Ordinal and disjoint sums of partially ordered patterns

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Abstract

Partially ordered patterns (POPs) extend the classical notion of permutation patterns within the framework of pattern avoidance. Building on recent work of Burstein, Han, Kitaev, and Zhang, who introduced the concept of shape-Wilf equivalence for sets of patterns, we develop the notions of *ordinal sums* and *disjoint sums* of labeled posets. This perspective allows us to reinterpret their main result as an ordinal-sum analogue of the classical theorem of Backelin, West, and Xin. We establish parallel results for disjoint sums of POPs and further extend their work by proving Wilf equivalence results for classes of POPs that include isolated vertices. In particular, we prove the shape-Wilf equivalence of the pattern sets $\{123, 213, 312\}$ and $\{132, 231, 321\}$, as well as $\{123, 132, 231\}$ and $\{213, 312, 321\}$. Our approach is based on a bijection using an encoding scheme for transversals avoiding these patterns. As an application of our results, we completely classify partially ordered patterns of sizes 3, 4, and 5 whose connected components are chains, thereby confirming a conjecture of Dimitrov posed at the problem session of the British Combinatorics Conference 2024 (BCC30).

Mathematics Subject Classifications: 05A05, 05A15

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1 Introduction

Let $\mathbb{N}$ denote the set of natural numbers. For $n \in \mathbb{N}$, let $[n] := \{1, \dots, n\}$ and $\mathfrak{S}_n$ denote the set of permutations of $[n]$. An occurrence of a (classical) permutation pattern $p = p_1, p_2, \dots, p_k$ in a permutation $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$ is a subsequence $\pi_{i_1} \cdots \pi_{i_k}$ that is order isomorphic to p (i.e. $\pi_{i_j} < \pi_{i_m}$ if and only if $p_j < p_m$ for all $1 \leqslant j < m \leqslant$

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k). Permutation patterns are very well studied and have many connections to computer science, algebra, geometry and mathematical biology (see the book by Kitaev [19] and the references therein).

Partially ordered patterns were introduced by Kitaev [17] and have been extensively studied in words and compositions (see, for example, [15], [18], [20]). A *partially ordered pattern* (POP) p of size k is defined by a k-element partially ordered set (poset) P labeled by the elements in $\{1, \ldots, k\}$. An occurrence of such a POP p in a permutation $\pi = \pi_1 \cdots \pi_n$ is a subsequence $\pi_{i_1} \cdots \pi_{i_k}$, where $1 \leqslant i_1 < \cdots < i_k \leqslant n$, such that for $1 \leqslant j < m \leqslant k$, $\pi_{i_j} < \pi_{i_m}$ if and only if $j < m$ in P. We will use the word "pattern" to mean a classical or partially ordered pattern. A permutation π is said to avoid a pattern p if it does not contain any occurrences of the pattern p. We denote by $\mathfrak{S}_n(p)$ the set of permutations in $\mathfrak{S}_n$ that avoid the pattern p. We say that two patterns p, q are *Wilf-equivalent*, denoted by $p \sim q$, if $|\mathfrak{S}_n(p)| = |\mathfrak{S}_n(q)|$ for all $n \geqslant 1$. An alternative and equivalent perspective, due to Bean et al. [5], interprets avoidance of a POP p as the simultaneous avoidance of all classical patterns arising as group-theoretic inverses of the linear extensions of p. Throughout this paper, we freely identify a POP p with the corresponding set of classical patterns generating $\mathfrak{S}_n(p)$.

To study shape-Wilf-equivalence, it is convenient to work with pattern-avoiding transversals of Ferrers boards, which generalize permutation matrices. We briefly recall the necessary terminology. Throughout the paper, we draw Ferrers boards in French notation with rows increasing in length from top to bottom. We number columns in increasing order from left to right and rows in increasing order from bottom to top. In this paper, both row and column indices start from 1. We use the pair of indices (i, j) to represent the square located in row i and column j. A transversal of a Ferrers board $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a 0–1 filling with exactly one 1 in each row and each column. A permutation $\pi = \pi_1\pi_2 \cdots \pi_n$ can be regarded as a transversal of the n by n square diagram, in which the square (i, π_i) is filled with a 1 for all $1 \leqslant i \leqslant n$ and all the other squares are filled with 0's. The transversal corresponding to the permutation π is also called the permutation matrix of π. Let $\mathfrak{S}_\lambda$ denote the set of transversals of the Ferrers board λ.

Figure 1 illustrates this correspondence. Let us consider the Ferrers board $\lambda = (5, 5, 4, 3, 3)$ and the permutation $\pi = 4, 5, 2, 3, 1$. Recall that row indices increase from bottom to top and columns indices from left to right. Then π can be regarded as the following transversal.

0	1	0		
1	0	0		
0	0	0	1	
0	0	1	0	0
0	0	0	0	1

Figure 1: Illustrating the transversal corresponding to the permutation $\pi = 4, 5, 2, 3, 1$

Given a permutation α of $\mathfrak{S}_m$, let M_α be its permutation matrix. A transversal T

of a Ferrers board λ with n columns and n rows will be said to contain the pattern α if there exists two subsets of the index set $[n]$, namely, $R = \{r_1, r_2, \ldots, r_m\}$ and $C = \{c_1, c_2, \ldots, c_m\}$, such that the matrix M' on the set R of rows and the set C of columns is a copy of M_α and each of the squares (r_i, c_j) falls within the Ferrers board. In this context, we say that the matrix M' is isomorphic to α or the matrix M' is an occurrence of α. If this does not happen for any choice of row and column indices R and C respectively, we say that T avoids the pattern α and T is α-avoiding. Given a set P of patterns, let $\mathfrak{S}_\lambda(P)$ denote the set of transversals of the Ferrers board λ that avoid each pattern in P. Given two sets P and Q of patterns, we say that P and Q are shape-Wilf-equivalent if $|\mathfrak{S}_\lambda(P)| = |\mathfrak{S}_\lambda(Q)|$ holds for any Ferrers board λ. In this context, we write $P \sim_s Q$. If P and Q are singleton sets with $P = \{\sigma\}$ and $Q = \{\tau\}$, then, instead of writing $\{\sigma\} \sim_s \{\tau\}$, we simply write $\sigma \sim_s \tau$. Clearly, shape-Wilf-equivalence would imply Wilf-equivalence. One of the main advantages of defining shape-Wilf-equivalence is the following.

Theorem 1 (Backelin, West and Xin). *Let C and D be two matrices of order t, and let M and M' be related by*

$$M = \begin{bmatrix} C & 0 \\ 0 & A \end{bmatrix}, \quad M' = \begin{bmatrix} D & 0 \\ 0 & A \end{bmatrix}.$$

If $C \sim_s D$ then $M \sim_s M'$.

To describe how larger POPs may be built from smaller ones, we introduce ordinal sums of labeled posets. We define the ordinal sum of two labeled posets. First, we recall the ordinal sum of two posets. Let $p = (T, \leqslant_p)$ and $q = (T', \leqslant_q)$ be two posets. The *ordinal sum* $p \oplus q$ is the relation on the disjoint union $T \bigsqcup T'$ that satisfies

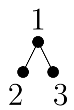
1. if $x, y \in T$ such that $x \leqslant_p y$, then $x \leqslant y$ in $p \oplus q$;

2. if $x, y \in T'$ such that $x \leqslant_q y$, then $x \leqslant y$ in $p \oplus q$;

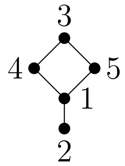
3. if $x \in T$ and $y \in T'$, then $x \leqslant y$ in $p \oplus q$.

To extend this notion to labeled posets, we specify what happens to the labels of p, q in $p \oplus q$. The labels $L_p \subset \mathbb{Z}_{>0}$ of the poset T in $p \oplus q$ are unchanged while the labels $L_q \subset \mathbb{Z}_{\geqslant 0}$ of the poset T' in $p \oplus q$ are all raised by $m = \max(L_p)$, that is, i is replaced by $i + \max(L_p)$ for all $i \in L_q$. In this work, the labeled posets will have labels from $\mathbb{Z}_{>0}$ and no two vertices will have the same label.

For example, if $p =$ 

and $q =$ 

, then the ordinal sum of p and q is $p \oplus q =$



. With these definitions in place, we proceed to study how ordinal and disjoint sums interact with Wilf-equivalence, leading to a complete classification of POPs of small size whose connected components are chains. Note that for a POP p on k elements, the set of labels of p will be $\{1, 2, \ldots, k\}$. With this language, Theorem 2.2 of [10] can be recast as follows.

Theorem 2 (Burstein, Han, Kitaev and Zhang)**.** *Let p, p' be POPs with $p \sim_s p'$ (i.e. p is shape-Wilf equivalent to p'). Then, for all POPs q, the POPs $p \oplus q$ and $p' \oplus q$ are shape-Wilf- equivalent i.e. $p \oplus q \sim_s p' \oplus q$.*

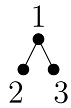
The Theorem 2 of Burstein et al. [10] is a POP analogue of the famous Theorem 1 of Backelin, West and Xin [2].

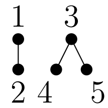
Similar to effect of ordinal sums on pattern avoidance, we study the effect of another poset operation on pattern avoidance. Given $p = (T, \leqslant_p)$ and $q = (T', \leqslant_q)$, the *disjoint sum* $p + q$ is the relation on $T \bigsqcup T'$ defined by

1. if $x, y \in T$ such that $x \leqslant_p y$, then $x \leqslant y$ in $p + q$;

2. if x, y in T' such that $x \leqslant_q y$, then $x \leqslant y$ in $p + q$.

To extend this to labeled posets, we specify how the labels are handled. The labels L_p in T are untouched, while the labels L_q of T' in $p + q$ are all raised by $\max(L_p)$. For instance, let $p =$ 

and $q =$ 

. The disjoint sum of p and q is $p + q =$ 

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We state our result concerning Wilf-equivalence of disjoint sums of POPs.

