The algebraic frustration dimension of a graph

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Abstract

We introduce a new minor monotone graph parameter, the algebraic frustration dimension frustdim(G) of a graph G, as the greatest minimum rank of an optimal solution in a certain family of embedding problems for signed graphs with underlying graph G. Our main results are forbidden minor characterizations of the classes of graphs with frustdim at most k for $k \in \{0, 1, 2\}$. The proofs establish connections to tree-width and related parameters and to minimum rank problems for graphs.

Mathematics Subject Classifications: 05C22, 05C50, 05C62, 05C83

1 Introduction

A signed graph is a pair (G, σ) with graph a $G = (V, E)$ and a map $\sigma\colon E \to \{-1, 1\}$ which we call an edge signature. The graphs in this paper are considered simple and undirected without loops or multiple edges. We usually assume that $V = \{1, \dots, n\}$ and denote the edge from vertex i to vertex j by ij (when writing $\{i, j\}$ we mean the set of vertices, regardless of connectivity). The edges of positive and negative sign are collected in the sets

$$E^+(\sigma) = \{ij \in E\colon \sigma_{ij} = 1\},\ E^-(\sigma) = \{ij \in E\colon \sigma_{ij} = -1\},$$

respectively. If σ is clear from the context we write E^+ and E^-. We sometimes write $V(G)$ and $E(G)$ to denote the vertex and edge sets of G if in danger of ambiguity.

The algebraic frustration dimension is defined on the following graph embedding problem. Given a graph $G = (V, E)$, non-negative vertex weights $s \in \mathbb{R}^V$ and non-negative edge lengths $l \in \mathbb{R}^E$ and an edge signature σ we define the embedding problem

$$\mathrm{Emb}(G, \sigma, s, l)\colon\ \max \sum_{i \in V} s_i \|y_i\|^2 \ \text{ s.t. } \begin{cases} y_i \in \mathbb{R}^k,\ k \in \mathbb{Z},\ k \geqslant 1 \\ \|y_i - y_j\| \leqslant l_{ij},\ ij \in E^+ \\ \|y_i + y_j\| \leqslant l_{ij},\ ij \in E^-. \end{cases} \tag{1}$$

Then frustdim(G, σ, s, l) is defined as the minimum rank of an optimal solution to Emb(G, σ, s, l) (if some exist) and frustdim(G) is the maximum of these ranks taken over

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all parameter sets s, l and σ. In this introductory section we gloss over a few details regarding the solvability of $\mathrm{Emb}(G, \sigma, s, l)$ which are given in Section 2. This definition is very similar in nature to that of the rotational dimension of [4]. It turns out that the frustration dimension is a minor monotone parameter and therefore, by the Graph Minor Theorem [9], the graphs of frustration dimension $\leqslant k$ for an integer $k \geqslant 0$ are characterized by a finite set of forbidden minors. As part of our main results we provide forbidden minor characterizations for $k \in \{0, 1, 2\}$.

Also very similar to the rotational dimension we establish a connection to the treewidth $tw(G)$ of a graph by showing that $\mathrm{frustdim}(G)$ is at most $tw(G)+1$. Furthermore we derive a bound in terms of the independence number $\alpha(G)$, namely $\mathrm{frustdim}(G) \leqslant n - \alpha(G)$. These results are obtained by geometric arguments.

Our techniques to construct solutions to $\mathrm{Emb}(G, \sigma, s, l)$ of high rank (in order to bound $\mathrm{frustdim}(G)$ from below) provide connections to minimum semidefinite rank problems for graphs: in

$$PSD(G) = \{X = (x_{ij}) \succeq 0 \colon ij \notin E(G) \implies x_{ij} = 0\}$$

find a matrix of minimum rank (possibly with additional constraints). This is due to a reformulation of the embedding problem as a semidefinite program whose dual is a generalized eigenvalue optimization problem involving the weighted (signed-graph-)Laplacian of (G, σ). The rows of an optimal embedding turn out to be generalized eigenvectors. The unconstrained minimum semidefinite rank problem is adressed in [1]. The paper [2] introduces a minor monotone graph invariant $\nu(G)$ which is the minimum rank of a matrix in $PSD(G)$ which additionally satisfies the Strong Arnold Property [8]. It turns out that for $k \leqslant 2$ we have

$$\{G \colon \mathrm{frustdim}(G) \leqslant k\} = \{G \colon \nu(G) \leqslant k\}.$$

Outline of the paper. This introductory section is concluded with some notation and preliminaries on signed graphs. Section 2 introduces the frustration dimension. Section 3 gives upper bounds in terms of tree width and related parameters. Lower bounds and the actual computation of frustdim of some graphs use semidefinite duality and are the topic of Sections 4 and 5, these results are used in 6 to prove the forbidden minor characterizations for graphs of frustration dimension at most 2.

Preliminaries and notation

We write $\mathbb{R}^V$ and $\mathbb{R}^E$ for the spaces of vectors indexed with the vertex V resp., edge set E. Similarly, we write $\mathbb{R}^{k \times V}$ for the space matrices Y with k rows whose columns are indexed by V. For a subset $U \subseteq V$ of the vertices we write $Y(U) = (y_i \colon i \in U)$. We denote by $\mathbf{1}_E$ the all-ones vector indexed with E. The adjacency matrix of a signed graph (G, σ) is the matrix $A(G, \sigma) = \{a_{ij}\}$ with $a_{ij} = \sigma_{ij}$ for edges ij and $a_{ij} = 0$ otherwise. Let $D(G)$ be the $V \times V$ diagonal matrix of vertex degrees of G and define the signed-graph Laplacian $L(G, \sigma) = D(G) - A(G, \sigma)$. For $\sigma = \mathbf{1}_E$ we get the (ordinary) Laplacian, for $\sigma = -\mathbf{1}_E$ the signless Laplacian. These matrices have edge weighted counterparts. We are particularly interested in the weighted signed-graph Laplacian which can be written as

$$L(G,\sigma,w) = \sum_{ij\in E^+} w_{ij}(e_i - e_j)(e_i - e_j)^\top + \sum_{ij\in E^-} w_{ij}(e_i + e_j)(e_i + e_j)^\top$$

or in more compact form

$$L(G,\sigma,w) = \sum_{ij\in E} w_{ij}E^\sigma_{ij} \text{ with } E^\sigma_{ij} = (e_i - \sigma_{ij}e_j)(e_i - \sigma_{ij}e_j)^\top.$$

On signed graphs we have the switching operation: we say that we switch σ at the vertex $i \in V$ if we multiply the sign of each edge incident with i by -1. We call an edge signature σ of G balanced if we obtain the all positive edge signature from σ by a sequence of vertex switchings. We say that we switch σ at a vertex set $C \subseteq V$ if we reverse the signs of all edges with one vertex in C and the other in its complement and leave all the remaining edge signs unchanged. This is equivalent to switching once at every single vertex of C. The sign of a cycle in G is the product of the signs of its edges. We call a cycle positive (resp., negative) if its sign is positive (resp., negative). Observe that the sign of a cycle is invariant under switching because every vertex switch switches 0 or 2 edges of that cycle. We will make frequent use of the following characterizations of balancedness [5].

Theorem 1. *Let (G,σ) be a signed graph. Then the following are equivalent.*

1. *σ is balanced.*
2. *There is a bipartition of the vertex set such that the negative edges connect vertices of different classes while positive edges connect vertices of the same class.*
3. *Every cycle in G is positive.*

Consequently, every edge signature of a forest can be switched to all positive.

In terms of the Laplacian, switching at the vertex i means that we conjugate $L(G,\sigma,w)$ with the diagonal matrix which has a -1 on the i-th diagonal entry and ones on the others. Hence switching does not alter the spectrum of the Laplacian and its eigenvectors simply have the signs of their i-th component reversed.

2 Frustration dimension

Given a signed graph (G,σ), non-negative vertex weights $s = (s_i : i \in V) \in \mathbb{R}^V$ and non-negative edge lengths $l = (l_{ij} : ij \in E) \in \mathbb{R}^E$ we define a graph embedding problem

$$\mathrm{Emb}(G,\sigma,s,l)\colon\ \max \sum_{i\in V} s_i\|y_i\|^2 \ \text{ s.t. } \begin{cases} y_i \in \mathbb{R}^k,\ k\in\mathbb{Z},\ k\geqslant 1 \\ \|y_i - y_j\| \leqslant l_{ij},\ ij\in E^+ \\ \|y_i + y_j\| \leqslant l_{ij},\ ij\in E^-. \end{cases}$$

Lemma 2. *1. If τ results from σ by switching at the vertex i and*

$$Y = (y_1, \dots, y_n)$$

is a feasible (resp., optimal) solution of $\mathrm{Emb}(G, \sigma, s, l)$ *then*

$$Y' = (y_1, \dots, y_{i-1}, -y_i, y_{i+1}, \dots y_n)$$

is a feasible (resp., optimal) solution of $\mathrm{Emb}(G, \tau, s, l)$.

