpha = (a_1, a_2, \ldots, a_n) \in PP_n$, with $U_\alpha = [2, k+1]$, define $\tilde{\alpha} = (n+2-a_1, n+2-a_2, \ldots, n+2-a_n)$ (this will be called the *complement* of α). Since $|\alpha|_1 = 0$ (by (iii) of Lemma 3), it follows immediately that $\tilde{\alpha} \in PP_n$. By definition of $\tilde{\alpha}$, we have that $|\tilde{\alpha}|_i = |\alpha|_{n+2-i}$, hence, for all $j > 2$:

$$\begin{aligned} u_{\tilde{\alpha}}(j) &= \sum_{i=j}^{n} |\tilde{\alpha}|_i - (n-j+1) = \sum_{i=j}^{n} |\alpha|_{n+2-i} - (n-j+1) \\ &= \sum_{i=1}^{n+2-j} |\alpha|_i - (n-j+2) + 1 = -u_\alpha(n-j+3) + 1 \end{aligned}$$

(the last equality follows from (i) of Lemma 2). As a consequence, we get that, when $j > 2$, $u_{\tilde{\alpha}}(j) \geqslant 1$ if and only if $u_\alpha(n-j+3) \leqslant 0$, i.e. $j \in U_{\tilde{\alpha}}$ if and only if $n-j+3 \notin U_\alpha = [2, k+1]$, or, equivalently, if and only if $j \leqslant n-k+1$. Moreover, $u_{\tilde{\alpha}}(2) = \sum_{i=2}^{n} |\tilde{\alpha}|_i - (n-2+1) =$

$n-(n-1)=1$. Therefore we have that $U_{\tilde{\alpha}}=[2,n-k+1]$, that is $\tilde{\alpha}$ is among the preferences counted by $\Theta^{(=)}_{n,n-k}$. Since the complement map is clearly an involution, we get that $\Theta^{(=)}_{n,k}=\Theta^{(=)}_{n,n-k}$, as desired. □

Next we further decompose the quantities $\Theta^{(=)}_{n,k}$ by introducing more parameters.

Let $n,m,h\geqslant 1$ and $k\geqslant 0$. Define $\vartheta_{n,k}(m,h)$ to be the number of parking preferences $\alpha\in PP_n$ such that $U_\alpha=[2,k+1]$, $|\alpha|_i=0$ for $i=1,\dots h-1$, and $|\alpha|_h=m$ (in particular, if $k=0$ we have that $U_\alpha=\emptyset$). Roughly speaking, the coefficients $\vartheta_{n,k}(m,h)$ refine the enumeration of parking preferences whose excess set is a single interval by looking at the smallest preference. More specifically

$$\Theta^{(=)}_{n,k}=\sum_{m\geqslant 1}\sum_{h\geqslant 1}\vartheta_{n,k}(m,h). \tag{6}$$

The expression (6) can be considerably simplified by observing that many of the coefficients $\vartheta_{n,k}(m,h)$ actually vanish.

Lemma 21. *Let $n,m,h\geqslant 1$ and $n>k\geqslant 0$.*

(i) $\vartheta_{n,k}(m,h)=0$ for all $h>k+1$; in particular, $\vartheta_{n,0}(m,h)=0$ for all $h>1$;

(ii) if $k\geqslant 1$, then $\vartheta_{n,k}(m,1)=0$;

(iii) if $k\geqslant 1$, then $\vartheta_{n,k}(m,k+1)=0$ for all $m<k+1$;

(iv) if $k\geqslant 1$ and $2\leqslant h\leqslant k$, then $\vartheta_{n,k}(m,h)=0$ for all $m\geqslant h$;

(v) in all the remaining cases, $\vartheta_{n,k}(m,h)\neq 0$.

