q-deformation of chromatic polynomials and graphical arrangements

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Abstract

We first observe a mysterious similarity between the braid arrangement and the arrangement of all hyperplanes in a vector space over the finite field $\mathbb{F}_q$. These two arrangements are defined by the determinants of the Vandermonde and the Moore matrix, respectively. These two matrices are transformed to each other by replacing a natural number n with q^n (q-deformation).

In this paper, we introduce the notion of "q-deformation of graphical arrangements" as certain subarrangements of the arrangement of all hyperplanes over $\mathbb{F}_q$. This new class of arrangements extends the relationship between the Vandermonde and Moore matrices to graphical arrangements. We show that many invariants of the "q-deformation" behave as "q-deformations" of invariants of the graphical arrangements. Such invariants include the characteristic (chromatic) polynomial, the Stirling number of the second kind, freeness, exponents, basis of logarithmic vector fields, etc.

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1 Introduction

It is classically known that there are similarities between subsets of $[n] = \{1, \ldots, n\}$ and linear subspaces of $\mathbb{F}_q^n$, which is sometimes called the "q-analogue" [7]. We start with pointing out further similarities between chromatic polynomials for graphs and characteristic polynomials for hyperplane arrangements over finite fields.

A central arrangement $\mathcal{A}$ is a finite collection of linear hyperplanes in a finite dimensional vector space. Define the **intersection lattice** $L(\mathcal{A})$ and the **characteristic**

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polynomial $\chi(\mathcal{A},t)$ by

$$L(\mathcal{A}) \coloneqq \left\{ \bigcap_{H\in\mathcal{B}} H \;\middle|\; \mathcal{B}\subseteq\mathcal{A} \right\}, \qquad \chi(\mathcal{A},t) \coloneqq \sum_{X\in L(\mathcal{A})} \mu(X) t^{\dim X},$$

where $L(\mathcal{A})$ is ordered by the reverse inclusion and μ denotes the **Möbius function** on the lattice $L(\mathcal{A})$. See [8] for details.

A typical example of an arrangement is the **braid arrangement** $\mathcal{B}_\ell$ in $\mathbb{R}^\ell$ whose defining polynomial is the **Vandermonde determinant**, i.e.,

$$Q(\mathcal{B}_\ell) = \prod_{1\leqslant i<j\leqslant \ell} (x_j - x_i) = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{\ell-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{\ell-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_\ell & x_\ell^2 & \cdots & x_\ell^{\ell-1} \end{vmatrix}.$$

The characteristic polynomial of the braid arrangement $\mathcal{B}_\ell$ is

$$\chi(\mathcal{B}_\ell,t) = t(t-1)(t-2)\cdots(t-\ell+1).$$

There are mysterious similarities between the braid arrangements and the arrangements consisting of all hyperplanes in vector spaces over finite fields. Let q be a prime power and $\mathbb{F}_q$ the finite field of order q. Define the arrangement $\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)$ as the set of all hyperplanes in $\mathbb{F}_q^\ell$. Its defining polynomial is the determinant of the **Moore matrix**, i.e.,

$$Q\left(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)\right) = \prod_{i=1}^{\ell} \prod_{c_1,\dots,c_{i-1}\in\mathbb{F}_q} (c_1x_1 + \cdots + c_{i-1}x_{i-1} + x_i) = \begin{vmatrix} x_1 & x_1^q & x_1^{q^2} & \cdots & x_1^{q^{\ell-1}} \\ x_2 & x_2^q & x_2^{q^2} & \cdots & x_2^{q^{\ell-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ x_\ell & x_\ell^q & x_\ell^{q^2} & \cdots & x_\ell^{q^{\ell-1}} \end{vmatrix}.$$

The characteristic polynomial of $\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)$ is

$$\chi\left(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell),t\right) = (t-1)(t-q)(t-q^2)\cdots(t-q^{\ell-1}).$$

By formally replacing q^k in the expressions of $Q\left(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)\right)$ and $\chi\left(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell),t\right)$ with k, we obtain the expressions for $Q(\mathcal{B}_\ell)$ and $\chi(\mathcal{B}_\ell,t)$. Note that $\chi(\mathcal{B}_\ell,\ell) = \ell! = |\mathfrak{S}_\ell|$ and $\chi\left(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell),q^\ell\right) = (q^\ell-1)(q^\ell-q)\cdots(q^\ell-q^{\ell-1}) = |\operatorname{GL}_\ell(\mathbb{F}_q)|$. It is worth mentioning that the permutation group $\mathfrak{S}_\ell$ is considered as the "$\mathbb{F}_1$-version" of the general linear group $\operatorname{GL}_\ell(\mathbb{F}_q)$ [11].

A similar phenomenon can be observed in the context of the freeness of arrangements. Let $\mathcal{A} = \{H_1,\dots,H_n\}$ be an arrangement of hyperplanes in $V=\mathbb{K}^\ell$. Let $\alpha_i : V \to \mathbb{K}$ be a linear form such that $H_i = \alpha_i^{-1}(0)$. The module of logarithmic polynomial vector fields $D(\mathcal{A})$ is defined as

$$D(\mathcal{A}) = \left\{ \theta = \sum_{i=1}^{\ell} f_i\partial_i \;\middle|\; \theta\alpha_i \in (\alpha_i),\ 1\leqslant i\leqslant n \right\},$$

where $f_i \in S = \mathbb{K}[x_1, \dots, x_\ell]$ and $\partial_i = \frac{\partial}{\partial x_i}$. An arrangement $\mathcal{A}$ is said to be **free** if $D(\mathcal{A})$ is a free module over the polynomial ring (See [8] for details). Both of $\mathcal{B}_\ell$ and $\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)$ are free with bases

$$\left\{ \sum_{i=1}^{\ell} x_i^k \partial_i \;\middle|\; 0 \leqslant k \leqslant \ell - 1 \right\} \text{ for } D(\mathcal{B}_\ell) \text{ and}$$
$$\left\{ \sum_{i=1}^{\ell} x_i^{q^k} \partial_i \;\middle|\; 0 \leqslant k \leqslant \ell - 1 \right\} \text{ for } D\left(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)\right)$$

(See [8, Example 4.22 and 4.24]).

