Gamma vectors of partitioned permutohedra

Tatsuya Horiguchi[a] Mikiya Masuda[b,c] Takashi Sato[b]
John Shareshian[d] Jongbaek Song[e]

Submitted: Aug 28, 2025; Accepted: Jul 29, 2026; Published: Sep 11, 2026

Abstract

We show that γ-vectors of partitioned permutohedra have nonnegative entries, thereby generalizing a result of Foata and Schützenberger. Our result is closely related to a result of Athanasiadis on the representation of the symmetric group on the cohomology of the permutohedral variety. We explain how to derive Athanasiadis' result from ours and vice versa.

Mathematics Subject Classifications: 05E10, 14M25, 20C30, 52B05

1 Introduction

We continue the study of partitioned permutohedra that was initiated by some of the present authors in [13]. We examine the γ-vectors of these polytopes, obtaining a generalization of a result of Foata and Schützenberger found in [7] and recovering a representation-theoretic result of Athanasiadis found in [1].

The symmetric group $\mathfrak{S}_n$ acts on Euclidean space $\mathbb{R}^n$ by permuting coordinates. This action preserves each affine hyperplane $E \subset \mathbb{R}^n$ consisting of vectors with constant coordinate sum. Given such E, fix $p = (p_1, \ldots, p_n) \in E$ such that the entries p_i are pairwise distinct. The *permutohedron* P_n is the convex hull of the $\mathfrak{S}_n$-orbit of p. (The combinatorial type of P_n does not depend on the choice of E or p.) For $k \in [n-1]$, we define the halfspace

$$H(k)^{\leqslant} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_k \leqslant x_{k+1}\}.$$

[a] National Institute of Technology, Akashi College, Hyogo 674-8501, Japan (tatsuya.horiguchi0103@gmail.com).
[b] Osaka Central Advanced Mathematical Institute, Osaka Metropolitan University, Osaka 558-8585, Japan (mikiyamsd@gmail.com, 00tkshst00@gmail.com).
[c] International Laboratory of Algebraic Topology and Its Applications, HSE University, Moscow 109028, Russia (mikiyamsd@gmail.com.)
[d] Department of Mathematics, Washington University, St Louis, MO 63130, U.S.A. (jshareshian@wustl.edu).
[e] Corresponding Author. Department of Mathematics Education, Pusan National University, Busan 46241, Republic of Korea (jongbaek.song@pusan.ac.kr).

The *partitioned permutohedron* associated to $K \subseteq [n-1]$ is

$$P_n(K) \coloneqq P_n \cap \bigcap_{k \in K} H(k)^{\leqslant}.$$

It was shown in [13] that the polytopes $P_n(K)$ are all simple of dimension $n-1$. Therefore, if we write f_i for the number of i-dimensional faces of $P_n(K)$ and set

$$h_{P_n(K)}(t) \coloneqq \sum_{i=0}^{n-1} h_i t^i = \sum_{i=0}^{n-1} f_i (t-1)^i, \tag{1}$$

then $h_{P_n(K)}(t)$ is palindromic, that is, $h_i = h_{n-1-i}$ for $0 \leqslant i \leqslant n-1$. (This is the content of the *Dehn–Sommerville equations*, see for example [12, Section 9.2].) It follows that there exist uniquely determined $\gamma_{n,j,K} \in \mathbb{Z}$ ($0 \leqslant j \leqslant \lfloor \frac{n-1}{2} \rfloor$) such that

$$h_{P_n(K)}(t) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,j,K} t^j (1+t)^{n-1-2j}. \tag{2}$$

In the main result of this article (Theorem 6), we show that each $\gamma_{n,j,K}$ appearing in (2) is non-negative by exhibiting subsets $\widehat{W^K}$ and $\widetilde{W^K}$ of $\mathfrak{S}_n$ such that

$$\gamma_{n,j,K} = \left|\{w \in \widehat{W^K} \mid \operatorname{des}(w) = j\}\right| = \left|\{w \in \widetilde{W^K} \mid \operatorname{des}(w) = j\}\right|,$$

where $\operatorname{des}(w)$ denotes the number of descents of w. So, the h-polynomial (1) of $P_n(K)$ can also be written as

$$h_{P_n(K)}(t) = \sum_{w \in \widehat{W^K}} t^{\operatorname{des}(w)} (1+t)^{n-1-2\operatorname{des}(w)}.$$

We refer the reader to Section 2 for definitions of $\widehat{W^K}$ and $\widetilde{W^K}$.

Remark 1. It is known that the h-polynomial $h_{P_n}(t)$ of P_n is the Eulerian polynomial,

$$h_{P_n}(t) = A_n(t) \coloneqq \sum_{w \in \mathfrak{S}_n} t^{\operatorname{des}(w)}. \tag{3}$$

(See for instance [18, Section 4.1].) We will see that $\widehat{W^{\emptyset}}$ is the set $\widehat{\mathfrak{S}_n}$ of permutations with no double descent and no final descent and

$$A_n(t) = \sum_{w \in \widehat{\mathfrak{S}_n}} t^{\operatorname{des}(w)} (1+t)^{n-1-2\operatorname{des}(w)},$$

as proved by Foata and Schützenberger in [7]. See also Theorem 5.

Remark 2. A simple polytope is called *flag* if every collection of pairwise intersecting faces of codimension 1 has a nonempty intersection. It was shown in [13] that each $P_n(K)$ is a simple flag polytope. It was conjectured by Gal in [11] that when the h-polynomial of a simple flag d-polytope is expanded in terms of the basis $t^j(1+t)^{d-2j}$, each coefficient is nonnegative. Theorem 6 confirms that Gal's Conjecture holds for partitioned permutohedra.

There is a topological counterpart of the main result, involving the cohomology of certain toric varieties. We consider toric variety $X_n(K)$ of complex dimension $n-1$ associated with the normal fan of $P_n(K)$. By work of Danilov and of Jurkiewicz (see [5, Theorem 10.8 and Remark 10.9]), we have

$$h_{P_n(K)}(t) = \sum_{i=0}^{n-1} \dim H^{2i}(X_n(K); \mathbb{C}) t^i.$$

By construction, P_n admits an $\mathfrak{S}_n$-action. With $X_n = X_n(\emptyset)$, this action determines an $\mathfrak{S}_n$-representation on each $H^{2i}(X_n; \mathbb{C})$. Such representations have been studied closely, for example by Procesi in [20], by Stanley in [22], by Stembridge in [25, 24], and by Dolgachev and Lunts in [6].

Theorem 1.1 of [13] says that the cohomology ring $H^*(X_n(K); \mathbb{C})$ is isomorphic (as a graded ring) to the ring $H^*(X_n; \mathbb{C})^{W_K}$ of W_K-invariant elements of $H^*(X_n; \mathbb{C})$. Every $\mathbb{C}[\mathfrak{S}_n]$-module V is completely determined by the dimension of fixed-point submodules V^{W_K} as K ranges over all subsets of $[n-1]$. (Indeed, one needs only one representative from each $\mathfrak{S}_n$-conjugacy class of parabolic subgroups.) Since each $h_{P_n(K)}(t)$ is palindromic, we recover from Theorem 6 the already-known isomorphism of $\mathbb{C}[\mathfrak{S}_n]$-modules

$$H^{2i}(X_n; \mathbb{C}) \cong_{\mathfrak{S}_n} H^{2(n-1-i)}(X_n; \mathbb{C})$$

for $0 \leqslant i \leqslant n-1$. (See [22, Corollary 4.4] and [24, Proposition 12].) We should therefore hope for a nice expansion of the representation $\mathfrak{S}_n$ on $H^*(X_n; \mathbb{C})$ using the palindromic polynomials $t^j(1+t)^{n-1-2j}$. Such an expansion was given by Athanasiadis, who expressed the irreducible decomposition of this representation in the desired form in [1]. (We remark here that Stembridge gave in [24, Theorem 4.2] a combinatorial formula for the irreducible decomposition of each $H^{2k}(X_n; \mathbb{C})$. A different such formula is a special case of [21, Theorem 6.3].)

Write $\mathcal{SYT}_n$ for the set of all standard Young tableaux with n boxes and arbitrary shape and $\widehat{\mathcal{SYT}_n}$ for the set of all such tableaux having no double descent and no descent at $n-1$. Given a tableau Q, write $\lambda(Q)$ for the shape of Q and $S^{\lambda(Q)}$ for the corresponding simple module, called the *Specht module*. The readers are referred to [10, Section 7] for this definition. Athanasiadis [1, Corollary 2.41, Equation (90)] proved that there is an isomorphism

$$\sum_{k=0}^{n-1} H^{2k}(X_n; \mathbb{C}) t^k \cong_{\mathfrak{S}_n} \sum_{Q \in \widehat{\mathcal{SYT}_n}} S^{\lambda(Q)} t^{\operatorname{des}(Q)} (1+t)^{n-1-2\operatorname{des}(Q)}.$$

of $\mathbb{C}[\mathfrak{S}_n]$-modules.