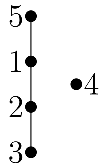
Theorem 3. *If $p \sim p'$ and $q \sim q'$, then $p + q \sim p' + q'$.*

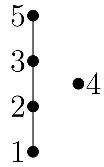
Our next result states that, up to Wilf-equivalence, the patterns $p + q$ and $q + p$ are the same.

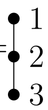
Theorem 4. *Let p, q be two POPs. Then, $p + q \sim q + p$.*

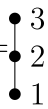
Our proofs of Theorems 3 and 4 appear in Section 2. We extend Wilf-equivalence results to POPs with isolated vertices. Our first main result in this direction is a Wilf-equivalence analogue of Theorem 2 when isolated vertices are present.

Theorem 5. *Let p, p' both be POPs of size k with isolated vertices $I = \{i_1 < i_2 < \cdots < i_s\}$ such that the vertices labeled $[i_1, k] \backslash I$ are greater than the vertices labeled $[i_1 - 1]$, that is, if $x \in [i_1, k] \backslash I$ and $y \in [i_1 - 1]$, then $x >_p y$ and $x >_{p'} y$. Let $I_p, I_{p'}$ be the subposets induced by $[i_1 - 1]$ and $J_p, J_{p'}$ be the subposets induced by $[i_1, k] \backslash I$. If $I_p \sim_s I_{p'}$ and $J_p = J_{p'}$, then $p \sim p'$.*

To illustrate this theorem, consider the POPs $p =$ 

and $p' =$ 

. Here $I = \{4\}$, $I_p =$ 

, $I_{p'} =$ 

and $J_p = J_{p'} =$ 

. As $I_p \sim_s I_{p'}$, we have $p \sim p'$.

Our second main result is the Wilf-equivalence of certain POPs with a specific form.

Theorem 6. *Let p, p' both be POPs of size k with the same isolated vertex labels I such that $k-3 \notin I, k-1 \in I$ and the vertices labeled $k-2, k$ are greater (in p, p') than the vertices labeled $[k-3]\backslash I$, that is, if $x \in [k-3]\backslash I$, then $k-2, k \geqslant_p x$ and $k-2, k \geqslant'_p x$.. Let $I_p, I_{p'}$ be the subposets induced by the vertices labeled $[k-3]\backslash I$ and $J_p, J_{p'}$ be the subposets induced by $[k-2, k]\backslash I$ in p, p' respectively. If $I_p = I_{p'}$ and*

$$J_p = \begin{matrix} k-2 \\ \bullet \\ | \\ \bullet \\ k \end{matrix}, \qquad J_{p'} = \begin{matrix} k \\ \bullet \\ | \\ \bullet \\ k-2 \end{matrix},$$

then $p \sim p'$.

We illustrate Theorem 6 through an example. Consider the POPs

$$p = \begin{matrix} 3\bullet \\ | \\ 5\bullet \\ | \\ 1\bullet \\ | \\ 2\bullet \end{matrix} \quad \bullet 4 \quad \text{and } p' = \begin{matrix} 5\bullet \\ | \\ 3\bullet \\ | \\ 1\bullet \\ | \\ 2\bullet \end{matrix} \quad \bullet 4 \; .$$

Here $I = \{4\}$, $I_p = I_{p'} = \begin{matrix} \bullet 1 \\ | \\ \bullet 2 \end{matrix}$. Hence, by Theorem 6, we get Wilf-equivalence of p and p'.

We give our proofs of Theorem 5 and Theorem 6 in Section 3.

We additionally show the following shape-Wilf-equivalence results. These are proved in Sections 7 and 8 respectively.

Theorem 7. *The POPs $p := 1\bullet \; \begin{matrix} \bullet 3 \\ | \\ \bullet 2 \end{matrix}$ and $p' := 1\bullet \; \begin{matrix} \bullet 2 \\ | \\ \bullet 3 \end{matrix}$ are shape-Wilf-equivalent.*

Theorem 8. *The POPs $p := \begin{matrix} 1\bullet \\ | \\ 2\bullet \end{matrix} \; \bullet 3$ and $p' := \begin{matrix} 2\bullet \\ | \\ 1\bullet \end{matrix} \; \bullet 3$ are shape-Wilf-equivalent.*

Note 9. We can write the POPs $1\bullet \; \begin{matrix} \bullet 3 \\ | \\ \bullet 2 \end{matrix}$, $1\bullet \; \begin{matrix} \bullet 2 \\ | \\ \bullet 3 \end{matrix}$, $\begin{matrix} 1\bullet \\ | \\ 2\bullet \end{matrix} \; \bullet 3$ and $\begin{matrix} 2\bullet \\ | \\ 1\bullet \end{matrix} \; \bullet 3$ as the sets of classical patterns $\{123, 213, 312\}$, $\{132, 231, 321\}$, $\{213, 312, 321\}$ and $\{123, 132, 231\}$ respectively.

Using Theorems 3, 4, 5, 6, we classify the Wilf-equivalence classes of POPs of size 3, 4, and 5 all of whose connected components are linearly ordered (chains). In Section 6, 7, 8, we completely classify the Wilf-equivalence classes of POPs whose components are all chains of size $3, 4, 5$ respectively.

2 Proofs of Theorems 3 and 4

We begin with our proof of Theorem 3.

Proof of Theorem 3. We will prove the Wilf-equivalences $p+q \sim p+q'$ and $p+q \sim p'+q$. Putting them together, we will get the Wilf-equivalences:

$$p + q \sim p + q' \sim p' + q'.$$

We only show the first equivalence and omit the proof of the second as it is very similar, though we highlight the differences at the end of the proof. Our proof is bijective. Let F be a square Ferrers board. Fix a transversal f that avoids $p + q$. Notice that if f avoids p, then f avoids occurrences of both $p + q$ and $p + q'$. If f avoids p, then map f to itself. Otherwise, perform the following. We start with the squares without any colour.

Step 1: Fix a bijection Λ from the transversals avoiding q to q'.

Step 2: For any square $(i, j) \in F$, if there is an occurrence of p in the subboard of F that is to the left of (i, j), then we color it gray. Otherwise, we colour it white.

Step 3: Find the 1's coloured white and colour their corresponding rows white.

Step 4: Remove all the white squares from F, denoting this new board, obtained by left and bottom justifying the squares, by F'. F' will be square as we start with a square Ferrers board. The induced transversal on F' is denoted by f'.

Step 5: Replace this transversal f' by $f'' := \Lambda(f')$ and restore all the white squares to the places they were removed from.

Step 6: The transversal that we have created avoids occurrences of $p + q'$.

Step 7: Map f to this transversal. This gives the required map.

Choose an occurrence of p in f in columns $c_1 < \cdots < c_k$ such that c_k is smallest. We can observe that this occurrence is untouched by the bijection described above and every cell to the right of this occurrence had to be gray and every cell weakly to the left of column c_k had to be white. Alternatively, the bijection can be described as applying Λ on the cells to the right of this occurrence. Therefore, since this occurrence is untouched and applying Λ will not create an occurrence of p to the left of the aforementioned occurrence, we can apply Λ^{-1} to exactly these cells and get back our original transversal f. Therefore, this map is invertible and hence, a bijection. This also explains why Step 6 gives a transversal avoiding $p + q'$ because an existence of an occurrence of $p + q'$ would imply that there is an occurrence of p strictly to the left of c_k but Λ does not change the contents of the white cells.

For the second equivalence, in our proof, we just have to look for occurrences of q' to the right of cells in Step 2. The rest of the proof is identical and is hence omitted. $\square$

Let p, q be two POPs of size k and l respectively with $k + l = n$. Thus, the size of $p+q$ is n. From the result of Gao and Kitaev [15, Theorem 1], the reverse $r(p)$ of any POP p of size n can be obtained by performing the complementation operation on its labels, that is, replacing a label i by $n + 1 - i$. Therefore, $r(p + q)$ is the reverse of the POP $p + q$.

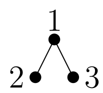
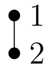
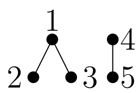
Let P, Q be the underlying sets of the POPs p, q respectively. Let $r_P(p+q)$ be the operation on $P \bigsqcup Q$ that replaces a label i of P by $k+1-i$ while keeping the labels on the set Q fixed. Similarly, $r_Q(p+q)$ is the operation on $P \bigsqcup Q$ that replaces a label i' of Q by $l+1-i'$ while keeping the labels on theset P fixed. Our next result explores the effect of a composition of these reversal operations.

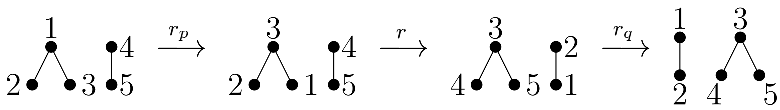
Remark 10. Informally, the underlying set of the poset $p+q$ is $P \bigsqcup Q$. r_P changes only the labels of P in $p+q$ and r_Q changes only the labels of Q in $p+q$. We do not change the Hasse diagram of the underlying poset.

Lemma 11. *$r_Q(r(r_P(p+q)) = q+p$.*

Proof. Before starting the proof, note that any two labels i and i' of P and Q in $p+q$ become $i+l$ and $i'-k$ in $q+p$ respectively. We prove the lemma in the following way. Let i, j be two labels of elements u, v in P such that $u <_p v$ in $p+q$. Consider $r_P(p+q)$ where the labels i, j become $k-i, k-j$. Now, consider $r(r_P(p+q))$. The labels become $n-k+i, n-k+j$ which is nothing but $l+i, l+j$. Lastly, if we consider r_Q, then it will not affect the labels of P. Hence, i, j becomes $l+i, l+j$ in $r_Q(r(r_P(p+q))$. Therefore, the labels of $u, v \in P$ change from i, j to $l+i, l+j$ respectively.