Proof. Let $\alpha\in PP_n$. If $\alpha\in PF_n$, then $|\alpha|_1\neq 0$, hence $\vartheta_{n,0}(m,h)=0$ for all $h>1$. For the rest of the proof we assume that $k\geqslant 1$ and that $U_\alpha=[2,k+1]$. By Lemma 3, we have that $|\alpha|_{k+1}\neq 0$, and so $\vartheta_{n,k}(m,h)=0$ for all $h>k+1$, thereby proving (i). Moreover, the same lemma also gives that $|\alpha|_1=0$, from which (ii) follows. Now suppose that $h=k+1$. This means that $|\alpha|_i=0$ for all $i\leqslant k$, and so, using Lemma 2 and the fact that $k+2\notin U_\alpha$,

$$0\geqslant u_\alpha(k+2)=k+1-\sum_{i=1}^{k+1}|\alpha|_i=k+1-|\alpha|_{k+1}.$$

Therefore $|\alpha|_{k+1}\geqslant k+1$, and this implies that $\vartheta_{n,k}(m,k+1)=0$ for all $m<k+1$, which is (iii). On the other hand, if $2\leqslant h\leqslant k$, then $h+1\in U_\alpha$, hence

$$1\leqslant u_\alpha(h+1)=h-\sum_{i=1}^{h}|\alpha|_i=h-|\alpha|_h,$$

i.e. $|\alpha|_h\leqslant h-1$, which proves (iv).

To conclude the proof, we now need to show that in all the remaining cases $\vartheta_{n,k}(m,h)\neq 0$. We will prove this by explicitly providing a parking preference for each single case.

- If $k = 0$ and $h = 1$, choose $\alpha = (\underbrace{1, \dots, 1}_{m}, \underbrace{2, \dots, 2}_{n-m})$.
- If $k \geqslant 1$ and $2 \leqslant h \leqslant k$, for $m < h$, choose $\alpha = (\underbrace{h, \dots, h}_{m}, \underbrace{k+1, \dots, k+1}_{n-m})$.
- If $k \geqslant 1$ and $h = k+1$, for $m \geqslant k+1$, choose $\alpha = (\underbrace{k+1, \dots, k+1}_{m}, \underbrace{k+2, \dots, k+2}_{n-m})$.

$\square$

Using Lemma 21, we can therefore simplify formula (6) by keeping only the nonzero summands, thus getting

$$\Theta^{(=)}_{n,k} = \sum_{h=2}^{k} \sum_{m=1}^{h-1} \vartheta_{n,k}(m, h) + \sum_{m=k+1}^{n} \vartheta_{n,k}(m, k+1).$$

To conclude our enumeration strategy, we now give a recurrence to compute the coefficients $\vartheta_{n,k}(m, h)$.

Proposition 22. *For $n \geqslant m \geqslant 1$, we have*

$$\vartheta_{n,0}(m, 1) = \binom{n-1}{m-1} n^{n-m}.$$

Proof. Observe that $\vartheta_{n,0}(m, 1)$ counts (classical) parking functions of length n in which exactly m cars have preference 1. The generating function of such coefficients can be found in [17] (Corollary 1.16). $\square$

Proposition 23. *Let $1 \leqslant k \leqslant n$ and $1 \leqslant h \leqslant n$. Then*

$$\vartheta_{n,k}(1, h) = n \cdot \sum_{h'=h}^{k} \sum_{m'=1}^{n-1} \vartheta_{n-1,k-1}(m', h') \tag{7}$$

and, for $m \geqslant 2$,

$$\vartheta_{n,k}(m, h) = \frac{n}{m} \vartheta_{n-1,k-1}(m-1, h-1). \tag{8}$$

Proof. The basic idea is to develop a construction showing how each of the k-Naples parking functions of length n counted by $\vartheta_{n,k}(m, h)$ can be obtained (not necessarily in a unique way) from one of length $n-1$.