2. *The feasible set of* $\mathrm{Emb}(G, \sigma, s, l)$ *is non-empty. It is unbounded if and only if* (G, σ) *has a balanced component.*

3. *If G is connected and σ is unbalanced then the set of optimal solutions to* $\mathrm{Emb}(G, \sigma, s, l)$ *is non-empty.*

Proof. 1) If Y is feasible for $\mathrm{Emb}(G, \sigma, s, l)$ and $ij \in E^+(\sigma)$ and $ik \in E^-(\sigma)$ then $l_{ij} \geqslant \|y_j - y_i\| = \|y_j + (-y_i)\|$ and $\|y_k + y_i\| = \|y_k - (-y_i)\|$ and thus Y' satisfies the constraints for $\mathrm{Emb}(G, \tau, s, l)$

2) We can always embed every vertex at 0 and thereby satisfy the constraints. So the feasible set always contains 0.

If (G, σ) has a balanced component P, by 1) we can assume that P has only positive edges. Then we obtain a feasible solution of $\mathrm{Emb}(G, \sigma, s, l)$ by embedding every vertex of P in the same point y and the remaining vertices (if any) in 0. We can choose $\|y\|$ arbitrarily large and therefore have an unbounded feasible set.

Conversely, assume G is connected and σ is unbalanced. Hence it has a cycle with an odd number of negative edges; after switching, we can assume that there is only one negative edge in that cycle. Assume that the cycle has the vertices $1, 2, \dots, k$ with $1k$ the negative edge. Let $(y_i, i \in V)$ be a feasible solution of $\mathrm{Emb}(G, \sigma, s, l)$. By the constraints and the triangle inequality the embedding points y_1 and y_k are at distance at most $\|y_1 - y_k\| \leqslant l_{12} + \dots + l_{k-1k} =: d$. Their sum has norm $\|y_1 + y_k\| \leqslant l_{1k}$ by the constraint for the edge $1k$. Therefore, their scalar product is bounded,

$$\begin{aligned} 4|\langle y_1, y_k \rangle| &= \left| \|y_1 + y_k\|^2 - \|y_1 - y_k\|^2 \right| \\ &\leqslant \max\left(\|y_1 + y_k\|, \|y_1 - y_k\|\right)^2 \leqslant \max(l_{1k}, d)^2 \end{aligned}$$

and thus $\|y_1\|^2$ and $\|y_k\|^2$ are bounded, as well because

$$l_{1k}^2 \geqslant \|y_1\|^2 + \|y_k\|^2 + 2\langle y_1, y_k \rangle \geqslant \|y_1\|^2 + \|y_k\|^2 - \frac{1}{2} \max(l_{1k}, d)^2$$

and thus

$$\max(\|y_1\|, \|y_k\|) \leqslant \frac{3}{2} \max(l_{1k}, l_{12} + \dots + l_{k-1k}).$$

Since every other vertex v is connected to v_1 we can consider a shortest v_1-v-path (shortest path with respect to the sum of the edge lengths from l). By switching, we can achieve that all edges on this path are positive and thus $\|v_1 - v\|$ is at most the sum of the edge lengths in a shortest path.

3) Optimal solutions exist by compactness of the feasible set and continuity of the objective function. $\square$

Definition 3. Let $G = (V, E)$ be a connected graph.

1. If G admits an unbalanced edge signature $\sigma \in \{-1, 1\}^E$ then define for edge lengths $l \in \mathbb{R}^E_+$ and vertex weights $s \in \mathbb{R}^V_+$ the parametrized frustration dimension as

$$\operatorname{frustdim}(G, \sigma, s, l) = \min\{\operatorname{rank}(Y) \colon Y \in \operatorname{argmax}(\operatorname{Emb}(G, \sigma, s, l))\}.$$

2. If G admits an unbalanced edge signature then define the frustration dimension of G as

$$\operatorname{frustdim}(G) = \max_{\sigma,s,l} \{\operatorname{frustdim}(G, \sigma, s, l)\}.$$

   where the maximum is taken over all unbalanced edge signatures $\sigma \in \{-1, 1\}^E$, vertex weights $s \in \mathbb{R}^V_+$, and edge lengths $l \in \mathbb{R}^E_+$.

3. If G is a tree with at least two vertices then define $\operatorname{frustdim}(G) = 1$.

4. The frustration dimension of K_1 is defined to be 0.

5. The frustration dimension of a disconnected graph is defined as the maximum of the frustration dimensions of its connected components.

Items 1 and 2 are well defined because of the previous lemma. The definitions in 3 and 4 for graphs which do not admit unbalanced edge signatures are chosen to match the upcoming results Theorem 16 and Corollary 18 on the frustration dimension of cliques and products of trees and cliques.

Observation 4. *The frustration dimension of a graph G is zero if and only if G has no edges (equivalently, has no K_2 minor).*

Proof. If G has an edge but no cycles this is true by definition: for a tree on at least two vertices every signature is balanced and therefore the frustration dimension is 1 by definition. So assume G has a component with a cycle. We can even assume that G is connected. Assuming that the vertex weights s and edge lengths l are all positive and the edge signature σ is unbalanced, Lemma 2 guarantees an optimal solution to $\operatorname{Emb}(G, \sigma, s, l)$. It must have at least one vertex embedded at a non-zero point and therefore $\operatorname{frustdim}(G, \sigma, s, l) \geqslant 1$. $\square$

Proposition 5. *The frustration dimension is a minor monotone graph parameter.*

Proof. Throughout the proof let G be obtained from $\hat{G}$ by a minor operation, i.e., by deleting an isolated vertex, by deleting an edge or contracting an edge and let σ, $s = (s_1, \ldots, s_n)$ and $l = (l_{ij} \colon ij \in E)$ be parameters such that $\operatorname{frustdim}(G, \sigma, s, l) = \operatorname{frustdim}(G)$. Observe further that by the previous lemma $\operatorname{frustdim}(G, \sigma, s, l)$ depends only on the switching class of σ.

Deletion of an isolated vertex. The graph $\hat{G} = G \cup K_1$ is disconnected and

$$\begin{aligned}\operatorname{frustdim}(\hat{G}) &= \max\{\operatorname{frustdim}(G), \operatorname{frustdim}(K_1)\} \\ &= \max\{\operatorname{frustdim}(G), 0\} = \operatorname{frustdim}(G).\end{aligned}$$

Deletion of an edge. Let G be obtained from $\hat{G}$ by deletion of the edge 12. We distinguish three cases.

First case. Assume that G has no cycles, i.e., every component C of G is a tree and $\operatorname{frustdim}(C) \leqslant 1$. Because of the previous observation $\operatorname{frustdim}(\hat{G}) \geqslant 1$.

Second case. Assume that G is connected and contains a cycle. Then we can choose an optimal parameter set such that the feasible region of $\operatorname{Emb}(G, \sigma, s, l)$ is bounded and $\operatorname{frustdim}(G) = \operatorname{frustdim}(G, \sigma, s, l)$. Define $\hat{l}_{12} = \sum_{ij \in E(G)} l_{ij} + 1$ and $\sigma_{12} = +1$ and choose the following parameter set for $\hat{G} : \hat{s} = s$, $\hat{l} = (l, \hat{l}_{12})$, $\hat{\sigma} = (\sigma, \sigma_{12})$. The embedding points of the vertices 1 and 2 cannot be further apart than $\sum_{ij \in E(G)} l_{ij}$ so that the constraint $\|y_1 - y_2\| \leqslant \hat{l}_{12}$ is redundant. Therefore, $\operatorname{Emb}(\hat{G}, \hat{\sigma}, \hat{s}, \hat{l})$ and $\operatorname{Emb}(G, \sigma, s, l)$ are equivalent and

$$\operatorname{frustdim}(G) = \operatorname{frustdim}(G, \sigma, s, l) = \operatorname{frustdim}(\hat{G}, \hat{\sigma}, \hat{s}, \hat{l}) \leqslant \operatorname{frustdim}(\hat{G}).$$

Third case. Assume that G has two connected components G_1 and G_2 with $1 \in V(G_1)$ and $2 \in V(G_2)$. If both graphs are trees we are in case one, so assume that G_1 has a cycle and $\operatorname{frustdim}(G) = \operatorname{frustdim}(G_1) \geqslant \operatorname{frustdim}(G_2)$. Choose an optimal parameter set s, l, σ of G_1 and extend it to a parameter set for G by setting s and l to zero on G_2 and σ to $+1$ on G_2. Set additionally $\hat{l}_{12} = 0$, $\hat{l} = (l, \hat{l}_{12})$, $\hat{\sigma}_{12} = +1$, $\hat{\sigma} = (\sigma, \hat{\sigma}_{12})$, $\hat{s} = s$. Then every optimal solution of $\operatorname{Emb}(\hat{G}, \hat{\sigma}, \hat{s}, \hat{l})$ yields an optimal solution for $\operatorname{Emb}(G_1, \sigma, s, l)$ and therefore frustdim does not decrease when passing from G to $\hat{G}$.