Moreover, there are mysterious similarities for subarrangements. Let G be a simple graph on $[\ell] = \{1, \dots, \ell\}$. Define the **graphical arrangement** $\mathcal{A}_G$ in $\mathbb{R}^\ell$ by

$$\mathcal{A}_G \coloneqq \{ \{x_i - x_j = 0\} \mid \{i, j\} \in E_G \}.$$

Note that every subarrangement of $\mathcal{B}_\ell$ is of the form $\mathcal{A}_G$ and it is well known that the chromatic polynomial $\chi(G, t)$ coincides with the characteristic polynomial $\chi(\mathcal{A}_G, t)$. We will see later that the characteristic polynomials of arrangements in $\mathbb{F}_q^\ell$ have properties similar to those of chromatic polynomials.

We will focus on specific subarrangements of $\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)$ that arise from simple graphs. Note that the braid arrangement $\mathcal{B}_\ell$ corresponds to the graphical arrangement associated with the complete graph K_ℓ. Thus, it seems natural to assign K_ℓ to $\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)$. Furthermore, it seems natural to assign every clique of G to the set of all hyperplanes using the coordinates corresponding to the vertices in the clique.

Definition 1. We define a ***q*-deformation of graphical arrangement** $\mathcal{A}_G^q$ in $\mathbb{F}_q^\ell$ as follows.

$$\mathcal{A}_G^q \coloneqq \bigcup_{\{i_1, \dots, i_r\}} \{ \{a_{i_1} x_{i_1} + \cdots + a_{i_r} x_{i_r} = 0\} \mid (a_{i_1}, \dots, a_{i_r}) \in \mathbb{F}_q^r \setminus \{0\} \},$$

where $\{i_1, \dots, i_r\}$ runs over all cliques of G.

It is expected that $\mathcal{A}_G^q$ has a lot of properties similar to the graphical arrangement $\mathcal{A}_G$. The organization of this paper is as follows. In Section 2, we will present that the characteristic polynomials of arrangements in $\mathbb{F}_q^\ell$ have properties similar to those of chromatic polynomials. In Section 3, we will focus on the q-deformations $\mathcal{A}_G^q$ of graphical arrangements. In Section 3.1, we will show that the characteristic polynomial of $\mathcal{A}_G^q$ determines the chromatic polynomial G when q is large enough (Theorem 14 and Corollary 15) and we will describe a relationship between the characteristic polynomial and the numbers of stable partitions of G (Theorem 18). In Section 3.2, we will show that $\mathcal{A}_G^q$ is free if and only if G is chordal as in the case of graphical arrangements (Theorem 23). In Section 3.3, we construct an explicit basis for $D(\mathcal{A}_G^q)$ for a chordal graph G (Theorem 27).

2 Characteristic polynomials of arrangements over finite fields

The following proposition for the chromatic polynomial is trivial by definition.

Proposition 2. *Let G be a simple graph. Suppose $\chi(G, k) = 0$ for some $k \in \mathbb{Z}_{\geqslant 0}$. Then $\chi(G, j) = 0$ for $0 \leqslant j \leqslant k$.*

There is a q-version of Proposition 2.

Proposition 3 ([12, Lemma 7])**.** *Let $\mathcal{A}$ be an arrangement in $\mathbb{F}_q^\ell$. If $\chi(\mathcal{A}, q^k) = 0$ for some $k \in \mathbb{Z}_{\geqslant 0}$, then $\chi(\mathcal{A}, q^j) = 0$ for any $0 \leqslant j \leqslant k$.*

Proof. Let $\mathcal{A} \otimes \mathbb{F}_q^k$ denote the subspace arrangement in $(\mathbb{F}_q^k)^\ell$ defined by

$$\mathcal{A} \otimes \mathbb{F}_q^k \coloneqq \left\{ H \otimes_{\mathbb{F}_q} \mathbb{F}_q^k \;\middle|\; H \in \mathcal{A} \right\}.$$

Then the intersection lattices $L(\mathcal{A})$ and $L(\mathcal{A} \otimes \mathbb{F}_q^k) = \left\{ X \otimes_{\mathbb{F}_q} \mathbb{F}_q^k \;\middle|\; X \in L(\mathcal{A}) \right\}$ are naturally isomorphic. By [2, Proposition 3.1],

$$\chi(\mathcal{A}, q^k) = \chi(\mathcal{A} \otimes \mathbb{F}_q^k, q^k) = \# \left((\mathbb{F}_q^k)^\ell \setminus \bigcup_{H \in \mathcal{A}} H \otimes_{\mathbb{F}_q} \mathbb{F}_q^k \right).$$

Thus, if $\chi(\mathcal{A}, q^k) = 0$ and $0 \leqslant j \leqslant k$, then $\chi(\mathcal{A}, q^j) = 0$. $\square$

A **stable partition** of a simple graph is a set partition of the vertex set such that no edge connects vertices within the same block. Define the **falling factorial** $t^{\underline{i}}$ for each $i \in \mathbb{Z}_{>0}$ by $t^{\underline{i}} \coloneqq t(t-1)\cdots(t-i+1)$. Note that the falling factorial $t^{\underline{\ell}}$ coincides with the characteristic polynomial $\chi(\mathcal{B}_\ell, t)$.

Proposition 4 ([9, Theorem 15])**.** *Let G be a simple graph and suppose $\chi(G, t) = \sum_{i=1}^{\ell} c_i t^{\underline{i}}$. Then c_i coincides with the number of stable partitions of G into i blocks. In other words, c_i coincides with the number of i-dimensional subspaces in $L(\mathcal{B}_\ell)$ that are not contained in any hyperplanes in $\mathcal{A}_G$.*

Next, define the polynomial $t_q^{\underline{i}}$ by

$$t_q^{\underline{i}} \coloneqq \chi(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^i), t) = (t-1)(t-q)\cdots(t-q^{i-1}).$$

There is a formula similar to Proposition 4 for arrangements in $\mathbb{F}_q^\ell$. To prove it the following lemma is required.