Let $\alpha \in PF_{n,k}$ such that $U_\alpha = [2, k+1]$, and let $h \leqslant n$ be such that $|\alpha|_i = 0$ for all $i < h$ and $|\alpha|_h \neq 0$. Choose an index $\tilde{i}$ such that $a_{\tilde{i}} = h$ and denote with J the set of the remaining indices (that is, $J = \{1, \dots, n\} \setminus \{\tilde{i}\}$). Now consider the parking preference $\tau_1(\alpha_{|_J})$ (of length $n-1$); roughly speaking, this is nothing else than removing $a_{\tilde{i}}$ from α and decreasing the remaining entries by 1. We now want to show that $\tau_1(\alpha_{|_J})$ is a $(k-1)$-Naples parking function of length $n-1$ counted by a suitable ϑ coefficient.

Since obviously $|\tau_1(\alpha_{|J})|_i = |\alpha|_{i+1}$ for all $i \neq h-1$, and $|\tau_1(\alpha_{|J})|_{h-1} = |\alpha|_h - 1$, an easy computation shows that, for $h \leqslant j \leqslant n-1$, we have $u_{\tau_1(\alpha_{|J})}(j) = u_\alpha(j+1)$. This fact, together with the fact that $h \leqslant k+1$ (which is a consequence of (i) of Lemma 21, since otherwise α would not exist), tells us that $[h,k] \subseteq U_{\tau_1(\alpha_{|J})}$, and also that $i \notin U_{\tau_1(\alpha_{|J})}$ for all $i > k$. Notice that, in the case $h = k+1$, the interval $[h,k]$ is empty, and so the inclusion $[h,k] \subseteq U_{\tau_1(\alpha_{|J})}$ trivially holds. Furthermore, a similarly easy computation as above shows that, for $2 \leqslant j \leqslant h-1$, we have $u_{\tau_1(\alpha_{|J})}(j) = u_\alpha(j+1) - 1$. Notice that, for $3 \leqslant i \leqslant h$, we have $u_\alpha(i) \geqslant 2$; in fact, if this were not the case, then the fact that $|\alpha|_{i-1} = 0$ would imply (using (iii) of Lemma 2) that $u_\alpha(i-1) = u_\alpha(i) + |\alpha|_{i-1} - 1 \leqslant 0$, against the fact that $i-1 \in U_\alpha$ (this is because $i-1 \in [2,h-1] \subset [2,k+1] = U_\alpha$). As a consequence, we have that, for all $2 \leqslant j \leqslant h-1$, $u_{\tau_1(\alpha_{|J})}(j) = u_\alpha(j+1) - 1 \geqslant 1$, and so $[2,h-1] \subseteq U_{\tau_1(\alpha_{|J})}$. Summing up, we have thus proved that $U_{\tau_1(\alpha_{|J})} = [2,k]$. This ensures that $\tau_1(\alpha_{|J})$ is a $(k-1)$-Naples parking function of length $n-1$.

In order to reverse the above construction, let β be a $(k-1)$-Naples parking function of length $n-1$ such that $U_\beta = [2,k]$, and let $h \leqslant k$ be such that $|\beta|_i = 0$ for all $i \leqslant h-2$. We can thus construct a k-Naples parking function α of length n by simply increasing all entries of β by 1 and then inserting somewhere a new entry equal to h. Using similar arguments as above it is possible to prove that indeed $\alpha \in PF_{n,k}$, $U_\alpha = [2,k+1]$, and of course $|\alpha|_i = 0$ for all $i \leqslant h-1$. However, several different parking preferences α can be obtained from the same β, since there are n possible positions where to insert the new entry: this explains the factor n appearing in the left hand side of both (7) and (8). Moreover, we need to distinguish two cases, depending on whether β contains some entry equal to $h-1$ or not. If $|\beta|_{h-1} = 0$, then the above "reverse construction" allows us to reconstruct α in a unique way, so we get (7). On the other hand, if $|\beta|_{h-1} = m-1$, with $m \geqslant 2$, then the same α can be obtained from m different preferences β, and this gives (8). $\square$

The recursion described in Propositions 22 and 23 allows us to effectively compute the coefficients $\vartheta_{n,k}(m,h)$, which in turn make it possible to effectively compute the coefficients $\Theta^{(=)}_{n,k}$ (by simply using (6) or, even better, its simplified version displayed before Proposition 22). The table below reports the first few values of such coefficients.

