Non-isomorphic Cayley Graphs with Identical Random Walk Distribution

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Abstract

We construct an infinite family of triples (G, S_1, S_2) each consisting of a group G and a pair (S_1, S_2) of distinct subsets of G with the following properties.

i The two Cayley graphs $\mathtt{Cay}(G, S_1)$ and $\mathtt{Cay}(G, S_2)$ are non-isomorphic.

ii The distributions of the simple random walks on $\mathtt{Cay}(G, S_1)$ and $\mathtt{Cay}(G, S_2)$ are the same if one applies an appropriate bijection between the two vertex sets at each step.

iii The spectral set of $\mathtt{Cay}(G, S_i)$ is decomposed into a disjoint union of two subsets A and B_i of equal size ($|A| = |B_i| = |G|/2$), which satisfies $B_2 = -B_1 = \{-\lambda \mid \lambda \in B_1\}$.

As a byproduct, an infinite family of pairs of isomorphic Cayley graphs on non-isomorphic groups is obtained.

Mathematics Subject Classifications: 05C50, 05C25, 60J10

1 Introduction

The simple random walks on non-isomorphic Cayley graphs generally have different total variation distances from the uniform distribution. However, rare exceptions exist. We present an infinite family of such exceptions, i.e., non-isomorphic pairs of Cayley graphs whose simple random walks have exactly the same total variation distance from the uniform distribution at each step. Each pair is given as a pair of distinct quotient graphs of the same graph, whose construction is similar to Terras's construction [7] of isospectral non-isomorphic graphs. (See also [3] and [2] for the details of the construction of isospectral graphs.) However, the pairs we construct are not isospectral pairs. Their spectral sets are half the identical and half opposite.

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To be more precise, the main aim of this paper is to give a simple and explicit construction of an infinite family of triples (G, S_1, S_2) each consisting of a group G and a pair (S_1, S_2) of distinct subsets of G with the following properties.

i The two Cayley graphs $\mathtt{Cay}(G, S_1)$ and $\mathtt{Cay}(G, S_2)$ are non-isomorphic.

ii The distributions of the simple random walks on $\mathtt{Cay}(G, S_1)$ and $\mathtt{Cay}(G, S_2)$ are the same if one applies an appropriate bijection between the two vertex sets at each step.

iii The spectral set of $\mathtt{Cay}(G, S_i)$ is decomposed into a disjoint union of two subsets A and B_i of equal size ($|A| = |B_i| = |G|/2$) which satisfies $B_2 = -B_1 = \{-\lambda \mid \lambda \in B_1\}$.

As a byproduct, an infinite family of pairs of isomorphic Cayley graphs on non-isomorphic groups is obtained.

Here we show a concrete example which illustrates our results in this paper. For a group G and its subset S, the *Cayley graph* $\mathtt{Cay}(G, S)$ is a directed graph, which has the vertex set G and has an edge from $g \in G$ to $h \in G$ if and only if there exists an element $k \in S$ such that $gk = h$. Though we call S the *generating set* of the graph $\mathtt{Cay}(G, S)$, we do not impose S to generate the group G, and hence $\mathtt{Cay}(G, S)$ may not be connected. Let $\sigma = (1, 3, 2)$ be the cyclic permutation of order 3 and $\tau = (1, 3)$ be the transposition in the symmetric group $\mathfrak{S}_3$ of degree 3. Then each of the two sets

$$S_1 = \{\sigma, \sigma^{-1}, \tau\}, \quad S_2 = \{\tau, \tau\sigma, \mathrm{id}\},$$

generates $\mathfrak{S}_3$. The Cayley graphs $\mathtt{Cay}(\mathfrak{S}_3, S_1)$ and $\mathtt{Cay}(\mathfrak{S}_3, S_2)$ are depicted in Figure 1.

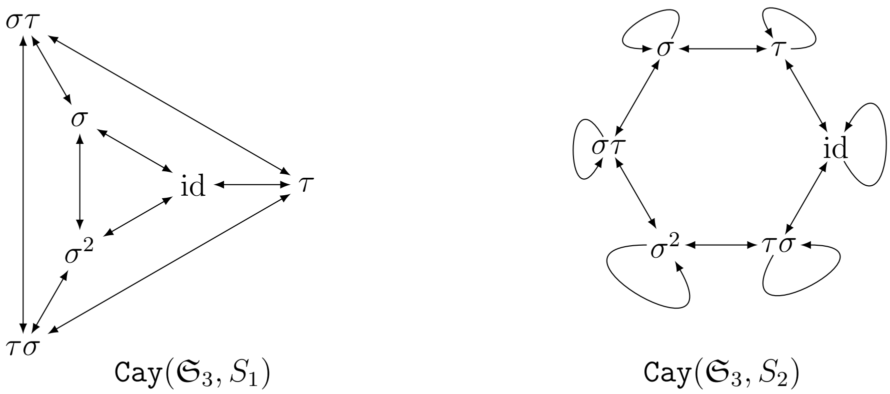


Figure 1: Cayley graphs of the symmetric group $\mathfrak{S}_3$ generated by S_1 (left) and S_2 (right)

The generating sets S_1 and S_2 give rise to visibly different Cayley graphs. In particular, the diameter of $\mathtt{Cay}(\mathfrak{S}_3, S_1)$ is 2, whereas that of $\mathtt{Cay}(\mathfrak{S}_3, S_2)$ is 3. Hence the two graphs are non-isomorphic.

We consider the simple random walks on these Cayley graphs. Namely we consider the two sequences $(\mu_1^{(0)}, \mu_1^{(1)}, \mu_1^{(2)}, \ldots)$ and $(\mu_2^{(0)}, \mu_2^{(1)}, \mu_2^{(2)}, \ldots)$ of probability measures over $\mathfrak{S}_3$, which are defined as follows,

$$\mu_i^{(0)}(g) = \begin{cases} 1 & g = \mathrm{id}, \\ 0 & g \neq \mathrm{id}, \end{cases}$$

$$\mu_i^{(t+1)}(g) = \frac{1}{3} \sum_{k \in S_i} \mu_i^{(t)}(gk^{-1}).$$

It can be easily checked that these random walks are ergodic and have the uniform distribution U as their stationary distributions, that is, $\lim_{t\to\infty} \mu_i^{(t)}(g) = U(g) = 1/6$ for all $g \in \mathfrak{S}_3$. The speed of the convergence is measured by the total variation distance $\mathrm{d}_{TV}(\mu_i^{(t)}, U)$ defined by

$$\mathrm{d}_{TV}(\mu_i^{(t)}, U) = \frac{1}{2} \sum_{g \in \mathfrak{S}_3} \left| \mu_i^{(t)}(g) - U(g) \right|.$$

Even though the graphs have different diameters, the distributions of the simple random walks on them approach the uniform distribution in exactly the same way. That is,

$$\mathrm{d}_{TV}(\mu_1^{(t)}, U) = \mathrm{d}_{TV}(\mu_2^{(t)}, U),$$

for $t = 0, 1, 2, \ldots$. This can be observed in Table 1, where the values of $\mu_1^{(t)}(g)$ and $\mu_2^{(t)}(g)$ for $0 \leqslant t \leqslant 5$ and $g \in \mathfrak{S}_3$ are shown.

$t \backslash g$	id	σ	σ^2	τ	$\sigma\tau$	$\tau\sigma$
0	1	0	0	0	0	0
1	0	1/3	1/3	1/3	0	0
2	3/9	1/9	1/9	0	2/9	2/9
3	2/27	6/27	6/27	7/27	3/27	3/27
4	19/81	11/81	11/81	8/81	16/81	16/81
5	30/243	46/243	46/243	51/243	35/243	35/243

$t \backslash g$	id	σ	σ^2	τ	$\sigma\tau$	$\tau\sigma$
0	1	0	0	0	0	0
1	1/3	0	0	1/3	0	1/3
2	3/9	1/9	1/9	2/9	0	2/9
3	7/27	3/27	3/27	6/27	2/27	6/27
4	19/81	11/81	11/81	16/81	8/81	16/81
5	51/243	35/243	35/243	46/243	30/243	46/243

Table 1: Tables of $\mu_1^{(t)}$ (top) and $\mu_2^{(t)}$ (bottom)

By comparing the two tables, we find that $\mu_1^{(t)}$ and $\mu_2^{(t)}$ are nearly the same. Let φ_0 and φ_1 be two permutations on $\mathfrak{S}_3$ defined by

$$\varphi_0 = \begin{pmatrix} \mathrm{id} & \sigma & \sigma^2 & \tau & \sigma\tau & \tau\sigma \\ \mathrm{id} & \sigma^2 & \sigma & \sigma\tau & \tau & \tau\sigma \end{pmatrix}, \quad \varphi_1 = \begin{pmatrix} \mathrm{id} & \sigma & \sigma^2 & \tau & \sigma\tau & \tau\sigma \\ \tau & \tau\sigma & \sigma\tau & \sigma & \mathrm{id} & \sigma^2 \end{pmatrix}. \tag{1}$$

Then, as we will prove later, we have

$$\mu_1^{(t)}(\varphi_{[t]}(g)) = \mu_2^{(t)}(g),$$

where $[t]$ is an integer in $\{0,1\}$ obtained by taking modulo 2 of t. The characteristic polynomials of the adjacency matrices of $\texttt{Cay}(\mathfrak{S}_3, S_1)$ and $\texttt{Cay}(\mathfrak{S}_3, S_2)$ are

$$P_1(x) = (x-3)(x-1)x^2(x+2)^2, \quad \text{and} \quad P_2(x) = (x-3)(x-2)^2x^2(x+1),$$

from which we can observe that they have the common factor $P_Z(x) = (x-3)x^2$ and

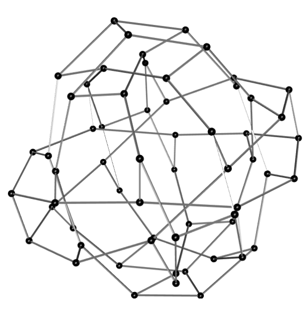
$$\frac{P_1(x)}{P_Z(x)} = -\frac{P_2(-x)}{P_Z(-x)}.$$

The spectral sets of the two graphs contain the same subset $\{3,0,0\}$, whose complements in two spectral sets $\{1,-2,-2\}$ and $\{-1,2,2\}$ have opposite signs.