Lemma 5 (Deletion-restriction formula [8, Corollary 2.57])**.** *Let $\mathcal{A}$ be an arrangement and suppose that $H \in \mathcal{A}$. Then*

$$\chi(\mathcal{A}, t) = \chi(\mathcal{A}', t) - \chi(\mathcal{A}^H, t),$$

where $\mathcal{A}'$ and $\mathcal{A}^H$ denote the ***deletion*** *and* ***restriction*** *defined by $\mathcal{A}' \coloneqq \mathcal{A} \setminus \{H\}$ and $\mathcal{A}^H \coloneqq \{ K \cap H \mid K \in \mathcal{A}' \}$.*

Proposition 6. *Let $\mathcal{A}$ be an arrangement in $\mathbb{F}_q^\ell$ and suppose $\chi(\mathcal{A}, t) = \sum_{i=0}^{\ell} c_i t_{\overline{q}}^{i}$. Then c_i is the number of i-dimensional subspaces in $\mathbb{F}_q^\ell$ that are not contained in any hyperplanes in $\mathcal{A}$.*

Proof. We proceed by double induction on ℓ and $|\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell) \setminus \mathcal{A}|$. When $\ell = 1$, $\mathcal{A}$ is the empty arrangement $\varnothing$ or the single-point arrangement $\{0\}$. The claim is true since

$$\begin{aligned} \chi(\varnothing, t) &= t = (t-1) + 1 = t_{\overline{q}}^{1} + t_{\overline{q}}^{0}, \\ \chi(\{0\}, t) &= t - 1 = t_{\overline{q}}^{1}. \end{aligned}$$

Suppose that $\ell \geqslant 2$. Since $\chi(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell), t) = t_{\overline{q}}^{\ell}$, the assertion is true for $\mathcal{A} = \mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)$. Assume that $\mathcal{A} \subsetneq \mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)$ and $H \in \mathcal{A}$. Then, by Lemma 5, we have

$$\chi(\mathcal{A}', t) = \chi(\mathcal{A}, t) + \chi(\mathcal{A}^H, t) = \sum_{i=0}^{\ell} c_i t_{\overline{q}}^{i} + \sum_{i=0}^{\ell-1} d_i t_{\overline{q}}^{i} = t_{\overline{q}}^{\ell} + \sum_{i=0}^{\ell-1} (c_i + d_i) t_{\overline{q}}^{i-1}.$$

By the induction hypothesis,

$$\begin{aligned} c_i &= \#\left\{\, X \in L(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)) \;\middle|\; \dim X = i,\ X \nsubseteq K \text{ for any } K \in \mathcal{A} \,\right\}, \\ d_i &= \#\{\, X \in L(\mathcal{A}_{\mathrm{all}}(H)) \mid \dim X = i,\ X \nsubseteq H \cap K \text{ for any } K \in \mathcal{A} \setminus \{H\} \,\}. \end{aligned}$$

Since

$$\begin{aligned} &\left\{\, X \in L(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)) \;\middle|\; \dim X = i,\ X \nsubseteq K \text{ for any } K \in \mathcal{A} \setminus \{H\} \,\right\} \\ &\qquad = \left\{\, X \in L(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)) \;\middle|\; \dim X = i,\ X \nsubseteq K \text{ for any } K \in \mathcal{A} \,\right\} \\ &\qquad\quad \sqcup \{\, X \in L(\mathcal{A}_{\mathrm{all}}(H)) \mid \dim X = i,\ X \nsubseteq H \cap K \text{ for any } K \in \mathcal{A} \setminus \{H\} \,\}, \end{aligned}$$

the claim follows. □

Some combinatorial sequences and their q-analogues have interpretations in terms of characteristic polynomials of hyperplane arrangements over $\mathbb{F}_q$.

Example 7. Consider the empty arrangement in $\mathbb{F}_q^\ell$. Then we have the following well-known formula (see [7, Expression 4.4]):

$$t^\ell = \sum_{i=0}^{\ell} \binom{\ell}{i}_q t_{\overline{q}}^{i},$$

where

$$\binom{\ell}{i}_q = \frac{[\ell]_q [\ell-1]_q \cdots [\ell-i+1]_q}{[i]_q [i-1]_q \cdots [1]_q}$$

is the q-binomial coefficient, which is equal to the number of i-dimensional subspaces of $\mathbb{F}_q^\ell$, and $[k]_q = \frac{q^k - 1}{q-1}$ denotes the q-integer. Note that t^ℓ is the characteristic polynomial of the empty arrangement in $\mathbb{F}_q^\ell$, and $t_{\overline{q}}^{i}$ is the characteristic polynomial of $\mathcal{A}_{\mathrm{all}}$ in $\mathbb{F}_q^i$.

Example 8. Consider the Boolean arrangement in $\mathbb{F}_q^\ell$, which consists of all coordinate hyperplanes. Define $S_q(\ell, i)$ by

$$(t-1)^\ell = \sum_{i=0}^{\ell} (q-1)^{\ell-i} S_q(\ell, i) t_q^{\underline{i}},$$

which is called the q-Stirling number of the second kind. It satisfies the following recurrence formula.

$$S_q(\ell, i) = S_q(\ell-1, i-1) + [i]_q S_q(\ell-1, i), \qquad S_q(0, i) = \delta_{0,i}.$$

It is proved as follows.

$$\begin{aligned}
(t-1)^\ell &= \sum_{i=0}^{\ell-1} (q-1)^{\ell-i-1} S_q(\ell-1, i) t_q^{\underline{i}} (t-1) \\
&= \sum_{i=0}^{\ell-1} (q-1)^{\ell-i-1} S_q(\ell-1, i) t_q^{\underline{i}} (t - q^i + q^i - 1) \\
&= \sum_{i=0}^{\ell-1} (q-1)^{\ell-i-1} S_q(\ell-1, i) t_q^{\underline{i+1}} + \sum_{i=0}^{\ell-1} (q-1)^{\ell-i-1} (q^i - 1) S_q(\ell-1, i) t_q^{\underline{i}} \\
&= t_q^{\underline{\ell}} + \sum_{i=0}^{\ell-1} (q-1)^{\ell-i} (S_q(\ell-1, i-1) + [i]_q S_q(\ell-1, i)) t_q^{\underline{i}}.
\end{aligned}$$

This is a q-version of the following well-known formula for the Stirling number of the second kind.

$$t^\ell = \sum_{i=0}^{\ell} S(\ell, i) t^{\underline{i}}, \qquad S(\ell, i) = S(\ell-1, i-1) + iS(\ell-1, i).$$

Remark 9. By Proposition 6,

$$S_q(\ell, i) = \frac{\#\left\{\, X \in L(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)) \;\middle|\; \dim X = i,\ X \not\subseteq \{x_j = 0\} \text{ for any } j \in [\ell] \,\right\}}{(q-1)^{\ell-i}}.$$

3 q-deformations of graphical arrangements

3.1 Characteristic polynomial

Let $G = (V, E)$ be a graph. One of the most important properties of chromatic polynomials is the following **deletion-contraction formula**

$$\chi(G, t) = \chi(G \setminus e, t) - \chi(G/e, t),$$

for $e \in E$.