$\Theta^{(=)}_{n,k}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=2$	1						
$n=3$	4	4					
$n=4$	27	21	27				
$n=5$	256	176	176	256			
$n=6$	3125	1995	1765	1995	3125		
$n=7$	46656	28344	23304	23304	28344	46656	
$n=8$	823543	482825	378007	351337	378007	482825	823543

Table 1: The first values of the coefficients $\Theta^{(=)}_{n,k}$.

As we already know from Lemma 20, the rows of Table 1 are symmetric. In general, rows and columns of such a table (as well as the whole table) seem not to be recorded in [13]. A notable exception is the first column, i.e. the values $\Theta^{(=)}_{n,1} = \Theta^{(=)}_{n,n-1} = (n-1)^{n-1}$. In the next proposition we will prove this fact in a bijective fashion, namely by showing that the set of 1-Naples parking functions counted by $\Theta^{(=)}_{n,1}$ are in bijection with prime parking functions of length n. A *prime parking function* is a parking function $\alpha = (a_1, a_2, \ldots, a_n)$ such that the sequence obtained from α by removing a 1 is still a parking function (of length $n-1$). Equivalently, α is a prime parking function when, denoting with $\beta = (b_1, b_2, \ldots, b_n)$ its nondecreasing rearrangement, we have that $b_i < i$ for all $i \neq 1$. According to Stanley [15], the notion of prime parking function was introduced by Gessel, who also gave the above mentioned enumerative formula (see also [9]).

Proposition 24. *For all $n \geqslant 2$, we have*

$$\Theta^{(=)}_{n,1} = (n-1)^{n-1}.$$

Proof. Let $\alpha = (a_1, a_2, \ldots, a_n)$ be a 1-Naples parking function such that $U_\alpha = \{2\}$. From Lemma 3 we get that $|\alpha|_1 = 0$ and $|\alpha|_2 \geqslant 2$. This implies that the sequence $\tilde{\alpha} = (a_1 - 1, a_2 - 1, \ldots, a_n - 1)$ is a parking preference of length n which contains at least two 1's. Moreover, given $j > 1$, since $j+1 \notin U_\alpha$ (and so $u_\alpha(j+1) \leqslant 0$), we can use (i) of Lemma 2 to get that $\sum_{i=1}^{j} |\alpha|_i = j - u_\alpha(j+1) \geqslant j$. As a consequence, in the nondecreasing rearrangement of α the j-th entry is at most j, hence in the nondecreasing rearrangement of $\tilde{\alpha}$ the j-th entry is strictly less than j. We have thus proved that $\tilde{\alpha}$ is a prime parking function. Since the above construction is clearly reversible, it is a bijection, and so $\Theta^{(=)}_{n,1}$ is equal to the number of prime parking functions of length n, which is $(n-1)^{n-1}$. ☐

An immediate corollary of Proposition 24 concerns complete parking preferences.

Corollary 25. *The number of complete parking preferences of length $n \geqslant 2$ is $(n-1)^{n-1}$.*

Proof. The number of complete parking preferences of length n is $\Theta^{(=)}_{n,n-1}$. Then the thesis follows from Lemma 20 and the above proposition. ☐

Referring again to Table 1, its partial row sums are clearly the the coefficients $\Theta^{(\leqslant)}_{n,k}$. Also in this case there are no correspondences in [13], except for $\Theta^{(\leqslant)}_{n,n-1}$, which seems to match sequence A071720. Such a sequence, whose closed form is $(n-1)(n+1)^{n-2}$, counts the number of spanning trees in the complete graph on $n+1$ vertices minus an edge. It would be nice to have a proof of this fact, which is still missing.

The results found in the present section allows us to efficiently compute several values of the coefficients $T_{n,k}$, which count permutation-invariant k-Naples parking functions of length n. They are reported in Table 2, where each row is eventually constant (and equal to its rightmost entry, which is the total number of parking preferences of the relevant length).