As we have discussed, Theorem 6 and the result of Athanasiadis stated above carry the same information; if one knows the dimensions of the fixed point spaces for all parabolic subgroups, then one knows the representation and the converse is clear. After proving Theorem 6 directly in Section 7, we show in Section 8 how to derive each of the theorems quickly assuming the other.

2 Preliminaries

We review here some standard definitions and known facts from algebraic combinatorics. We also introduce some notation. It is reasonable to skip much of this section and refer back as necessary, but note the use of $\widehat{X}$ (resp. $\widetilde{X}$) to indicate the set of all elements of X having no double descent and no final descent (resp. no double descent and no initial descent) when X is a set on whose elements descents are defined.

2.1 Words, permutations, and descents

By a *word of length* n we mean an ordered list $v = v(1)\cdots v(n)$ of n positive integers. So, the set of words of length n is $\mathbb{Z}_{>0}^n$. A *descent* of $v \in \mathbb{Z}_{>0}^n$ is any $i \in [n-1]$ such that $v(i) > v(i+1)$. We write $\mathrm{Des}(v)$ for the set of descents of v and $\mathrm{des}(v)$ for $|\mathrm{Des}(v)|$. We say that v has a *double descent* if there is some $i \in [n-2]$ such that $\{i, i+1\} \subseteq \mathrm{Des}(v)$ and that v has an *initial descent* if $1 \in \mathrm{Des}(v)$. Similarly, we say that v has a *final descent* if $n-1 \in \mathrm{Des}(v)$. The *multiplicity* of $i \in \mathbb{Z}_{>0}$ in v is the number of $j \in [n]$ such that $v(j) = i$, which will be denoted by $m_i(v)$. For example, $v = 4223185$ is a word of length seven, $\mathrm{Des}(v) = \{1, 4, 6\}$, $\mathrm{des}(v) = 3$, and (for example) $m_2(v) = 2$ and $m_8(v) = 1$. In this case, v has no double descent but does have an initial descent and a final descent. We write $\widetilde{\mathbb{Z}_{>0}^n}$ (resp. $\widehat{\mathbb{Z}_{>0}^n}$) for the set of all words of length n having no double descent and no initial descent (resp. no double descent and no final descent).

For a subset X of $\mathbb{Z}_{>0}^n$, we write $\widetilde{X}$ and $\widehat{X}$ for $X \cap \widetilde{\mathbb{Z}_{>0}^n}$ and $X \cap \widehat{\mathbb{Z}_{>0}^n}$, respectively. For example, a permutation $w = w(1)w(2)\cdots w(n) \in \mathfrak{S}_n$, written in one-line notation, can be thought of as a word of length n in which $m_i(v) = 1$ for all $i \in [n]$. So, we write $\widetilde{\mathfrak{S}_n}$ (resp. $\widehat{\mathfrak{S}_n}$) for the set of permutations with no double descent and no initial descent (resp. no double descent and no final descent).

2.2 Parabolic subgroups of $\mathfrak{S}_n$

A subgroup of $\mathfrak{S}_n$ is *parabolic* if it is generated by transpositions. Every parabolic subgroup is conjugate in $\mathfrak{S}_n$ to a *standard parabolic subgroup*, which is a subgroup generated by some adjacent transpositions $(k, k+1)$. For a subset $K \subseteq [n-1]$, we write W_K for the standard parabolic subgroup generated by $\{(k, k+1) : k \in K\}$. If $1 \leqslant i < j \leqslant n$ then i, j lie in the same W_K-orbit on $[n]$ if and only if $r \in K$ whenever $i \leqslant r < j$.

The *length* $\ell(w)$ of $w \in \mathfrak{S}_n$ is the smallest number of adjacent transpositions whose product is w, which is equal to the number of pairs (i, j) from $[n]$ such that $i < j$

and $w(i) > w(j)$ (see for example [2, Proposition 1.5.2]). It is known (see for example [2, Proposition 2.4.4]) that for $K \subseteq [n-1]$, each left coset wW_K for $w \in \mathfrak{S}_n$ has a unique representative of minimal length. We compose permutations from left to right. So, applying $x \in wW_K$ to $[n]$ is the same as applying w and then permuting elements within W_K-orbits. Therefore, w is the representative of minimal length in wW_K if and only if i appears to the left of j in w whenever $i < j$ and i, j are in the same W_K-orbit.

We write W^K for the set of all minimal length coset representatives for W_K in $\mathfrak{S}_n$. In other words, if we decompose a subset $K \subseteq [n-1]$ into maximal consecutive strings $K = [a_1, b_1] \sqcup \cdots \sqcup [a_m, b_m]$, then W^K is the set of elements of $\mathfrak{S}_n$ satisfying the following condition:

$$\begin{array}{l}\text{for each } i \text{ with } 1 \leqslant i \leqslant m, \text{ the numbers } a_i, a_i+1, \cdots, b_i, b_i+1 \\ \text{appear in the one-line notation of } w \text{ as a subsequence in increasing order.}\end{array} \tag{4}$$

Finally, we define
$$\widetilde{W^K} \coloneqq \widetilde{\mathfrak{S}_n} \cap W^K \quad \text{and} \quad \widehat{W^K} \coloneqq \widehat{\mathfrak{S}_n} \cap W^K.$$

Example 3. Consider the case $n = 5$ and $K = \{1, 3\}$. The orbits of W_K are $\{1, 2\}$, $\{3, 4\}$ and $\{5\}$. Note that the condition (4) is satisfied by $w \in \mathfrak{S}_5$ if and only if $w^{-1}(1) \leqslant w^{-1}(2)$ and $w^{-1}(3) \leqslant w^{-1}(4)$. We see that

$$\begin{aligned} W^K = \{&12345, 12354, 12534, 13245, 13254, 13425, 13452, 13524, 13542, 15234, \\ &15324, 15342, 31245, 31254, 31425, 31452, 31524, 31542, 34125, 34152, \\ &34512, 35124, 35142, 35412, 51234, 51324, 51342, 53124, 53142, 53412\}. \end{aligned}$$

To calculate $\widetilde{W^K}$, observe that there are 6 elements of W^K satisfying $w(1) = 5$ and thus having an initial descent. There are another 6 satisfying $w(1) = 3$ and $w(2) = 1$. Among the remaining 18 permutations, 13542, 15324, and 35412 each have a double descent. The remaining 15 elements of $\widetilde{W^K}$ are the following, where the descents are indicated by underlines.

- 12345 with no descent;
- $123\underline{5}4$, $12\underline{5}34$, $1\underline{3}245$, $13\underline{4}25$, $134\underline{5}2$, $13\underline{5}24$, $1\underline{5}234$, $3\underline{4}125$, $34\underline{5}12$, and $3\underline{5}124$ with one descent; and
- $1\underline{3}2\underline{5}4$, $1\underline{5}3\underline{4}2$, $3\underline{4}1\underline{5}2$, and $3\underline{5}1\underline{4}2$ with two descents.

2.3 Fans, polytopes, toric varieties, and h-polynomials

A toric variety is an algebraic variety containing an algebraic torus as a dense open subset such that the natural torus action on itself extends to the whole variety. We will not discuss at length the rich theory of toric varieties and related combinatorial objects, which can be found in various textbooks, including [4, 9, 15]. Rather, we record a little more than we need herein.

A d-dimensional convex polytope P in $\mathbb{R}^d$ is *simple* if each vertex of P is contained in exactly d faces of dimension $(d-1)$. Given a simple polytope P, we write $f_i = f_i(P)$ for the number of i-dimensional faces of P $(0 \leqslant i \leqslant d)$. The *$f$-polynomial* of P is

$$f_P(t) \coloneqq \sum_{i=0}^{d} f_i t^i,$$

and the h-polynomial of P is

$$h_P(t) \coloneqq f_P(t-1).$$

So, for example, if P is a 4-gon in $\mathbb{R}^2$ then $f_P(t) = 4 + 4t + t^2$ and $h_P(t) = 1 + 2t + t^2$.

Given a simple polytope $P \subseteq \mathbb{R}^n$ with rational vertices, the fan Δ_P dual to P is simplicial and determines a toric variety X_P. The key point here, mentioned in the introduction, is the result proved in the projective case by Jurkiewicz and in the general case by Danilov,

$$\sum_{k=0}^{d} \dim H^{2k}(X_P; \mathbb{C}) t^k = h_P(t).$$

We remark that X_P has trivial rational cohomology in odd degrees, hence $h_P(t)$ tells us everything about the Betti numbers of X_P.