We obtain the following q-version of this formula for q-deformations of graphical arrangements.

Proposition 10. *Let $e = \{i_1, i_2\} \in E$ be an edge such that*

(1) $\{i_1, i_2\}$ is a maximal clique, and

(2) The contraction G/e does not result in a new clique.

Then

$$\chi(\mathcal{A}^q_G, t) = \chi(\mathcal{A}^q_{G\setminus e}, t) - (q-1)\chi(\mathcal{A}^q_{G/e}, t).$$

Proof. Without loss of generality, we may assume that $e = \{\ell-1, \ell\}$. Then the difference between $\mathcal{A}^q_{G\setminus e}$ and $\mathcal{A}^q_G$ is $(q-1)$ hyperplanes of the form $\{x_{\ell-1} = \alpha x_\ell\}$ with $\alpha \in \mathbb{F}_q^\times$. Under the hypothesis (1) and (2), the restricted arrangement on such a hyperplane is $\mathcal{A}^q_{G/e}$. Applying Lemma 5, $(q-1)$ times, we obtain the formula. □

Example 11. Let P_ℓ be the path graph on the vertex set $[\ell]$. Then a clique is either a vertex or an edge. Hence $\mathcal{A}^q_{P_\ell}$ consists of $1 + (\ell-1)q$ hyperplanes of the form $\{ax_i + bx_{i+1} = 0\}$, $i = 1, \dots, \ell-1$, $(a, b) \in \mathbb{F}_q^2 \setminus \{(0,0)\}$. Using Proposition 10, we have $\chi(\mathcal{A}^q_{P_\ell}, t) = (t-1)(t-q)^{\ell-1}$ (Note that $\chi(\mathcal{A}_{P_\ell}, t) = t(t-1)^{\ell-1}$).

There is a well-known explicit formula for the chromatic polynomial of a cycle graph C_ℓ for $\ell \geqslant 3$.

$$\chi(\mathcal{A}_{C_\ell}, t) = (t-1)^\ell + (-1)^\ell(t-1).$$

The following lemma is a q-version of this formula.

Lemma 12. *Let $\ell \geqslant 4$. Then*

$$\chi(\mathcal{A}^q_{C_\ell}, t) = (t-q)^\ell + (-1)^\ell(q-1)^{\ell-1}(t-q).$$

Proof. First consider the arrangement $\mathcal{B}$ in $\mathbb{F}_q^3$ consisting of the following hyperplanes.

$$x_1 = ax_2,\ x_2 = ax_3,\ x_3 = ax_1 \quad (a \in \mathbb{F}_q).$$

Then $\mathcal{B}$ is supersolvable with exponents $(1, q, 2q-1)$. Using the deletion-restriction theorem [8, Corollary 2.57] $q-1$ times, we have

$$\begin{aligned}
\chi(\mathcal{A}^q_{C_4}, t) &= \chi(\mathcal{A}^q_{P_4}, t) - (q-1)\chi(\mathcal{B}, t)\\
&= (t-1)(t-q)^3 - (q-1)(t-1)(t-q)(t-2q+1)\\
&= (t-1)(t-q)\left((t-q)^2 - (q-1)(t-2q+1)\right)\\
&= (t-1)(t-q)\left((t-q)^2 - (q-1)(t-q) + (q-1)^2\right)\\
&= (t-1)(t-q)\frac{(t-q)^3 + (q-1)^3}{(t-q) + (q-1)}\\
&= (t-q)^4 + (-1)^4(q-1)^3(t-1).
\end{aligned}$$

Now suppose that $\ell \geqslant 5$. Using Proposition 10, we have

$$\begin{aligned}
\chi(\mathcal{A}^q_{C_\ell}, t) &= \chi(\mathcal{A}^q_{P_\ell}) - (q-1)\chi(\mathcal{A}^q_{C_{\ell-1}}, t)\\
&= (t-1)(t-q)^{\ell-1} - (q-1)(t-q)^{\ell-1} - (-1)^{\ell-1}(q-1)^{\ell-1}(t-q)\\
&= (t-q)^\ell + (-1)^\ell(q-1)^{\ell-1}(t-q).
\end{aligned}$$

□

In many concrete graphs G, we observe that $\chi(\mathcal{A}_G^q, t)$ is a polynomial in q and t. It is expected that the chromatic polynomial $\chi(G, t)$ is recovered from the polynomial $\chi(\mathcal{A}_G^q, t)$.

Conjecture 13. The function $\chi(\mathcal{A}_G^q, t)$ is a polynomial in q and in t, and the equality

$$\lim_{q\to 1} \frac{\chi(\mathcal{A}_G^q, q^t)}{(q-1)^\ell} = \chi(G, t)$$

holds.

The following results are weaker versions of this conjecture.

Theorem 14. *For any $k \in \mathbb{Z}_{\geqslant 0}$, and prime power q,*

$$\frac{\chi(\mathcal{A}_G^q, q^k)}{(q-1)^\ell} \equiv \chi(G, k) \pmod{q-1}.$$

Proof. Let $V = \mathbb{F}_q^k$. Define

$$X_G^q(V) := \left\{ (v_1, \dots, v_\ell) \in V^\ell \;\middle|\; \begin{array}{l} \{v_{i_1}, \dots, v_{i_p}\} \text{ is linearly independent} \\ \text{over } \mathbb{F}_q \text{ if } \{i_1, \dots, i_p\} \text{ is a clique of } G. \end{array} \right\}.$$

Note that $\chi(\mathcal{A}_G^q, q^k) = \#X_G^q(V)$ since $\chi(\mathcal{A}_G^q, q^k)$ equals the cardinality of the complement of the subspace arrangement $\mathcal{A}_G^q \otimes \mathbb{F}_q^k$ as mentioned in the proof of Proposition 3. The group $(\mathbb{F}_q^\times)^\ell$ acts on $X_G^q(V)$ by

$$(a_1, \dots, a_\ell) \cdot (v_1, \dots, v_\ell) := (a_1 v_1, \dots, a_\ell v_\ell).$$