The entries of Table 2 also do not appear in the Online Encyclopedia of Integer Sequences [13], except of course for the diagonals $T_{n,n-1} = n^n$ and $T_{n,n-2} = n^n - (n-1)^{n-1}$, counting parking preferences and noncomplete parking preferences, respectively.

$T_{n,k}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=2$	4						
$n=3$	23	27					
$n=4$	192	229	256				
$n=5$	2077	2558	2869	3125			
$n=6$	27808	35154	40000	43531	46656		
$n=7$	444411	572470	662519	726668	776887	823543	
$n=8$	8266240	10815697	12693504	14055341	15097600	15953673	16777216

Table 2: The first values of the coefficients $T_{n,k}$.

6 On the enumeration of complete Naples parking functions

The relevance of complete parking preferences in characterizing Naples parking functions and in counting the permutation-invariant ones (as shown in the previous section) leads us naturally to address the problem of their enumeration. To this aim, we need to introduce a few further notations.

Given $n, k, m \geqslant 1$, denote with $v_{n,k}(m)$ the number of k-Naples parking functions α of length n such that $u_\alpha(i) \geqslant 1$ for all $m+1 \leqslant i \leqslant n$, and $u_\alpha(i) \geqslant 0$ for all $1 \leqslant i \leqslant m$. Moreover, $v^{(0)}_{n,k}(m)$ counts the special subset of the above described preferences with the additional constraint that $u_\alpha(m) = 0$.

Notice that, since $u_\alpha(1) = 0$ for all parking preferences α, the number of complete k-Naples parking functions of length n is given by $v_{n,k}(1) = v^{(0)}_{n,k}(1)$. Moreover, when $n \leqslant k+1$, Corollary 25 implies that $v_{n,k}(1) = (n-1)^{n-1}$.

The next three propositions give a recursive description of the coefficients $v^{(0)}_{n,k}(1)$.

Proposition 26. *Given $n, k, m \geqslant 1$, with $n \geqslant k$, we have*

$$v^{(0)}_{n+1,k}(1) = \sum_{m=1}^{k} v_{n,k}(m). \tag{9}$$

Proof. Let $\alpha = (a_1, a_2, \ldots, a_n, a_{n+1}) \in PF_{n+1,k}$ be complete. This implies that $|\alpha|_1 = 0$ (by Lemma 3) and, using Lemma 14, that $a_{n+1} \leqslant k+1$. We can then consider the parking preference $\tilde{\alpha} = (a_1 - 1, a_2 - 1, \ldots, a_n - 1) \in PP_n$ (obtained from α by removing the last entry and subtracting 1 to the remaining ones). Notice that, again by Lemma 14, $\psi^{(\alpha)}_k(n+1) = 1$, i.e., when parking according to the preference α, the $(n+1)$-th car parks in spot 1. As a consequence, also recalling that none of the first n cars have preference 1 in α, when the first n cars park according to $\tilde{\alpha}$, each of them parks in the spot to the left of the spot it occupies when parking according to α, i.e., $\psi^{(\tilde{\alpha})}_k(i) = \psi^{(\alpha)}_k(i) - 1$. This implies that indeed $\tilde{\alpha} \in PF_{n,k}$.

Setting $m = a_{n+1} - 1 \leqslant k$, it is an easy computation to check that $u_{\tilde{\alpha}}(i) = u_\alpha(i+1) \geqslant 1$ when $i > m$, whereas $u_{\tilde{\alpha}}(i) = u_\alpha(i+1) - 1 \geqslant 0$ when $i \leqslant m$. As a consequence, $\tilde{\alpha}$ is one of the k-Naples parking functions of length n counted by $v_{n,k}(m)$. Since the above construction of $\tilde{\alpha}$ from α is reversible, we get formula (9). $\square$