A larger example of such pairs of Cayley graphs is given by $\texttt{Cay}(A_5, S_1)$ and $\texttt{Cay}(A_5, S_2)$, where A_5 is the alternating group of degree 5 and

$$S_1 = \{(1,2,3,4,5),(1,2,3,4,5)^{-1},(1,2)(3,4)\}, \; S_2 = \{(1,2)(3,5),(1,2)(4,5),(1,3)(4,5)\}.$$

See Figure 2. The Cayley graph $\texttt{Cay}(A_5, S_1)$ is of diameter 6, while $\texttt{Cay}(A_5, S_2)$ is of diameter 9.



$\texttt{Cay}(A_5, S_1)$ $\qquad$ $\texttt{Cay}(A_5, S_2)$

Figure 2: Another example of Cayley graphs simple random walks on which have the same total variation distance from the uniform distribution. All edges are bi-directed.

Let $P_i(x)$ be the characteristic polynomial of the adjacency matrix of the Cayley graph $\mathtt{Cay}(A_5, S_i)$ for $i = 1, 2$. Then $P_1(x)$ and $P_2(x)$ have the common factor

$$P_Z(x) = (x-3)(x-1)^9(x+2)^4(x^2-x-3)^5(x^2+3x+1)^3,$$

whose degree is $|A_5|/2 = 30$, and

$$\frac{P_1(x)}{P_Z(x)} = \frac{P_2(-x)}{P_Z(-x)} = (x^2+x-4)^4(x^2+x-1)^5(x^4-3x^3-2x^2+7x+1)^3.$$

Thus, as multiset, the spectral set of $\mathtt{Cay}(A_5, S_i)$ is decomposed into a disjoint union of two subsets A and B_i of equal size ($|A| = |B_i| = |A_5|/2$), which satisfies $B_2 = -B_1 = \{-\lambda \mid \lambda \in B_1\}$.

Our main aim in this paper is to construct an infinite family of triples (G, S_1, S_2) consisting of a group G and a pair (S_1, S_2) of distinct subsets of G with the properties **i**, **ii**, and **iii**. Our construction uses the sliding block puzzles defined on a certain family of graphs, which can be considered as a generalization of the one studied in [4].

The outline of the paper is as follows. In Section 2, we show a construction of an infinite family of triples. In Section 3, we show that the triples constructed in Section 2 have property **ii**. In Section 4, by applying the theory of the zeta functions of finite graph covering, we show the triples have properties **i** and **iii**.

2 Construction of pairs

Let Γ be a finite undirected simple graph equipped with the vertex set V of cardinality $n+1$ and the edge set E. A *position* of the *sliding block puzzle* defined on Γ is a bijection $f : V \to \{1, 2, \ldots, n, 0\}$. We say $v \in V$ is *blank* or *unoccupied* in a position f if $f(v) = 0$, and hence, the vertex $f^{-1}(0)$ is blank. A position f is transformed into another position g by a *move* if there exist two mutually adjacent vertices v, w in Γ, such that, v or w is blank in f, and

$$g = f \circ (v, w),$$

where (v, w) denotes the transposition of v and w. Since Γ is undirected and simple, every path p on Γ can be represented as a sequence $(v_0, v_1, \ldots, v_l)$ of the vertices. We define the permutation σ_p of the vertices by

$$\sigma_p = (v_0, v_1)(v_1, v_2) \cdots (v_{l-1}, v_l).$$

If $g = f \circ \sigma_p$ and v_0 is blank in f, then v_l is blank in g. That is, σ_p sends the blank from v_0 to v_l through the path p.

Let $\mathrm{puz}(\Gamma)$ be the graph whose vertex set consists of the positions of the puzzle. Two vertices (or positions) f and g of $\mathrm{puz}(\Gamma)$ are connected by an edge $\{f, g\}$ if and only if there is a move transforming f into g.

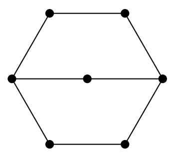


Figure 3: The graph θ_0

Wilson [8] shows the following fundamental theorem for the sliding block puzzles:

Theorem 1. ([8, Theorem 1]) *Let* Γ *be a finite simple connected graph other than a polygon or the graph* θ_0 *shown in Figure* 3. *Then* $\mathrm{puz}(\Gamma)$ *is connected unless* Γ *is bipartite, in which case* $\mathrm{puz}(\Gamma)$ *has exactly two components. In this latter case, positions* f, g *on* Γ *having blank vertices at even (resp. odd) distance in* Γ *are in the same component of* $\mathrm{puz}(\Gamma)$ *if and only if there exists an even (resp. odd) permutation* σ *of* V *such that* $f \circ \sigma = g$.

Definition 2. Let X be an undirected simple graph. Let v be a vertex of degree two, $\{v_1, v_2\}$ be the neighboring vertices of v, and e_1 and e_2 be the two edges incident to v. By replacing e_1 and e_2 with a single edge e connecting v_1 and v_2 vertices neighboring v and removing v from X, we obtain a smaller graph. By repeatedly applying this procedure, we obtain a graph $\widetilde{X}$ without vertices of degree two. We call $\widetilde{X}$ the *path contraction* of X.

For a positive integer a and a non-negative integer b, the *theta graph* $\theta_{a,b}$ is defined as follows. The theta graph $\theta_{a,b}$ has the vertex set $V = \{v_0, v_1, \ldots, v_n\}$ where $n = 2a+b+1$, and the edge set E defined by

$$E = \{\{v_i, v_{i+1}\} \mid i \in \{0, 1, \ldots, n-2\} \backslash \{2a+1\}\} \\ \cup \{\{v_0, v_{2a+1}\}, \{v_0, v_{2a+2}\}, \{v_{2a+b+1}, v_{a+1}\}\}.$$

where $\{v, w\}$ denotes the unique edge connecting v and w in $\theta_{a,b}$. Figure 4 shows $\theta_{2,3}$.

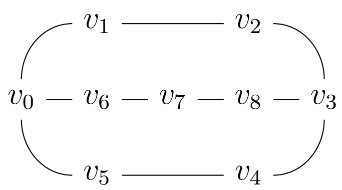


Figure 4: $\theta_{2,3}$

Let $\rho : V \to V$ be the bijection defined by

$$\rho(v_i) = \begin{cases} v_{(i+a+1) \bmod (2a+2)} & i = 0, 1, \ldots, 2a+1, \\ v_{4a+b+3-i} & i = 2a+2, 2a+3, \ldots, 2a+b+1. \end{cases}$$

Then ρ induces a graph automorphism of $\theta_{a,b}$, which can be considered as the 180° rotation. Let $\psi : V \to V$ be the bijection defined by

$$\psi(v_i) = \begin{cases} v_{a+1-i} & i = 0, 1, \dots, a+1, \\ v_{3a+3-i} & i = a+2, a+3, \dots, 2a+1, \\ v_{4a+b+3-i} & i = 2a+2, 2a+3, \dots, 2a+b+1. \end{cases}$$

Then ψ induces another graph automorphism of $\theta_{a,b}$, which can be considered as a vertical flip. These two automorphisms ρ and ψ generate a subgroup of the automorphism group $\mathrm{Aut}(\theta_{a,b})$ of $\theta_{a,b}$, which is isomorphic to the Klein four-group. The images of each vertex v_i in $\theta_{2,3}$ of ρ (resp. ψ) are shown in the left (resp. right) side of Figure 5.

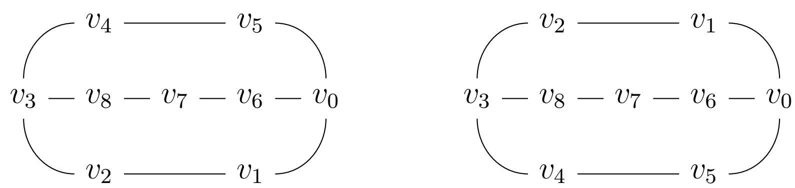


Figure 5: Images of ρ (left) and ψ (right) for $\theta_{2,3}$

There exist three simple paths p_1, p_2, p_3 from v_0 and v_{a+1}, defined by

$$p_1 = (v_0, v_1, \dots, v_{a+1}), \quad p_2 = (v_0, v_{2a+2}, \dots, v_{2a+b+1}, v_{a+1}), \quad p_3 = (v_0, v_{2a+1}, \dots, v_{a+1}).$$

Each path p_k defines a vertex permutation σ_{p_k} of $\theta_{a,b}$. The vertex permutation $\sigma_{p_k}\rho$ fixes the vertex v_0, which gives the permutation $\sigma_k \in \mathfrak{S}_n$ such that

$$\sigma_{p_k}\rho(v_i) = v_{\sigma_k(i)}, \tag{2}$$

for $i = 1, 2, \dots, n$. We define S_1 by

$$S_1 = \{\sigma_1, \sigma_2, \sigma_3\}. \tag{3}$$

In the same manner, we define $\tau_k \in \mathfrak{S}_n$ by using ψ instead of ρ, that is,

$$\sigma_{p_k}\psi(v_i) = v_{\tau_k(i)},$$

for $k = 1, 2, 3$, and we define S_2 by

$$S_2 = \{\tau_1, \tau_2, \tau_3\}. \tag{4}$$

Let S be a subset of a finite group G. The *Cayley graph* $\mathtt{Cay}(G, S)$ of the group G with the generating set S is a directed graph, whose vertex set is G and there exists an edge from g to h in G if and only if there exists an element $k \in S$ such that $h = gk$. For an edge e of $\texttt{Cay}(G,S)$, the starting vertex of e is denoted $o(e)$, and the terminal vertex is denoted $t(e)$.