Therefore $\#X_G^q(\mathbb{P}(V)) = \dfrac{\chi(\mathcal{A}_G^q, q^k)}{(q-1)^\ell}$, where

$$X_G^q(\mathbb{P}(V)) := \left\{ (\overline{v}_1, \dots, \overline{v}_\ell) \in \mathbb{P}(V)^\ell \;\middle|\; \begin{array}{l} \{v_{i_1}, \dots, v_{i_p}\} \text{ is linearly independent} \\ \text{over } \mathbb{F}_q \text{ if } \{i_1, \dots, i_p\} \text{ is a clique of } G. \end{array} \right\}.$$

Let D denote the subgroup of $(\mathbb{F}_q^\times)^k$ defined by $D := \left\{ (a, \dots, a) \in (\mathbb{F}_q^\times)^k \;\middle|\; a \in \mathbb{F}_q^\times \right\}$. Let $T := (\mathbb{F}_q^\times)^k / D$, that is, the quotient of $(\mathbb{F}_q^\times)^k$ by the diagonal action of $\mathbb{F}_q^\times$. Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \neq 0$. Then T acts on $\mathbb{P}(V)$ by

$$\overline{(a_1, \dots, a_k)} \cdot \overline{x} = \overline{\begin{pmatrix} a_1 x_1 \\ \vdots \\ a_k x_k \end{pmatrix}}.$$

Let $\nu(x) := \# \{ i \in [k] \mid x_i \neq 0 \}$. Then it is easily seen that

$$\#(T \cdot \overline{x}) = (q-1)^{\nu(x)-1}.$$

In particular, $\overline{x} \in \mathbb{P}(V)$ is a fixed point if and only if $\overline{x} = \overline{e}_i$, the standard-basis vector, for some i. Consider the diagonal T-action on $\mathbb{P}(V)^\ell$. This induces a T-action on $X_G^q(\mathbb{P}(V))$. The fixed point set is

$$X_G^q(\mathbb{P}(V))^T = \left\{ (\overline{e}_{s_1}, \dots, \overline{e}_{s_\ell}) \in \mathbb{P}(V)^\ell \;\middle|\; \begin{array}{l} \{e_{s_{i_1}}, \dots, e_{s_{i_p}}\} \text{ is linearly independent} \\ \text{over } \mathbb{F}_q \text{ if } \{i_1, \dots, i_p\} \text{ is a clique of } G. \end{array} \right\}.$$

Note that $\{e_{s_{i_1}}, \dots, e_{s_{i_p}}\}$ is linearly independent if and only if they are mutually different. Therefore we have

$$\# X_G^q(\mathbb{P}(V))^T = \chi(G, k).$$

Any other orbits have cardinalities divisible by $q - 1$. This completes the proof. □

Corollary 15. *If $q > \ell^\ell$, then $\chi(\mathcal{A}_G^q, t)$ determines $\chi(G, t)$.*

Proof. Note that $\chi(G, t)$ is determined by the values $\chi(G, 1), \chi(G, 2), \dots, \chi(G, \ell)$. For $k \in [\ell]$, we have $\chi(G, k) \leqslant k^\ell < q - 1$. Hence $\chi(G, k)$ is the remainder of $\dfrac{\chi(\mathcal{A}_G^q, q^k)}{(q-1)^\ell}$ divided by $q - 1$. □

Recall that a graph G is **triangle-free** if G has no cycle of length 3. For example, bipartite graphs and cycle graphs C_ℓ ($\ell \geqslant 4$) are triangle-free. When G is triangle-free, the characteristic polynomial $\chi(\mathcal{A}_G^q, t)$ is equivalent to the chromatic polynomial of G as follows. Also, note that the following formula is consistent with Lemma 12 for C_ℓ ($\ell \geqslant 4$).

Theorem 16. *Let G be a triangle-free graph. Then,*

$$\chi(\mathcal{A}_G^q, t) = (q-1)^\ell \chi\left(G, \frac{t-1}{q-1}\right).$$

Proof. Let $V = \mathbb{F}_q^k$, and consider $\# X_G^q(\mathbb{P}(V))$. Note that nonzero vectors $v_1, v_2 \in V$ are linearly independent if and only if $\overline{v}_1 \neq \overline{v}_2$ in $\mathbb{P}(V)$. Since a clique of a triangle-free graph G is either a vertex or an edge,

$$X_G^q(\mathbb{P}(V)) = \left\{ (\overline{v}_1, \dots, \overline{v}_\ell) \in \mathbb{P}(V)^\ell \;\middle|\; \overline{v}_i \neq \overline{v}_j \text{ for each edge } \{i, j\} \in E \right\}.$$

In this case, $X_G^q(\mathbb{P}(V))$ is equivalent to the set of vertex coloring by the set $\mathbb{P}(V)$. Hence, the cardinality is expressed as

$$\# X_G^q(\mathbb{P}(V)) = \chi(G, \#\mathbb{P}(V)) = \chi\left(G, \frac{q^k - 1}{q - 1}\right).$$

Thus we have

$$\frac{\chi(\mathcal{A}_G^q, q^k)}{(q-1)^\ell} = \chi\left(G, \frac{q^k - 1}{q - 1}\right).$$

Once we fix a prime power q, both $\chi(\mathcal{A}_G^q, t)$ and $(q-1)^\ell \chi(G, \frac{t-1}{q-1})$ are polynomials in t of degree ℓ. These polynomials take same value for $t = q^k$ ($k \geqslant 1$). Hence, they are identical. □

Remark 17. There exists a pair of graphs (G, H) depicted in the following picture such that $\chi(\mathcal{A}_G^2, t) = \chi(\mathcal{A}_H^2, t)$ but $\chi(G, t) \neq \chi(H, t)$.

The characteristic polynomials are as follows.

$$\begin{aligned}\chi(\mathcal{A}_G^2, t) = \chi(\mathcal{A}_H^2, t) &= t^7 - 30t^6 + 376t^5 - 2545t^4 + 9934t^3 - 21880t^2 + 24384t - 10240, \\ \chi(G, t) &= t^7 - 14t^6 + 83t^5 - 265t^4 + 474t^3 - 441t^2 + 162t, \\ \chi(H, t) &= t^7 - 14t^6 + 83t^5 - 264t^4 + 468t^3 - 430t^2 + 156t.\end{aligned}$$

The characteristic polynomial $\chi(\mathcal{A}_G^q, t)$ can also determine the numbers of stable partitions of G by the following theorem, which is a q-version of Proposition 4 and a special case of Proposition 6.