2.4 Partitioned permutohedra

Let $p_1, \ldots, p_n$ be pairwise distinct real numbers. The *permutohedron* $P_n(p)$ associated to $p = (p_1, \ldots, p_n)$ is the convex hull of the $n!$ points $(p_{w(1)}, \ldots, p_{w(n)})$, with w running through $\mathfrak{S}_n$. We note that $P_n(p)$ lies in the affine hyperplane $E \subset \mathbb{R}^n$ consisting of vectors whose entries sum to $\sum_{i=1}^{n} p_i$. It is known that $P_n(p)$ is an $(n-1)$-dimensional simple polytope (see for instance [17]). Under our assumption that the coordinates in p are pairwise distinct, the combinatorial type of $P_n(p)$ does not depend on the choice of p. Therefore, we write P_n for $P_n(p)$ for simplicity.[1] The toric variety associated with P_n is called the *permutohedral variety*.

For each $k \in [n-1]$, we set

$$H(k)^{\leqslant} \coloneqq \{(x_1, \ldots, x_n) \in E \mid x_k \leqslant x_{k+1}\}.$$

The *partitioned permutohedron* $P_n(K)$ associated to $K \subseteq [n-1]$ is

$$P_n(K) := P_n \cap \bigcap_{k \in K} H(k)^{\leqslant}.$$

See Figure 1 for the case where $n = 3$ with $p = (1, 2, 3)$. The polytopes $P_n(K)$ were first studied in [13], where it is proved that $P_n(K)$ is a simple flag polytope (see [13,

[1] A canonical choice of p is $(1, 2, \ldots, n)$, but alternative choices can sometimes simplify computations. Indeed, we will take $p = (p_1, \ldots, p_n)$ with $p_{n+1-i} = -p_i$ for $i = 1, \ldots, n$ in Section 7 for the proof of the main theorem.

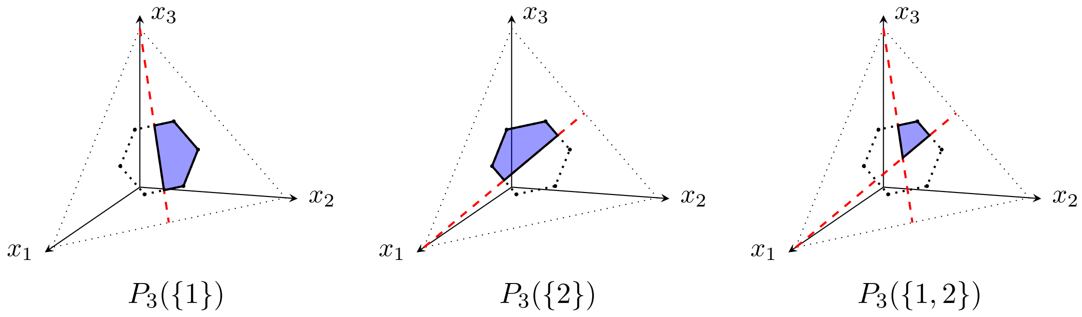


Figure 1: Partitioned permutohedra

Proposition 4.6]). Note that $P_n(\emptyset) = P_n$. On the other hand, $P_n([n-1])$ is combinatorially equivalent to a cube (see [3] for an algebraic proof). Using work of Precup on Hessenberg varieties in [19], the following result was proved in [13].

Proposition 4 ([13], Proposition 7.4)**.** *Given $K \subseteq [n-1]$, set*

$$W(K) := \{w \in \mathfrak{S}_n \mid w^{-1}(k) - w^{-1}(k+1) \leqslant 1 \text{ for all } k \in K\}.$$

For each such K,

$$h_{P_n(K)}(t) = \sum_{w \in W(K)} t^{\mathrm{des}(w)}.$$

3 Main theorem

The main result of this paper gives a combinatorial formula for expanding $h_{P_n(K)}(t)$ in terms of the gamma basis $\{t^j(1+t)^{n-1-2j}\}_{0\leqslant j\leqslant\lfloor\frac{n-1}{2}\rfloor}$. Foata and Schützenberger proved the following result.

Theorem 5. *[7, Theorem 5.6] The Eulerian polynomial* (3) *satisfies*

$$A_n(t) = h_{P_n}(t) = \sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor} \gamma_{n,j}\, t^j(1+t)^{n-1-2j},$$

where the coefficient $\gamma_{n,j}$ is equal to

$$\gamma_{n,j} = |\{w \in \widetilde{\mathfrak{S}}_n \mid \mathrm{des}(w) = j\}| = |\{w \in \widehat{\mathfrak{S}}_n \mid \mathrm{des}(w) = j\}|.$$

The last equality in Theorem 5 follows from the observation that the map from $\widetilde{\mathfrak{S}}_n$ to $\widehat{\mathfrak{S}}_n$ sending w to the permutation w^* satisfying $w^*(i) = n+1-w(n+1-i)$ for all $i \in [n]$ is a descent-number-preserving bijection. Our main theorem generalizes Theorem 5 as follows.

Theorem 6. *For a subset $K \subseteq [n-1]$, the h-polynomial of the partitioned permutohedron $P_n(K)$ satisfies*

$$h_{P_n(K)}(t) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,j,K}\, t^j (1+t)^{n-1-2j},$$

where for each j,

$$\gamma_{n,j,K} = |\{w \in \widetilde{W^K} \mid \operatorname{des}(w) = j\}| = |\{w \in \widehat{W^K} \mid \operatorname{des}(w) = j\}|.$$

After making necessary preparations in the next three sections, we complete the proof of Theorem 6 in Section 7.

Example 7. We continue to consider the case $n = 5$ and $K = \{1, 3\}$ from Example 3. Using Theorem 6, we conclude that

$$h_{P_5(\{1,3\})} = (1+t)^4 + 10t(1+t)^2 + 4t^2 = 1 + 14t + 30t^2 + 14t^3 + t^4. \tag{5}$$

We can confirm that (5) is correct using Proposition 4.

4 Valley hopping

We begin by reviewing the *valley-hopping* operation on a permutation defined by Foata and Strehl in [8]. One may also refer to the description by Petersen in [16].

Definition 8. For a permutation $w = w(1)w(2)\cdots w(n) \in \mathfrak{S}_n$ in one-line notation, we say

1. $w(i)$ is a *peak* if $w(i-1) < w(i) > w(i+1)$;
2. $w(i)$ is a *valley* if $w(i-1) > w(i) < w(i+1)$;
3. $w(i)$ is a *free letter* if $w(i)$ is neither peak nor valley.

Here we take $w(0) = w(n+1) = \infty$ by convention, so that neither $w(1)$ nor $w(n)$ is a peak but each might be a valley.

Example 9. If $w = 13245 \in \mathfrak{S}_5$, then 3 is the peak, $1, 2$ are valleys, and $4, 5$ are free letters. We draw a permutation as a *mountain range* following [16, Section 4]. See Figure 2 for such a depiction of w.

Let $w(i) = j$ be a free letter. We denote by $H_j(w)$ the permutation obtained by moving j directly across the adjacent valley(s) to the nearest mountain slope while maintaining its height. More precisely, we define $H_j(w)$ as follows.

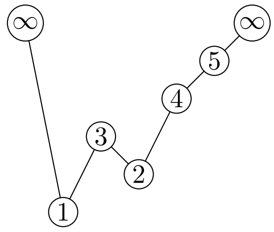


Figure 2: Peaks, valleys and free letters for $w = 13245$.

- If $w(i) = j$ lies on a downslope, i.e., $w(i-1) > w(i) > w(i+1)$, then we define

$$H_j(w) = w(1)\cdots w(i-1)w(i+1)\cdots w(k)\,j\,w(k+1)\cdots w(n)$$

for the smallest $k > i$ such that $w(k) < j < w(k+1)$.

- If $w(i) = j$ lies on an upslope, i.e., $w(i-1) < w(i) < w(i+1)$, then we define

$$H_j(w) = w(1)\cdots w(k-1)\,j\,w(k)\cdots w(i-1)w(i+1)\cdots w(n)$$

for the largest $k < i$ such that $w(k-1) > j > w(k)$.

For any collection of free letters $J = \{j_1, \ldots, j_k\}$, we define

$$H_J(w) \coloneqq H_{j_1}\cdots H_{j_k}(w).$$

Note that H_{j_1} and H_{j_2} commute for all free letters j_1 and j_2. Hence, the operator H_J is well-defined. We observe that $H_J(w)$ has the same peaks and valleys as w. The hop equivalence relation $\sim$ on $\mathfrak{S}_n$ is defined by

$$w \sim u \text{ if and only if } u = H_J(w) \text{ for some collection of free letters } J = \{j_1, \ldots, j_k\}.$$

We denote by $\mathrm{Hop}(w)$ the hop equivalence class of w.