Contraction of an edge. Edge contractions do not change the number of components so we may assume that G is connected and obtained from $\hat{G}$ by contraction of the edge $01 \in E(\hat{G})$. If G has no cycle then, by definition, $\operatorname{frustdim}(G) \leqslant 1$. On the other hand, $\operatorname{frustdim}(\hat{G}) \geqslant 1$ because $\hat{G}$ has an edge. So assume that G has a cycle and that s, l, σ is an optimal parameter set for G with an unbalanced σ. Define parameters $\hat{s}$, $\hat{l}$ and $\hat{\sigma}$ for $\hat{G}$ by setting

$$\hat{s}_i = \begin{cases} 0 & \text{if } i = 0, \\ s_i & \text{otherwise} \end{cases} \qquad (i \in V(\hat{G}) = V(G) \cup \{0\})$$

and

$$\hat{l}_{ij} = \begin{cases} 0 & \text{if } ij = 01 \\ l_{1j} & \text{if } i = 0 \text{ and } 0j \in E(\hat{G}) \\ l_{ij} & \text{otherwise.} \end{cases}$$

and

$$\hat{\sigma}_{ij} = \begin{cases} +1 & \text{if } ij = 01 \\ \sigma_{1j} & \text{if } i = 0 \text{ and } 0j \in E(\hat{G}) \\ \sigma_{ij} & \text{otherwise.} \end{cases}$$

for $ij \in E(\hat{G})$. Since σ is unbalanced we have that $\hat{\sigma}$ is unbalanced, as well, and thus $\mathrm{Emb}(\hat{G}, \hat{\sigma}, \hat{s}, \hat{l})$ is bounded. Observe that in $\mathrm{Emb}(\hat{G}, \hat{\sigma}, \hat{s}, \hat{l})$ we have the constraint

$$\|y_1 - y_0\| \leqslant 0$$

and thus y_0 and y_1 must coincide in every feasible solution to $\mathrm{Emb}(\hat{G}, \hat{\sigma}, \hat{s}, \hat{l})$. Thus the feasible sets of $\mathrm{Emb}(G, \sigma, s, l)$ and $\mathrm{Emb}(\hat{G}, \hat{\sigma}, \hat{s}, \hat{l})$ are in bijection via

$$(y_1, \dots, y_n) \mapsto (y_1, y_1, \dots, y_n)$$

and because $\hat{s}_0 = 0$ this also defines a bijection of the optimal sets. Consequently, $\mathrm{frustdim}(G, \sigma, s, l) = \mathrm{frustdim}(\hat{G}, \hat{\sigma}, \hat{s}, \hat{l})$ which implies $\mathrm{frustdim}(G) \leqslant \mathrm{frustdim}(\hat{G})$. □

The following Lemma is borrowed from [4].

Lemma 6. *Let G be a connected graph with at least one cycle and let $\sigma \in \{-1, 1\}^E$ be an unbalanced edge signature for which* $\mathrm{frustdim}(G)$ *is attained, i.e.,* $\mathrm{frustdim}(G, \sigma, s, l) = \mathrm{frustdim}(G)$ *for some $s \geqslant 0$ and $l \geqslant 0$. Let furthermore $\mathcal{S} \subset \mathbb{R}_+^V$ and $\mathcal{L} \subset \mathbb{R}_+^E$ be dense subsets. Then we have that*

$$\mathrm{frustdim}(G) = \max \left\{ \mathrm{frustdim}(G, \sigma, s, l) \colon s \in \mathcal{S}, l \in \mathcal{L} \right\}.$$

In particular, there exist $s > 0$ and $l > 0$ with $\mathrm{frustdim}(G, \sigma, s, l) = \mathrm{frustdim}(G)$.

Proof. The proof is an almost verbatim copy of the proof of the analogous result for the rotational dimension in [4] and is omitted. □

As a corollary, we can choose the parameters s and l even as positive integer vectors because

$$\mathrm{frustdim}(G, \sigma, s, l) = \mathrm{frustdim}(G, \sigma, \mu s, \nu l)$$

for any positive real scalars μ and ν.

3 Upper bounds: tree decompositions and tree width bound

In this section we consider $\mathrm{Emb}(G, \sigma, s, l)$ for a connected graph $G = (V, E)$, some edge length parameters l, vertex weights s and an imbalanced edge signature σ (thus assuming that G contains a cycle) and use geometric arguments to bound $\mathrm{frustdim}(G, \sigma, s, l)$ from above.

Recall that a tree decomposition of G is pair $(T, \mathcal{V})$ in which T is a tree and $\mathcal{V}$ is a family of vertex sets (referred to as “bags” in what follows) indexed with the vertex set of T, $\mathcal{V} = (V_t \colon V_t \subseteq V(G), t \in T)$ such that the following are true.

1. $\bigcup_{t \in T} V_t = V$.

2. For every edge $ij \in E$ there is a bag V_t with $\{i,j\} \subseteq V_t$.

3. If $v \in V_s \cap V_t$ for some $s \neq t$ then $v \in V_x$ for every vertex $x \in T$ on the unique s-t-path in T.

The width of $(T, \mathcal{V})$ is the size of the largest bag minus one. The tree width $\mathrm{tw}(G)$ is the minimum width that a tree decomposition of G can attain.

The goal of this section is to prove the following.

Theorem 7. *Let G be a connected graph with at least one cycle and $(T, \mathcal{V})$ be a tree decomposition of G of width d. Let further σ be an imbalanced edge signature, $s \in \mathbb{R}^V$ a vector of non-negative vertex weights and $l \in \mathbb{R}^E$ be non-negative edge lengths.*

1. *It holds that* $\mathrm{frustdim}(G, \sigma, s, l) \leqslant d + 1$ *and*

$$\mathrm{frustdim}(G) \leqslant \mathrm{tw}(G) + 1.$$

2. *If additionally every bag $B \in \mathcal{V}$ with $B = d + 1$ contains two distinct vertices v, w such that the set $\{v, w\}$ is not contained in any other bag $C \in \mathcal{V} \setminus \{B\}$ then it holds that* $\mathrm{frustdim}(G, \sigma, s, l) \leqslant w$. *If additionally $(T, \mathcal{V})$ is of minimal width $d = \mathrm{tw}(G)$ with that property then*

$$\mathrm{frustdim}(G) \leqslant \mathrm{tw}(G).$$

Both bounds are attained.

We remark that above bounds also hold by definition for graphs without cycles. Two very useful applications of part 2 of the Theorem are given at the end of this section.

The main tools in the proof of Theorem 7 are the following two lemmas.

Lemma 8 (Rotation Lemma)**.** *Let V, W some subspaces of a Euclidean vectorspace E with $\dim W \leqslant \dim V$ and let $v \in V$ and $w \in W$. Then there is an orthogonal transformation Q of E satisfying*

1. *$QW \subseteq V$.*

2. *$Qx = x$ for every $x \in V \cap W$.*

3. *$\|Qw - v\| \leqslant \|w - v\|$.*

The proof is postponed to Section 7. The second main ingredient is the following fact on separation properties of tree decompositions (see [3]):

Lemma 9. *If $t_1 t_2$ is an edge of T and T_1, T_2 are the trees constituting $T - t_1 t_2$ with $t_i \in T_i$ ($i \in \{1, 2\}$) then the vertex sets*

$$U_i = \bigcup_{t \in T_i} V_t \quad (i \in \{1, 2\})$$

of G are separated by $V_{t_1} \cap V_{t_2}$

The sets $V_{t_1} \cap V_{t_2}$ will serve as "hinges" for rotation operations which successively rotate "small rank bags into adjacent large rank bags around their intersection". Part 1 of Theorem 7 follows immediately from the next result.