We then also consider the Cayley graph of the group $G\times C_2$, where $C_2=\mathbb{Z}/2\mathbb{Z}=\{0,1\}$ is the cyclic group of order 2. For the sake of simplicity of the notation, we write

$$X(G,S)=\texttt{Cay}(G,S),\quad Y(G,S)=\texttt{Cay}(G\times C_2,S\times\{1\}).$$

Then the Cayley graph $Y(G,S)$ is a bipartite graph, that is, the vertex set $G\times C_2$ of $Y(G,S)$ is divided into two disjoint parts $B=G\times\{0\}$ and $W=G\times\{1\}$ and every edge is from B to W or W to B.

Let f_0 be the *initial* position which is defined by $f_0(v_i)=i$ for $i\in\{0,1,\dots,n\}$. We are interested in the connected component $\mathrm{puz}_0(\theta_{a,b})$ of $\mathrm{puz}(\theta_{a,b})$, which contains the initial position f_0. It is clear that the maps

$$f\mapsto f\rho,\quad f\mapsto f\psi,$$

can be considered as two automorphisms of $\widetilde{\mathrm{puz}}(\theta_{a,b})$, since they are the 180° rotation and the vertical flip of $\theta_{a,b}$ respectively. Thus, we can consider the group $K=\{\mathrm{id},\rho,\psi,\rho\psi\}$ acting also on $\widetilde{\mathrm{puz}}(\theta_{a,b})$ as a subgroup of the automorphisms group $\mathrm{Aut}(\widetilde{\mathrm{puz}}(\theta_{a,b}))$. When $a+b$ is an even integer, by Theorem 1, $\mathrm{puz}(\theta_{a,b})$ has two connected components. The following lemma shows the conditions for each of ρ and ψ to be contained in $\mathrm{Aut}(\widetilde{\mathrm{puz}}_0(\theta_{a,b}))$.

Lemma 3. *If $a\equiv 0 \pmod 2$ and $b\equiv 0 \pmod 4$, then*

$$\rho,\psi\in\mathrm{Aut}\left(\widetilde{\mathrm{puz}}_0(\theta_{a,b})\right),\quad \textit{and}\quad S_1,S_2\subset A_n.$$

If $a\equiv 0 \pmod 2$ and $b\equiv 2 \pmod 4$, then

$$\rho,\psi\notin\mathrm{Aut}\left(\widetilde{\mathrm{puz}}_0(\theta_{a,b})\right),\quad \textit{and}\quad S_1,S_2\subset \mathfrak{S}_n\backslash A_n.$$

If $a\equiv 1 \pmod 2$ and $b\equiv 1 \pmod 4$, then

$$\rho\in\mathrm{Aut}\left(\widetilde{\mathrm{puz}}_0(\theta_{a,b})\right),\quad \psi\notin\mathrm{Aut}\left(\widetilde{\mathrm{puz}}_0(\theta_{a,b})\right),\quad S_1\subset A_n,\quad \textit{and}\quad S_2\subset \mathfrak{S}_n\backslash A_n.$$

If $a\equiv 1 \pmod 2$ and $b\equiv 3 \pmod 4$, then

$$\rho\notin\mathrm{Aut}\left(\widetilde{\mathrm{puz}}_0(\theta_{a,b})\right),\quad \psi\in\mathrm{Aut}\left(\widetilde{\mathrm{puz}}_0(\theta_{a,b})\right),\quad S_1\subset \mathfrak{S}_n\backslash A_n,\quad \textit{and}\quad S_2\subset A_n.$$

Proof. First note that ρ can be expressed as

$$\rho=\prod_{i=0}^{a}(v_i,v_{a+1+i})\prod_{j=1}^{\lfloor b/2\rfloor}(v_{2a+1+j},v_{n+1-j})$$

which is the product of $a+1+\lfloor b/2\rfloor$ disjoint transpositions of the vertices of $\theta_{a,b}$, and ψ can be expressed as

$$\psi=(v_0,v_{a+1})\prod_{i=1}^{\lfloor a/2\rfloor}(v_i,v_{a+1-i})(v_{a+1+i},v_{2a+2-i})\prod_{j=1}^{\lfloor b/2\rfloor}(v_{2a+1+j},v_{n+1-j})$$

which is the product of an even number of disjoint transpositions if and only if $\lfloor b/2 \rfloor$ is an odd integer. If $a \equiv 0 \pmod 2$ and $b \equiv 0 \pmod 4$, then the distance from v_0 to v_{a+1} is odd and ρ can be expressed as the product of an odd number of vertex transpositions, and $f\rho$ is a vertex of $\widetilde{\text{puz}}_0(\theta_{a,b})$ for every vertex f of $\widetilde{\text{puz}}_0(\theta_{a,b})$. Since ψ is also the product of the odd number $(a+1+b/2)$ of disjoint transpositions of the vertices, and $f\psi$ is also a vertex of $\widetilde{\text{puz}}_0(\theta_{a,b})$ for every vertex f of $\widetilde{\text{puz}}_0(\theta_{a,b})$. Since σ_{p_k} is an odd permutation of the vertices for $k = 1, 2, 3$, $\sigma_{p_k}\rho$ and $\sigma_{p_k}\psi$ are both even permutations fixing v_0. Thus we have $S_1, S_2 \subset A_n$. The other cases can be shown in the same manner. ☐

In the following theorem, we consider the graph $\widetilde{\text{puz}}(\theta_{a,b})$ as a bi-directed graph, that is, if there is an edge $\{f, g\}$ connecting f and g in $\widetilde{\text{puz}}(\theta_{a,b})$, we consider there are two edges of opposite directions, from f to g and from g to f. Therefore, the symbol $\cong$ in the theorem stands for the isomorphism of two directed graphs.

Theorem 4. *Let a be a positive integer, and b a non-negative integer such that $(a, b) \neq (2, 1)$. Let S_1 and S_2 be defined by (3) and (4) respectively. Then, we have*

$$\widetilde{\text{puz}}(\theta_{a,b}) \cong Y(\mathfrak{S}_n, S_1) \cong Y(\mathfrak{S}_n, S_2), \tag{5}$$

where $n = 2a + b + 1$ and $\widetilde{\text{puz}}(\theta_{a,b})$ is the path contraction of $\text{puz}(\theta_{a,b})$.

If $a + b$ is an odd integer, then each of S_1 and S_2 generates the symmetric group $\mathfrak{S}_n$ of degree n, and $\widetilde{\text{puz}}(\theta_{a,b})$ is a strongly connected graph.

If $a \equiv 0 \pmod 2$ and $b \equiv 0 \pmod 4$, then

$$\widetilde{\text{puz}}_0(\theta_{a,b}) \cong Y(A_n, S_1) \cong Y(A_n, S_2),$$

and each of S_1 and S_2 generates A_n.

If $a \equiv 0 \pmod 2$ and $b \equiv 2 \pmod 4$, then

$$\widetilde{\text{puz}}_0(\theta_{a,b}) \cong X(\mathfrak{S}_n, S_1) \cong X(\mathfrak{S}_n, S_2),$$

and each of S_1 and S_2 generates $\mathfrak{S}_n$.

If $a \equiv 1 \pmod 2$ and $b \equiv 1 \pmod 4$, then

$$\widetilde{\text{puz}}_0(\theta_{a,b}) \cong Y(A_n, S_1) \cong X(\mathfrak{S}_n, S_2),$$

S_1 generates A_n, and S_2 generates $\mathfrak{S}_n$.

If $a \equiv 1 \pmod 2$ and $b \equiv 3 \pmod 4$, then

$$\widetilde{\text{puz}}_0(\theta_{a,b}) \cong X(\mathfrak{S}_n, S_1) \cong Y(A_n, S_2),$$

S_1 generates $\mathfrak{S}_n$, and S_2 generates A_n.