Example 10. For $w = 13245 \in \mathfrak{S}_5$ of Example 9, we see that

$$\mathrm{Hop}(w) = \{13245, 41325, 51324, 54132\}$$

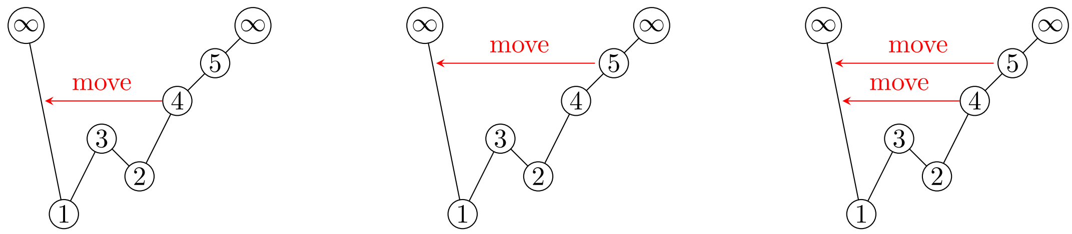
as $H_4(w) = 41325$, $H_5(w) = 51324$ and $H_{\{4,5\}}(w) = 54132$. See Figure 3.

For any $w \in \mathfrak{S}_n$, one can see that

$$\mathrm{des}(w) = |\{\text{peaks of } w\}| + |\{\text{free letters of } w \text{ which lie on a downslope}\}|.$$

If w has r peaks, then it has $r+1$ valleys and $n-1-2r$ free letters. So, writing $\mathrm{pk}(w)$ as the number of peaks of $w \in \mathfrak{S}_n$, we have

$$\sum_{u \in \mathrm{Hop}(w)} t^{\mathrm{des}(u)} = t^{\mathrm{pk}(w)}(1+t)^{n-1-2\,\mathrm{pk}(w)}. \tag{6}$$



$H_4(w) = 41325$ $\qquad$ $H_5(w) = 51324$ $\qquad$ $H_{\{4,5\}}(w) = 54132$

Figure 3: Valley-hopping.

Recall that for a subset $K \subseteq [n-1]$, we define

$$\widetilde{W^K} = \{w \in \widetilde{\mathfrak{S}}_n \mid w \text{ satisfies (4)}\}.$$

By summing (6) over all $w \in \widetilde{W^K}$, we get

$$\sum_{w \in \widetilde{W^K}} \sum_{u \in \mathrm{Hop}(w)} t^{\mathrm{des}(u)} = \sum_{w \in \widetilde{W^K}} t^{\mathrm{pk}(w)}(1+t)^{n-1-2\,\mathrm{pk}(w)}. \tag{7}$$

Hence, in order to prove Theorem 6, it suffices to show that the left-hand side of (7) is equal to $h_{P_n(K)}(t)$.

Recall from Proposition 4 that a combinatorial formula for $h_{P_n(K)}(t)$ is given by

$$h_{P_n(K)}(t) = \sum_{v \in W(K)} t^{\mathrm{des}(v)}. \tag{8}$$

By definition, $v \in W(K)$ if and only if in the one-line notation for v, we see either

$$\text{(i)} \ \cdots k \cdots k+1 \cdots \qquad \text{or} \qquad \text{(ii)} \ \cdots k+1\ k \cdots \tag{9}$$

for each $k \in K$. We note that the cases (i) and (ii) above mean $v^{-1}(k) < v^{-1}(k+1)$ and $v^{-1}(k) - 1 = v^{-1}(k+1)$, respectively.

It is not true in general that $W(K)$ is the union of the hop equivalence classes of the elements of $\widetilde{W^K}$. So, we cannot use a straightforward counting argument to prove Theorem 6. Instead, we claim that there is a bijection

$$\Theta\colon \{(w,u) \in \widetilde{W^K} \times \mathfrak{S}_n \mid u \in \mathrm{Hop}(w)\} \to W(K)$$

such that if $\Theta(w,u) = v$, then $\mathrm{des}(u) = \mathrm{des}(v)$. Then, the bijection Θ together with (8) shows that $h_{P_n(K)}(t)$ agrees with the left-hand side of (7). To construct Θ, we introduce the following operators.

Definition 11. 1. For a permutation $u \in \mathfrak{S}_n$ and $k \in [n-1]$, if the one-line notation of u does not satisfy the condition (9), i.e., $u^{-1}(k) > u^{-1}(k+1)+1$, then we define $\mathfrak{J}_k(u)$ to be the permutation obtained from u by moving $k+1$ to the right so that it is of the form (ii) of (9). Otherwise, we define $\mathfrak{J}_k(u) = u$.

2. In general, for $K = \{i_1, i_2, \ldots, i_m\}$ with $i_1 < i_2 < \cdots < i_m$, we define

$$\mathfrak{J}_K(u) \coloneqq \mathfrak{J}_{i_m} \cdots \mathfrak{J}_{i_2} \mathfrak{J}_{i_1}(u). \tag{10}$$

We observe that $\mathfrak{J}_K(u) \in W(K)$ for all $u \in \mathfrak{S}_n$ by definition.

Proposition 12. *For a subset $K \subseteq [n-1]$, the map*

$$\Theta \colon \{(w,u) \in \widetilde{W^K} \times \mathfrak{S}_n \mid u \in \mathrm{Hop}(w)\} \to W(K) \tag{11}$$

defined by $\Theta(w,u) = \mathfrak{J}_K(u)$ satisfies $\mathrm{des}(\mathfrak{J}_K(u)) = \mathrm{des}(u)$ and is bijective.

We will prove the above proposition in the next two sections.

Example 13. For $n = 5$, $K = \{1,3\}$ and $w = 13245 \in \widetilde{W^K}$ as in Examples 3 and 7, we have $\mathrm{Hop}(w) = \{13245, 41325, 51324, 54132\}$. Applying the operator $\mathfrak{J}_K$ to permutations in $\mathrm{Hop}(w)$ above, we obtain

$$\{13245, 14325, 51324, 51432\},$$

which is a subset of $W(K)$. Here, the first and the third elements of $\mathrm{Hop}(w)$ are already elements of $W(K)$ so the operator $\mathfrak{J}_K$ does nothing, while the second and the fourth elements are changed. One can verify that $\bigsqcup_{w \in \widetilde{W^K}} \{\mathfrak{J}_K(u) \mid u \in \mathrm{Hop}(w)\} = W(K)$.

5 Preserving the descent number

In this section we prove the first assertion of Proposition 12, namely that the operator $\mathfrak{J}_K$ introduced in Definition 11 preserves the number of descents. We first observe that some of the operators $\mathfrak{J}_{i_1}, \ldots, \mathfrak{J}_{i_m}$ in (10) act trivially in the construction of $\mathfrak{J}_K(u)$. For instance, when $n = 5, K = \{1,3\}$ and $u = H_4(w) = 41325 \in \mathrm{Hop}(w)$ of Example 13, one can see that

$$\mathfrak{J}_K(u) = 14325 = \mathfrak{J}_3(u) \neq u.$$

Hence, we introduce the following definition to make computations simpler.

Definition 14. Let K be a subset of $[n-1]$. For a permutation $u \in \mathfrak{S}_n$ with $\mathfrak{J}_K(u) \neq u$, one can write

$$\mathfrak{J}_K(u) = \mathfrak{J}_{K_m} \cdots \mathfrak{J}_{K_1}(u) \tag{12}$$

for some closed intervals $K_j \coloneqq [k_j, k_j + \ell_j - 1]$ in K such that

- $k_i < k_i + \ell_i < k_{i+1}$ for $i \in [m-1]$ and $\ell_m > 0$;
- $\eth_{k_j+i}\eth_{k_j+i-1}\cdots\eth_{k_j}(u_j) \neq \eth_{k_j+i-1}\eth_{k_j+i-2}\cdots\eth_{k_j}(u_j)$ for all $0 \leqslant i \leqslant \ell_j - 1$ with $u_1 := u$ and $u_j := \eth_{K_{j-1}}\cdots\eth_{K_1}(u)$ for $2 \leqslant j \leqslant m$.

We call the expression in (12) the *effective expression for* $\eth_K(u)$. Such an expression exists for every $\eth_K(u)$. Note that closed intervals $K_j = [k_j, k_j + \ell_j - 1]$ in (12) depend on u.