Lemma 10. *Let Y be an optimal embedding for* $\mathrm{Emb}(G,\sigma,s,l)$. *Then there is another optimal embedding Z for* $\mathrm{Emb}(G,\sigma,s,l)$ *such that*

$$\mathrm{rank}(Z) = \max\{\mathrm{rank}\, Y(V_t)\colon t \in T)\}.$$

Proof. The proof is by induction and follows directly from the following assertion.

Let S be a subtree of T and set $U = \bigcup_{t \in S} V_t$. Let $t_1 t_2$ be an edge of T with $t_1 \in S$ and $t_2 \notin S$. Let Y be an optimal embedding for $\mathrm{Emb}(G,\sigma,s,l)$. Then there is an optimal embedding Z such that

$$\mathrm{rank}(Z(U \cup V_{t_2})) = \max\{\mathrm{rank}\, Y(U),\ \mathrm{rank}\, Y(V_{t_2})\}.$$

With the notation of Lemma 9 we first observe that $U \subseteq U_1$ and and $V_{t_2} \subseteq U_2$. Now we apply the rotation lemma 8 to the spaces $A = \mathrm{span}(Y(U))$ and $B = \mathrm{span}(Y(V_{t_2}))$ (with the vectors v, w in 8 both zero) to obtain an orthogonal transformation Q which maps the smaller space into the larger one: assume first that $\dim(A) \geqslant \dim(B)$. Then the rotation lemma provides Q with $QB \subseteq A$ and $Qx = x$ for every $x \in A \cap B$. Observe that $\mathrm{span}(Y(V_{t_1} \cap V_{t_2})) \subseteq A \cap B$. The desired embedding Z is obtained as

$$Z(U_1 \setminus (V_{t_1} \cap V_{t_2})) = Y(U_1 \setminus (V_{t_1} \cap V_{t_2})) \text{ and } Z(U_2) = QY(U_2).$$

Observe that the edge length constraints are still all satisfied: there are no edges between $U_1 \setminus U_2$ and $U_2 \setminus U_1$ because $U_1 \cap U_2 = V_{t_1} \cap V_{t_2}$ separates U_1 and U_2. For an edge ab with $a \in V_{t_1} \cap V_{t_2}$ and $b \in U_2$ we have

$$\|z_a - \sigma_{ab} z_b\| = \|Qy_a - \sigma_{ab} Qy_b\| = \|y_a - \sigma_{ab} y_b\| \leqslant l_{ab}$$

and for an edge cd with $d \in V_{t_1} \cap V_{t_2}$ and $c \in U_1$ we have

$$\|z_d - \sigma_{cd} z_c\| = \|Qy_d - \sigma_{cd} y_c\| = \|y_d - \sigma_{cd} y_c\| \leqslant l_{cd}.$$

In the case that $\dim(A) < \dim(B)$ one would get analogously

$$Z(U_1) = QY(U_1) \text{ and } Z(U_2 \setminus (V_{t_1} \cap V_{t_2})) = Y(U_2 \setminus (V_{t_1} \cap V_{t_2})).$$

To finish the proof of the proposition, root the tree at some vertex t_0, set $S_0 = \{t_0\}$ and $Z_0 = Y$ For $i \geqslant 0$ obtain S_{i+1} from S_i by adding a vertex of $T - S$ adjacent to S and obtain a new embedding Z_{i+1} from Z_i according to the assertion, with Z_i and Z_{i+1} playing the roles of Y and Z, respectively. $\square$

Now, in order to prove part 2 of Theorem 7 we reduce the dimensions of the single bags. Here the property 3 of the Rotation Lemma 8 is required:

Lemma 11. *Let $(T, \mathcal{V})$ be a tree decomposition of G and let V_{t_0} be one of its bags with $|V_{t_0}| = k+1$. Assume that V_{t_0} has two distinct vertices v, w such that the set $\{v, w\}$ is not contained in any other bag $V_t \in \mathcal{V}$ with $V_t \neq V_{t_0}$. Assume that Y is a an optimal solution to* $\operatorname{Emb}(G, \sigma, s, l)$*. Then there is an optimal solution Z of* $\operatorname{Emb}(G, \sigma, s, l)$ *with*

$$\operatorname{rank}(Z(V_{t_0})) \leqslant k \textit{ and } \operatorname{rank}(Z(V_t)) = \operatorname{rank}(Y(V_t)) \textit{ for } t \neq t_0.$$

Proof. Let $Y = \{y_1, \ldots, y_n\}$ be an optimal embedding for $\operatorname{Emb}(G, \sigma, s, l)$ and assume that $\operatorname{rank}(Y(V_{t_0})) = k+1$ for otherwise we can simply choose $Z = Y$. Observe that this implies that the edge length parameters on the edges induced on V_{t_0} are positive because $l_{ij} = 0$ would yield that $y_i = y_j$ or $y_i = -y_j$.

After a renumbering of vertices we can assume that $V_{t_0} = \{1, \ldots, k+1\}$ and $v = 1$ and $w = 2$. Denote by $C_1 = \{1, 3, \ldots, k+1\}$ and $C_2 = \{2, 3, \ldots, k+1\}$.

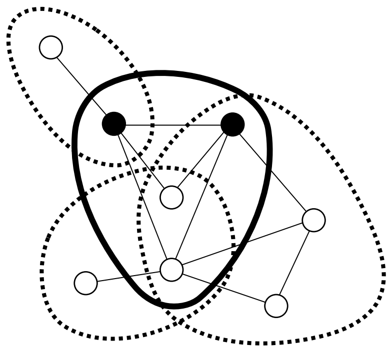

Figure 1: A tree decomposition as in Lemma 11. The black vertices constitute the set $\{v, w\}$ as in the Lemma.

We now apply the rotation lemma 8 with the spaces $R = \operatorname{span}(Y(C_1))$ and $S = \operatorname{span}(Y(C_2))$ and the points $y_1 \in R$ and $\sigma_{12} y_2 \in S$ where we assume for the moment that $12 \in E(G)$. Thus we obtain an orthogonal transformation Q which maps S into R with

$$\|y_1 - \sigma_{12} Q y_2\| \leqslant \|y_1 - \sigma_{12} y_2\| \leqslant l_{12} \tag{2}$$

and $Qy_i = y_i$ for $i \in C_1 \cap C_2$ because Q fixes $R \cap S = \operatorname{span}(Y(C_1 \cap C_2))$ pointwise. If $12 \notin E(G)$ we do not have to take care of the distance constraint.

By the assumption on the vertices v, w it is true that for every edge $t_0 u$ of T we have $V_{t_0} \cap V_u \subseteq C_1$ or $V_{t_0} \cap V_u \subseteq C_2$. Now, loosely speaking, we apply this Q to the embedding points of the vertex set $U_2 \in V$ consisting of C_2 and "everything which is connected to V_{t_0} via C_2" and leave the embedding points of $U_1 = C_1 \cup (V \setminus U_2)$ unchanged. To make that precise, we apply Lemma 9 to the edges of T incident with t_0. For such an edge $t_0 u$ of T denote by T_u the subtree of $T - t_0 u$ that contains u. There is a not necessarily unique assignment of the edges

$$f \colon \{t_0 u \colon t_0 u \in T\} \to \{C_1, C_2\}$$

such that

$$f(t_0 u) = C_i \implies V_{t_0} \cap V_u \subseteq C_i \quad (i = 1, 2).$$

Define two (possibly empty) forests in $T - t_0$

$$F_i = \bigcup_{u\colon f(t_0u)=C_i} T_u \quad (i = 1, 2)$$

and accordingly two vertex sets of G

$$U_i = C_i \cup \bigcup_{t \in F_i} V_t \quad (i = 1, 2).$$

These are the vertex sets connected to V_{t_0} "along C_i." We observe that $U_1 \cup U_2 = V$, $U_1 \cap U_2 = C_1 \cap C_2$ and define a new embedding Z by

$$\begin{aligned} Z(U_1 \setminus C_1 \cap C_2) &= Y(U_1 \setminus C_1 \cap C_2), \\ Z(C_1 \cap C_2) = Y(C_1 \cap C_2) &= QY(C_1 \cap C_2), \\ Z(U_2 \setminus C_1 \cap C_2) &= QY(U_2 \setminus C_1 \cap C_2). \end{aligned}$$

We observe that Z satisfies the edge length constraints of $\mathrm{Emb}(G, \sigma, s, l)$: the constraint for the edge 12 is satisfied by (2). For all other edges ij we either have $z_i = y_i$ and $z_j = y_j$ or $z_i = Qy_i$ and $z_j = Qy_j$. In particular, for all bags $V_t \neq V_{t_0}$ we have that

$$\mathrm{rank}(Z(V_t)) = \mathrm{rank}(Y(V_t)). \qquad \square$$

To finish the proof of part 2 of Theorem 7, first apply Lemma 11 successively to all bags of maximum size and then apply Lemma 10.