Proof. Given a position f, we define the permutation $\sigma_f \in \mathfrak{S}_n$ as follows. If f has the blank at v_0, that is $f(v_0) = 0$, then σ_f is defined by

$$f(v_i) = \sigma_f(i),$$

for $i = 1, 2, \ldots, n$. If f has the blank at v_{a+1}, then $f\rho(v_0) = 0$ and σ_f is the unique permutation which satisfies

$$f\rho(v_i) = \sigma_f(i),$$

for $i = 1, 2, \ldots, n$. Let $c_f \in \{0, 1\}$ be defined by

$$c_f = \begin{cases} 0 & \text{if} \quad f(v_0) = 0, \\ 1 & \text{if} \quad f(v_{a+1}) = 0. \end{cases}$$

Then the map

$$f \mapsto (\sigma_f, c_f)$$

is a bijection from the vertex set of $\widetilde{\mathrm{puz}}(\theta_{a,b})$ to $\mathfrak{S}_n \times C_2$. Let f be a vertex of $\widetilde{\mathrm{puz}}(\theta_{a,b})$. Then, since $\widetilde{\mathrm{puz}}(\theta_{a,b})$ is obtained by path contraction, $f^{-1}(0) \in \{v_0, v_{a+1}\}$. If $f^{-1}(0) = v_0$ and there exists an edge from f to a position g, there exists $k \in \{1, 2, 3\}$ such that $g = f\sigma_{p_k}$ and $g(v_{a+1}) = 0$. Therefore, using relation (2) we obtain $g\rho(v_i) = f\sigma_{p_k}\rho(v_i) = f(v_{\sigma_k(i)})$. This implies

$$\sigma_g = \sigma_f \sigma_k. \tag{6}$$

If $f(v_{a+1}) = 0$ and there exists an edge from f to a position g, there exists $k \in \{1, 2, 3\}$ such that $g = f\rho\sigma_{p_k}\rho$ and $g(v_0) = 0$. Then,

$$\sigma_g(i) = g(v_i) = f\rho\sigma_{p_k}\rho(v_i) = f\rho(v_{\sigma_k(i)}) = \sigma_f(\sigma_k(i)),$$

and (6) holds. Thus we have obtained the first isomorphism in (5). The second isomorphism in (5) can be obtained by using ψ instead of ρ.

Let $V_{a,b}$ be the vertex set of $\mathrm{puz}(\theta_{a,b})$ and $\widetilde{V}_{a,b}$ the vertex set of $\widetilde{\mathrm{puz}}(\theta_{a,b})$. It is clear that $|V_{a,b}| = (2a + b + 2)!$, since $\theta_{a,b}$ has $2a + b + 2$ vertices. Since a vertex in $\widetilde{V}_{a,b}$ can be identified with a vertex in $V_{a,b}$ of degree three, we have

$$|V_{a,b}| = |\widetilde{V}_{a,b}| + \frac{1}{2}\left(2a|\widetilde{V}_{a,b}| + b|\widetilde{V}_{a,b}|\right) = \frac{2a + b + 2}{2}|\widetilde{V}_{a,b}|.$$

Thus we have

$$|\widetilde{V}_{a,b}| = 2n!.$$

If $a + b$ is an odd integer, then $\mathrm{puz}(\theta_{a,b})$ is a connected graph. Therefore both $S_1 \times \{1\}$ and $S_2 \times \{1\}$ generate the group $\mathfrak{S}_n \times C_2$, and hence each of S_1 and S_2 generates $\mathfrak{S}_n$.

If $a + b$ is an even integer, then $\theta_{a,b}$ is bipartite and $\mathrm{puz}(\theta_{a,b})$ has two isomorphic connected components, $\mathrm{puz}_0(\theta_{a,b})$ and the other one. Let $\widetilde{V}^0_{a,b}$ be the vertex set of $\widetilde{\mathrm{puz}}_0(\theta_{a,b})$. Then it is clear that

$$\left|\widetilde{V}^0_{a,b}\right| = \frac{1}{2}\left|\widetilde{V}_{a,b}\right| = n!.$$

If $a \equiv 0 \pmod 2$ and $b \equiv 0 \pmod 4$, then, by Lemma 3, we have $S_1, S_2 \subset A_n$. Since $Y(\mathfrak{S}_n, S_i)$ has two connected components of size $n!$, one of the two components has the vertex set $A_n \times C_2$, and the other $(\mathfrak{S}_n \backslash A_n) \times C_2$ for $i = 1, 2$. Thus each of two components is isomorphic to $Y(A_n, S_i)$, which implies that each of S_1 and S_2 generates A_n.

If $a \equiv 0 \pmod 2$ and $b \equiv 2 \pmod 4$, then, by Lemma 3, we have $S_1, S_2 \subset \mathfrak{S}_n \backslash A_n$. One of the two connected components of $Y(\mathfrak{S}_n, S_i)$ contains the vertex $(\mathrm{id}, 0)$, and the other contains $(\mathrm{id}, 1)$. Each component is isomorphic to $X(\mathfrak{S}_n, S_i)$ for $i = 1, 2$, which implies that each of S_1 and S_2 generates $\mathfrak{S}_n$.

If $a \equiv 1 \pmod 2$ and $b \equiv 1 \pmod 4$, then, by Lemma 3, we have $S_1 \subset A_n$ and $S_2 \subset \mathfrak{S}_n \backslash A_n$. Hence $Y(\mathfrak{S}_n, S_1)$ has two connected components, each of which is isomorphic to $Y(A_n, S_1)$, and $Y(\mathfrak{S}_n, S_2)$ has two connected components, each of which is isomorphic to $X(\mathfrak{S}_n, S_2)$. This implies that S_1 generates A_n and S_2 generates $\mathfrak{S}_n$.

If $a \equiv 1 \pmod 2$ and $b \equiv 3 \pmod 4$, then, by Lemma 3, we have $S_1 \subset \mathfrak{S}_n \backslash A_n$ and $S_2 \subset A_n$, and the statement for this case can be proved in the same manner as the previous case. □

Remark 5. Theorem 4 gives examples of pairs of isomorphic Cayley graphs on the non-isomorphic groups $\mathfrak{S}_n$ and $A_n \times C_2$. For instance, when $a = b = 1$, we have

$$Y(A_4, \{(1,3,2), (1,3)(2,4), (1,2,3)\}) \cong X(\mathfrak{S}_4, \{(1,2), (2,4), (2,3)\}).$$

Example 6. Let $a = 1$ and $b = 2m$ for non-negative integer m. Then $\widetilde{\mathrm{puz}}(\theta_{a,b})$ is connected and we have

$$\begin{aligned}
\sigma_1 &= \begin{cases} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 3 & 1 & 2 & n & n-1 & \cdots & 4 \end{pmatrix} & m > 0, \\ (1,3,2) & m = 0. \end{cases} \\
\sigma_2 &= \begin{cases} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 3 & 4 & 1 & 2 & n & \cdots & 5 \end{pmatrix} & m > 0, \\ (1,3) & m = 0. \end{cases} \\
\sigma_3 &= \sigma_1^{-1}.
\end{aligned}$$

$$\begin{aligned}
\tau_1 &= (2,3)\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 2 & 1 & 3 & n & n-1 & \cdots & 4 \end{pmatrix}. \\
\tau_2 &= (1,3)\sigma_2 = \begin{cases} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 1 & 4 & 3 & 2 & n & \cdots & 5 \end{pmatrix} & m > 0, \\ \mathrm{id} & m = 0. \end{cases} \\
\tau_3 &= (1,2)\sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 1 & 3 & 2 & n & n-1 & \cdots & 4 \end{pmatrix}.
\end{aligned}$$

The diameters of the Cayley graphs $X(\mathfrak{S}_{2m+3}, S_i)$ for $i = 1, 2$ and $m = 0, 1, 2$ are listed in the following table.

$i \backslash m$	0	1	2
1	2	10	17
2	3	9	17

The first example shown in Section 1 is obtained from the case when $m = 0$. (See Figure 6.)

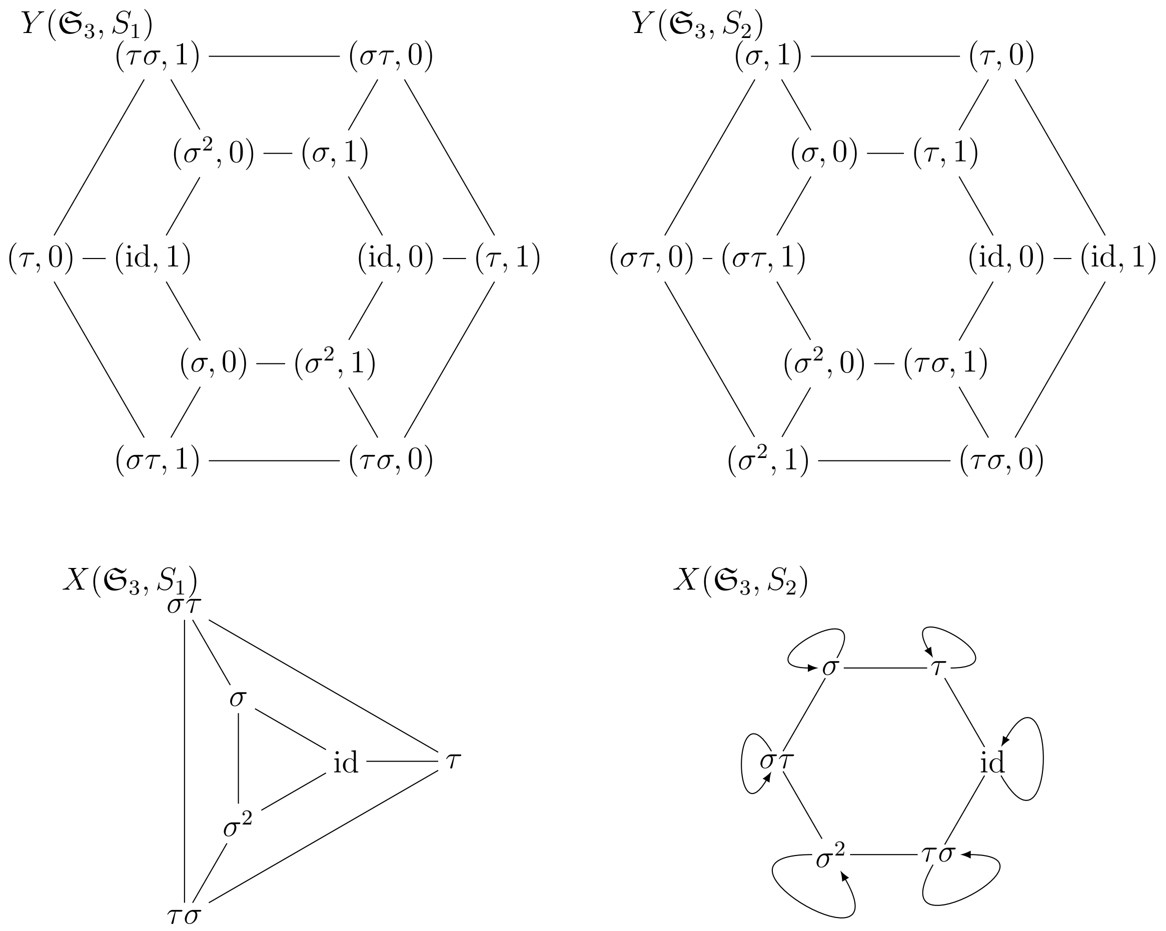


Figure 6: $X(\mathfrak{S}_3, S_1), X(\mathfrak{S}_3, S_2), Y(\mathfrak{S}_3, S_1)$ and $Y(\mathfrak{S}_3, S_2)$ with $\sigma = (1, 2, 3), \tau = (1, 2) \in \mathfrak{S}_3$. We can observe the isomorphism (5).