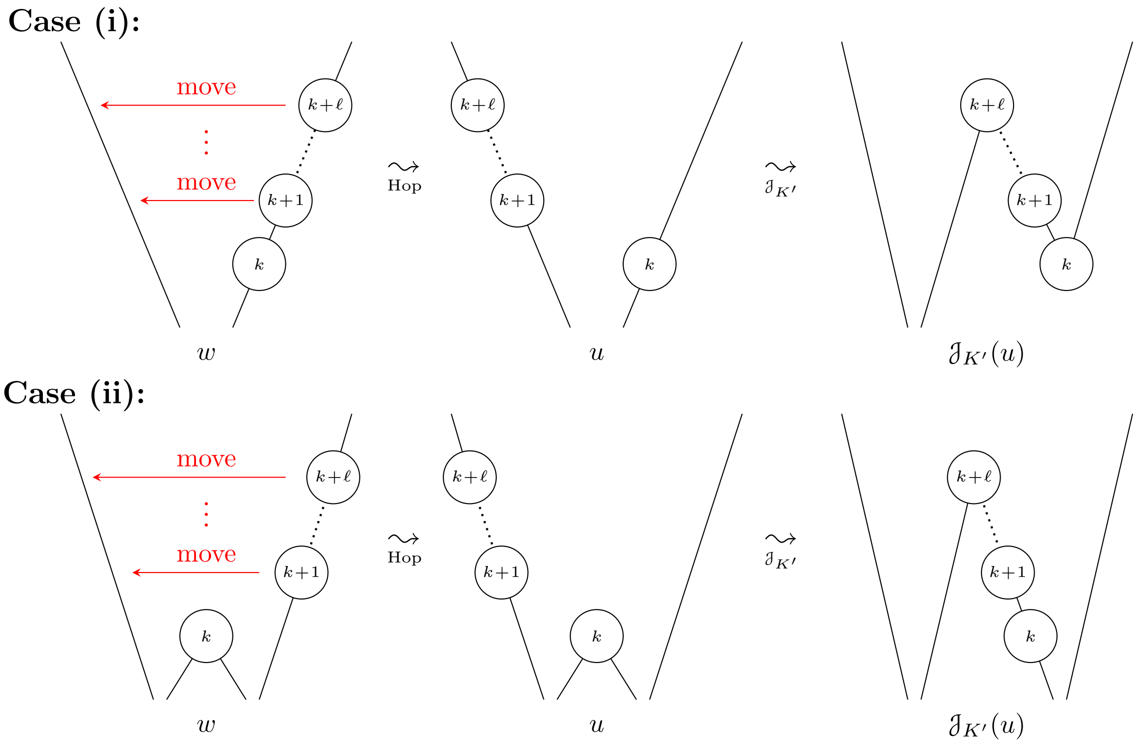


Figure 4: Two cases of the effective expression $\eth_{K'}(u)$.

Proposition 15. *Let K be an arbitrary subset of $[n-1]$. Let $w \in \widetilde{W^K}$ and $u \in \mathrm{Hop}(w)$. Then we have*

$$\mathrm{des}(\eth_K(u)) = \mathrm{des}(u).$$

Proof. It suffices to show that $\mathrm{des}(\eth_{K_m}\cdots\eth_{K_1}(u)) = \mathrm{des}(u)$ for the effective expression for $\eth_K(u)$. First we consider the case where the effective expression of $\eth_K(u)$ consists of only one closed interval, say $K' = [k, k+\ell-1]$.

For $w \in \widetilde{W^{K'}}$ and $u \in \mathrm{Hop}(w)$, due to the description (4) and Definition 11 together with the assumption on the effective expression, we have the two cases for the positions of the letters $k, k+1, \ldots, k+\ell$ in the mountain range descriptions for w, u

and $\mathfrak{J}_{K'}(u)$. See Figure 4. Observe in both cases $\mathfrak{J}_{K'}$ does not change the descent number, i.e., $\operatorname{des}(\mathfrak{J}_{K'}(u)) = \operatorname{des}(u)$.

If the effective expression consists of several closed intervals, say $K_1, \ldots, K_m$ as in (12) with $u_1 \coloneqq u$ and $u_j \coloneqq \mathfrak{J}_{K_{j-1}} \cdots \mathfrak{J}_{K_1}(u)$ for $2 \leqslant j \leqslant m$, then we consider $w_j \in \widetilde{W^{K_j}}$ obtained by moving all free letters of u_j on a downslope by crossing valleys. This implies that $u_j \in \operatorname{Hop}(w_j)$, so that we apply the previous result repeatedly to conclude

$$\operatorname{des}(\mathfrak{J}_K(u)) = \operatorname{des}(\mathfrak{J}_{K_m}(u_m)) = \operatorname{des}(u_m) = \operatorname{des}(\mathfrak{J}_{K_{m-1}}(u_{m-1})) = \cdots = \operatorname{des}(u_1) = \operatorname{des}(u).$$

□

6 Inverse operator

In this section we prove the second claim of Proposition 12, that the map Θ of (11) is bijective, by constructing its inverse explicitly.

Regarding the closed intervals $K_j = [k_j, k_j + \ell_j - 1]$ for $i = 1, \ldots, m$ in the effective expression (12) of $\mathfrak{J}_K(u)$, we first observe that

- $k_j + \ell_j$ is a peak of $\mathfrak{J}_K(u)$, and
- k_j is either a valley or a free letter on a downslope (see cases (i) and (ii) of Figure 4, respectivley).

We also have the following lemma.

Lemma 16. *Let K, w and u be as above. If $(\mathfrak{J}_K(u))^{-1}(k_j - 1) = (\mathfrak{J}_K(u))^{-1}(k_j) + 1$ (which happens only in Case (ii) of Figure 4), then $k_j - 1 \notin K$.*

Proof. If $k_j - 1 \in K$, then we have $u^{-1}(k_j - 1) = u^{-1}(k_j) + 1$. Indeed, if we write $\mathfrak{J}_K(u) = \mathfrak{J}_{K_m} \cdots \mathfrak{J}_{K_1}(u)$ as the effective expression discussed in (12), then the assumption $(\mathfrak{J}_K(u))^{-1}(k_j - 1) = (\mathfrak{J}_K(u))^{-1}(k_j) + 1$ implies that $k_j - 1$ does not belong to any closed interval K'_i. Since $w \in \operatorname{Hop}(u)$, one has $w^{-1}(k_j) < w^{-1}(k_j - 1)$, which contradicts $w \in \widetilde{W^K}$ (see (4)). Hence, we conclude $k_j - 1 \notin K$. □

Based on these facts, we now construct a map $\mathfrak{L}_K \colon W(K) \to \mathfrak{S}_n$. For $K \subset [n-1]$ and $v \in W(K)$, we denote by

$$(k + \ell; k + \ell - 1, \ldots, k) \tag{13}$$

a consecutive string in K consisting of a peak $k + \ell$ of v and letters $k + \ell - 1, \ldots, k$ immediately followed by $k + \ell$ such that $\{k + \ell - 1, \ldots, k\} \subset K$ and $k - 1 \notin K$ whenever $v^{-1}(k-1) - v^{-1}(k) = 1$.

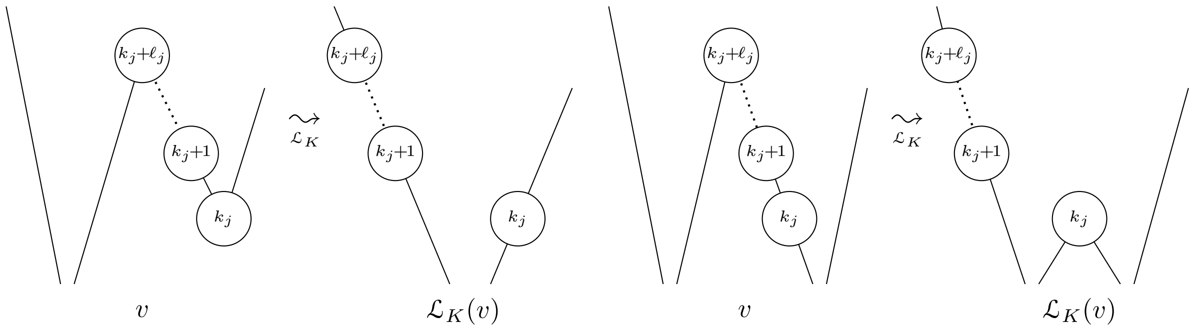
Definition 17. For $v \in W(K)$, consider all strings $(k_j + \ell_j; k_j + \ell_j - 1, \ldots, k_j)$ in v of the form (13). If v has no such a string, then we define $\mathfrak{L}_K(v) = v$. Otherwise, assuming $1 \leqslant j \leqslant m$ and $k_1 < k_1 + \ell_1 < k_2 < k_2 + \ell_2 < \cdots < k_m < k_m + \ell_m$ without loss of generality, we define $\mathfrak{L}_K(v)$ inductively as follows.

1. $v_{m+1} \coloneqq v$.
2. Given v_{j+1} with $j \leqslant m$, let $c_j \coloneqq v_{j+1}^{-1}(k_j + \ell_j)$ and a_j the largest integer such that $a_j < c_j$ and $v_{j+1}(a_j) > k_j + \ell_j$. We define v_j by the following one-line notation:

$$v_{j+1}(1)\cdots v_{j+1}(a_j)\ k_j+\ell_j\ \ k_j+\ell_j-1\ \ \cdots\ \ k_j+1$$
$$v_{j+1}(a_j+1)\cdots v_{j+1}(c_j-1)\ \ k_j\ \ v_{j+1}(c_j+\ell_j+1)\cdots v_{j+1}(n),$$

where we omit the first part $v_{j+1}(1)\cdots v_{j+1}(a_j)$ if $a_j = 0$.
3. We define $\mathcal{L}_K(v) \coloneqq v_1$.