Proposition 27. *Given $n, k, m \geqslant 1$, with $n \geqslant k$, we have*

$$v_{n,k}(m) = \sum_{i=1}^{m} v^{(0)}_{n,k}(i). \tag{10}$$

Proof. Let $\alpha = (a_1, a_2, \dots, a_n) \in PF_{n,k}$ be such that $u_\alpha(i) \geqslant 1$ for all $m+1 \leqslant i \leqslant n$, and $u_\alpha(i) \geqslant 0$ for all $1 \leqslant i \leqslant m$. Then either $u_\alpha(m) = 0$ (and there are $v^{(0)}_{n,k}(m)$ such preferences) or ($m \geqslant 2$ and) $u_\alpha(m) > 0$ (and there are $v_{n,k}(m-1)$ such preferences). We thus have that

$$v_{n,k}(m) = v^{(0)}_{n,k}(m) + v_{n,k}(m-1),$$

and iterating the above formula gives (10). □

Proposition 28. *Given $n, k, m \geqslant 1$, with $n \geqslant k$, we have*

$$v^{(0)}_{n,k}(m) = \binom{n}{m-1} v_{m-1,k}(m-1) v^{(0)}_{n-m+1,k}(1). \tag{11}$$

Proof. Let $\alpha = (a_1, a_2, \dots, a_n) \in PF_{n,k}$ be such that $u_\alpha(i) \geqslant 1$ for all $m+1 \leqslant i \leqslant n$, and $u_\alpha(i) \geqslant 0$ for all $1 \leqslant i \leqslant m$, with the additional requirement that $u_\alpha(m) = 0$. Setting $J = \{i \leqslant n \mid a_i \geqslant m\}$, by Lemma 11 we get that $\tau_{m-1}(\alpha_{|J})$ is a k-Naples parking function, which is complete since $u_{\tau_{m-1}(\alpha_{|J})}(i) = u_\alpha(i+m-1) \geqslant 1$ for all $2 \leqslant i \leqslant n-m+1$.

Moreover, setting as usual $J^c = [1, n] \setminus J$, we have that $u_{\alpha_{|J^c}}(i) = u_\alpha(i) \geqslant 0$ for all $1 \leqslant i \leqslant m-1$. Observe that, in α, all cars park by driving (weakly) backwards, otherwise (by Lemma 7) there would exist a position having negative excess. This implies that $\alpha_{|J^c}$ is a k-Naples parking function as well, and in particular it is counted by $v_{m-1,k}(m-1)$.

Conversely, given k-Naples parking functions α_1 and α_2 counted by $v_{m-1,k}(m-1)$ and $v^{(0)}_{n-m+1,k}(1)$ (respectively), if we fix $J^c \subseteq [1, n]$ with $|J^c| = m-1$, we can consider the parking preference α constructed by filling the indices in J^c with α_1 and the indices in $J = [1, n] \setminus J^c$ with $\tau_{-m+1}(\alpha_2)$. Note since α_2 is a complete Naples parking function, by Lemma 14 spot 1 is the last to be filled during its parking process. In the same way, since $\tau_{-m+1}(\alpha_2)$ has the same behavior of α_2 on the translated set of positions $[m, n]$, spot m is the last spot to be checked during the parking process: in particular, the cars belonging to $\tau_{-m+1}(\alpha_2)$ never check any spot strictly lower than m, and fill spots $[m, n]$. On the other hand, since α_1 is a k-Naples parking function, it trivially fills spots $[1, m-1]$ even when put in the larger preference α, since there is no interference from cars belonging to $\tau_{-m+1}(\alpha_2)$. Thus, the parking preference α so constructed is a k-Naples parking function, and it is counted by $v^{(0)}_{n,k}(m)$ by our assumptions on α_1 and α_2, and by Lemma 16.