Now, we will follow the same steps for the labels of q. Consider two labels i', j' from elements u, v in Q such that $u <_q v$ in $p+q$. If we consider r_P, it will not affect the labels of elements in Q. Now, do the reverse r of $p+q$. We have $n+1-i', n+1-j'$ for u, v in $r(r_P(p+q))$. Lastly, consider the operation r_Q. We have the labels $i'-k, j'-k$ for the elements u, v in $r_Q(r(r_P(p+q))$. Therefore, the labels of $u, v \in Q$ change from i', j' to $i'-k, j'-k$ respectively. As this is the labeling of the labeled poset $q+p$, this completes our proof. $\square$

We illustrate Lemma 11. Let $p =$  and $q =$  . Hence, $p+q =$  .

 $= q+p$. We are now ready to prove Theorem 4.

Proof of Theorem 4. Let $\text{rev}(\pi)$ be the reverse of the permutation π. If a permutation π avoids a POP p, then $\text{rev}(\pi)$ will avoid $r(p)$. Hence, $p \sim r(p)$. By Theorem 3, we have the following.

$$p+q \sim r(p)+q = r_P(p+q) \sim r(r_P(p+q))$$

Now, let $r|_P, r|_Q$ be the effect of r on the labels on elements of P, Q respectively. We have

$$r(r_P(p+q)) = r|_P(r_P(p)) + r|_Q(r_P(q)) \sim r|_P(r_P(p)) + r_Q(r|_Q(r_P(q)).$$

Since r_Q does not affect the labels on P, we have

$$r|_P(r_P(p)) + r_Q(r|_Q(r_P(q)) = r_Q(r(r_P(p)) + r_Q(r(r_P(q)) = r_Q(r(r_P(p+q)).$$

By Lemma 11, we have $r_Q(r(r_P(p+q)) = q+p$. Hence, $p+q \sim q+p$, completing the proof. $\square$

3 Proofs of Theorems 5 and 6

To prove Theorem 5, we follow an idea presented in many sources (see [2], [4], [3], [10]). The key difference is presented in the next lemma.

Before we jump into the proofs, we define the standardisation of a labeled poset. Let P be a labeled poset with positive integer labels $x_1, \dots, x_r \in \mathbb{Z}_{>0}$ such that $x_1 < \cdots < x_r$. Then, the standardisation of P is the labeled poset $\mathrm{STD}(P)$ with the same underlying vertices but with labels x_i replaced by i for each $i \in [r]$. Note that standardisation makes any labeled poset a POP.

Lemma 12. *Let p be a POP of size r and $I = \{i_1, \dots, i_s\}$ be the set of isolated vertices. An occurrence of p in a transversal of a square Ferrers board F is equivalent to the existence of two subsets $R = \{x_1, \dots, x_{r-s}\}$ and $C = \{y_1, \dots, y_r\}$ such that the submatrix*

$$\begin{pmatrix} F[x_1, y_1] & F[x_1, y_2] & \cdots & F[x_1, y_r] \\ F[x_2, y_1] & F[x_2, y_2] & \cdots & F[x_2, y_r] \\ \vdots & \vdots & \ddots & \vdots \\ F[x_{r-s}, y_1] & F[x_r, y_2] & \cdots & F[x_{r-s}, y_r] \end{pmatrix}$$

contains an occurrence of $\mathrm{STD}(p \backslash I)$ in the columns indexed by $[r] \backslash I$ and each cell (x_i, y_j) lies inside F for $1 \leqslant i \leqslant r-s$ and $1 \leqslant j \leqslant r$.

Proof. Let M be a submatrix that is an occurrence of p in F. Let M' be the matrix obtained by deleting from M the rows that contains 1's in the columns indexed by I.

In M, the columns indexed by $[r]\backslash I$ contain an occurrence of $\mathrm{STD}(p\backslash I)$. By deleting these rows that contains 1's in the columns indexed by I, we have not affected ones that appear in the columns in $[r]\backslash I$. Therefore, the columns indexed by $[r]\backslash I$ in M' contain an occurrence of $\mathrm{STD}(p\backslash I)$.

Similarly, given a $(r-s) \times s$ submatrix such that the columns indexed by $[r]\backslash I$ contain an occurrence of $\mathrm{STD}(p\backslash I)$, we can look for the rows that contain occurrence of 1's in the columns indexed by I and add the corresponding cells to the matrix M' to uniquely recover M. Since F is a square Ferrers board, this newly constructed M will lie completely inside F, thereby completing the proof. $\square$

We call such a $(r-s) \times s$ submatrix, obtained in Lemma 12, *an essential occurrence* of p. We are now ready to prove Theorem 5.

Proof of Theorem 5. Suppose F is a square Ferrers board and a transversal $f \in F(p)$. Fix a bijection g that maps $\mathfrak{S}_F(I_p)$ to $\mathfrak{S}_F(I'_p)$. This exists as these patterns are shape-Wilf-equivalent. We use the bijection g in the following algorithmic manner. We start with all the squares initially being uncoloured.

Step 1: For a square $(i, j) \in F$, if the subboard of F that is above and to the right of (i, j) contains an essential occurrence of J_p (the submatrix lies entirely to the top-right of the cell), then we colour the square white; otherwise, we colour it gray.

Step 2: Find the 1's coloured gray, and colour the corresponding rows and columns gray.

Step 3: Remove the gray squares from F and denote by F' the shape formed by the white squares, after bottom-left justifying it. Denote by f' the induced transversal of F'. Note that F' is a Ferrers board and the transversal f' avoids J_p.

Step 4: Since $I_p \sim_s I_{p'}$, we can map f' into f'' bijectively, using g which we know exists, having the same shape and avoiding $I_{p'}$.

Step 5: Put the gray squares back in their original places, bijectively, to get a transversal $f''' \in F(p')$.

Therefore, we have $p \sim p'$. The proof is complete. $\square$

Before we proceed to our proof of Theorem 6, we need a couple of definitions. Let p be a POP of size k. For $1 \leqslant r \leqslant k$, define p_r to be the labeled subposet induced by the vertices labeled $[r]$. A *prefix* of the POP p is an occurrence of the POP p_r for some $1 \leqslant r \leqslant k$.

Given a POP p and a permutation w, define the *p-rank* of an entry w_j in the permutation w to be the longest $w_{i_1} \cdots w_{i_k}$ such that $w_j = w_{i_k}$, that is a prefix of the POP p. For a permutation in the set $\mathfrak{S}_n(p)$ where p is a POP of size k, the *p-rank* of an entry can range anywhere between 1 and $k-1$.

Example 13. Let us consider the classical pattern $p = 231$, i.e., $k = 3$. Let $w \in \mathfrak{S}_4(p)$. The relevant prefixes of p are the patterns of length 1 and length 2 (12 or 21). Let $w = 4312$. The p-rank of $w_1 = 4$ is 1, $w_2 = 3$ is 2, $w_3 = 1$ is 2 and $w_4 = 2$, is also 2.

The maximum rank observed is 2. A p-rank of 3 would require a subsequence of length 3 ending at w_j that contains the full pattern p. Since w is defined as p-avoiding, no such subsequence exists, confirming that $1 \leqslant p\text{-rank}(w_j) \leqslant 2$.

The following lemma can also be proved using Theorem 5 by the reversal operation, but we give a different proof. The advantage of the bijection described here is that it is clear that the entries with q-rank and q'-rank lesser than $k-2$ are fixed. We will use this fact in the proof of Theorem 6.

Lemma 14. *Let q, q' be two POPs of size k with the same isolated vertex labels I such that $k-2 \notin I$ and the vertices labeled $k-1, k$ are greater than the vertices labeled $[k-2] \backslash I$ in both q, q', that is, if $x \in [k-2] \backslash I$, then $k-1, k \geqslant_q x$ and $k-1, k \geqslant_{q'} x$. Let $I_q, I_{q'}$ be the subposets induced by the vertices labeled $[k-2] \backslash I$ and $J_q, J_{q'}$ be the subposets induced by $k-1, k$ in q, q' respectively. If $I_q = I_{q'}$ and*

$$J_q = \begin{array}{c} k-1 \\ \bullet \\ | \\ \bullet \\ k \end{array}, \qquad J_{q'} = \begin{array}{c} k \\ \bullet \\ | \\ \bullet \\ k-1 \end{array},$$

then $q \sim q'$.

Proof. We will use the idea from the bijection of West [25] to give a bijection between $\mathfrak{S}_n(q)$ and $\mathfrak{S}_n(q')$. Let $w \in \mathfrak{S}_n(q)$.

Step 1: Leave the entries of w' of q-rank $k-2$ or less in their original position

Step 2: Let W, X be defined as above. Remove the elements in X from their positions in W. Put all the elements of X back in the positions in W in decreasing order.

Let $\phi(w)$ be the obtained permutation. Note that $\phi(w)$ avoids q' since the existence of such a pattern in $\phi(w)$ would mean that the last two entries of that pattern were not placed according to the rule specified above. To prove that this is a bijection, we give the inverse of ϕ. Let $w' \in \mathfrak{S}_n(q')$.

Step 1: Leave the entries of w with q'-rank $k-2$ or less untouched and in their position.

Step 2: Let W be the set of positions of w in which an entry of q'-rank $k-1$ is located, and let X be the set of entries of w that are of q'-rank $k-1$. Now fill the positions of W with the entries in X from left to right, so that each position $i \in W$ is filled with the smallest entry x of X that has not been placed yet and that is larger than the closest entry of rank $k-2$ on the left of position i.

We omit the proof that these are inverses of each other because the argument is same as the one given in West's thesis [25] or this article of Bóna [6, Lemma 2.2]. One just has to use the notion of q-rank (resp. q'-rank) of entries instead of rank. $\square$

We will use the bijection in Lemma 14 to define a bijection to prove Theorem 6. Define $q := \mathrm{STD}(p \backslash \{k-1\})$ and $q' := \mathrm{STD}(p' \backslash \{k-1\})$. Let $w \in \mathfrak{S}_n(p)$.

To finish the proof of Theorem 6, we need to prove that the function Φ defined above is a bijection from $\mathfrak{S}_n(p)$ to $\mathfrak{S}_n(p')$.