We describe $\mathcal{L}_K(v)$ pictorially in Figure 5. For instance, regarding $v = 14325 = \mathcal{J}_K(u) \in W(K) \subset \mathfrak{S}_5$ for $u = 41325$ and $K = \{1, 3\}$ as in Example 13, we have only one string $(4; 3)$ of (13) which is the case of Figure 5-(B). Hence we have $\mathcal{L}_K(v) = 41325$, and observe that $\mathcal{J}_K(\mathcal{L}_K(v)) = v$. Indeed, we have the following proposition.



(A) k_j is a valley in v.

(B) k_j is a free letter on a downslope in v.

Figure 5: $\mathcal{L}_K(v)$ in two cases.

Proposition 18. *Let K be a subset of $[n-1]$.*

1. *For $v \in W(K)$, the permutation $\mathcal{L}_K(v) \in \mathfrak{S}_n$ determines the unique permutation $w \in \widetilde{W^K}$ such that $\mathcal{L}_K(v) \in \mathrm{Hop}(w)$.*
2. *The map*

$$W(K) \to \{(w,u) \in \widetilde{W^K} \times \mathfrak{S}_n \mid u \in \mathrm{Hop}(w)\}$$

defined by $v \mapsto (w, \mathcal{L}_K(v))$, where w is the element determined by (1)*, is the inverse of Θ.*

Proof. (1) Let w be the canonical representative obtained from $\mathcal{L}_K(v)$ by moving free letters on a downslope to on an upslope by valley hopping. Then, w is the unique element of $\widetilde{\mathfrak{S}_n}$ such that $\mathcal{L}_K(v) \in \mathrm{Hop}(w)$. To show that $w \in \widetilde{W^K}$, we verify that

$$w^{-1}(k) - w^{-1}(k+1) < 0 \tag{14}$$

for each $k \in K$ by (4). Since v is an element of $W(K)$, we have $v^{-1}(k) - v^{-1}(k+1) \leqslant 1$ following (9). If $v^{-1}(k) - v^{-1}(k+1) < 0$, then (14) follows from the fact that $\mathcal{L}_K$ and the construction of w do not affect the positions of k and $k+1$. For the case of $v^{-1}(k) - v^{-1}(k+1) = 1$, the claim (14) follows from the definition of $\mathcal{L}_K$ and the construction of w as $k+1$ is a free letter.

(2) We first assert that $\mathcal{L}_K\big(\mathfrak{J}_K(u)\big) = u$ for $w \in \widetilde{W^K}$ and $u \in \mathrm{Hop}(w)$. Consider first the case $\mathfrak{J}_K(u) = u$. If p is a peak of u with $u^{-1}(p-1) = u^{-1}(p)+1$, then we have $p - 1 \notin K$ by a similar argument to that in the proof of Lemma 16 because the permutation $w \in \mathrm{Hop}(u)$ belongs to $\widetilde{W^K}$. Hence we have $\mathcal{L}_K(u) = u$ by Definition 17, which implies $\mathcal{L}_K\big(\mathfrak{J}_K(u)\big) = \mathcal{L}_K(u) = u$.

For the case $\mathfrak{J}_K(u) \neq u$, consider the effective expression $\mathfrak{J}_K(u) = \mathfrak{J}_{K_m} \cdots \mathfrak{J}_{K_1}(u)$ as in (12) with intervals $K_j = [k_j, k_j + \ell_j - 1]$. It follows from Lemma 16 together with the definition of the effective expression that $(k_j + \ell_j; k_j + \ell_j - 1, \ldots, k_j)$ for $1 \leqslant j \leqslant m$ are strings of (13) in $\mathfrak{J}_K(u)$. Moreover, by an argument similar to that in the previous case, one can see that any string (13) in $\mathfrak{J}_K(u)$ is $(k_j + \ell_j; k_j + \ell_j - 1, \ldots, k_j)$ for some $1 \leqslant j \leqslant m$. Then, putting $v = \mathfrak{J}_K(u)$, we get $v_j = u_j$ for all $1 \leqslant j \leqslant m$ by Definition 17. Hence we have $\mathcal{L}_K\big(\mathfrak{J}_K(u)\big) = \mathcal{L}_K(v) = v_1 = u_1 = u$.

Secondly, we claim that $\mathfrak{J}_K(\mathcal{L}_K(v)) = v$ for $v \in W(K)$. Let

$$\{(k_j + \ell_j; k_j + \ell_j - 1, \ldots, k_j) \mid 1 \leqslant j \leqslant m\}$$

be the set of strings in K corresponding to v as defined in (13). Without loss of generality, we may assume that $k_1 < k_1 + \ell_1 < k_2 < k_2 + \ell_2 < \cdots < k_m < k_m + \ell_m$. Then, following Definition 17, we have a sequence of permutations

$$v = v_{m+1}, v_m, v_{m-1}, \ldots, v_1 = \mathcal{L}_K(v).$$

Also, we have $w \in \widetilde{W^K}$ with $\mathcal{L}_K(v) \in \mathrm{Hop}(w)$ by the result of (1).

Writing $u \coloneqq \mathcal{L}_K(v)$ and $K_j = [k_j, k_j + \ell_j - 1]$ for each $j = 1, \ldots, m$, one can show that

$$\mathfrak{J}_K(u) = \mathfrak{J}_{K_m} \cdots \mathfrak{J}_{K_1}(u). \tag{15}$$

Indeed, if $\mathfrak{J}_K(u) \neq \mathfrak{J}_{K_m} \cdots \mathfrak{J}_{K_1}(u)$, then there exists $k \in K$ with $k_{j-1} + \ell_{j-1} < k < k_j$ for some $j \in \{1, \ldots, m+1\}$ such that $\mathfrak{J}_k(u_j) \neq u_j$ for $u_j = \mathfrak{J}_{K_{j-1}} \cdots \mathfrak{J}_{K_1}(u)$. (Here, the inequality $k_{j-1} + \ell_{j-1} < k < k_j$ means $k < k_1$ for $j = 1$ and $k_m + \ell_m < k$ for $j = m+1$, respectively.) This implies that the mountain range description of $u \in \mathrm{Hop}(w)$ is one of the following two.

By the definition of $u = \mathcal{L}_K(v)$, the relative positions of k and $k+1$ remain unchanged in the mountain range description of v (see Figure 5). So, v satisfies $v^{-1}(k) - v^{-1}(k+1) > 1$, which contradicts $v \in W(K)$. Thus, (15) holds, and we conclude that

$$v = v_{m+1} = \mathfrak{J}_{K_m}(v_m) = \mathfrak{J}_{K_m}\mathfrak{J}_{K_{m-1}}(v_{m-1}) = \cdots = \mathfrak{J}_{K_m}\cdots\mathfrak{J}_{K_1}(v_1)$$
$$= \mathfrak{J}_{K_m}\cdots\mathfrak{J}_{K_1}(u) = \mathfrak{J}_K(u) = \mathfrak{J}_K(\mathcal{L}_K(v)). \quad \square$$

Proof of Proposition 12. The claim follows immediately from Propositions 15 and 18. $\square$

7 Proof of Theorem 6

Recall from (6) that we have

$$\sum_{u \in \mathrm{Hop}(w)} t^{\mathrm{des}(u)} = t^{\mathrm{pk}(w)}(1+t)^{n-1-2\,\mathrm{pk}(w)}$$

for any $w \in \mathfrak{S}_n$. By summing the equality above over all $w \in \widetilde{W^K}$, we obtain

$$\sum_{w \in \widetilde{W^K}} \sum_{u \in \mathrm{Hop}(w)} t^{\mathrm{des}(u)} = \sum_{w \in \widetilde{W^K}} t^{\mathrm{pk}(w)}(1+t)^{n-1-2\,\mathrm{pk}(w)} = \sum_{w \in \widetilde{W^K}} t^{\mathrm{des}(w)}(1+t)^{n-1-2\,\mathrm{des}(w)}. \tag{16}$$

Here, the last equality follows because $\mathrm{pk}(w) = \mathrm{des}(w)$ for all $w \in \widetilde{\mathfrak{S}_n}$. By Propositions 4 and 12 the leftmost term of (16) is equal to

$$\sum_{v \in W(K)} t^{\mathrm{des}(v)} = h_{P_n(K)}(t).$$

Therefore, we conclude that

$$h_{P_n(K)}(t) = \sum_{w \in \widetilde{W^K}} t^{\mathrm{des}(w)}(1+t)^{n-1-2\,\mathrm{des}(w)}. \tag{17}$$

It remains to prove the equality

$$|\{w \in \widetilde{W^K} \mid \mathrm{des}(w) = j\}| = |\{w \in \widehat{W^K} \mid \mathrm{des}(w) = j\}|. \tag{18}$$

We set

$$K^* := \{n - k \mid k \in K\},$$

and for $w \in \mathfrak{S}_n$ let w^* be the permutation satisfying $w^*(i) = n + 1 - w(n + 1 - i)$ for all $i \in [n]$. One can confirm that $\mathrm{Des}(w^*) = \{n - i : i \in \mathrm{Des}(w)\}$ and that

$$\widetilde{W^{K^*}} = \{w^* \mid w \in \widehat{W^K}\}.$$

It follows now from (17) that

$$h_{P_n(K^*)}(t) = \sum_{w \in \widehat{W^K}} t^{\mathrm{des}(w)}(1+t)^{n-1-2\,\mathrm{des}(w)}.$$

The equality (18) will follow once we prove that $P_n(K^*)$ and $P_n(K)$ are isomorphic. To do this, we observe that we may assume that P_n is the convex hull of $\{w(p) : w \in \mathfrak{S}_n\}$, where $p = (p_1, \ldots, p_n)$ satisfies $p_{n+1-i} = -p_i$ for all $i \in [n]$. Under this assumption, it is straightforward to show that if w_0 is the permutation matrix mapping the standard basis vector e_i to e_{n+1-i} for all $i \in [n]$, then $P_n(K) = -w_0(P_n(K^*))$. The claim follows. This completes the proof of Theorem 6.