We now give two applications of Theorem 7 used in subsequent sections.

Independence number

Theorem 12. *In the situation of Theorem 7 let $B \subseteq V(G)$ be an independent set. Then it holds that*

$$\mathrm{frustdim}(G, \sigma, s, l) \leqslant |V| - |B|.$$

In particular it is true that

$$\mathrm{frustdim}(G) \leqslant |V| - \alpha(G).$$

where $\alpha(G)$ the independence number of G.

Proof. Let $A = V \setminus B$. We get a tree decomposition of G by choosing the bags as $\mathcal{V} = \{A \cup \{v\}\colon v \in B\}$ and the tree T as a path. The bags have all size $|A| + 1$ and they all satisfy the assumption of item 2 in Theorem 7 with a vertex $w \in A$ and $v \in B$. Thus $\mathrm{frustdim}(G, \sigma, s, l) \leqslant |A| = |V| - |B|$. $\square$

Colin de Verdiere's *la* parameter

Our characterization of graphs of small frustration dimension relies on another upper bound in terms of the parameter $la(G)$ [2] which is the smallest integer k such that G is a minor of a graph $T \times K_k$ for some tree T. Recall that for two graphs G and H their product $G \times H$ is the graph with vertex set $V(G) \times V(H)$ and the following edge set: a vertex (a_1, b_1) is connected to (a_2, b_2) if and only if $a_1 = a_2$ and $b_1 b_2 \in E(H)$ or $b_1 = b_2$ and $a_1 a_2 \in E(G)$. By definition la is minor monotone.

Theorem 13. *frustdim*$(G) \leqslant la(G)$.

Proof. Assume that $la(G) = k$ for some integer. So G is a minor of $K_k \times T$ for some tree T and Proposition 5 implies that

$$\operatorname{frustdim}(G) \leqslant \operatorname{frustdim}(K_k \times T).$$

Hence it suffices to show that $\operatorname{frustdim}(K_k \times T) \leqslant k$. This is achieved by providing a tree decomposition of $K_k \times T$ of width k which satisfies property 2 of Theorem 7.

The graph $K_k \times T$ is a clique sum of several copies $K_k \times K_2$ each copy corresponding to an edge ab of T and copies of incident edges ab and bc intersect in the k-clique corresponding to vertex b of T. So it suffices to give a tree decomposition $(D, \mathcal{V})$ of $K_k \times K_2$. We write $V(K_k \times K_2) = \{1, \ldots, k\} \times \{a, b\}$ and define $\mathcal{V} = \{V_i \colon 1 \leqslant i \leqslant k\}$ with

$$V_i = \{(r, b) \colon 1 \leqslant r \leqslant i\} \cup \{(s, a) \colon i \leqslant s \leqslant k\}$$

and D as the path $1 - 2 - \cdots - k$. Clearly the union of the bags is V and every edge has both endpoints in some bag. Moreover, if $i < l < j$ we have

$$V_i \cap V_j = \{(1, b), \ldots, (i, b), (j, a), \ldots, (k, a)\} \subseteq V_l$$

so $(D, \mathcal{V})$ is indeed a tree decomposition. Its width is k because every bag has size $k + 1$. Additionally, the condition of part 2 of Theorem 7 is satisfied because every bag V_i has a pair of vertices, namely $\{(i, a), (i, b)\}$, such that $\{(i, a), (i, b)\}$ is not a subset of any other bag $V_j, j \neq i$. Building these bags for every edge ab of T we obtain a tree decomposition of $K_k \times T$ with the same properties. $\square$

4 Semidefinite formulation and duality

For matrices A, B of the same dimensions we write

$$\langle A, B \rangle = \sum_{i,j} A_{ij} B_{ij} = \operatorname{Tr}(B^\top A).$$

Recall the following facts about a positive semidefinite matrices:

$$(A \succeq 0 \Leftrightarrow \forall B \succeq 0 \colon \langle A, B \rangle \geqslant 0) \quad \text{and } (\forall A, B \succeq 0 \colon \langle A, B \rangle = 0 \Leftrightarrow AB = 0). \qquad (3)$$

The embedding problem (1) is equivalent to the semidefinite program

$$\text{maximize } \langle S, X\rangle \quad \text{s.t. } \begin{cases} l_{ij}^2 - \langle E_{ij}^\sigma, X\rangle \geqslant 0 \quad (ij \in E), \\ X \succeq 0, \end{cases} \tag{4}$$

where $S = \operatorname{diag}(s_i,\, i \in V)$ because if X is feasible (resp., optimal) for (4) and $X = Y^\top Y$ is a Gram-decomposition of X then Y is feasible (resp., optimal) for (1) and vice versa.

We introduce Lagrange-multipliers $(w_{ij} \colon ij \in E)$ and lift the constraints to the objective to obtain the Lagrange dual of (4) by interchanging sup and inf.

$$\sup_{X\succeq 0} \inf_{w\geqslant 0} \sum_{ij\in E} w_{ij}(l_{ij}^2 - \langle E_{ij}^\sigma, X\rangle) + \langle S, X\rangle \leqslant \inf_{w\geqslant 0} \sup_{X\succeq 0} \sum_{ij\in E} l_{ij}^2 w_{ij} - \langle \sum_{ij\in E} w_{ij} E_{ij}^\sigma - S, X\rangle. \tag{5}$$

The first line is equivalent to (4) because the inner inf is $> -\infty$ if and only if the linear constraints of (4) are satisfied. In that case the inner inf equals $\langle S, X\rangle$. With the notation $L(G,\sigma,w) = \sum_{ij\in E} w_{ij} E_{ij}^\sigma$ and a similar argument the second line is equivalent to the Lagrange dual program

$$\mathrm{P}(G,\sigma,s,l)\colon \quad \text{minimise } \sum_{ij\in E} l_{ij}^2 w_{ij} \quad \text{s.t. } \begin{cases} L(G,\sigma,w) - S \succeq 0, \\ w_{ij} \geqslant 0 \ (ij \in E). \end{cases} \tag{6}$$

because the inner sup is $< \infty$ if and only if the semidefinite constraint of (6) holds.

The $\leqslant$ in (5) is the inf-sup inequality or weak duality. Regarding strong duality (i.e., equality) we have

Proposition 14 (Karush-Kuhn-Tucker (KKT) Conditions)**.** *Assume that the parameter vectors s and l in $\mathrm{P}(G,\sigma,s,l)$ and $\mathrm{Emb}(G,\sigma,s,l)$ (resp., (4)) are strictly positive and σ is unbalanced. Then both problems are solvable and their objective values coincide. Furthermore, feasible solutions $Y = (y_i, i \in V) \in \mathbb{R}^{r\times V}$ of $\mathrm{Emb}(G,\sigma,s,l)$ and $w = (w_{ij}, ij \in E) \in \mathbb{R}^E$ of $\mathrm{P}(G,\sigma,s,l)$ are optimal if and only if*

$$(S - L(G,\sigma,w))\, Y^\top = 0, \tag{7}$$

$$w_{ij}(\|y_i - \sigma_{ij} y_j\|^2 - l_{ij}^2) = 0 \ (ij \in E) \tag{8}$$

hold. In coordinates, (7) *reads*

$$\left(s_i - \sum_{j\colon ij\in E} w_{ij}\right) y_i + \sum_{j\colon ij\in E} w_{ij}\sigma_{ij} y_j = 0. \tag{9}$$

Proof. We have already seen in Lemma 2 that (1) has optimal solutions and hence (4) attains its optimal value. Furthermore, because all of the edge-length-parameters are positive, we can find a *strictly feasible point* of (4), i.e., a positive definite matrix $X \succ 0$ which satisfies the linear inequalities with inequality: just take a diagonal matrix with sufficiently small positive entries on the diagonal. Therefore, by a central result in convex optimisation [6] *strong duality* holds, i.e., $\mathrm{P}(G, \sigma, s, l)$ and (4) have the same optimal values and additionally $\mathrm{P}(G, \sigma, s, l)$ attains its optimal value. The following Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for optimality. The KKT conditions are primal and dual feasibility together with the following complementarity conditions

$$\left\langle \sum_{ij \in E} w_{ij} E_{ij}^{\sigma} - S, X \right\rangle = 0, \tag{10}$$

$$w_{ij} \left(l_{ij}^2 - \left\langle E_{ij}^{\sigma}, X \right\rangle \right) = 0 \quad (ij \in E). \tag{11}$$