Proof of Theorem 6. We will now give a bijection Φ from the set $\mathfrak{S}_n(p)$ to the set $\mathfrak{S}_n(p')$. Start with a permutation $w \in \mathfrak{S}_n(p)$. Let s_i be the transposition $(i, i+1)$ for $i \in [n-1]$.

Step 1: Let $i_1, \ldots, i_r$ be the positions of q- rank $k-1$. Multiply on the right of w by $s_{i_1-1} \cdots s_{i_r-1}$.

Step 2: Applying ϕ from Lemma 14 on the resulting permutation.

Step 3: Multiply on the right by $s_{i_1-1} \cdots s_{i_r-1}$.

The resulting permutation is our required permutation.

We show that the inverse of the map Φ exists. The inverse Ψ is as follows: Start from a permutation in $\mathfrak{S}_n(p')$.

Step 1: Let $i_1, \ldots, i_r$ be the positions of q'- rank $k-1$. Multiply on the right by $s_{i_1-1} \cdots s_{i_r-1}$.

Step 2: Applying ϕ^{-1} (as it is a bijection) from Lemma 14 on the resulting permutation.

Step 3: Multiply on the right by $s_{i_1-1}\cdots s_{i_r-1}$.

We CLAIM that if σ is a permutation in $\mathfrak{S}_n(p)$ (resp. $\mathfrak{S}_n(p')$) such that σ_i, σ_{i+1} are entries having q-rank (resp. q'-rank) $k-2$, then $\sigma \cdot s_i$ has q-rank (resp. q'-rank) $k-2$ at position i and $k-1$ at position $i+1$.

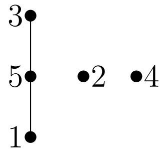
We first prove the claim for $\sigma \in \mathfrak{S}_n(p)$. Both σ_i, σ_{i+1} have q-rank $k-2$. If $\sigma_i > \sigma_{i+1}$, since σ_{i+1} is part of a sequence $\sigma_{i_1}\cdots\sigma_{i_{k-3}}\sigma_{i+1}$ order isomorphic to a prefix of q, such that $\sigma_{i_{k-3}}$ is to left of σ_i, we have q-rank of σ_{i+1} equal to $k-1$ because of the sequence $\sigma_{i_1}\cdots\sigma_{i_{k-3}}\sigma_i\sigma_{i+1}$, a contradiction. Therefore, $\sigma_i < \sigma_{i+1}$. Since σ_i has q-rank $k-2$, it has to be greater than an entry of rank $k-3$ to its left as the elements labeled $k-2$ are larger than the element labeled $k-3$ as $k-3 \notin I$ in both p, p' and hence, q, q'. Therefore, $\sigma \cdot s_i$ satisfies the claim.

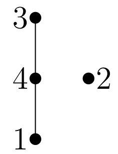
Now, we prove the claim for $\sigma \in \mathfrak{S}_n(p')$. Both σ_i, σ_{i+1} have q'-rank $k-2$. Suppose $\sigma_i < \sigma_{i+1}$, similar to the previous argument, we have a contradiction. This implies that $\sigma_i > \sigma_{i+1}$. Since σ_{i+1} has q'-rank $k-2$, it is part of a $\sigma_{i_1}\cdots\sigma_{i_{k-3}}\sigma_{i+1}$ that is order isomorphic to a prefix of q' with $\sigma_{i_{k-3}}$. Therefore, in $\sigma \cdot s_i$, $(\sigma \cdot s_i)_i = \sigma_{i+1}$ would have q'-rank $k-2$ and $(\sigma \cdot s_i)_{i+1} = \sigma_i$ would have q'-rank $k-1$. This proves our claim.

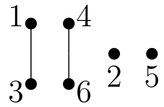
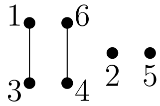
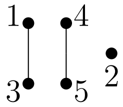
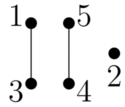
Now, we return to why the maps described are bijections. Clearly, given a permutation in $\mathfrak{S}_n(p)$, the entries of q-rank $k-1$ are well-defined. Therefore, the map itself is well-defined. We need to show that the image lies in $\mathfrak{S}_n(p')$. First, we show that Step 1 produces a permutation that avoids q. Each entry x of q-rank $k-1$ is part of an occurrence of q which involves x and the entry y immediately to its left. If not, this occurrence would be an occurrence of p. This also forces each entry immediately to the left of an entry of q-rank $k-1$ to be of q-rank $k-2$. This implies that two entries of q-rank $k-1$ are not adjacent. Hence, applying the transpositions in Step 1 strictly reduces the rank of x by one while keeping the rank of y constant. If we do these transpositions for every entry that has q-rank $k-1$, we create a permutation whose entries have q-rank at most $k-2$. Such a permutation avoids q. A similar argument for Ψ shows that Step 1 does indeed create a permutation that avoids q'.

Applying Step 2 of either Φ or Ψ is validated and since ϕ, ϕ^{-1} do not alter the elements of q-rank (resp. q'-rank) at most $k-3$, our claim from earlier shows that we produce a permutation that avoids p' (resp. p) after performing Step 3.

The argument also shows that the map Ψ is the inverse map of Φ described earlier as Step 1 decreases the q-rank of the entries at positions $i_1, \ldots, i_r$ and Step 3 increases the q'-rank of entries at positions $i_1, \ldots, i_r$. $\square$

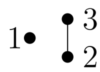
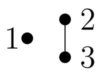
Example 15. We give an example where the claim mentioned in the proof of Theorem 6 is not true because $k-3 \in I$. Let $p =$ . Consider the permutation

425316 $\in \mathfrak{S}_n(p)$. Let $q =$  . The q-ranks are $1, 2, 3, 3, 2, 3$. However, applying s_3 gives the permutation 423516. But the q-ranks here are $1, 2, 2, 3, 2, 3$.

Example 16. We now present an example illustrating the algorithm of Theorem 6. Let $p =$  and $p' =$  . Let $w \in \mathfrak{S}_n(p)$, $w = 81723546$. Therefore, $q =$  and $q' =$  . and w contains q. The q-ranks of $8, 1, 7, 2, 3, 5, 4, 6$ are $1, 2, 3, 4, 4, 4, 5, 4$ respectively. Therefore $i_1 = 7$. Hence $w \cdot s_6 = 81723456$, which avoids q. Now we can apply Lemma 14 on $w \cdot s_6$.

The q-ranks of $8, 1, 7, 2, 3, 4, 5, 6$ are $1, 2, 3, 4, 4, 4, 4, 4$ respectively. Thus we keep the first three entries $8, 1, 7$ in their original positions rearrange the remaining entries $2, 3, 4, 5, 6$. Here $W = \{4, 5, 6, 7, 8\}$ and $X = \{2, 3, 4, 5, 6\}$. To avoid the POP q' the elements of X must be placed in decreasing order from left to right. Therefore, $\phi(w \cdot s_6) = 81765432 \in \mathfrak{S}_p(q')$. Again, we multiply s_6 on the right of $\phi(w \cdot s_6)$ to get $\phi(w \cdot s_6) \cdot s_6 = 81765342 \in \mathfrak{S}_n(p')$.

4 Proof of Theorem 7

We move on to the proof of Theorem 7. Recall that p is the POP  and p' is the POP  . We give an encoding scheme that maps a transversal of a Ferrers board F avoiding p (similarly, p') to certain words on the alphabet $\{0, 1, 2\}$. We will use this encoding scheme to give a bijection between $\mathfrak{S}(p)$ and $\mathfrak{S}(p')$ for all Ferrers boards.

To obtain the encoding word from a transversal, we employ the following process. We remove all the 1's from the board and put them back in their original position row by row starting from the top row.

Step 1: Start with an empty Ferrers board whose cells are all white.

Step 2: In the first step, we put back the 1 in the topmost row. We color the top row and the column containing the 1 gray.

Step 3: Recursively, once the top k rows of F have 1's in them, we place the next 1 in the topmost row of the remaining white cells. Subsequently, the row and column where it was placed will be colored gray.

When we insert in this manner to obtain a transversal that avoids p or p', we will find that 1 cannot be inserted in all positions in a row as it would inevitably cause occurrences of p or p' with future placements. Call a position/cell *suitable* if placing 1 in the position does not form an occurrence of p or p' with past or potential future placements.

Lemma 17. *At any step of the process, in the topmost row of white cells, there are at most two suitable cells to place* 1 *avoiding* p *or* p'*. For* p*, it is the first and second cells on the top row. For* p'*, it is the first and last cells on the top row.*

Proof. At the current step, if the topmost row has size more than 3, to avoid p, we cannot place the 1 outside of the first two columns. If we did, the 1's that would be placed in the first two columns in the future, along with this 1 would form an occurrence of p. Similarly, to avoid p', we cannot place 1 outside the first or last column. □

Next, we examine the case when there is at most 1 suitable position available in the top row. An instance is when there is exactly one white cell in the top row. Another one is described in the following lemma.

Lemma 18. *Let* F *be a Ferrers board and* $\ell(c)$ *denote the size of the column* c *in* F*. At some step of the process for* p *or* p'*, let* F' *be the board made up of the remaining white cells. Suppose that the topmost row of* F' *comes from row* r *and the columns* $c_1 < \cdots < c_l$ *of* F*. If the rectangle with corners* $(1, r+1), (c_1-1, r+1), (c_1-1, \ell(c_2)), (1, \ell(c_2))$ *(corners included) contains a* 1*, then there is at most one suitable position. To avoid* p*, it is the leftmost position on the top row. To avoid* p'*, it is the rightmost position on the top row.*

Proof. This follows from the previous lemma, as the 1 to the left of c_1 in the rectangle will form an occurrence of p or p' if placed in the other suitable position. □

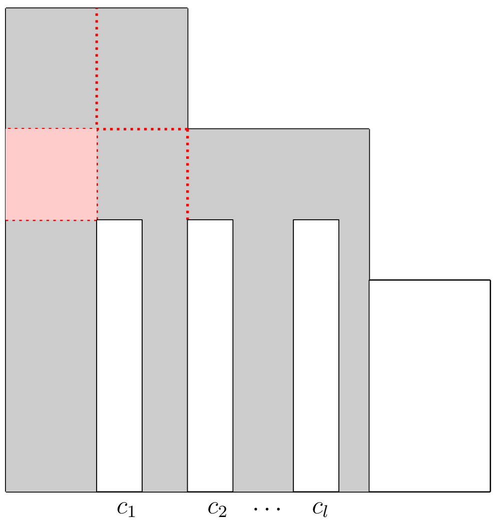


Figure 2: The rectangle (red) described in Lemma 18

Remark 19. The encoding is well defined because, at each step, one of the following situations occurs. In the i-th step if there is only one available cell, then the placement is forced, and we set $w_i = 0$. Otherwise, there are multiple available cells. In this case, two possibilities arise. If there exists a rectangle of the type described in Lemma 18, then the placement is again forced, so $w_i = 0$. If no such rectangle exists, then by Lemma 17 there are exactly two valid positions for placement. Accordingly, we set $w_i = 1$ or $w_i = 2$, depending on which of the two valid positions is chosen.