8 Athanasiadis' result

The aim of this section is to compare our main result (Theorem 6) with Athanasiadis' result [1, Corollary 2.41] on gamma vectors of permutohedra. We adhere to the notation of Section 2 and assume the reader is familiar with material in [23, Chapter 7] or [10, Parts I and II].

For a positive integer n, we write $\mathcal{SYT}_n$ and $\mathcal{SSYT}_n$ for the set of all standard Young tableaux and the set of all semistandard Young tableaux, respectively, whose underlying Young diagram has n squares. Given $T \in \mathcal{SYT}_n$, a *descent* of T is any $i \in [n-1]$ such that i appears in a higher row of T than $i+1$ does. We write $\operatorname{Des}(T)$ for the set of descents of T and $\operatorname{des}(T)$ for $|\operatorname{Des}(T)|$. Double, initial and final descents are defined as they were for words (see Section 2.1). We write $\widetilde{\mathcal{SYT}_n}$ (resp. $\widehat{\mathcal{SYT}_n}$) for the subset of $\mathcal{SYT}_n$ consisting of tableaux with no double descent and no initial descent (resp. no double descent and no final descent). In what follows, given an arbitrary semistandard Young tableau T, we write $\lambda(T)$ for the partition from which the Young diagram for T is obtained, and $c(T) = (c_1, c_2, \dots)$ for its content, namely the sequence $(c_1, c_2, \dots)$ such that c_i is the number of entries of T equal to i.

8.1 The Robinson–Schensted–Knuth (RSK) algorithm

We will not describe the RSK algorithm here. Rather, in addition to providing an example below, we will describe its key consequences. The interested reader can find details in [23, Chapter 7]. With the notation used in (7.38) on page 318 of [23], we assume at all times that $i_k = k$ for all $k \in [m]$. Under this assumption, the algorithm produces from a word v of length n a pair $(P_v, Q_v) \in \mathcal{SSYT}_n \times \mathcal{SYT}_n$ such that P_v and Q_v are obtained from the same partition and have two other essential properties.

Proposition 19. *For every positive integer n, there is a bijection*

$$\mathbf{RSK} : \mathbb{Z}^n_{>0} \to \{(P, Q) \in \mathcal{SSYT}_n \times \mathcal{SYT}_n \mid \lambda(P) = \lambda(Q)\}$$

such that given $\mathbf{RSK}(v) = (P_v, Q_v)$, both

(A) if P_v has content $c(P_v) = (c_1, c_2, \dots)$, then $c_i = m_i(v)$ for all i, and

(B) $\operatorname{Des}(Q_v) = \operatorname{Des}(v)$

hold.

For example, applying RSK to the word $v = 23132$, we obtain the following sequence of pairs of tableaux, terminating with (P_v, Q_v).

$$\left(\begin{array}{|c|}\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}\right) \to \left(\begin{array}{|c|c|}\hline 2&3\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\right) \to \left(\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right) \to \left(\begin{array}{|c|c|c|}\hline 1&3&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}\right) \to \left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 2&3\\\cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}\right).$$

We observe that $Q_v \in \widetilde{\mathcal{SYT}_n}$ (resp. $Q_v \in \widehat{\mathcal{SYT}_n}$) if and only if $v \in \widetilde{\mathbb{Z}^n_{>0}}$ (resp. $v \in \widehat{\mathbb{Z}^n_{>0}}$) and that $P_v \in \mathcal{SYT}_n$ if and only if $v \in \mathfrak{S}_n$. The fact that (A) holds follows immediately from the definition of the RSK algorithm. The fact that (B) holds follows from the discussion following (7.43) on page 321 along with Lemma 7.23.1 in [23].

8.2 Modules for $\mathbb{C}[\mathfrak{S}_n]$ and Kostka numbers

The simple modules of $\mathbb{C}[\mathfrak{S}_n]$ are parameterized by partitions of n. We write S^λ for the module associated to the partition λ. Every $\mathbb{C}[\mathfrak{S}_n]$-module is a direct sum of simple modules, and each S^λ is a summand in some permutation module M^μ. Here $\mu = \mu(K)$ for some $K \subseteq [n-1]$, and M^μ has a $\mathbb{C}$-basis indexed by the cosets of W_K in $\mathfrak{S}_n$. The elements of $\mathfrak{S}_n$ permute this basis as they permute the cosets by translation. The *Kostka number* $\mathcal{K}_{\lambda,\mu}$ is the number of semistandard Young tableaux of shape λ and content μ. The first equality in the next theorem is well-known (see for example [23, Theorem 7.18.7]) and the second follows directly from Frobenius reciprocity (see for example [14, Lemma 5.2]).

Theorem 20. *The Kostka number $\mathcal{K}_{\lambda,\mu}$ is equal to the number of summands isomorphic to S^λ in a direct sum decomposition of M^μ into simple submodules, which is equal in turn to the dimension $\dim(S^\lambda)^{W_K}$ of the subspace of S^λ consisting of vectors fixed by a parabolic subgroup W_K satisfying $\mu(K) = \mu$.*

8.3 Graded $\mathfrak{S}_n$-representation formula for gamma vectors

Consider the permutohedral variety X_n, which is the toric variety associated with the normal fan Δ_n of a permutohedron P_n. Since the fan Δ_n is invariant under the reflection action of $\mathfrak{S}_n$, this induces an action of $\mathfrak{S}_n$ on X_n, which determines a representation of $\mathfrak{S}_n$ on $H^*(X_n;\mathbb{C})$. This representation is palindromic as noted in [22, Corollary 4.4] and [24, Proposition 12], which implies that there exist virtual modules $R_{n,j}$ ($0 \leqslant j \leqslant \lfloor \frac{n-1}{2} \rfloor$) such that

$$\sum_{k=0}^{n-1} H^{2k}(X_n;\mathbb{C})t^k \cong_{\mathfrak{S}_n} \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} R_{n,j} t^j (1+t)^{n-1-2j},$$

where $\cong_{\mathfrak{S}_n}$ denotes isomorphism of $\mathbb{C}[\mathfrak{S}_n]$-modules. Athanasiadis proved the following result.

Theorem 21. *([1, Corollary 2.41]) For $0 \leqslant j \leqslant \lfloor \frac{n-1}{2} \rfloor$,*

$$R_{n,j} \cong_{\mathfrak{S}_n} \sum_{\substack{Q \in \widehat{\mathcal{SYT}_n} \\ \mathrm{des}(Q)=j}} S^{\lambda(Q)} = \sum_{\substack{Q \in \widetilde{\mathcal{SYT}_n} \\ \mathrm{des}(Q)=j}} S^{\lambda(Q)}.$$

Equivalently,

$$\begin{aligned}\sum_{k=0}^{n-1} H^{2k}(X_n;\mathbb{C})t^k &\cong_{\mathfrak{S}_n} \sum_{Q \in \widehat{\mathcal{SYT}_n}} S^{\lambda(Q)} t^{\mathrm{des}(Q)} (1+t)^{n-1-2\,\mathrm{des}(Q)} \\ &= \sum_{Q \in \widetilde{\mathcal{SYT}_n}} S^{\lambda(Q)} t^{\mathrm{des}(Q)} (1+t)^{n-1-2\,\mathrm{des}(Q)}.\end{aligned}$$

8.4 A relation between our theorem and Athanasiadis' result

The aim of this section is to explain how to derive each of Theorems 6 and 21 assuming the other. We begin with a result from [13]. For a subset $K \subseteq [n-1]$, we write $X_n(K)$ for the toric orbifold associated with the partitioned permutohedron $P_n(K)$.