3 Random walks

In the rest of the paper we assume the following: a is a positive integer and b is a non-negative integer satisfying one of the following two conditions **1** and **2**.

1 $a+b$ is odd and $(a,b) \neq (2,1)$.

2 a is even and b is divisible by four.

The group G is defined by

$$G = \begin{cases} \mathfrak{S}_n & \textbf{1} \text{ holds}, \\ A_n & \textbf{2} \text{ holds}, \end{cases} \tag{7}$$

where $n = 2a+b+1$. The subsets S_1 and S_2 of G are defined by (3) and (4) respectively. Under these assumptions, by Theorem 4, we have

$$Y(G, S_1) \cong Y(G, S_2). \tag{8}$$

This section shows that the triple (G, S_1, S_2) has property **ii**.

We first consider the simple random walk on $X(G, S)$ for an arbitrary subset S of G, which starts at the identity element id of G. Namely we consider the sequence $(\mu^{(0)}, \mu^{(1)}, \mu^{(2)}, \ldots)$ of the probability measures on G defined as follows.

$$\mu^{(0)}(g) = \begin{cases} 1 & g = \mathrm{id}, \\ 0 & g \neq \mathrm{id}, \end{cases}$$

$$\mu^{(t+1)}(g) = \frac{1}{d} \sum_{k \in S} \mu^{(t)}(gk^{-1}),$$

where $d = |S|$. Therefore $\mu^{(t)}$ has another expression

$$\mu^{(t)}(g) = \frac{1}{d^t} \left| \mathcal{P}^{(t)}(\mathrm{id}, g) \right|$$

where $\mathcal{P}^{(t)}(g, h)$ is the set of paths of length t from g to h in $X(G, S)$. If $X(G, S)$ is connected and aperiodic, then $\mu^{(t)}$ converges to the uniform distribution U_G on G,

$$\lim_{t \to \infty} \mu^{(t)}(g) = U_G(g) = \frac{1}{|G|},$$

for all $g \in G$. (See [6] for the details.) The speed of convergence is measured by the total variation distance $\mathrm{d}_{TV}(\mu^{(t)}, U_G)$ defined by

$$\mathrm{d}_{TV}(\mu^{(t)}, U_G) = \frac{1}{2} \sum_{g \in G} \left| \mu^{(t)}(g) - U_G(g) \right|.$$

For an edge e from (g, c) to $(h, c+1)$ in $Y(G, S)$, we define $\pi_E(e)$ to be the edge from g to h in $X(G, S)$. Let $\pi_V : G \times C_2 \to G$ be defined by $\pi_V(g, c) = g$. We state the following trivial fact as a lemma, which is frequently used later.

Lemma 7. *Let π_V and π_E be defined as above. Then the pair $\pi = (\pi_V, \pi_E)$ is a double covering map from $Y(G, S)$ to $X(G, S)$. That is,*

1. *π is a graph morphism: $\pi_V(o(e)) = o(\pi_E(e))$ and $\pi_V(t(e)) = t(\pi_E(e))$, where $o(e)$ is the starting vertex of e and $t(e)$ is the terminal vertex of e.*

2. *π_V and π_E are two-to-one surjections.*

3. *The restriction $\pi_E|_{E_x}$ of π_E is a bijection from E_x to $E_{\pi_V(x)}$ for every $x \in G \times C_2$, where E_x is the set of edges starting from a vertex x.*

By Lemma 7, π_E can induce the map π_P from the set of paths in $Y(G,S)$ to those of $X(G,S)$. That is, if $p = (e_1, e_2, \ldots, e_t)$ is a path of length t in $Y(G,S)$ where each e_i is an edge, then $\pi_P(p) = (\pi_E(e_1), \pi_E(e_2), \ldots, \pi_E(e_t))$ is a path in $X(G,S)$. We define $\mathcal{P}_X^{(t)}(g,h)$ as the set of the paths of length t starting from a vertex g and terminating at a vertex h in a graph X.

Lemma 8. *Let $X(G,S)$, $Y(G,S)$ and π_P be defined as above. Then the restriction of π_P to $\mathcal{P}_Y^{(t)}((g,c),(h,[c+t]))$ is a bijection to $\mathcal{P}_X^{(t)}(g,h)$ for every $g \in G$, where $[c+t] \in \{0,1\}$ satisfies $[c+t] \equiv c+t (\mathrm{mod} 2)$.*

Proof. Let $p = (e_1, e_2, \ldots, e_t)$ be a path in $X(G,S)$, which corresponds to a sequence $(s_1, s_2, \ldots, s_t)$ of the elements of S. That is, if $o(e_i) = g$ and $t(e_i) = h$, then $h = gs_i$. Then there exists a unique path $\bar{p}$ in $Y(G,S)$ which starts from the vertex (g,c) and corresponds to the sequence $((s_1,1),(s_2,1),\ldots,(s_t,1))$ of the elements of $S \times \{1\}$. It is clear that this $\bar{p}$ is the unique path in $Y(G,S)$ of length t such that $\bar{p}$ starts from (g,c) and $\pi_P(\bar{p}) = p$. ☐

Lemma 9. *Let S be a generating set of a finite group G and suppose that $S \times \{1\}$ generates the direct product $G \times C_2$. Let $(\mu^{(0)}, \mu^{(1)}, \ldots)$ be the simple random walk on $X(G,S)$ starting from the identity element id of G, and let $(\nu^{(0)}, \nu^{(1)}, \ldots)$ be the simple random walk on $Y(G,S)$ starting from the identity element $(\mathrm{id},0) \in G \times C_2$. Then,*

$$\mu^{(t)}(g) = \nu^{(t)}(g,[t]), \tag{9}$$

and therefore

$$\nu^{(t)}(g,[t+1]) = 0.$$

Proof. It is clear that

$$\mu^{(t)}(g) = \frac{1}{d^t}\left|\mathcal{P}_X^{(t)}(\mathrm{id},g)\right|, \qquad \nu^{(t)}(g,[t]) = \frac{1}{d^t}\left|\mathcal{P}_Y^{(t)}((\mathrm{id},0),(g,[t]))\right|.$$

From Lemma 8, (9) follows. ☐

By (8) $X(G,S_1)$ and $X(G,S_2)$ have a common double covering, and there exists a graph isomorphism $\varphi = (\varphi_V, \varphi_E)$ from $Y(G,S_1)$ to $Y(G,S_2)$, which satisfies $\varphi_V(\mathrm{id},0) = (\mathrm{id},0)$. Then it is obvious that

$$\varphi_V(G \times \{0\}) = G \times \{0\}, \quad \varphi_V(G \times \{1\}) = G \times \{1\},$$

which induces two bijections $\varphi_0 : G \to G$ and $\varphi_1 : G \to G$ such that

$$\varphi_V(g,0) = (\varphi_0(g),0), \quad \varphi_V(g,1) = (\varphi_1(g),1). \tag{10}$$

Theorem 10. *There exists a bijection between* $\mathcal{P}^{(t)}_{X(G,S_1)}(g,h)$ *and* $\mathcal{P}^{(t)}_{X(G,S_2)}(\varphi_0(g),\varphi_{[t]}(h))$ *and hence*

$$\left|\mathcal{P}^{(t)}_{X(G,S_1)}(g,h)\right| = \left|\mathcal{P}^{(t)}_{X(G,S_2)}(\varphi_0(g),\varphi_{[t]}(h))\right|. \tag{11}$$

Proof. By Lemma 8, there is a bijection between $\mathcal{P}^{(t)}_{X(G,S_1)}(g,h)$ and $\mathcal{P}^{(t)}_{Y(G,S_1)}((g,0),(h,[t]))$. Since condition (8) is satisfied, there exists a graph isomorphism $\varphi : Y(G,S_1) \to Y(G,S_2)$. Let φ_0 and φ_1 be defined by (10). Then, we obtain the bijection

$$\mathcal{P}^{(t)}_{Y(G,S_1)}((g,0),(h,[t])) \to \mathcal{P}^{(t)}_{Y(G,S_2)}(\varphi(g,0),\varphi(h,[t])) = \mathcal{P}^{(t)}_{Y(G,S_2)}((\varphi_0(g),0),(\varphi_{[t]}(h),[t])).$$

By applying Lemma 8, we obtain the bijection,

$$\mathcal{P}^{(t)}_{Y(G,S_2)}((\varphi_0(g),0),(\varphi_{[t]}(h),[t])) \to \mathcal{P}^{(t)}_{X(G,S_2)}(\varphi_0(g),\varphi_{[t]}(h)).$$

By composing these three bijections, we obtain a bijection

$$\mathcal{P}^{(t)}_{X(G,S_1)}(g,h) \to \mathcal{P}^{(t)}_{X(G,S_2)}(\varphi_0(g),\varphi_{[t]}(h)).$$

□

Corollary 11. *Let* $(\mu_i^{(0)},\mu_i^{(1)},\ldots)$ *be the simple random walk on the Cayley graph* $X(G,S_i)$ *for* $i=1,2$. *Then,*

$$\mu_1^{(t)}(g) = \mu_2^{(t)}(\varphi_{[t]}(g)) \tag{12}$$

for all $g \in G$ *and* $t \geqslant 0$, *and*

$$\mathrm{d}_{TV}(\mu_1^{(t)},U_G) = \mathrm{d}_{TV}(\mu_2^{(t)},U_G),$$

where U_G *is the uniform probability measure over* G.