Theorem 18. *Let $\chi(\mathcal{A}_G^q, t) = \sum_{i=0}^{\ell} c_i t_q^i$. Then $c_i/(q-1)^{\ell-i}$ is a non-negative integer and congruent to the number of stable partitions of G into i blocks modulo $q-1$.*

Proof. By Proposition 6, the coefficient c_i is equal to the number of subspaces that are not contained in any hyperplanes in $\mathcal{A}_G^q$. Let $V = \mathbb{F}_q^i$. Since every i-dimensional subspace in $\mathbb{F}_q^\ell$ corresponds to a matrix in reduced row echelon form with ℓ columns and rank i, we have

$$c_i = \# \left\{ (v_1, \dots, v_\ell) \in X_G^q(V) \;\middle|\; \begin{array}{l} (v_1, \dots, v_\ell) \text{ is in reduced row} \\ \text{echelon form with rank } i. \end{array} \right\}.$$

Therefore

$$\frac{c_i}{(q-1)^{\ell-i}} = \# \left\{ (\overline{v}_1, \dots, \overline{v}_\ell) \in X_G^q(\mathbb{P}(V)) \;\middle|\; \begin{array}{l} (v_1, \dots, v_\ell) \text{ is in reduced row} \\ \text{echelon form with rank } i. \end{array} \right\}.$$

Using the same group action in the proof of Theorem 14, we can show that $c_i/(q-1)^{\ell-i}$ is congruent to the cardinality of the set

$$\left\{ (\overline{e}_{s_1}, \dots, \overline{e}_{s_\ell}) \in X_G^q(\mathbb{P}(V))^T \;\middle|\; \begin{array}{l} (e_{s_1}, \dots, e_{s_\ell}) \text{ is in reduced row} \\ \text{echelon form with rank } i. \end{array} \right\}$$

modulo $q-1$. Each matrix $(e_{s_1}, \dots, e_{s_\ell})$ in the set above corresponds with a stable partition of G consisting of i blocks $\{ j \in [\ell] \mid s_j = k \}$ $(1 \leqslant k \leqslant i)$. $\square$

Some of properties of the chromatic polynomials for finite graphs have q-analogues. For example, let us denote by $G + K_m$ the join of a graph G and the complete graph K_m. Then it is easily seen that $\chi(G + K_m, t) = t(t-1)\cdots(t-m+1)\chi(G, t-m)$. As a q-analogue of this formula, we can prove the following.

Proposition 19. *Let G be a graph on $[\ell]$. Then*

$$\chi(\mathcal{A}^q_{G+K_m}, t) = (t-1)(t-q)\cdots(t-q^{m-1})q^{m\ell}\chi(\mathcal{A}^q_G, q^{-m}t).$$

For the proof, we first note that $G + K_m = (G + K_{m-1}) + K_1$. It suffices to show the case $m = 1$, to be specific,

$$\chi(\mathcal{A}^q_{G+K_1}, t) = (t-1)q^\ell\chi(\mathcal{A}^q_G, q^{-1}t).$$

To show this, we shall introduce a new kind of deformation of graphical arrangement, and then complete the proof following a lemma.

Definition 20. Let G be a graph on $[\ell]$. Define $\widetilde{\mathcal{A}}^q_G$ of G over the finite field $\mathbb{F}_q$ by

$$\widetilde{\mathcal{A}}^q_G \coloneqq \bigcup_{\{i_1,\ldots,i_r\}} \left\{\, \{a_{i_1}x_{i_1} + \cdots + a_{i_r}x_{i_r} = b\} \;\middle|\; (a_{i_1}, \ldots, a_{i_r}) \in \mathbb{F}_q^r \setminus \{0\},\, b \in \mathbb{F}_q \,\right\}$$

where $\{i_1, \ldots, i_r\}$ runs over all cliques of G.

The following lemma illustrates the characteristic polynomial of this kind of deformation of graphical arrangement.

Lemma 21. $\chi(\widetilde{\mathcal{A}}^q_G, t) = q^\ell\chi(\mathcal{A}^q_G, q^{-1}t)$.

Proof. By Whitney's theorem [8, Lemma 2.55], the characteristic polynomial of $\widetilde{\mathcal{A}}^q_G$ is represented as

$$\chi(\widetilde{\mathcal{A}}^q_G, t) = \sum_{\mathcal{B}} (-1)^{|\mathcal{B}|} t^{\dim(\bigcap_{H\in\mathcal{B}} H)},$$

where $\mathcal{B}$ runs over the subsets of $\widetilde{\mathcal{A}}^q_G$ which is central, that is, $\bigcap_{H\in\mathcal{B}} H \neq \varnothing$. For such a subset $\mathcal{B}$, by translating each hyperplane in $\mathcal{B}$ by the same vector in the intersection $\bigcap_{H\in\mathcal{B}} H$, we obtain another subset of $\mathcal{A}^q_G$ whose intersection is isomorphic to $\bigcap_{H\in\mathcal{B}} H$ as affine spaces.

Therefore any subset $\mathcal{S}$ of $\mathcal{A}^q_G$ contributes $(-1)^{|\mathcal{S}|} q^{\ell-\dim H_\mathcal{S}} t^{\dim H_\mathcal{S}}$ to the characteristic polynomial of $\widetilde{\mathcal{A}}^q_G$, where $H_\mathcal{S} = \bigcap_{H\in\mathcal{S}} H$ and $H_\varnothing$ is the whole space. Then we can compute the characteristic polynomial of $\widetilde{\mathcal{A}}^q_G$ as follows.

$$\begin{aligned}
\chi(\widetilde{\mathcal{A}}^q_G, t) &= \sum_{\mathcal{S}\subseteq\mathcal{A}^q_G} (-1)^{|\mathcal{S}|} q^{\ell-\dim H_\mathcal{S}} t^{\dim H_\mathcal{S}} = \sum_{\mathcal{S}\subseteq\mathcal{A}^q_G} (-1)^{|\mathcal{S}|} q^\ell (q^{-1}t)^{\dim H_\mathcal{S}} \\
&= q^\ell \sum_{\mathcal{S}\subseteq\mathcal{A}^q_G} (-1)^{|\mathcal{S}|} (q^{-1}t)^{\dim H_\mathcal{S}} = q^\ell\chi(\mathcal{A}^q_G, q^{-1}t).
\end{aligned}$$