Lemma 20. *At any step of the process, in the topmost row of white cells, there is at least one suitable position to place* 1 *avoiding* p *or* p'.

Proof. To avoid p, we can always place 1 in the leftmost position on the top row of the white cells. To avoid p', we can always place 1 in the rightmost position on the top row of the white cells. This position will be suitable. □

Using the above scheme, we create an encoding that maps transversals that avoid p to those that avoid p'. The encoding will be a word with the letters $\{0, 1, 2\}$, obtained in the following way. Let T be a transversal of F that avoids p. Let $w = w_1 \cdots w_n$ be the encoding word.

1. Remove all the 1's and place them back on the board according the process described above in 4.
2. While placing a 1 on the row i from the top, if there are two suitable positions on the top row and 1 appears in the first suitable position from the left in T, let $w_i = 1$. If it appears in the second suitable position from the left in T, let $w_i = 2$.
3. While placing a 1 in row i, if there is only one suitable position on the top row, then let $w_i = 0$.
4. Once we have placed all the 1's, return $w = w_1 \cdots w_n$.

Now, to obtain a transversal of F that avoids p', we invert the encoding as follows. Let $w = w_1 \cdots w_n$ be the encoding word.

1. If $w_i = 1$, place 1 in the first suitable position from the left, in the topmost row of white cells. If $w_i = 2$, place 1 in the second suitable position, from the left, in the topmost row of white cells.
2. If $w_i = 0$, place it in the only suitable position available in the top row of white cells.
3. Return the transversal after all the 1's are placed.

Similarly, we can start with a transversal that avoids p', obtain an encoding word and then, invert it to get a transversal that avoids p. We will prove that these are inverses.

We will often talk about the Ferrers board of white cells. Even though, they are not technically Ferrers boards, we use it to mean the Ferrers board obtained by squashing and left aligning the white cells. In the insertion process for p (or for p'), the shape of the board of white cells after i iterations will be the same (up to squashing and left aligning). Therefore, if there were only one white cell on the top row for p, it would be the case for p' and vice versa. We will call an insertion step *forced* if there is only one suitable cell in which to insert.

The next two lemmas give the structure of the encoding words based on the shape of F. The following lemma describes when placements are forced in the insertion process.

Lemma 21. *At some step i, if $w_i = 1$ and the size of the top most row of white cells is strictly greater than 2, the subsequent placements are forced until a step is encountered where the size of the top row of white cells is exactly 2. The step after that is not forced unless there is only one white cell in the top row. This happens at the same step for p and p'.*

Proof. We first prove the lemma for p. Let F' be the Ferrers board of white cells at step i. If the top row of F' has 3 or more cells, then so do subsequent rows. Now, if we place 1 in the first column in the first row, then we would have to place 1 in the second column in the second row. If not, when we place a 1 in the second column of F' in the future, we would have an occurrence of p. Arguing similarly, every placement is forced until we encounter a step where the size of the top row of white cells is 2. Let this be step j.

At step $j-1$, let $\hat{F'}$ be the board of white cells. By definition, the top two rows have 3 cells. At step $j-1$, we would have placed in the first column of $\hat{F'}$ as this was forced. At step j, we would have to place in the second column of $\hat{F'}$. This is also forced. If the third row of $\hat{F'}$ has 3 cells, then there would only be one cell for step $j+1$. Else, if there are more than 3 cells in the third row of $\hat{F'}$, we can place 1 in the fourth column on the third row and it would not participate any occurrence of p with pre-existing $1'$s as the previous rows do not have cells in this column. This proves the lemma for p.

Now, for p', let F'' be Ferrers board of white cells. If we place 1 in the first column in the first row, then in the second row, we cannot place 1 in the second column, by a similar reasoning as before. Therefore, we are forced to place in the last cell on the second row. Arguing similarly, every placement is forced until we reach a row where only the second and third columns of F'' have white cells and the rest of the row is gray. Let step j be the step when this happens.

At step $j-1$, let $\hat{F''}$ be the board of white cells. We have three cells in the top two rows, which correspond to the second, third and fourth columns of F''. In step $j-1$, we have to place 1 in the rightmost column (fourth column). In step j, once again, we have to place 1 in the rightmost cell of that row (third column). If the third row of $\hat{F''}$ had 3 cells to begin with, then, in step $j+1$, there would be only one cell to place in. Else, we can place 1 in the cell that is in the second column of F'' or in the rightmost column on the third row of $\hat{F''}$. By way of contradiction, suppose this step were forced. Then, there would need to be a 1 to the left of the first column in the rectangle described in Lemma 18. However, the rectangle described in Lemma 18 would not have any 1's to the left of the first column of $\hat{F''}$ as the rectangle lies entirely below the top j rows. This is because the fourth column of the third row of $\hat{F''}$ has no cells in the higher rows of F''. If there were cells in higher rows, there would also be a cell in the second row of $\hat{F''}$ in this column but the second row had strictly smaller size than the third row. This proves that this step is not forced. $\square$

We will now show that if $w_i = 2$, then the subsequent placement is not forced.

Lemma 22. *If $w_i = 2$, then the step $i + 1$ is not forced for p or p' unless the top row has only one white cell.*

Proof. We prove the lemma first for p. Let F' be the board of white cells at step i. The top row of F' has at least 2 cells. If the first and second row both have only 2 cells, then the top row at step $i + 1$ will have only one cell. For the general case, suppose there were 1 to the left of the first column in the rectangle described in Lemma 18 for step $i + 1$, then this 1 would also be in the rectangle for insertion at step i, thereby forcing $w_i = 0$, which is a contradiction.

Now, for p', let F'' be the board of white cells at step i. The analysis for the special case is similar. For the general case, suppose there were a 1 to the left of the first column in the rectangle described in Lemma 18 for step $i + 1$, then this would also be to the left of the first column in the rectangle for the insertion at step i. This would mean that insertion at step i was forced, which is a contradiction. $\square$

Remark 23. When we reach a step i where there is only one white cell in the top row, it means that, at step $i + 1$, the first i columns and top i rows of the board are gray. The remaining white board is completely to the right and below the already placed 1's and no 1 placed in the current white board, in the future, can form an occurrence of p or p' with a 1 in the first i columns. This is because the $(i + 1)$-th column has no cells in the i-th row from the top.

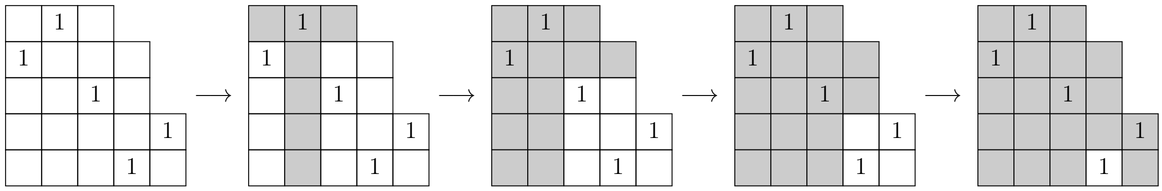
With the help of these lemmas, we are ready to show that the encoding scheme gives a bijection between the set of transversals avoiding p and the set of transversals avoiding p' for all Ferrers boards.

Proof of Theorem 7. We claim that the encoding scheme gives a bijection. To do this, we first note that the encoding is injective and given an encoding, we can easily reverse the process and obtain a transversal. Therefore, it is enough to show that the encoding processes for both p, p' give the same encoding words. We proceed by induction on the size of the Ferrers board. It is easily verified for Ferrers boards of size 3. For smaller boards, all transversal avoid p and p'. Now, suppose it is true for all Ferrers board of size $< k$. We prove for boards of size k. Let F be a Ferrers board of size k. If the top most row of F has one cell, then we argue as follows. Let F' be the board obtained by deleting the first column of F. Any transversal of F' avoiding p or p' can be extended to a transversal of F avoiding p or p' and each transversal can be restricted to a transversal of F'. If the encoding words for F' match, then they match for F by induction.

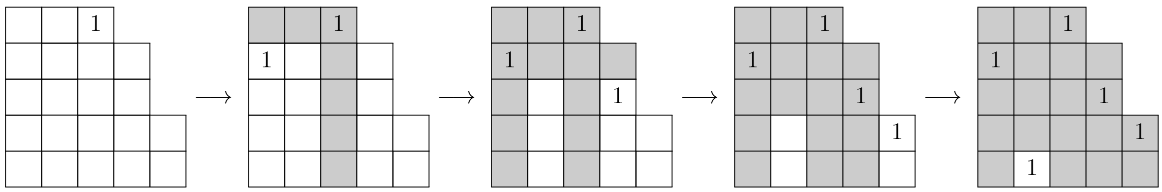
Suppose the first row has more than 1 cell, then Lemmas 21, 22 ensure that we reach a position where we have filled the first i columns and top i rows for some i (see the Remark preceding the proof). Let F_t be the Ferrers board formed by the first i columns and top i

rows. By the lemmas, we know encoding words have been the same so far. Here, similar to our previous argument, we have that the white cells form a smaller Ferrers board F' and since, by induction, the encoding words match for F' and the encoding words for F are concatenations of words for F_t with those of F', we are through by induction. $\square$

Example 24. We work out an example for the board $(5, 5, 4, 4, 3)$. We can see that the transversal $4, 5, 3, 1, 2$ avoids p. We obtain the encoding word from the board and transversal.