We can therefore conclude that every preference α as specified at the beginning of the proof can be uniquely determined by choosing the subset J (of cardinality $n-m+1$) from $[1, n]$, then choosing a complete k-Naples parking function $\tau_{m-1}(\alpha_{|J})$ of length $n-m+1$ and a k-Naples parking function $\alpha_{|J^c}$ of length $m-1$ among those counted by $v_{m-1,k}(m-1)$ (i.e., such that $u_{\alpha_{|J^c}}(i) \geqslant 0$ for all $i \in [m-1]$). This gives precisely formula (11). □

Corollary 29. *With the notations of Proposition 28, when $m \leqslant k$ recursion (11) becomes*

$$v_{n,k}^{(0)}(m) = \binom{n}{m-1} m^{m-2} v_{n-m+1,k}^{(0)}(1). \tag{12}$$

Proof. If $m \leqslant k$, keeping the same notations as in the previous proposition, we have that $\alpha_{|_{J^c}}$ is an arbitrary parking preference only subject to the condition that $u_{\alpha_{|_{J^c}}}(i) = u_\alpha(i) \geqslant 0$ for all $1 \leqslant i \leqslant m-1$ (as the length of $\alpha_{|_{J^c}}$ is smaller than the maximum number of backward steps the cars can take).

Let $\alpha_{|_{J^c}} = (\hat{a}_1, \hat{a}_2, \dots, \hat{a}_{m-1})$, and define $\beta = (m - \hat{a}_1, m - \hat{a}_2, \dots, m - \hat{a}_{m-1})$. Clearly $|\beta|_i = |\alpha_{|_{J^c}}|_{m-i}$, and so, for all $2 \leqslant j \leqslant m-1$ (using (i) of Lemma 2)

$$\begin{aligned} u_\beta(j) &= \sum_{i=j}^{m-1} |\beta|_i - ((m-1) - j + 1) = \sum_{i=j}^{m-1} |\alpha_{|_{J^c}}|_{m-i} - ((m-1) - j + 1) \\ &= \sum_{i=1}^{m-j} |\alpha_{|_{J^c}}|_i - ((m-j+1) - 1) = -u_{\alpha_{|_{J^c}}}(m-j+1) \leqslant 0, \end{aligned}$$

i.e. β is a parking function of length $m-1$. Also, notice that the above combinatorial construction (from $\alpha_{|_{J^c}}$ to β) is clearly an involution, which allows us to conclude. □

We are now ready to state our final recursion.

Theorem 30. *Let $n \geqslant 2$ and $k \geqslant 1$. The number of complete k-Naples parking functions of length n satisfies the following recursion:*

$$v_{n,k}^{(0)}(1) = \begin{cases} (n-1)^{n-1} & n \leqslant k+1, \\ \displaystyle\sum_{i=1}^{k} \binom{n-1}{i-1} i^{i-2} (k-i+1) v_{n-i,k}^{(0)}(1) & n \geqslant k+2. \end{cases} \tag{13}$$

Proof. We have already observed that, when $n \leqslant k+1$, $v_{n,k}^{(0)}(1) = v_{n,k}(1) = (n-1)^{n-1}$. When $n \geqslant k+2$, formulas (9) and (10) give

$$v_{n,k}^{(0)}(1) = \sum_{m=1}^{k} \sum_{i=1}^{m} v_{n-1,k}^{(0)}(i) = \sum_{i=1}^{k} (k-i+1) v_{n-1,k}^{(0)}(i),$$

and we conclude using formula (12). □

Also in the case of complete Naples parking functions, our recurrence allows us to efficiently find many of the coefficients $v_{n,k}^{(0)}(1)$. Some of them are recorded in Table 3 (where, as in Table 2, rows are eventually constant).

Sequence $v_{n,n-2}^{(0)}(1)$ seems to match sequence A101334 in [13]. This is indeed the case, as we prove in our last proposition.

$v^{(0)}_{n,k}(1)$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$n=2$	1							
$n=3$	1	4						
$n=4$	1	11	27					
$n=5$	1	38	131	256				
$n=6$	1	131	783	1829	3125			
$n=7$	1	490	5136	15634	29849	46656		
$n=8$	1	1897	34623	148321	332869	561399	823543	
$n=9$	1	7714	251817	1505148	4102015	7735566	11994247	16777216

Table 3: The first values of the coefficients $v^{(0)}_{n,k}(1)$.