Hence we get the formula as required. $\square$

Proof of Proposition 19. We observe that for any clique in G, it is also a clique in $G+K_1$ by adding the vertex in K_1, so $\widetilde{\mathcal{A}}^q_G$ is the deconing of $\mathcal{A}^q_{G+K_1}$ with respect to the additional hyperplane $H_0 = \{\ker(x_0)\}$ where x_0 is the indeterminate corresponding to the added vertex in K_1. It is well known that for coning and deconing, the characteristic polynomials differ by a factor $t-1$, which means

$$\begin{aligned}\chi(\mathcal{A}^q_{G+K_1}, t) &= (t-1)\chi(\widetilde{\mathcal{A}}^q_G, t) \\ &= (t-1)q^\ell \chi(\mathcal{A}^q_G, q^{-1}t)\end{aligned}$$

as desired. □

3.2 Freeness

An arrangement $\mathcal{A}$ is **supersolvable** if and only if there exists a filtration

$$\varnothing = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}_\ell = \mathcal{A}$$

such that, for each $i \in [\ell]$, $\operatorname{rank} \mathcal{A}_i = i$ and for any distinct hyperplanes $H, H' \in \mathcal{A}_i \setminus \mathcal{A}_{i-1}$ there exists H'' such that $H \cap H' \subseteq H''$ [1, Theorem 4.3]. Every supersolvable arrangement is inductively free by [6, Theorem 4.2]. When $\mathcal{A}$ is supersolvable with the filtration above, the characteristic polynomial decomposes as

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (t - |\mathcal{A}_i \setminus \mathcal{A}_{i-1}|).$$

A vertex v of a simple graph G is called **simplicial** if the neighborhood $N_G(v) \coloneqq \{\, u \in V_G \mid \{u, v\} \in E_G \,\}$ is a clique. An ordering $(v_1, \ldots, v_\ell)$ of the vertices of G is a **perfect elimination ordering** if v_i is simplicial in the subgraph of G induced by $\{v_1, \ldots, v_i\}$ for each $i \in [\ell]$.

Theorem 22 (Stanley (See [4, Theorem 3.3]), Dirac [3], Fulkerson-Gross [5])**.** *The following are equivalent.*

(1) G has a perfect elimination ordering.

(2) $\mathcal{A}_G$ is supersolvable.

(3) $\mathcal{A}_G$ is free.

(4) G is chordal.

Moreover, when $(v_1, \ldots, v_\ell)$ is a perfect elimination ordering of G

$$\chi(\mathcal{A}_G, t) = \prod_{i=1}^{\ell} (t - |N_{G_i}(v_i)|),$$

where $G_i \coloneqq G[\{v_1, \ldots, v_i\}]$ and $N_{G_i}(v_i)$ is the set of adjacent vertices of v_i in G_i.

Theorem 23. *The following conditions are equivalent to the conditions in Theorem 22.*

(5) $\mathcal{A}_G^q$ *is supersolvable.*

(6) $\mathcal{A}_G^q$ *is free.*

Moreover, when $(v_1, \dots, v_\ell)$ *is a perfect elimination ordering of* G

$$\chi(\mathcal{A}_G^q, t) = \prod_{i=1}^{\ell} \left(t - q^{|N_{G_i}(v_i)|}\right),$$

where $G_i \coloneqq G[\{v_1, \dots, v_i\}]$ *and* $N_{G_i}(v_i)$ *is the set of adjacent vertices of* v_i *in* G_i.

Proof. The proof is very similar to the proof of Theorem 22.

To show (1) $\Rightarrow$ (5), let $(v_1, \dots, v_\ell)$ be a perfect elimination ordering and $(x_1, \dots, x_\ell)$ the corresponding coordinates. Let $\mathcal{A}_i$ be the subarrangement of $\mathcal{A}_G^q$ consisting hyperplanes whose defining linear form contains $x_1, \dots, x_i$. Then the filtration

$$\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}_\ell = \mathcal{A}_G^q$$

guarantees that $\mathcal{A}_G^q$ is supersolvable and $|\mathcal{A}_i \setminus \mathcal{A}_{i-1}| = q^{|N_{G_i}(v_i)|}$.

The implication (5) $\Rightarrow$ (6) follows since every supersolvable arrangement is inductively free. To prove (6) $\Rightarrow$ (4) it suffices to show that $\chi(\mathcal{A}_{C_\ell}^q, t)$ does not factor into the product of linear forms over $\mathbb{Z}$ when $\ell \geqslant 4$ by Terao's factorization theorem. Since $\chi(\mathcal{A}_{C_\ell}^q, t) = (t-q)^\ell + (-1)^\ell (q-1)^{\ell-1}(t-q)$ by Lemma 12, we have

$$\chi(\mathcal{A}_{C_\ell}, t) = (q-1)^\ell (x^\ell + (-1)^\ell) = (q-1)^\ell x (x^{\ell-1} + (-1)^\ell),$$

where $x = \frac{t-q}{q-1}$. Since $x^\ell + (-1)^\ell$ has an imaginary root, $\chi(\mathcal{A}_{C_\ell}^q, t)$ does not factor over $\mathbb{Z}$. $\square$

Example 24. Let T be a tree on $[\ell]$. Since T is chordal, both $\mathcal{A}_T$ and $\mathcal{A}_T^q$ are free. Moreover, since every leaf of T is simplicial, we can compute inductively

$$\chi(\mathcal{A}_T, t) = t(t-1)^{\ell-1} \qquad \text{and} \qquad \chi(\mathcal{A}_T^q, t) = (t-1)(t-q)^{\ell-1}.$$

Note that these can be computed using the deletion-contraction formula, as in Example 11.