In the topmost row, there are two valid positions, first and second columns. As the 1 lies in the second valid position (from left), we get $w_1 = 2$. In the next row first and third columns are valid, as the 1 lies in the first valid position (from left), we get $w_2 = 1$. In the next row (third row), the placement is forced by Lemma 18, hence $w_3 = 0$. Next we have last two columns are valid, as 1 lies in the second valid position (from left), we get $w_4 = 2$. The bottom most row has only one possible position, hence $w_5 = 0$. Therefore, the encoding word is $2, 1, 0, 2, 0$.
The boards below display the insertion process for the transversal avoiding p'.



In the topmost row there are two valid positions, first and third columns. As $w_1 = 2$, we place 1 in the third column. In the next row the valid positions are first and fourth columns. As $w_2 = 1$, we place 1 in the first column. In the next row placement is forced by Lemma 18 and $w_3 = 0$, so the placement is compatible. Hence we place 1 in the fourth column. Next we have second and last two columns are valid. As $w_4 = 2$, we place 1 in the fourth column. The bottom most row has only one possible position and $w_5 = 0$. Therefore the transversal obtained is $4, 1, 5, 3, 2$ which avoids p'.

5 Proof of Theorem 8

In this Section, we present our proof of Theorem 8. The lemmas in this section are analogous to the lemmas from the last section and we only present proofs when the difference is substantial. The following are analogues of Lemmas 17 and Lemma 18.

Lemma 25. *At any step of the process, in the topmost row of white cells, there are at most two suitable cells to place 1 avoiding p or p'. For p, it is the last two cells on the top row. For p', it is the first and last cell on the top row.*

Lemma 26. *At some step of the process for p or p', let F' be the board made up of the remaining white cells. Suppose that the top row of F' comes from the row r and the columns $c_1 < \cdots < c_l$ of F. If the cells in F that are above and to the right of (c_l, r) already contain a 1, then there is at most one suitable position. To avoid p, it is the rightmost position on the top row. To avoid p', it is the leftmost position on the top row.*

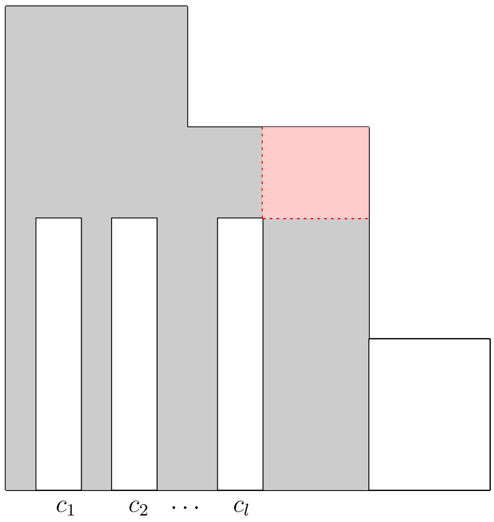


Figure 3: The cells (red) described in Lemma 26

Lemma 27. *At any step of the process, in the topmost row of white cells, there is at least one suitable position to place 1 avoiding p or p'.*

Proof. To avoid p, we can always place 1 in the rightmost position on the top row of the white cells. To avoid p', we can always place 1 in the leftmost position on the top row of the white cells. This position will be suitable. □

Using the above scheme, we create an encoding that maps transversals that avoid p to those that avoid p'. The encoding will be a word with the letters $\{0, 1, 2\}$, obtained in the following way. Let T be a transversal of F that avoids p. Let $w = w_1 \cdots w_n$ be the encoding word.

1. Remove all the 1's and place them back on the board according the process described above.

2. While placing 1 on the row i from the top, if there were two suitable positions on the top row and 1 appears in the first suitable position from the right in T, let $w_i = 1$. If it appears in the second suitable position from the right in T, let $w_i = 2$.

3. While placing 1 in row i, if there were only one suitable position on the top row, then let $w_i = 0$.

4. Once we have placed all the 1's, return $w = w_1 \cdots w_n$.

Now, to obtain a transversal of F that avoids p', we invert the encoding as follows. Let $w = w_1 \cdots w_n$ be the encoding word.

1. If $w_i = 1$, place 1 in the first suitable position from the right, in the topmost row of white cells. If $w_i = 2$, place 1 in the second suitable position, from the right, in the topmost row of white cells.
2. If $w_i = 0$, place it in the only suitable position available in the top row of white cells.
3. Return the transversal once all 1's are placed.

Similarly, we may start with a transversal that avoids p', obtain an encoding word and then, invert it to get a transversal that avoids p.

In the insertion process for p or for p', the shape of the board of white cells after i iterations will be the same (up to squashing and left aligning). Therefore, if there were only one white cell on the top row for p, it would be the case for p' and vice versa. We will often talk about the Ferrers board of white cells. Even though, they are not technically Ferrers boards, we use it to mean the Ferrers board obtained by squashing and left aligning the white cells. We will call an insertion step *forced* if there is only one suitable cell to insert at.

Lemma 28. *Let r_k denote the length of k-th row from the top. At some step i, if $w_i = 1$, then subsequent placements are forced for both p, p' till we reach a step j where $r_j > r_i$.*

Proof. Let $j > i$ with $r_j = r_i$. The top-right cells described in Lemma 26 will have the one placed in step i. Therefore, this step will be forced. If $r_j > r_i$, then, at step j, the last column will not have cells in the previous rows. Therefore, there will be no 1's in the cells described in Lemma 26 and will not be forced. □

Lemma 29. *If $w_i = 2$, then the step $i + 1$ is not forced for p or p' unless the top row has only one white cell.*

Proof. Let r_i, r_{i+1} be the row lengths of i-th and $(i + 1)$-th row from the top. If $r_i = r_{i+1}$, then a 1 in the top right cells described in Lemma 26 for step $i + 1$ would have been present in the top right cells described in Lemma 26 for step i. This forces $w_i = 0$, a contradiction. If $r_{i+1} > r_i$, then the last column of $(i + 1)$-th row from the top does not have cells in the i-th row from the top. Therefore, by Lemma 26, this step cannot be forced. □

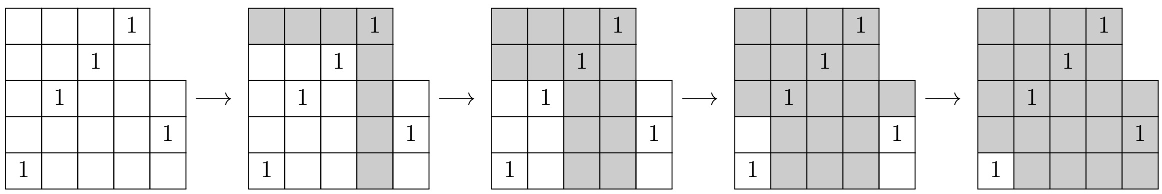
Remark 30. When we reach a step i where there is only one white cell in the top row, it means that, at step $i + 1$, the last i columns and top i rows of the board are gray. The remaining white board is completely to the left and below the already placed 1's and no 1 placed in the current white board, in the future, can form an occurrence of p or p' with a 1 in the last i columns. This is because the $(l - i)$-th column has no cells in the i-th row from the top.

With the help of these lemmas, we are ready to prove that this is a bijection.

Proof of Theorem 8. We claim that the encoding scheme gives a bijection. To do this, we first note that the encoding is injective and given an encoding, we can easily reverse the process and obtain a transversal. Therefore, it is enough to show that the encoding processes for both p, p' give the same encoding words. We proceed by induction on the size of the Ferrers board. It is easily verified for Ferrers boards of size 3. For smaller boards, all transversal avoid p and p'. Now, suppose it is true for all Ferrers board of size $< k$. We prove for boards of size k. Let F be a Ferrers board of size k. If the top most row of F has one cell, then we argue as follows. Let F' be the board obtained by deleting the first column of F. Any transversal of F' avoiding p or p' can be extended to a transversal of F avoiding p or p' and each transversal can be restricted to a transversal of F'. If the encoding words for F' match, then they match for F by induction.

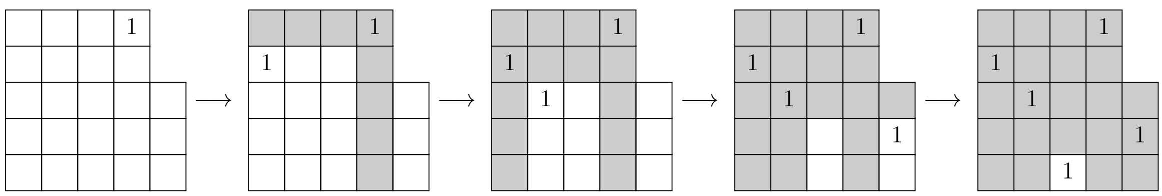
Suppose the first row has more than 1 cell, then Lemmas 28, 29 ensure that we reach a position where we have filled the first i columns and top i rows for some i (see the Remark preceding the proof). Let F_t be the Ferrers board formed by the first i columns and top i rows. By the lemmas, we know encoding words have been the same so far. Here, similar to our previous argument, we have that the white cells form a smaller Ferrers board F' and since, by induction, the encoding words match for F' and the encoding words for F are concatenations of words for F_t with those of F', we are through by induction. $\square$

Example 31. We work out an example for the board $(5, 5, 5, 4, 4)$. We can see that the transversal $1, 3, 4, 5, 2$ avoids p. We obtain the encoding word from the board and transversal.