Proof. Since we suppose the random walks start from the identity element $\mathrm{id} \in G$,

$$\mu_1^{(t)}(g) = \frac{1}{d^t}\left|\mathcal{P}^{(t)}_{X(G,S_1)}(\mathrm{id},g)\right|, \quad \mu_2^{(t)}(\varphi_{[t]}(g)) = \frac{1}{d^t}\left|\mathcal{P}^{(t)}_{X(G,S_1)}(\mathrm{id},\varphi_{[t]}(g))\right|.$$

Since $\varphi_0(\mathrm{id}) = \mathrm{id}$, Theorem 10 implies

$$\left|\mathcal{P}^{(t)}_{X(G,S_1)}(\mathrm{id},g)\right| = \left|\mathcal{P}^{(t)}_{X(G,S_1)}(\mathrm{id},\varphi_{[t]}(g))\right|.$$

Thus we obtain (12).

$$\begin{aligned}\mathrm{d}_{TV}(\mu_1^{(t)},U_G) &= \frac{1}{2}\sum_{g\in G}\left|\mu_1^{(t)}(g) - \frac{1}{|G|}\right| = \frac{1}{2}\sum_{g\in G}\left|\mu_2^{(t)}\left(\varphi_{[t]}(g)\right) - \frac{1}{|G|}\right| \\ &= \frac{1}{2}\sum_{g\in G}\left|\mu_2^{(t)}(g) - \frac{1}{|G|}\right| = \mathrm{d}_{TV}(\mu_2^{(t)},U_G).\end{aligned}$$

□

4 Spectral structure

This section is about properties **i** and **iii**. Recall from Theorem 4 that $\mathrm{puz}_0(\theta_{a,b})$ stands for a connected component of $\mathrm{puz}(\theta_{a,b})$, whose path contraction $\widetilde{\mathrm{puz}}_0(\theta_{a,b})$ is isomorphic to $Y(G, S_i)$ where G is as in (7), and the generating sets S_1 and S_2 have been defined by (3) and (4) respectively. We apply the theory of the zeta functions of the finite graph coverings to show that the Cayley graphs $X(G, S_1)$ and $X(G, S_2)$ have properties **i** and **iii**. The readers who are not familiar with the theory of the zeta functions of the finite graphs are referred to [7]. A remark is in order here.

Remark 12. We will introduce the zeta function and L-function of a finite graph Γ, which is considered to be a *bi-directed* graph. That is, every edge e of Γ is directed and there exists a unique edge $\overline{e}$, the *reverse* of e. Among all the Cayley graphs of the forms $X(G, S)$ and $Y(G, S)$, the only exception which does not fit into this formulation is our running example $\widetilde{\mathrm{puz}}_0(\theta_{1,0})$ since $X(\mathfrak{S}_3, S_2)$ with $S_2 = \{(1, 2), (2, 3), \mathrm{id}\}$ has loops and the loops do not have their reverses. Nevertheless the statements of our main results Theorem 17 and 19 perfectly hold for this only exception (shown in Figure 9).

As we have explained in the previous section, the automorphism group $\mathrm{Aut}(\theta_{a,b})$ contains a subgroup $K = \{\mathrm{id}, \rho, \psi, \rho\psi\}$ isomorphic to the Klein four-group, which acts also on $Y = \widetilde{\mathrm{puz}}_0(\theta_{a,b})$ from the right as a subgroup of the automorphism group of Y, namely we can regard K as

$$K \subset \mathrm{Aut}(Y).$$

The group K has three non-trivial subgroups $\langle\rho\rangle, \langle\psi\rangle$, and $\langle\rho\psi\rangle$. Let $Y = \widetilde{\mathrm{puz}}_0(\theta_{a,b}) = (V_Y, E_Y)$, where V_Y is the vertex set of Y and E_Y is the set of (directed) edges of Y. For each subgroup H of K, we obtain the *quotient graph* $X_H = Y/H$ as follows. The subgroup acts on V_Y and E_Y from the right. The vertex set of X_H is the set of H-orbits V_Y/H, and the edge set of X_H is the set of H-orbits E_Y/H. Thus we obtain the natural covering map $\pi = (\pi_V, \pi_E)$ consisting of

$$\pi_V : V_Y \ni v \mapsto vH \in V_{X_H} = V_Y/H, \quad \pi_E : E_Y \ni e \mapsto eH \in E_{X_H} = E_Y/H.$$

Then this *covering* is *normal* (or Galois), that is, H acts transitively and freely on each fiber $\pi_V^{-1}(v)$ (and also on $\pi_E^{-1}(e)$). When a graph Γ is a normal cover of a graph Δ, the covering transformation group (or the Galois group) of Γ/Δ is denoted $\mathrm{Gal}(\Gamma/\Delta)$. Thus we have

$$H = \mathrm{Gal}(Y/X_H).$$

Let $Z = Y/K$. Then $\mathrm{Gal}(Y/Z) = K$ and by applying Theorem 14.3 of [7], we have three intermediate coverings of Y/Z,

$$X_{\langle\rho\rangle} = Y/\langle\rho\rangle, \quad X_{\langle\psi\rangle} = Y/\langle\psi\rangle, \quad X_{\langle\rho\psi\rangle} = Y/\langle\rho\psi\rangle,$$

each of which is a double (and therefore Galois) covering of Z. Figure 7 illustrates this covering relation.

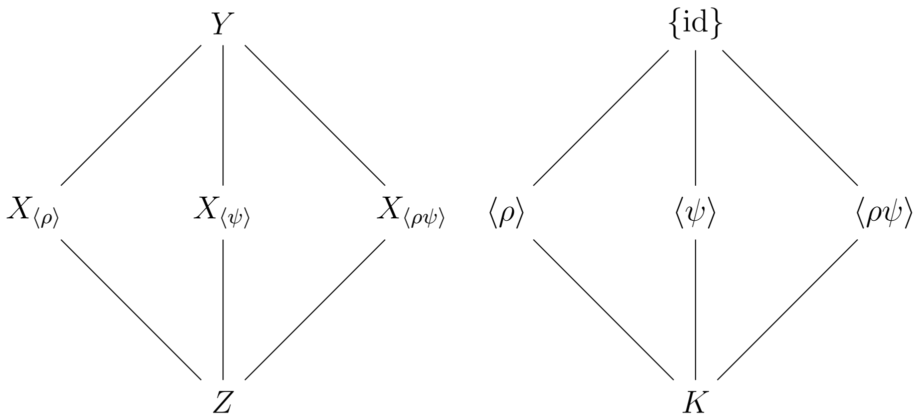


Figure 7: Covering relation of $\widetilde{\mathrm{puz}}(\theta_{a,b})$, $X(G,S_1)$, $X(G,S_2)$, and X_3, Z

Note that we have

$$X_{\langle\rho\rangle} = X(G,S_1), \quad X_{\langle\psi\rangle} = X(G,S_2).$$

The following is an example of $X_{\langle\rho\psi\rangle}$.

Example 13.

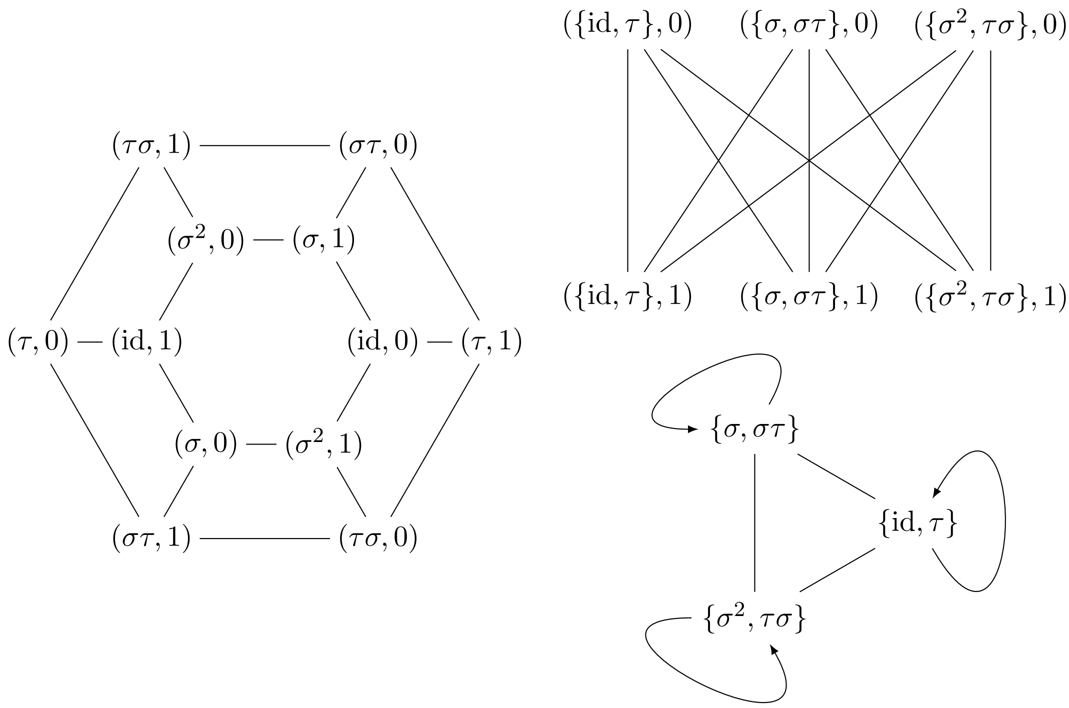


Figure 8: $Y(\mathfrak{S}_3,S_1) \cong Y = \widetilde{\mathrm{puz}}(\theta_{1,0})$(left), $X_{\langle\rho\psi\rangle}$(upper right) and Z(lower right)