Proposition 31. *For $n \geqslant 3$, we have*

$$v^{(0)}_{n,n-2}(1) = (n-1)^{n-1} - n^{n-2}.$$

Proof. Thanks to Theorem 30, we obtain

$$v^{(0)}_{n,n-2}(1) = \sum_{i=1}^{n-2} \binom{n-1}{i-1} i^{i-2}(n-i-1)v^{(0)}_{n-i,n-2}(1) = \sum_{i=0}^{n-3} \binom{n-1}{i}(i+1)^{i-1}(n-i-2)^{n-i-1}.$$

We now use the well-known Abel's generalization of Newton's binomial formula, which has several equivalent formulations. One of those is the following (see for instance [12]):

$$(z+w+m)^m = \sum_{i=0}^{m} \binom{m}{i} w(w+i)^{i-1}(z+m-i)^{m-i}.$$

Setting $m = n-1$, $z = -1$ and $w = 1$ gives

$$(n-1)^{n-1} = \sum_{i=0}^{n-1} \binom{n-1}{i}(i+1)^{i-1}(n-i-2)^{n-i-1}.$$

As a consequence, subtracting from the above expression for $(n-1)^{n-1}$ the terms corresponding to $i = n-1$ and $i = n-2$ gives the desired formula for $v^{(0)}_{n,n-2}(1)$. □

7 Conclusions and further work

In the present work we have introduced the notion of *complete parking preference*, which then has been used to provide a characterization of Naples parking functions. Theorem 12 and Proposition 10 indicate a method to check if a given parking preference is indeed a Naples parking function which, as a consequence (Corollary 15), allows us to determine if a Naples parking function is permutation-invariant in a way that is totally intrinsic to the preference, and only requires checking if it has some specific properties. We can summarize the results obtained in Theorem 12, Proposition 10 and Corollary 15 as follows.

Theorem 32. *Let $\alpha \in PP_n$ and suppose that all cars follow the k-Naples parking rule, with $k \geqslant 1$. Let $[p, q] \subseteq U_\alpha$ be a maximal interval. The following are equivalent:*

(i) restricting to cars having preferences at least p, spot $p-1$ is occupied by some car (by driving backwards, of course);

(ii) there exists $J \subseteq [n]$, with $|J| \geqslant q-p+2$, such that $\tau_{p-2}(\alpha_{|J}) \in PF_{|J|,k}$ is a complete k-Naples parking function.

Each of the above stated conditions is guaranteed to be satisfied when $|[p, q]| \leqslant k$. Moreover, $\alpha \in PF_{n,k}$ if and only if all maximal intervals of U_α having size at least $k+1$ satisfy any of the above conditions.

We would like to point out that the second author has also found an intrinsic characterization of complete Naples parking function, which is contained in his master thesis [16] (specifically in Section 2.2, in particular Theorems 2.2.1 and 2.2.5). However, we have decided not to include it in the present paper, since it is extremely technical and rather difficult to apply in practice.

From the enumerative point of view, once complete parking preferences have been introduced, a natural task is to count complete k-Naples parking functions (with respect to the length), a problem that we addressed in Section 6. Moreover, since the notion of excess function of a parking preference is useful to characterize permutation-invariant Naples parking functions, we also obtain enumerative results concerning this last class of Naples parking function in Section 5, thus enhancing and improving the results achieved in [5]. In both cases, our enumerations are effective, and follow from a recursive description of the class of objects to be counted. The problem of finding a direct (closed) formula for such quantities remains open.

To conclude, we would like to propose a far reaching generalization of Naples parking functions that in our opinion deserves to be investigated.

The classical k-Naples parking rule requires that *all* cars check up to k spots preceding the preferred one before proceeding forward. However, one may ask that *each* car c_i has to check up to a certain number k_i (depending on i) of spots by driving backwards. In other words, we can consider a parking rule in which each car has its own number of spots to check by driving backwards. The resulting notion of parking function seems not to have been previously considered in the literature, and we plan to define and study it in a forthcoming paper.

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