Example 25. Let G be the diamond graph on $[4]$ with edge set $E_G = \binom{[4]}{2} \setminus \{\{2,4\}\}$ (See Figure 1). Then G has a perfect elimination ordering $(1,2,3,4)$ and hence

$$\chi(\mathcal{A}_G, t) = t(t-1)(t-2)^2 \qquad \text{and} \qquad \chi(\mathcal{A}_G^t, t) = (t-1)(t-q)(t-q^2).$$

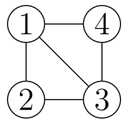

Figure 1: Diamond graph

3.3 Basis construction

Let G be a chordal graph with a perfect elimination ordering $(v_1, \dots, v_\ell)$ and $(x_1, \dots, x_\ell)$ the corresponding coordinates. Define the sets $C_{\geqslant k}$ and $E_{<k}$ by

$$\begin{aligned} C_{\geqslant k} &:= \{k\} \cup \left\{ i \in [\ell] \;\middle|\; \begin{array}{c} \text{there exists a path } v_k v_{j_1} \cdots v_{j_n} v_i \\ \text{such that } k < j_1 < \cdots < j_n < i \end{array} \right\}, \\ E_{<k} &:= \{\, j \in [\ell] \mid j < k \text{ and } \{v_j, v_k\} \in E_G \,\}. \end{aligned}$$

Let $\Delta(x_1, \dots, x_k)$ denote the Vandermonde determinant:

$$\Delta(x_1, \dots, x_k) := \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-1} \\ 1 & x_2 & x_2^2 & \cdots & x_k^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^{k-1} \end{vmatrix} = \prod_{1 \leqslant i < j \leqslant k} (x_j - x_i).$$

When $E_{<k} = \{j_1, \dots, j_m\}$ with $j_1 < \cdots < j_m$,

$$\Delta(E_{<k}) := \Delta(x_{j_1}, \dots, x_{j_m}) \quad \text{and} \quad \Delta(E_{<k}, x_i) := \Delta(x_{j_1}, \dots, x_{j_m}, x_i).$$

Theorem 26 ([10, Theorem 4.1])**.** *Suppose that G is a chordal graph and the symbols are as above. Let*

$$\theta_k := \sum_{i \in C_{\geqslant k}} \frac{\Delta(E_{<k}, x_i)}{\Delta(E_{<k})} \partial_i \qquad (1 \leqslant k \leqslant \ell).$$

Then $\{\theta_1, \dots, \theta_\ell\}$ forms a basis for $D(\mathcal{A}_G)$.

Let $\Delta_q(x_1, \dots, x_k) \in \mathbb{F}_q[x_1, \dots, x_k]$ denote the determinant of the Moore matrix. Namely

$$\Delta_q(x_1, \dots, x_k) = \begin{vmatrix} x_1 & x_1^q & x_1^{q^2} & \cdots & x_1^{q^{k-1}} \\ x_2 & x_2^q & x_2^{q^2} & \cdots & x_k^{q^{k-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ x_k & x_k^q & x_k^{q^2} & \cdots & x_k^{q^{k-1}} \end{vmatrix} = \prod_{i=1}^{k} \prod_{c_1, \dots, c_{i-1} \in \mathbb{F}_q} (c_1 x_1 + \cdots + c_{i-1} x_{i-1} + x_i).$$

The following theorem is a q-version of Theorem 26.

Theorem 27. *Suppose that G is a chordal graph and the symbols are as above. Let*

$$\theta_k \coloneqq \sum_{i\in C_{\geqslant k}} \frac{\Delta_q(E_{<k}, x_i)}{\Delta_q(E_{<k})}\partial_i \qquad (1 \leqslant k \leqslant \ell).$$

Then $\{\theta_1, \dots, \theta_\ell\}$ forms a basis for $D(\mathcal{A}_G^q)$.

Proof. Let $K = \{i_1, \dots, i_m\}$ be a clique of G with $i_1 < \cdots < i_m$. Let $\alpha = c_{i_1}x_{i_1} + c_{i_2}x_{i_2} + \cdots + c_{i_m}x_{i_m}$ be a nonzero linear form over $\mathbb{F}_q$. If $K \cap C_{\geqslant k} = \varnothing$, then $\theta_k(\alpha) = 0$.

Suppose that $K \cap C_{\geqslant k} \neq \varnothing$ and take $i_s \in K \cap C_{\geqslant k}$ with minimal s. Then one can show that $\{i_1, \dots, i_{s-1}\} \subseteq E_{<k}$ and $\{i_s, i_{s+1}, \dots, i_m\} \subseteq C_{\geqslant k}$. Then

$$\theta_k(\alpha) = \sum_{u=s}^{m} \frac{\Delta_q(E_{<k}, x_{i_u})}{\Delta_q(E_{<k})} \cdot c_{i_u} = \frac{\Delta_q(E_{<k},\ c_{i_s}x_{i_s} + \cdots + c_{i_m}x_{i_m})}{\Delta_q(E_{<k})} = \frac{\Delta_q(E_{<k},\ \alpha)}{\Delta_q(E_{<k})} \in (\alpha).$$

Using Saito's criterion [8, Theorem 4.19], we can prove $\{\theta_1, \dots, \theta_\ell\}$ is a basis. □

Example 28. Let G be the diamond graph shown in Figure 1. Then

$$\begin{aligned} C_{\geqslant 1} &= \{1,2,3,4\}, & E_{<1} &= \varnothing, \\ C_{\geqslant 2} &= \{2,3,4\}, & E_{<2} &= \{1\}, \\ C_{\geqslant 3} &= \{3,4\}, & E_{<3} &= \{1,2\}, \\ C_{\geqslant 4} &= \{4\}, & E_{<4} &= \{1,3\}. \end{aligned}$$

Therefore, $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ forms a basis for $D(\mathcal{A}_G^q)$, where

$$\begin{aligned} \theta_1 &= x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4, \\ \theta_2 &= \prod_{a\in\mathbb{F}_q}(ax_1 + x_2)\partial_2 + \prod_{a\in\mathbb{F}_q}(ax_1 + x_3)\partial_3 + \prod_{a\in\mathbb{F}_q}(ax_1 + x_4)\partial_4, \\ \theta_3 &= \prod_{a,b\in\mathbb{F}_q}(ax_1 + bx_2 + x_3)\partial_3 + \prod_{a,b\in\mathbb{F}_q}(ax_1 + bx_2 + x_4)\partial_4, \\ \theta_4 &= \prod_{a,b\in\mathbb{F}_q}(ax_1 + bx_3 + x_4)\partial_4. \end{aligned}$$

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