Using (3) the first equation becomes

$$(S - L(G, \sigma, w))\, X = 0$$

and substituting X by a Gram-decomposition $X = Y^{\top} Y$ with $Y = (y_i, i \in V)$ yields (7) and (9). The same substitution in (11) leads to (8). $\square$

If we drop the assumption of positive edge lengths then $\mathrm{P}(G, \sigma, s, l)$ does not necessarily have a solution:

Example of an unsolvable $\mathrm{P}(\boldsymbol{G, \sigma, s, l})$: Consider the complete graph $G = K_3$ with the vertex weights $s_i = 1$ $(i = 1, 2, 3)$, edge lengths $l_{12} = 1$ and $l_{23} = l_{13} = 0$ and the edge signature $\sigma_{12} = -1$ and $\sigma_{23} = \sigma_{13} = 1$. From the edge length constraints of $\mathrm{Emb}(G, \sigma, s, l)$ for edges 13 and 23 it follows that every feasible embedding satisfies $y_1 = y_2 = y_3$. Consequently, (4) is not strictly feasible because it has no positive definite feasible solutions. The constraint for 12 yields $\|v_1\| \leqslant \frac{1}{2}$ and hence the optimal value of $\mathrm{Emb}(G, \sigma, s, l)$ is $\frac{3}{4}$. On the other hand, $\mathrm{P}(G, \sigma, s, l)$ is strictly feasible and therefore strong duality holds. If $\mathrm{P}(G, \sigma, s, l)$ had an optimal solution it would follow that $w_{12} = \frac{3}{4}$ because the objective function of $\mathrm{P}(G, \sigma, s, l)$ is simply w_{12}. We show that the semidefinite constraint of $\mathrm{P}(G, \sigma, s, l)$ is not satisfied whenever $w_{12} = \frac{3}{4}$. Observe that by symmetry and convexity there is an optimal solution to $\mathrm{P}(G, \sigma, s, l)$ in which $w_{13} = w_{23} = b$ for some $b > 0$ because if $w = (w_{12}, w_{23}, w_{13})$ is optimal then so is $w' = (w_{12}, w_{13}, w_{23})$ and thus $(w + w')/2$. With $w = (3/4, b, b)$ we have

$$A := L(G, \sigma, w) - S = \begin{pmatrix} b - 1/4 & 3/4 & -b \\ 3/4 & b - 1/4 & -b \\ -b & -b & 2b - 1 \end{pmatrix}$$

whose determinant is $(1 - b)/2$ and thus necessarily $b \leqslant 1$. On the other hand, with $z = (0, 1, 1)^{\top}$ we have $z^{\top} A z = b - 5/4$ and thus necessarily $b \geqslant 5/4 > 1$, a contradiction. $\square$

Theorem 15. *Consider optimal solutions w of $\mathrm{P}(G, \sigma, s, l)$ and $Y = (y_i, i \in V)$ of $\mathrm{Emb}(G, \sigma, s, l)$. For $i \in V$ define b_i as*

$$b_i = \sum_{j : ij \in E} l_{ij}^2 w_{ij}.$$

and $b = (b_i, i \in V)$*. Define further* $z = (\|y_i\|^2, i \in V)$*. Then* z *satisfies the linear system of equations*

$$(2S - L(G, \mathbf{1}_E, w))z = b.$$

In particular, if $2S - L(G, \mathbf{1}_E, w)$ *is invertible then the norms of the embedding points are uniquely determined by* w*.*

Proof. Consider the dual constraint corresponding to an edge ij with $w_{ij} > 0$. By the KKT-conditions it is satisfied with equality,

$$l_{ij}^2 = \|y_i - \sigma_{ij} y_j\|^2.$$

Expand the right side to obtain

$$l_{ij}^2 = \|y_i\|^2 + \|y_j\|^2 - 2\langle y_i, \sigma_{ij} y_j\rangle.$$

Fix $i \in V$, multiply the previous equation by w_{ij} and sum over all neighbours j of i to get

$$\sum_{j\colon ij\in E} w_{ij} l_{ij}^2 = \sum_{j\colon ij\in E} w_{ij}\|y_i\|^2 + \sum_{j\colon ij\in E} w_{ij}\|y_j\|^2 - 2\langle y_i, \sum_{j\colon ij\in E} \sigma_{ij} w_{ij} y_j\rangle. \tag{12}$$

Observe that by the KKT conditions

$$\left(\sum_{j\colon ij\in E} w_{ij} - s_i\right) y_i = \sum_{j:ij\in E} w_{ij}\sigma_{ij} y_j.$$

Substitute this into (12) to get the assertion

$$\sum_{j\colon ij\in E} w_{ij} l_{ij}^2 = 2s_i\|y_i\|^2 - \sum_{j\colon ij\in E} w_{ij}\|y_i\|^2 + \sum_{j\colon ij\in E} w_{ij}\|y_j\|^2. \tag{13}$$

□

5 Frustration dimension of complete graphs and k-trees

The use of Theorem 15 is in finding parameter vectors s, l and σ which force a large $\operatorname{frustdim}(G, \sigma, s, l)$:

1. Choose w, σ and s such that $L(G, \sigma, w) - S \succeq 0$ and $\operatorname{rank}(L(G, \sigma, w) - S)$ is as small as possible.
2. Choose a matrix $Y = (y_i, i \in V)$ (y_i column vectors) whose rows contain a basis of $\ker(L(G, \sigma, w) - S)$.
3. For $ij \in E$ compute $l_{ij} = \|y_i - \sigma_{ij} y_j\|$.

With these choices of s and l we have that w and Y are optimal solutions to $\mathrm{P}(G, \sigma, s, l)$ and $\mathrm{Emb}(G, \sigma, s, l)$, respectively, because by construction they are feasible and satisfy the KKT system. The rank of Y is clearly an upper bound for frustdim. Theorem 15 may now help provide a lower bound.

Theorem 16. *The frustration dimension of a graph G on $n \geqslant 3$ vertices is $n-1$ if and only if G is complete.*

Proof. We assume that $V(K_n) = \{1, \dots, n\}$. We first show that $\operatorname{frustdim}(K_n) = n-1$. We have $\operatorname{frustdim}(K_n) \leqslant n-1$ by Theorem 12. We follow the procedure outlined above.

1. Choose $L(G, \sigma, w) - S$ as the all ones matrix. This means that $\sigma \equiv -1$, $w \equiv 1$ and $s \equiv n-2$.

2. Choose Y as
$$\left[\begin{array}{c|c} -1 & \\ \vdots & I_{n-1} \\ -1 & \end{array}\right].$$
Clearly, $\operatorname{rank}(Y) = n-1$ and $(L(G, \sigma, w) - S)Y^\top = 0$.

3. Compute $l_{1j}^2 = \|y_1 + y_j\|^2 = n-2$, $l_{ij}^2 = \|y_i + y_j\|^2 = 2$ for $i, j = 2, \dots, n$.

4. The matrix $L(G, \mathbf{1}, w)$ used in Theorem 15 is in this case the ordinary (unsigned, unweighted) Laplacian $L(K_n)$ with columns
$$(-1, \dots, -1, n-1, -1, \dots, -1)^\top$$
(in column i the $n-1$ is in the i-th position). Its eigenvalues are 0 (simple) and n ($n-1$-fold). Therefore, if $n \neq 4$, we have that $2S - L(G, \mathbf{1}, w) = (2n-4)I - L(K_n)$ is invertible. Now by Theorem 15 we know that if Y' is another optimal solution to $\operatorname{Emb}(G, \sigma, s, l)$ we must have that $\|y_i'\| = \|y_i\|$ for $i = 1, \dots, n$. By the KKT conditions Y' must fulfill all constraints with equality and so the scalar products
$$\langle y_i', y_j' \rangle = \frac{l_{ij}^2 - \|y_i'\|^2 - \|y_j'\|^2}{2\sigma_{ij}}$$
and hence the Gram matrix are also uniquely determined, namely as
$$(Y')^\top Y' = Y^\top Y = \left[\begin{array}{c|ccc} n-1 & -1 & \cdots & -1 \\ \hline -1 & & & \\ \vdots & & I_{n-1} & \\ -1 & & & \end{array}\right]$$
which obviously has rank $n-1$. Consequently, $\operatorname{frustdim}(K_n) = n-1$ if $n \neq 4$.