As we have seen in Example 6, $\widetilde{\mathrm{puz}}(\theta_{1,0})$ is isomorphic to the Cayley graph $Y(\mathfrak{S}_3, S_1)$ where $S_1 = \{\sigma = (1,3,2), \tau = (1,3)\}$. Then the vertex permutation $\rho\psi$ is the transposition just exchanging the vertices v_1 and v_3. The quotient graphs $X_{\langle\rho\psi\rangle}$ and X_K are shown in Figure 8.

Let $C = (e_0, e_1, \ldots, e_{l-1})$ be a cycle in a graph Γ. Then C is *non-backtracking* if

$$e_i \neq e_{i+1}$$

for every $i \in \mathbb{Z}/l\mathbb{Z}$. The *length* $\nu(C)$ of a cycle C is defined by

$$\nu(C) = l.$$

A non-backtracking cycle C is a *prime cycle* if there exists no pair of a cycle D and an integer $k > 1$ such that

$$C = D^k.$$

Let $C = (e_0, e_1, \ldots, e_{l-1})$ be a cycle in a graph Γ. We introduce an equivalence relation $\sim$ to the set of prime cycles in Γ as follows. Let C and C' be two prime cycles in Γ. Then we define that $C \sim C'$ if there exists a $k \in \mathbb{Z}/l\mathbb{Z}$ such that

$$C' = (e_k, e_{k+1}, \ldots, e_{k+l-1}).$$

That is, C' is obtained from C by changing the starting vertex. This relation defines an equivalence relation on the set of all prime cycles in Γ, and we denote the equivalence class $[C]$ to which C belongs. We call $[C]$ a *prime* in Γ.

The *zeta function* ζ_Γ of a graph Γ is defined by

$$\zeta_\Gamma(u) = \prod_{[C]} \left(1 - u^{\nu(C)}\right)^{-1},$$

where $[C]$ runs through the primes in Γ. The following theorem shows how the zeta functions are related to the spectra of Γ.

Theorem 14. ([5, 1]) *Let A be the adjacency matrix of Γ, and let Q be the diagonal matrix whose diagonal entry corresponding to vertex v is $\deg(v) - 1$. Then we have*

$$\zeta_\Gamma(u) = \left((1-u^2)^{r-1} \det\left(I - uA + u^2 Q\right)\right)^{-1},$$

where $r - 1 = \frac{1}{2}\mathrm{Tr}\,(Q - I)$.

When we apply Theorem 14 to $X_H = Y/H$, we have $Q = 2I$ and

$$\begin{aligned} \zeta_{X_H}(u) &= (1-u^2)^{-|G|/|H|} u^{-2|G|/|H|} \det\left(\frac{2u^2+1}{u} I - A\right)^{-1} && (13) \\ &= (1-u^2)^{-|G|/|H|} u^{-2|G|/|H|} P_{X_H}\left(2u + \frac{1}{u}\right)^{-1}, && (14) \end{aligned}$$

where $P_\Gamma(x)$ is the characteristic polynomial of the adjacency matrix of a graph Γ. We use this fact frequently to translate the relations of the zeta functions into the relations of the characteristic polynomials.

If Γ is a normal covering of a graph Δ where $\mathrm{Gal}(\Gamma/\Delta)$ is abelian, we say Γ/Δ is an *abelian* covering. In what follows, we state theorems on Galois coverings in [7] in the forms restricted to the abelian cases. Let Γ/Δ be an abelian covering with the covering map $\pi = (\pi_V, \pi_E)$. Let $C = (e_0, e_1, \ldots, e_{l-1})$ be a prime cycle of Δ, and $\widetilde{C} = (f_0, f_1, \ldots, f_{l-1})$ be a lift of C in Γ, that is, $\widetilde{C}$ is a path in Γ whose projection $\pi(\widetilde{C})$ onto Δ is C. Let $o(\widetilde{C})$ be the starting vertex of $\widetilde{C}$ and $t(\widetilde{C})$ the terminal vertex. Then, since Γ/Δ is a normal covering, and both $o(\widetilde{C})$ and $t(\widetilde{C})$ are projected onto the same vertex v, there exits a unique automorphism g in $\mathrm{Gal}(\Gamma/\Delta)$ such that $o(C)g = t(C)$. This unique automorphism g is denoted

$$\left(\frac{\Gamma/\Delta}{C}\right).$$

The following proposition corresponds to the parts (1) and (3) of Proposition 16.5 of [7, p.137] restricted to the abelian covering case.

Proposition 15. *The automorphism* $\left(\frac{\Gamma/\Delta}{C}\right)$ *is determined by the prime* $[C]$ *not depending on the choice of* $\widetilde{C}$.

We call the automorphism $\left(\frac{\Gamma/\Delta}{C}\right)$ the *Frobenius automorphism* of C associated with the abelian covering Γ/Δ. Let Γ/Δ be an abelian covering, and let χ be a character of $\mathrm{Gal}(\Gamma/\Delta)$. Then the *Artin L-function* $L(u, \chi, \Gamma/\Delta)$ of Γ/Δ associated with the character χ is defined by

$$L(u, \chi, \Gamma/\Delta) = \prod_{[C]} \left(1 - \chi\left(\frac{\Gamma/\Delta}{C}\right) u^{\nu(C)}\right)^{-1}, \tag{15}$$

where $[C]$ in the product runs through the primes of Δ. The following proposition corresponds to Proposition 18.10 and Corollary 18.11 of [7, p.154–155].

Proposition 16. *Let* Γ/Δ *be an abelian covering, and* $\widetilde{\Delta}$ *be an intermediate covering of* Γ/Δ. *Then, a character* $\widetilde{\chi}$ *of* $\mathrm{Gal}(\widetilde{\Delta}/\Delta)$ *can be lifted to the character* χ *of* $\mathrm{Gal}(\Gamma/\Delta)$ *and we have*

$$L(u, \chi, \Gamma/\Delta) = L(u, \widetilde{\chi}, \widetilde{\Delta}/\Delta). \tag{16}$$

The zeta function ζ_Γ *is factorized into the products of L-functions:*

$$\zeta_\Gamma(u) = \prod_{\chi \in \widehat{\mathrm{Gal}(\Gamma/\Delta)}} L(u, \chi, \Gamma/\Delta), \tag{17}$$

where $\widehat{\mathrm{Gal}(\Gamma/\Delta)}$ *is the set of the characters of* $\mathrm{Gal}(\Gamma/\Delta)$.

Theorem 17. *Let* $Y = \widetilde{\mathrm{puz}}(\theta_{a,b})$ *with* $(a, b) \neq (1, 0)$. *Let* $X_{\langle\rho\rangle}, X_{\langle\psi\rangle}, X_{\langle\rho\psi\rangle}$ *and* Z *be defined as above. Then*

$$P_Y(u)P_Z(u)^2 = P_{X_{\langle\rho\rangle}}(u)P_{X_{\langle\psi\rangle}}(u)P_{X_{\langle\rho\psi\rangle}}(u)$$

Proof. We apply Proposition 16 to our case where $\Gamma = Y = \widetilde{\mathrm{puz}}(\theta_{a,b})$ and $\Delta = Z = X_K$. The Galois group $K = \mathrm{Gal}(Y/Z) = \{\mathrm{id}, \rho, \psi, \rho\psi\}$ has the four characters, which is summarized in the following table.

$$\begin{array}{c|cccc} & \mathrm{id} & \rho & \psi & \rho\psi \\ \hline 1 & 1 & 1 & 1 & 1 \\ \chi_\rho & 1 & 1 & -1 & -1 \\ \chi_\psi & 1 & -1 & 1 & -1 \\ \chi_{\rho\psi} & 1 & -1 & -1 & 1 \end{array} \tag{18}$$

Then χ_ρ is the lift of the non-trivial character $\widetilde{\chi}_\rho$ of $K/\langle\rho\rangle$ and it follows from (16) that

$$L(u, \chi_\rho, Y/Z) = L(u, \widetilde{\chi}_\rho, X_{\langle\rho\rangle}/Z).$$

In the same manner we have

$$L(u, \chi_\psi, Y/Z) = L(u, \widetilde{\chi}_\psi, X_{\langle\psi\rangle}/Z),$$

and

$$L(u, \chi_{\rho\psi}, Y/Z) = L(u, \widetilde{\chi}_{\rho\psi}, X_{\langle\rho\psi\rangle}/Z).$$