In the topmost row there are two valid positions, last two columns. As 1 lies in the first valid position (from right), we get $w_1 = 1$. In the next row, the placement is forced by Lemma 28, hence $w_2 = 0$. In the next row (third) the valid positions are second and fifth columns. As 1 lies in the second valid position (from right), we get $w_3 = 2$. Next we have first and last two columns are valid, as 1 lies in the first valid position (from right), we get $w_4 = 1$. The bottom most row has only one possible position, hence $w_5 = 0$. Therefore the encoding word is $1, 0, 2, 1, 0$.

The boards below display the insertion process for the transversal avoiding p'.



In the topmost row there are two valid positions, first and last columns. As $w_1 = 1$, we place 1 in the last column, first valid position from right. In the next row, the placement is forced by Lemma 28 and $w_2 = 0$, so the placement is compatible. Hence we place 1 in the first column. In the next row (third) we have two valid positions, second and last columns. As $w_3 = 2$, we place 1 in the second column, second valid position from right. Next we have two valid positions, third and last columns. As $w_4 = 1$, we place 1 in the last column, first valid position from right. The bottom most row has only one possible position and $w_5 = 0$. Therefore the transversal obtained is $4, 3, 1, 5, 2$ which avoids p'.

6 Classification of POPs of size 3 with chain components

Before we give our classification of POPs of size 3, 4 and 5 all of whose components are chains, we recall the following result [15, Theorem 4] of Gao and Kitaev that describes the effect of having isolated vertices with labels that are completely smaller or larger than the labels of the remaining vertices.

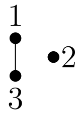
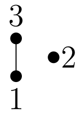
Theorem 32 (Gao and Kitaev). *Let p be a POP of size $k \geqslant 1$, and suppose that the set of labels of isolated vertices includes*

$$I = \{1, 2, \ldots, i\} \cup \{k - s + i + 1, k - s + i + 2, \ldots, k\}$$

for integers $0 \leqslant i \leqslant s \leqslant k$. Let q be the POP obtained from p by removing the elements corresponding to the labels in I. Then,

$$|\mathfrak{S}_n(p)| = \begin{cases} \dfrac{n!}{(n-s)!}\,|\mathfrak{S}_{n-s}(q)| & \text{if } n < k, \\ 0 & \text{if } n \geqslant k. \end{cases}$$

The connected POPs of size 3 are the classical patterns of size 3 and have received significant attention in previous studies. The number of permutations avoiding the pattern 123, $|\mathfrak{S}_n(123)|$, was done by MacMahon [22] and the number of permutations avoiding the pattern 132, $|\mathfrak{S}_n(132)|$, was done by Knuth [21]. Both cardinalities turn out to be the n-th Catalan number C_n. All other classical patterns of size 3 can be obtained from these two by applying the reverse and complement operations. Therefore, for any classical pattern p of size 3 we have $|\mathfrak{S}_n(p)| = C_n = \frac{1}{n+1}\binom{2n}{n}$. The shape-Wilf-equivalence of the classical patterns of size 3 $\begin{array}{c}2\\1\\3\end{array} \sim_s \begin{array}{c}1\\3\\2\end{array}$ was proved by Stankova, West [23] and $\begin{array}{c}1\\2\\3\end{array} \sim_s \begin{array}{c}3\\1\\2\end{array} \sim_s \begin{array}{c}3\\2\\1\end{array}$ was proved by Babson, West [1] and Backelin, West, Xin [2].

We consider the POPs of size 3 with 2 linear components. Theorem 7 shows that $1\bullet\ \begin{array}{c}2\\3\end{array} \sim_s 1\bullet\ \begin{array}{c}3\\2\end{array}$. By Theorem 32, we have $1\bullet\ \begin{array}{c}b\\c\end{array} \sim \begin{array}{c}b-1\\c-1\end{array}\ \bullet 3$ for $\{b, c\} = \{2, 3\}$, but they are not shape-Wilf-equivalent. The POP  is the reverse of . So, they are Wilf-equivalent but not shape-Wilf-equivalent. A detailed classification of these POPs is provided in Table1 below.

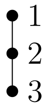
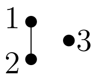
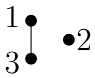
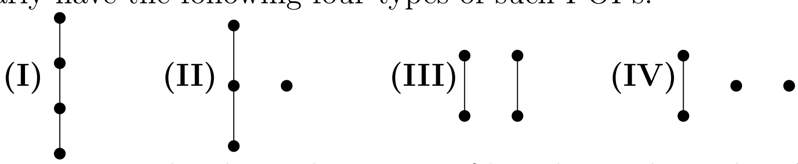
No.	POP (p)	$\lvert\mathfrak{S}_n(p)\rvert$ $(n = 1, 2, \ldots, 8, \ldots)$	OEIS
1		$1, 2, 5, 14, 42, 132, 429, 1430, \ldots$	A000108
2		$1, 2, 3, 4, 5, 6, 7, 8, \ldots$	A000027
3		$1, 2, 3, 5, 8, 13, 21, 34, \ldots$	A000045

Table 1: Classification of POPs of size 3

We will apply the Theorems 3, 4, 5, 6, 32 to classify patterns of size 4 and 5 consisting of chain components. For a given pattern p, our computations for $\mathfrak{S}_n(p)$ reveal multiple non-trivial equivalence classes, each represented by a row. Adjacent rows without separating lines indicate Wilf-equivalent classes, as discussed below. POPs in different rows are **not** Wilf-equivalent. Rows separated by a thick line represent patterns with distinct isolated vertices. Whenever the sequence is present on OEIS, we provide the OEIS entry. This convention will be followed throughout the rest of the paper.

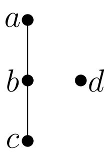
7 Classification of size 4 POPs whose components are chains

In this section, we will classify POPs of size 4 all of whose components are chains. We clearly have the following four types of such POPs.



Type I are the classical patterns of length 4 and are classified in [7], [8], [9], [14], [16], [24], while type IV POPs are classified in [15]. Thus, we classify only the type II and III POPs here.

7.1 Type II POPs

Let the POP  be denoted by $(a, b, c; d)$.

There are total of 24 POPs of type II. The complement and reverse of $(a, b, c; d)$ clearly are $(c, b, a; d)$ and $(5-a, 5-b, 5-c; 5-d)$ respectively. Therefore, POPs with $d = 1, 2$ are reverse of the POPs with $d = 4, 3$ respectively. As complementation and reverse will give Wilf-equivalent POPs, it is enough to consider the POPs with $d = 3, 4$.

As all the classical patterns of size 3, that is, all the connected linear POPs of size 3 are Wilf-equivalent, the POPs $(a, b, c; d)$ where $d = 1, 4$ are Wilf-equivalent, by Theorem 32. In [15, Theorem 24], the POP $p = (4, 1, 2; 3)$ was studied, and we will prove that all POPs $(a, b, c; d)$ with $d = 3$ are Wilf-equivalent to p. Let $p = (4, 1, 2; 3)$, $p' = (4, 2, 1; 3)$, $I = \{3\}$, $I_p = \begin{array}{c}1\\ \bullet\\ |\\ \bullet\\ 2\end{array}$, $I_{p'} = \begin{array}{c}2\\ \bullet\\ |\\ \bullet\\ 1\end{array}$ and $J_p = J_{p'} = \bullet\, 4$. As $I_p \sim_s I_{p'}$, we have $p \sim p'$ by Theorem 5. $(4, 2, 1; 3) \sim (2, 4, 1; 3)$ by Theorem 6.

Therefore, for type II, there are 2 Wilf-equivalence classes. A detailed classification of type II POPs is provided below Table 2.

No.	POP of Type II (p)	$\lvert\mathfrak{S}_n(p)\rvert$ $(n = 1, 2, \ldots, 8, \ldots)$	Ref and OEIS
1	$(1, 2, 3; 4)$ $(1, 3, 2; 4)$ $(3, 1, 2; 4)$	$1, 2, 6, 20, 70, 252, 924, 3432, \ldots$	Theorem 32 A000984
2	$(4, 1, 2; 3)$ $(4, 2, 1; 3)$ $(2, 4, 1; 3)$	$1, 2, 6, 20, 71, 264, 1015, 4002, \ldots$	Theorem 5, 6 A049124

Table 2: Classification of Type II POPs of size 4

7.2 Type III POPs

Let the POP $\begin{array}{c}a\bullet\\ |\\ b\bullet\end{array}\ \ \begin{array}{c}\bullet c\\ |\\ \bullet d\end{array}$ be denoted by $(a, b; c, d)$. The total number of Type III POPs is 24. The complement and reverse of $(a, b; c, d)$ are $(b, a; d, c)$ and $(5-a, 5-b; 5-c, 5-d)$ respectively. As $(a, b; c, d) = (c, d; a, b)$, it is enough to consider the POPs with $(a, b) \in \{(1, 2), (1, 3), (1, 4)\}$.

The POPs $(1, 2; 4, 3), (1, 3; 4, 2)$ and $(1, 4; 3, 2)$ were studied by Gao and Kitaev in [15, Theorem 15,16,18] respectively. Each of these POPs is in a separate Wilf-equivalence class. As $\begin{array}{c}1\bullet\\ |\\ 2\bullet\end{array} \sim \begin{array}{c}2\bullet\\ |\\ 1\bullet\end{array}$, we have $(1, 2; 3, 4) \sim (1, 2; 4, 3)$ (by Theorem 3). The POPs $(1, 3; 2, 4)$ and $(1, 4; 2, 3)$ are in 2 different Wilf-equivalence classes.

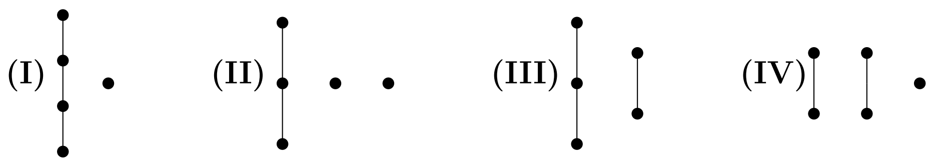
Thus, we get 5 Wilf-equivalent classes. It is easy to check that the remaining POPs are compositions of complement and reverse of POPs mentioned so far. Therefore, there are total 5 Wilf-equivalence classes of type III. A detailed classification of type III POPs is provided in Table 3.