For $n = 4$ the above argument does not work. Indeed, a one-dimensional optimal embedding for $\operatorname{Emb}(K_4, \sigma \equiv -1, s \equiv 2, l \equiv 2)$ is given by
$$x_1 = -\frac{3\sqrt{2}}{2},\ x_2 = x_3 = x_4 = \frac{\sqrt{2}}{2}$$
and hence $\operatorname{frustdim}(K_4, \sigma \equiv -1, s \equiv 2, l \equiv 2) = 1$.

The above procedure works however if we choose

$$L(G,\sigma,w)-S=\begin{bmatrix}4&2&2&2\\2&1&1&1\\2&1&1&1\\2&1&1&1\end{bmatrix}$$

i.e., for the parameter set with vertex weights $s_1=2$, $s_2=s_3=s_4=3$, $\sigma\equiv-1$, edge weights

$$w_{1j}=2 \text{ and } w_{ij}=1,\ i,j\in\{2,3,4\}.$$

For these weights $L(G,\sigma,w)-S$ has rank one. A basis of the kernel is in the rows of

$$Y=\begin{bmatrix}-1/2&1&0&0\\-1/2&0&1&0\\-1/2&0&0&1\end{bmatrix}.$$

The columns yield the edge lengths

$$l_{1j}^2=\frac{3}{4} \text{ and } l_{ij}^2=2,\ i,j\in\{2,3,4\}.$$

The matrix

$$2S-L(G,\mathbf{1},w)=\begin{bmatrix}-2&2&2&2\\2&2&1&1\\2&1&2&1\\2&1&1&2\end{bmatrix}$$

is invertible and we can argue as above that the Gram matrix of any optimal solution is

$$Y^\top Y=\begin{bmatrix}3/4&-1/2&-1/2&-1/2\\-1/2&1&0&0\\-1/2&0&1&0\\-1/2&0&0&1\end{bmatrix}.$$

We get that $\operatorname{frustdim}(K_4)=3$.

Any non-complete graph has an independence number of at least 2 and therefore a frustration dimension of at most $n-2$. ☐

Corollary 17. *The frustration dimension of the complete bipartite graph $K_{m,n}$ with $m\leqslant n$ is m.*

Proof. Because $K_{m,n}$ has a K_{m+1} minor we have $\operatorname{frustdim}(K_{m,n})\geqslant m$. And because $\alpha(K_{m,n})=n$ we have $\operatorname{frustdim}(K_{m,n})\leqslant m$ because of Theorem 12. ☐

Corollary 18. *If T is a tree with at least two vertices then the frustration dimension of $K_k\times T$ is k.*

Proof. We have already seen in the proof of Theorem 13 that $\operatorname{frustdim}(K_k\times T)\leqslant k$. Because T has at least one edge $K_k\times T$ has a K_{k+1} minor and thus equality holds. ☐

Corollary 19. *The frustration dimension of a graph G is at most one if and only if it is a forest (equivalently, does not have a K_3 minor).*

Proof. Forests have frustration dimension at most 1 by definition. If G is not a forest then it contains a cycle and therefore a K_3 minor. It follows that $\operatorname{frustdim}(G) \geqslant \operatorname{frustdim}(K_3) = 2 > 1$. □

Definition 20. 1. Let $k > 0$. A graph G is called a k-tree if there is a finite sequence $G_0, \dots, G_r$ ($r \geqslant 0$) of graphs with $G_0 = K_{k+1}$, $G_r = G$ and G_{i+1} is obtained by connecting a new vertex to a k-clique in G_i. Equivalently, G is obtained by pasting K_{k+1}'s along K_k's.

2. Any graph obtained by deleting edges from a k-tree is a called a partial k-tree.

3. For $k \geqslant 2$ the k-tree TP_k on $k+4$ vertices resulting from connecting three distinct vertices to three pairwise distinct k-cliques of a K_{k+1} is called a k-tripod. More precisely,

$$\begin{aligned} V(TP_k) =& \{1, \dots, k+1, k+2, k+3, k+4\} \\ E(TP_k) =& \{ij \colon 1 \leqslant i < j \leqslant k+1\} \\ & \cup \{i(k+1+j) \colon 1 \leqslant i \leqslant k+1, 1 \leqslant j \leqslant 3, i \neq j\}. \end{aligned} .$$

Lemma 21. *The building process of a k-tree yields a tree decomposition $(T, \mathcal{V})$ in which the family of bags $\mathcal{V}$ is the family of $k+1$-cliques of G.*

Proof. Induction on the number of vertices of G. If G is a $k+1$ clique take $\mathcal{V} = \{V\}$ with an empty edge set. Otherwise remove a degree-k-vertex u from G. By induction $G - u$ has tree-decomposition as desired. Extend it to one of G with an additional bag V_u which is simply the $k+1$-clique of G which contains u. The bag V_u is added to the vertices of T and is connected to some bag V_t in T that contains the k-clique to which u is connected in G. □

Observe that this tree decomposition of TP_k fails the assumption of the second part of Theorem 7. Indeed we have

Lemma 22. *The frustration dimension of TP_k is $k+1$.*

Proof. The vertices $k+2$, $k+3$ and $k+4$ form an independent set and therefore $\operatorname{frustdim}(TP_k) \leqslant k+1$. We show that equality holds. We build a candidate $W = L(TP_k, \sigma, w) - S$ of rank 3 by performing simultaneous row and column operations starting from a diagonal matrix with diagonal entries $(0, \ldots, 0, 1, 1, 1)$ and get

$$W = \left(\begin{array}{ccc|ccc|ccc}
2 & 1 & 1 & 2 & \cdots & 2 & 0 & 1 & 1 \\
1 & 2 & 1 & 2 & \cdots & 2 & 1 & 0 & 1 \\
1 & 1 & 2 & 2 & \cdots & 2 & 1 & 1 & 0 \\
\hline
2 & 2 & 2 & 3 & \cdots & 3 & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & 3 & \cdots & 3 & 1 & 1 & 1 \\
\hline
0 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & \cdots & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & \cdots & 1 & 0 & 0 & 1
\end{array}\right). \tag{14}$$

By construction M is positive semi-definite and $\sigma = -\mathbf{1}$. The diagonal of the weighted Laplacian with the off-diagonal elements of W is

$$(2k, 2k, 2k, 3k+3, \ldots, 3k+3, k, k, k)$$

and thus the vector of vertex weights is

$$s = (2k-2, 2k-2, 2k-2, 3k, \ldots, 3k, k-1, k-1, k-1).$$

For the matrix $2S - L(TP_K, \mathbf{1}, w)$ we have

$$2S - L(TP_k, \mathbf{1}, w) = L(TP_k, \sigma, w) - S + C = W + C$$

where C is the diagonal matrix with diagonal

$$(k-3) \cdot (2, 2, 2, 3, \ldots, 3, 1, 1, 1).$$

So for $k \geqslant 4$ the matrix $2S - L(TP_k, \mathbf{1}, w)$ is indeed invertible because it is positive definite. If $k = 2$ one checks that the determinant is -4. Assume for now that $k \neq 3$. A basis of its kernel is in the rows of

$$Y = \left(\begin{array}{ccc|cccc|ccc}
1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & -1 & -1 \\
0 & 1 & 0 & 0 & \cdots & \cdots & 0 & -1 & 0 & -1 \\
0 & 0 & 1 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & \cdots & 0 & -1 & -1 & -1 \\
\vdots & \vdots & \vdots & 0 & \ddots & & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & & \ddots & 0 & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 & -1 & -1
\end{array}\right).$$

This leads to the choice $l_{ij}^2 = 2$ for $i, j \in \{1, \ldots, k+1\}$ and $l_{ij}^2 = k-1$ on the remaining edges. By Theorem 15 the norms of the embedding points of any optimal embedding $Z = (z_1, \ldots, z_{k+4})$ for the given parameters s, l are uniquely determined because $2S -$

$L(TP_k, \mathbf{1}_{E(TP_k)}, w)$ is invertible. Therefore $\|z_i\|^2 = \|y_i\|^2$ for $i \in V$. Because $w_{ij} > 0$ for $i, j \in \{1, \dots, k+1\}$ and the KKT conditions we have that

$$2 = l_{ij}^2 = \|z_i + z_j\|^2 = \|z_i\|^2 + \|z_j\|^2 + 2\langle z_i, z_j\rangle = 2 + 2\langle z_i, z_j\rangle.$$

In particular, the Gram matrix of the first $k+1$ vertices is the $(k+1) \times (k+1)$ identity matrix. Therefore, $\mathrm{rank}(Z) \geqslant k+1$. This proves that $\mathrm{frustdim}(W, \sigma, s, l) \geqslant k+1$ and thus $\mathrm{frustdim}(TP_k) = k+1$ for $k \neq 3$.