Then by applying the factorization formula (17) to abelian covers $X_{\langle\rho\rangle}/Z, X_{\langle\psi\rangle}/Z$ and $X_{\langle\rho\psi\rangle}/Z$, we have

$$\zeta_{X_{\langle\rho\rangle}}(u) = \zeta_Z(u)L(u, \chi_\rho, Y/Z), \tag{19}$$

$$\zeta_{X_{\langle\psi\rangle}}(u) = \zeta_Z(u)L(u, \chi_\psi, Y/Z), \tag{20}$$

and

$$\zeta_{X_{\langle\rho\psi\rangle}}(u) = \zeta_Z(u)L(u, \chi_{\rho\psi}, Y/Z).$$

Consequently, by applying (17) to the abelian cover Y/Z, we obtain

$$\begin{aligned} \zeta_Y(u) &= \zeta_Z(u)L(u, \chi_\rho, Y/Z)L(u, \chi_\psi, Y/Z)L(u, \chi_{\rho\psi}, Y/Z) \\ &= \zeta_Z(u)\frac{\zeta_{X_{\langle\rho\rangle}}(u)}{\zeta_Z(u)}\frac{\zeta_{X_{\langle\psi\rangle}}(u)}{\zeta_Z(u)}\frac{\zeta_{X_{\langle\rho\psi\rangle}}(u)}{\zeta_Z(u)}, \end{aligned}$$

from which we have

$$\zeta_Y(u)\zeta_Z(u)^2 = \zeta_{X_{\langle\rho\rangle}}(u)\zeta_{X_{\langle\psi\rangle}}(u)\zeta_{X_{\langle\rho\psi\rangle}}(u).$$

□

Lemma 18. *Let $Y = \widetilde{\mathrm{puz}}_0(\theta_{a,b})$, let $Z = Y/K$ and let C be a prime cycle of Z. Then we have*

$$\nu(C) \text{ is even } \iff \left(\frac{Y/Z}{C}\right) \in \{\mathrm{id}, \rho\psi\}.$$

Proof. As we have explained in Section 2, $Y \cong Y(G, S_i)$ is a bipartite graph. If two vertices (or positions) f_1 and f_2 are adjacent in Y, then the blanks of f_1 and f_2 do not coincide, that is, $f_1^{-1}(0) \neq f_2^{-1}(0)$. Let $\widetilde{C}$ be the lift of C to Y. If $\nu(C)$ is even and $\widetilde{C}$ starts from f and terminates at g, then $f^{-1}(0) = g^{-1}(0)$ and $f \circ \left(\frac{Y/X}{C}\right) = g$, which means $\left(\frac{Y/X}{C}\right) \in \{\mathrm{id}, \rho\psi\}$ since ρ and ψ move the blank. The converse is clear. □

Theorem 19.

$$\frac{P_{X_{\langle\rho\rangle}}(x)}{P_Z(x)} = (-1)^{|G|/2}\frac{P_{X_{\langle\psi\rangle}}(-x)}{P_Z(-x)} \tag{21}$$

$$P_{X_{\langle\rho\psi\rangle}}(x) = (-1)^{|G|/2} P_Z(x)P_Z(-x) \tag{22}$$

Proof. By (14),(19) and (20), to prove (21), it suffices to show

$$L(u, \chi_\rho, Y/Z) = L(-u, \chi_\psi, Y/Z). \tag{23}$$

Let C be a prime cycle in Z and $\widetilde{C}$ its lift to Y. If $\nu(C)$ is an even integer, then by Lemma 18, we have

$$\left(\frac{Y/Z}{C}\right) \in \{\mathrm{id}, \rho\psi\}.$$

Further, by the table (18), we have $\chi_\rho\left(\frac{Y/Z}{C}\right) = \chi_\psi\left(\frac{Y/Z}{C}\right)$ and the corresponding factors in (15) of the L-functions coincide:

$$1 - \chi_\rho\left(\frac{Y/Z}{C}\right)u^{\nu(C)} = 1 - \chi_\psi\left(\frac{Y/Z}{C}\right)(-u)^{\nu(C)}. \tag{24}$$

If $\nu(C)$ is an odd integer, then by Lemma 18, we have

$$\left(\frac{Y/Z}{C}\right) \in \{\rho, \psi\}.$$

Further, by the table (18), we obtain $\chi_\rho\left(\frac{Y/Z}{C}\right) = -\chi_\psi\left(\frac{Y/Z}{C}\right)$ and (24) holds. Thus we obtain (23). The relation (22) can be obtained in the same manner. □

Theorem 20. *Let G, S_1 and S_2 be as above. Then $X(G, S_1)$ is not isomorphic to $X(G, S_2)$.*

Proof. As we have defined $X_{\langle\rho\rangle} \cong X(G, S_1)$ (resp. $X_{\langle\psi\rangle} \cong X(G, S_2)$) as the quotient $Y/\langle\rho\rangle$ (resp. $Y/\langle\psi\rangle$), if two vertices of Y are on the same ρ-orbit (resp. ψ-orbit), they are at odd distance. Hence paths connecting two vertices on a ρ-orbit (resp. ψ-orbit) are projected onto cycles of odd length in $X_{\langle\rho\rangle}$ (resp. $X_{\langle\psi\rangle}$). Thus $X_{\langle\rho\rangle}$ and $X_{\langle\psi\rangle}$ are non-bipartite. For the same reason, $Z = Y/\langle\rho, \psi\rangle$ is also a non-bipartite graph containing prime cycles of odd length. Therefore we have

$$L(u, \chi_\rho, Y/Z) \neq L(-u, \chi_\rho, Y/Z) = L(u, \chi_\psi, Y/Z).$$

Hence

$$\zeta_{X_{\langle\rho\rangle}}(u) = \zeta_Z(u)L(u, \chi_\rho, Y/Z) \neq \zeta_Z(u)L(u, \chi_\psi, Y/Z) = \zeta_{X_{\langle\psi\rangle}}(u),$$

which completes the proof. □

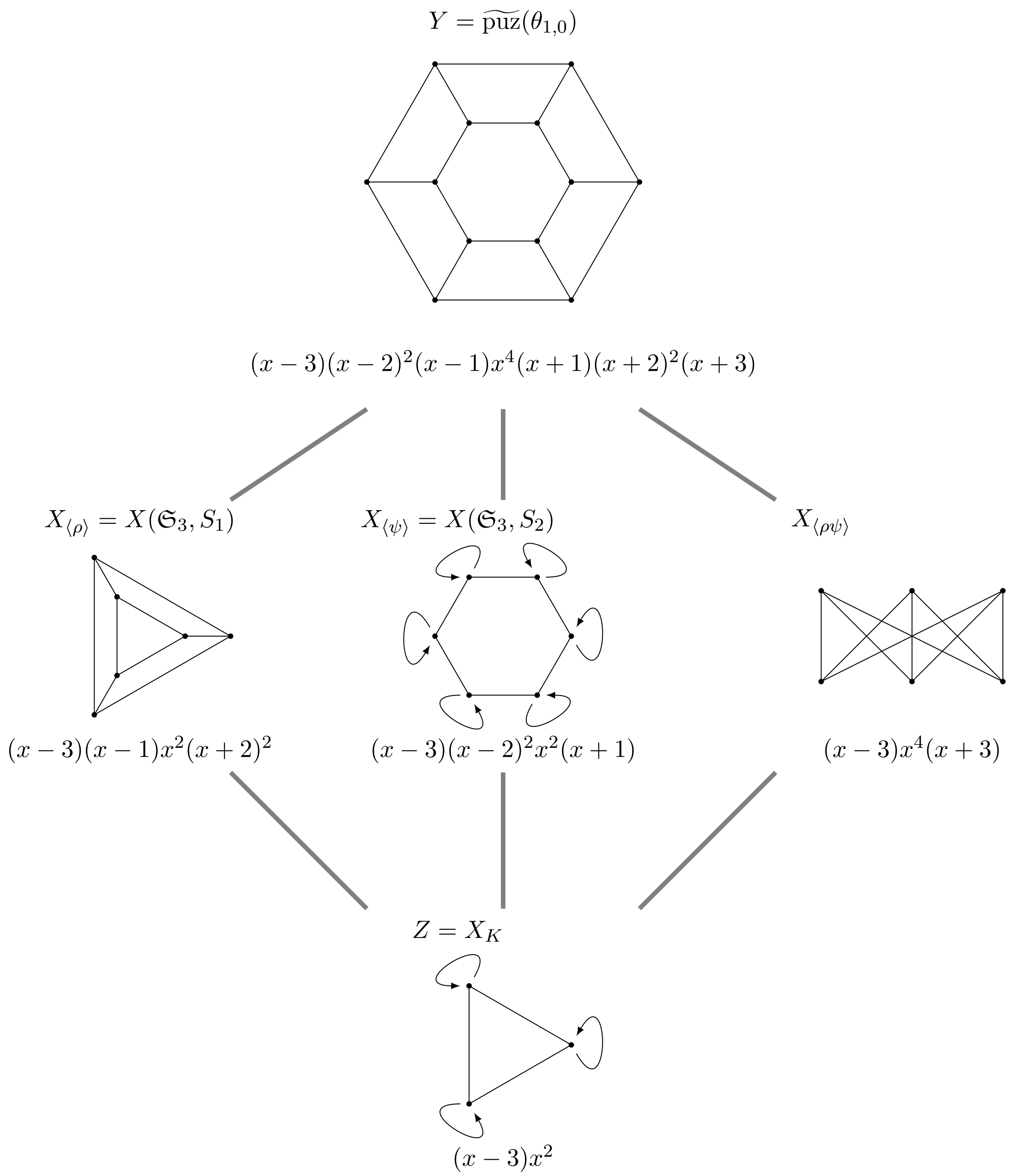


Figure 9: The statements of Theorem 17 and 19 hold for this exceptional case $\theta_{1,0}$.

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