Theorem 22 ([13], Theorem 1.1)**.** *For any subset $K \subseteq [n-1]$, the cohomology ring $H^*(X_n(K); \mathbb{C})$ is isomorphic to the ring $H^*(X_n; \mathbb{C})^{W_K}$ of W_K-fixed elements of $H^*(X_n; \mathbb{C})$. In particular, we obtain*

$$h_{P_n(K)}(t) = \sum_{\ell=0}^{n-1} \dim H^{2\ell}(X_n(K); \mathbb{C}) t^\ell = \sum_{\ell=0}^{n-1} \dim(H^{2\ell}(X_n; \mathbb{C})^{W_K}) t^\ell. \tag{19}$$

For $K \subseteq [n-1]$, write $\mu(K) = (\mu_1, \ldots, \mu_m)$ for the composition of n defined to be $(|O_1|, \ldots, |O_{n-|K|}|)$ where $O_1, \ldots, O_{n-|K|}$ are W_K-orbits in $[n]$. For example, if $n = 5$ and $K = \{1, 3\}$, we have $\mu(K) = (2, 2, 1)$. Then, we define

$$\widehat{[m]}^n(K) \coloneqq \{v \in \widehat{\mathbb{Z}^n_{>0}} \mid m_i(v) = \mu_i \text{ for all } i \in [m]\}.$$

So, the set $\widehat{[m]}^n(K)$ consists of all words v of length n with entries in $[m]$ such that v has no double descent, no final descent, and μ_i letters equal to i for each $i \in [m]$ (and no letter greater than m). We observe that **RSK** maps $\widehat{[m]}^n(K)$ onto the set of pairs (P, Q) from $\mathcal{SSYT}_n \times \widehat{\mathcal{SYT}_n}$ such that $\lambda(P) = \lambda(Q)$ and P has content $\mu(K)$. Therefore,

$$\sum_{v \in \widehat{[m]}^n(K)} t^{\operatorname{des}(v)}(1+t)^{n-1-2\operatorname{des}(v)} = \sum_{Q \in \widehat{\mathcal{SYT}_n}} \mathcal{K}_{\lambda(Q), \mu(K)} t^{\operatorname{des}(Q)}(1+t)^{n-1-2\operatorname{des}(Q)}. \tag{20}$$

Here, one can see that there is a descent-preserving bijection from $\widehat{[m]}^n(K)$ to $\widehat{W^K}$. Given $v \in \widehat{[m]}^n(K)$ and $i \in [m]$, define

$$Y_i(v) \coloneqq \{j \in [n] \mid v(j) = i\}$$

and let $\phi(v) \in \mathfrak{S}_n$ be the permutation such that if j is the k^{th} smallest element of $Y_i(v)$ then

$$\phi(v)_j = k + \sum_{\ell=1}^{i-1} \mu_\ell.$$

For instance, an element $v = 23112 \in \widehat{[3]}^5(K)$ with $K = \{1, 3\}$ gives $\phi(v) = 35124 \in \widehat{W^K}$.

It is straightforward to confirm that ϕ is the desired bijection. Hence, the left hand side of (20) is described as $\sum_{w \in \widehat{W^K}} t^{\operatorname{des}(w)}(1+t)^{n-1-2\operatorname{des}(w)}$. On the other hand, the right hand side of (20) is $\sum_{Q \in \widehat{\mathcal{SYT}_n}} \dim(S^{\lambda(Q)})^{W_K} t^{\operatorname{des}(Q)}(1+t)^{n-1-2\operatorname{des}(Q)}$ by Theorem 20. Thus, we can rewrite (20) as

$$\sum_{w \in \widehat{W^K}} t^{\operatorname{des}(w)}(1+t)^{n-1-2\operatorname{des}(w)} = \sum_{Q \in \widehat{\mathcal{SYT}_n}} \dim(S^{\lambda(Q)})^{W_K} t^{\operatorname{des}(Q)}(1+t)^{n-1-2\operatorname{des}(Q)}. \tag{21}$$

Therefore, we conclude from (19) and (21) that the equation

$$h_{P_n(K)}(t) = \sum_{w \in \widehat{W^K}} t^{\operatorname{des}(w)}(1+t)^{n-1-2\operatorname{des}(w)}$$

is equivalent to

$$\sum_{\ell=0}^{n-1} \dim(H^{2\ell}(X_n;\mathbb{C})^{W_K})t^\ell = \sum_{Q \in \widehat{\mathcal{SYT}_n}} \dim(S^{\lambda(Q)})^{W_K} t^{\operatorname{des}(Q)}(1+t)^{n-1-2\operatorname{des}(Q)}.$$

If we pick for each partition ν of n some K_ν such that $\mathsf{par}(\mu(K_\nu)) = \nu$, then any $\mathbb{C}[\mathfrak{S}_n]$-module M is determined up to isomorphism by the dimensions of all of the spaces of W_{K_ν}-invariant vectors. (This is well-known and follows from, for example [23, Propositions 7.10.5 and 7.18.7].) Now we know that Theorem 6 follows from Theorem 21 and vice versa.

Acknowledgements

The authors are grateful to Christos A. Athanasiadis for his helpful comments on an earlier version of this paper. We also thank the anonymous referees for their valuable suggestions that helped improve the presentation of the paper. This research was partly supported by Osaka Central Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics). The first author is supported in part by JSPS Grant-in-Aid for Young Scientists: 19K14508. The second author was supported in part by Grant-in-Aid for Scientific Research 22K03292 and the HSE University Basic Research Program. The fourth author was supported in part by National Science Foundation grant DMS-1518389. The last named author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (RS-2025-00555914).

References

[1] C. A. Athanasiadis, Gamma-positivity in combinatorics and geometry, *Sém. Lothar. Combin.* 77 ([2016–2018]), Art. B77i, 64 pp.

[2] A. Björner and F. Brenti, *Combinatorics of Coxeter Groups*, Graduate Texts in Mathematics 231, Springer, New York, 2005.

[3] G. Burrull, G. Tao, and H. Hongsheng. Strongly dominant weight polytopes are cubes, arXiv:2311.16022 (2023).

[4] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011.

[5] V. I. Danilov. The geometry of toric varieties, *Uspekhi Mat. Nauk* 33 (1978), no. 2, 85–134.

[6] I. Dolgachev and V. Lunts, A character formula for the representation of a Weyl group in the cohomology of the associated toric variety, *J. Algebra* 168 (1994), no. 3, 741–772.

[7] D. Foata and M. P. Schützenberger, Théorie géométrique des polynômes eulériens, Lecture Notes in Math., Vol. 138 Springer-Verlag, Berlin-New York, 1970, v+94 pp.

[8] D. Foata and V. Strehl, Rearrangements of the symmetric group and enumerative properties of the tangent and secant numbers, *Math. Z.* 137 (1974), 257–264.

[9] W. Fulton, *Introduction to toric varieties.* No. 131. Princeton university press, 1993.

[10] W. Fulton, *Young tableaux. With applications to representation theory and geometry.* London Mathematical Society Student Texts, 35. Cambridge University Press, Cambridge, 1997.

[11] Ś. R. Gal, Real root conjecture fails for five- and higher-dimensional spheres, *Discrete Comput. Geom.* 34 (2005), no. 2, 269–284.

[12] B. Grünbaum, *Convex polytopes.* Grad. Texts in Math., 221, Springer-Verlag, New York, 2003. xvi+468 pp.

[13] T. Horiguchi, M. Masuda, J. Shareshian, and J. Song, Toric orbifolds associated with partitioned weight polytopes in classical types, *Selecta Math. (N.S.)* 30 (2024), no. 5, Paper No. 84.

[14] I. M. Isaacs, *Character Theory of Finite Groups*, Dover, Mineola, NY, 1994.

[15] T. Oda, *Convex bodies and algebraic geometry*, Ergeb. Math. Grenzgeb. (3), 15, Springer-Verlag, Berlin, 1988, viii+212 pp.

[16] T. K. Petersen, *Eulerian Numbers*, Birkhäuser Adv. Texts Basler Lehrbücher, Birkhäuser/Springer, New York, 2015, xviii+456 pp.

[17] A. Postnikov, Permutohedra, Associahedra, and Beyond, *Int. Math. Res. Not.* IMRN, 2009(6), 1026–1106.

[18] A. Postnikov, V. Reiner, L. Williams, Faces of generalized permutohedra. *Doc. Math.* 13 (2008), 207–273.

[19] M. Precup, Affine pavings of Hessenberg varieties for semisimple groups, *Selecta Math. (N.S.)* 19 (2013), no. 4, 903–922.

[20] C. Procesi, The toric variety associated to Weyl chambers, *Lang. Raison. Calc.*, Editions Hermès, Paris, 1990, 153–161.

[21] J. Shareshian and M. L. Wachs, Chromatic quasisymmetric functions, *Adv. Math.* 295 (2016), 497–551.

[22] R. P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, *Ann. New York Acad. Sci.* 576, New York Academy of Sciences, New York, 1989, 500–535.

[23] R. P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Stud. Adv. Math., 208, Cambridge University Press, Cambridge, 2024, xvi+783 pp.

[24] J. Stembridge, Eulerian numbers, tableaux, and the Betti numbers of a toric variety, *Discrete Math.* 99 (1992), no. 1-3, 307–320.

[25] J. Stembridge, Some permutation representations of Weyl groups associated with the cohomology of toric varieties, *Adv. Math.* 106 (1994), no. 2, 244–301.