In the case $k = 3$ we replace W by $W' = TWT$ where T is the diagonal matrix with diagonal $(2, 1, 1, 1, 1, 1, 1)$. In this case we obtain vertex weights $s = (4, 5, 5, 8, 2, 3, 3)$ and the matrix $2S - L(TP_3, \mathbf{1}_{E(TP_3)}, w')$ turns out to have determinant -240. A basis of the kernel of W' is in the rows of

$$Y = \left(\begin{array}{ccc|c|ccc} 1 & 0 & 0 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right)$$

and we argue as above. □

Theorem 23. *Let G be a k-tree. Then* $\mathrm{frustdim}(G) \in \{k, k+1\}$ *and* $\mathrm{frustdim}(G) = k+1$ *if and only if G contains a k-tripod.*

Proof. Because of Theorem 16 we have $\mathrm{frustdim}(G) \geqslant k$ and from part 1 of Theorem 7 we have $\mathrm{frustdim}(G) \leqslant k+1$ thanks to the tree decomposition of Lemma 21. The k-tripod has by Proposition 22 frustration dimension $k+1$ and hence this is true for a k-tree with a tripod subgraph. It remains to show that for a k-tree G without a k-tripod subgraph the problem $\mathrm{Emb}(G, \sigma, s, l)$ admits an optimal solution of rank k. This follows from part two of Theorem 7. Consider a bag V_t of the tree decomposition of Lemma 21. It contains at most two distinct k-cliques C_1 and C_2 along which other $k+1$-cliques are connected (otherwise there was a k-tripod). Then the set $\{v, w\} := V_t \setminus C_1 \cap C_2$ is not contained in any other bag and part 2 of the Theorem applies. □

6 Graphs of frustration dimension at most 2

This section is devoted to the proof of the following theorem.

Theorem 24. *Denote by $\mathcal{F}_k = \{G\colon \mathrm{frustdim}(G) \leqslant k\}$ the class of graphs of frustration dimension at most k. Then the following hold.*

1. *For $k \in \{0, 1\}$: $G \in \mathcal{F}_k$ if and only if G has no K_{k+2} minor.*
2. *$G \in \mathcal{F}_2$ if and only if G has no minor in the set $\{K_4, TP_2\}$.*

The Graph Minor Theorem guarantees the existence of a set of forbidden minors characterizing $\mathcal{F}_k$ and the assertions under 1 hold by Observation 4 and Corollary 19.

We have seen already that K_4 is a minimal graph of frustration dimension 3 because each of its subgraphs and minors has a frustration dimension less than 3. So K_4 is a minimal forbidden minor for the class $\mathcal{F}_2 = \{G\colon \mathrm{frustdim}(G) \leqslant 2\}$. Furthermore, we have seen in Theorem 23 that the 2-tripod TP_2 [1] with

$$V(TP_2) = \{1, \dots, 6\}, \quad E(TP_2) = \{12, 23, 13, 15, 16, 24, 26, 34, 35\}$$

(see Figure 2) is a forbidden minor for $\mathcal{F}_2$, as well. Indeed it is a minimal forbidden minor:

Lemma 25. *Every subgraph and every minor of TP_2 has frustration dimension strictly less than 3.*

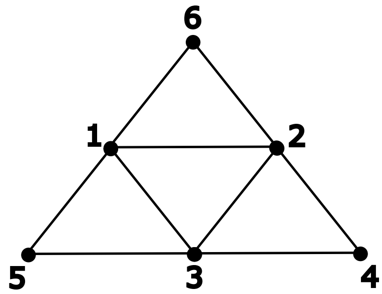

Figure 2: The graph TP_2

Proof. We show that the removal (resp., contraction), of any single edge leads to a smaller frustration dimension. First observe that up to isomorphism there are only two different edges to consider, namely 12 and 16.

In the graph $TP_2 - 12$ we can add the edge 36 and thus obtain a 2-tree A without a TP_2-subgraph. Similarly, in $TP_2 - 16$ we add an edge 46 to obtain another 2-tree B without a TP_2 subgraph.

In the graph $TP_2/12$ we add the edge 56 and obtain a 2-tree C, the graph $TP_2/16$ is indeed a 2-tree isomorphic to C and clearly has no TP_2 subgraph.

From Theorem 23 we know that A, B and C can have frustration dimension at most 2 and consequently this is true for both subgraphs and both minors under consideration. $\square$

Theorem 24 follows from Theorem 13 ($\mathrm{frustdim}(G) \leqslant la(G)$) and the following characterization of the graphs with $la(G)$ at most two:

Theorem 26 ([7])**.** *For a graph G the following are equivalent.*

1. *$la(G) \leqslant 2$.*
2. *$\nu(G) \leqslant 2$.*
3. *G has no minors in $\{K_4, TP_2\}$.*

[1] which goes under various names in the literature, the k-tripods are yet another family it belongs to

7 Proof of the Rotation Lemma

This section provides the postponed proof of the rotation lemma.

Lemma 27 (Rotation Lemma)**.** *Let V, W some subspaces of a Euclidean vectorspace E with $\dim W \leqslant \dim V$ and let $v \in V$ and $w \in W$. Then there is an orthogonal transformation Q of E satisfying*

1. *$QW \subseteq V$,*
2. *$Qx = x$ for every $x \in V \cap W$,*
3. *$\|Qw - v\| \leqslant \|w - v\|$.*

Proof. We construct two orthonormal bases of E and define Q by the mapping taking the one to the other. Denote by $U = V \cap W$, $A = (U^\perp \cap V)$ and $B = (U^\perp \cap W)$. Then we can write E as orthogonal sums, namely and

$$E = U + A + V^\perp$$

and

$$E = U + B + W^\perp.$$

Accordingly define two orthonormal bases (ONBs) of E

$$\mathcal{B}_1 = (u_1, \ldots, u_k, a_1, \ldots, a_p, v_1, \ldots, v_q)$$

with $u_i \in U, i = 1, \ldots, k$, $a_i \in A, i = 1, \ldots, p$ and

$$\mathcal{B}_2 = (u_1, \ldots, u_k, b_1, \ldots, b_s, w_1, \ldots, w_t)$$

with $b_i \in B, i = 1, \ldots, s$. An orthogonal transformation Q which satisfies 1 and 2 is obtained if we define

$$Qu_i = u_i,\ i = 1, \ldots, k, \tag{15}$$
$$Qb_j = a_j,\ j = 1, \ldots, s, \tag{16}$$
$$Qw_k = a_{s+k},\ k = 1, \ldots, p - s, \tag{17}$$
$$Qw_{p-s+l} = v_l,\ l = 1, \ldots, q. \tag{18}$$

In order to make Q satisfy 3, as well, we first observe that vectors $v \in V, w \in W$ decompose as $v = u_v + a_v$ and $w = u_w + b_w$ with $u_v, u_w \in U$, $a_v \in A$ and $b_w \in B$. If both a_v and b_w are non-zero we can choose the ONBs of A (resp., B) such that

$$a_1 = \frac{a_v}{\|a_v\|} \text{ and } b_1 = \frac{b_w}{\|b_w\|}.$$

Then we have

$$\begin{aligned}\|Qw - v\|^2 &= \|Qu_w - u_v\|^2 + \|Qb_w - a_v\|^2 \\ &= \|u_w - u_v\|^2 + \|\frac{\|b_w\|}{\|a_v\|}a_v - a_v\|^2 \\ &= \|u_w - u_v\|^2 + \|\|b_w\| - \|a_v\|\|^2 \\ &\leqslant \|u_w - u_v\|^2 + \|b_w - a_v\|^2 \\ &= \|w - v\|^2\end{aligned}$$

and Q also satisfies 3. If $a_v = 0$ or $b_w = 0$ it is easy to see that Q as defined in (15) satisfies 3 with equality. □

8 Conclusion

We have discovered a new minor monotone graph invariant and established connections to other minor monotone graph invariants, most strikingly Colin de Verdière's ν-invariant. It would be interesting to study the general relation between ν and frustdim. In [10] the characterization of graphs G with $\nu(G) \leqslant 3$ is given and it is furthermore shown that ν is large if and only if the tree width is large. Both questions can be asked for frustdim, as well. We point out that the results in [10] are derived for multigraphs in which multiple edges are allowed but loops are not. This leads to a slightly different forbidden minor characterization of graphs with $\nu \leqslant 2$ than the one given in [7] for simple graphs. Such a multigraph version of frustdim should also be considered. Perhaps the results of [10] can be utilized to establish a characterization of graphs with frustdim $\leqslant 3$.

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