No.	POP of Type III (p)	$\lvert\mathfrak{S}_n(p)\rvert$ $(n = 1, 2, \ldots, 8, \ldots)$	Ref and OEIS
1	$(1, 2; 4, 3)$ $(1, 2; 3, 4)$	$1, 2, 6, 18, 50, 130, 322, 770, \ldots$	Theorem 3 A048495
2	$(1, 3; 4, 2)$	$1, 2, 6, 18, 52, 152, 444, 1296, \ldots$	A077835
3	$(1, 3; 2, 4)$	$1, 2, 6, 18, 52, 147, 413, 1159, \ldots$	
4	$(1, 4; 3, 2)$	$1, 2, 6, 18, 50, 134, 358, 962, \ldots$	A271897
5	$(1, 4; 2, 3)$	$1, 2, 6, 18, 53, 156, 460, 1357, \ldots$	

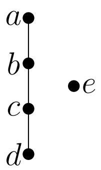
Table 3: Classification of Type III POPs of size 4

8 Classification of size 5 POPs whose components are chains

In this section, we classify all POPs of size 5 all of whose components are chains. Linear ordered POPs of size 5 are the classical patterns of size 5 that have already been classified by Clisby, Conway, Guttmann and Inoue [12]. Therefore, only the following four types of POPs remain to be classified.



8.1 Type I

Let the POP 

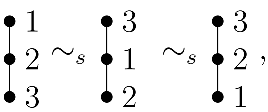
be denoted by $(a, b, c, d; e)$. The total number of Type I POPs is 120.

The complement and reverse of $(a, b, c, d; e)$ are $(d, c, b, a; e)$ and $(6-a, 6-b, 6-c, 6-d; 6-e)$ respectively. Therefore, the POPs with $e = 1, 2$ are the reverse of the POPs with $e = 5, 4$ respectively. Hence, it is enough to consider the POPs with $e = 3, 4, 5$.

Classical patterns of size 4 have been thoroughly studied, and their complete classification has been obtained in [7],[24], [26]. There are the following non-trivial Wilf-equivalences:

$$\{1234, 2143, 1243, 1432, 1423\}, \{1342, 2413\}, \{1324\}.$$

Other classical patterns of size 4 are trivially related to the previously mentioned patterns. Thus, there are 3 distinct classes of classical patterns of length 4. Therefore when $e = 5$, by Theorem 32, we have 3 distinct classes of POPs of type I.

By Theorem 6, we have $(5, 3, 2, 1; 4) \sim (3, 5, 2, 1; 4)$ and $(5, 3, 1, 2; 4) \sim (3, 5, 1, 2; 4)$. As 

, we have $(5, 3, 2, 1; 4) \sim (5, 3, 1, 2; 4) \sim (5, 1, 2, 3; 4)$ (by Theorem 5). Therefore, $(5, 3, 2, 1; 4)$, $(3, 5, 2, 1; 4)$, $(5, 3, 1, 2; 4)$, $(3, 5, 1, 2; 4)$ and $(5, 1, 2, 3; 4)$ are in the same Wilf-equivalence class. As $\begin{smallmatrix}2\\ \vert\\ 1\\ \vert\\ 3\end{smallmatrix} \sim_s \begin{smallmatrix}1\\ \vert\\ 3\\ \vert\\ 2\end{smallmatrix}$, we have $(5,2,1,3;4)\sim(5,1,3,2;4)$ (by Theorem 5). As $\begin{smallmatrix}2\\ \vert\\ 1\end{smallmatrix} \sim_s \begin{smallmatrix}1\\ \vert\\ 2\end{smallmatrix}$, by Theorem 5 we have $(5,4,2,1;3)\sim(5,4,1,2;3)$ and $(4,5,2,1;3)\sim(4,5,1,2;3)\sim(5,4,1,2;3)$. The POPs $(5,2,3,1;4)$, $(2,5,3,1;4)$, $(1,5,3,2;4)$, $(2,5,1,3;4)$, $(1,5,2,3;4)$, $(5,2,4,1;3)$, $(4,2,5,1;3)$, $(2,5,4,1;3)$, $(2,4,5,1;3)$, $(2,5,1,4;3)$ are in 10 different Wilf-equivalence classes.

So far, we have 16 Wilf-equivalent classes. It is easy to check that the remaining POPs are trivially related to the ones mentioned so far. Therefore, there are a total of 16 Wilf-equivalence classes of type I. Detailed classification of type I POPs is provided in Table 4.

Remark 33. This proves a conjecture of Dimitrov [13, Conjecture 8.1], which was also presented at the problem session of the British Combinatorial Conference, 2024 [11].

1. $1\square 234 \sim 1\square 243 \sim 2\square 143$.

2. $12\square 34 \sim 12\square 43 \sim 21\square 43$.

3. $123\square 4 \sim 124\square 3 \sim 214\square 3$.

Using the notation of POPs, we have $1\square 234 = (5,4,3,1;2)$, $1\square 243 = (4,5,3,1;2)$, $2\square 143 = (4,5,1,3;2)$. These are trivially related to $(5,3,2,1;4), (5,3,1,2;4), (3,5,1,2;4)$ respectively. These are in the same class and correspond to Row 4 of Table 4.

Similarly, $12\square 34 = (5,4,2,1;3)$, $12\square 43 = (4,5,2,1;3) \sim (5,4,1,2;3)$, and we have $21\square 43 = (4,5,1,2;3)$. These are in the same class and correspond to Row 11 of Table 4.

Finally, $123\square 4 = (5,3,2,1;4)$, $124\square 3 = (3,5,2,1;4)$, $214\square 3 = (3,5,1,2;4)$. This corresponds to Row 4 of Table 4.

8.2 Type II

Let the POP $\begin{smallmatrix}a\\ \vert\\ b\\ \vert\\ c\end{smallmatrix} \quad \bullet d \quad \bullet e$ be denoted by $(a,b,c;d;e)$. We clearly have 60 POPs of type II.

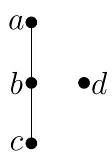
The complement and reverse of $(a,b,c;d,e)$ are $(c,b,a;d,e)$ and $(6-a,6-b,6-c;6-d,6-e)$ respectively. Therefore, POPs with $(d,e) = (4,1), (3,1), (2,1), (4,2), (3,2)$ are the reverse of POPs with $(d,e) = (2,5)$, $(3,5)$, $(4,5)$,$(2,4)$ and $(3,4)$ respectively.

As $(a,b,c;d;e) = (a,b,c;e;d)$, it is enough to consider the POPs with $(d,e) = (2,5)$, $(3,5)$,$(4,5)$,$(2,4)$,$(3,4)$.

Theorem 32 tells us that the POPs $(a,b,c;d;e)$ where $(d;e) = (4;5)$ and $(1;2)$ are Wilf-equivalent as all classical patterns of size 3 are Wilf-equivalent. In Section 7.1, we proved

No.	POP of Type I (p)	$\lvert\mathfrak{S}_n(p)\rvert$ $(n = 1, 2, \ldots, 8, \ldots)$	Ref and OEIS
1	$(1,2,3,4;5)$ $(2,1,4,3;5)$ $(2,1,3,4;5)$ $(3,2,1,4;5)$ $(3,1,2,4;5)$	$1,2,6,24,115,618,3591,22088,\ldots$	Theorem 32 A128088
2	$(2,3,1,4;5)$ $(3,1,4,2;5)$	$1,2,6,24,115,618,3584,21920,\ldots$	Theorem 32
3	$(1,3,2,4;5)$	$1,2,6,24,115,618,3591,22096,\ldots$	Theorem 32
4	$(5,3,2,1;4)$ $(5,3,1,2;4)$ $(5,1,2,3;4)$ $(3,5,2,1;4)$ $(3,5,1,2;4)$	$1,2,6,24,115,619,3612,22386,\ldots$	Theorem 5, 6
5	$(5,2,1,3;4)$ $(5,1,3,2;4)$	$1,2,6,24,115,618,3592,22102,\ldots$	Theorem 5
6	$(5,2,3,1;4)$	$1,2,6,24,115,619,3613,22412,\ldots$	
7	$(2,5,3,1;4)$	$1,2,6,24,115,619,3607,22257,\ldots$	
8	$(2,5,1,3;4)$	$1,2,6,24,115,618,3587,22000,\ldots$	
9	$(1,5,3,2;4)$	$1,2,6,24,115,619,3608,22293,\ldots$	
10	$(1,5,2,3;4)$	$1,2,6,24,115,619,3606,22232,\ldots$	
11	$(4,5,1,2;3)$ $(5,4,2,1;3)$ $(5,4,1,2;3)$	$1,2,6,24,115,619,3614,22425,\ldots$	Theorem 5
12	$(5,2,4,1;3)$	$1,2,6,24,115,619,3615,22457,\ldots$	
13	$(4,2,5,1;3)$	$1,2,6,24,115,619,3608,22272,\ldots$	
14	$(2,5,4,1;3)$	$1,2,6,24,115,619,3609,22297,\ldots$	
15	$(2,5,1,4;3)$	$1,2,6,24,115,619,3601,22147,\ldots$	
16	$(2,4,5,1;3)$	$1,2,6,24,115,619,3614,22426,\ldots$	

Table 4: Classification of Type I POPs of size 5

that all POPs of the shape 

where $d = 2$ or 3 are Wilf-equivalent. By Theorem 32, we see that the POPs $(a, b, c; d; e)$ where $\{d, e\} = \{2, 5\}, \{3, 5\}$ are Wilf-equivalent.

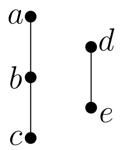
The POPs $(1, 3, 5; 2; 4)$, $(1, 5, 3; 2; 4)$, $(1, 2, 5; 3; 4)$ and $(1, 5, 2; 3; 4)$ are in 4 different Wilf-equivalence classes. We have $(1, 2, 5; 3; 4) \sim (5, 2, 1; 3; 4)$ and by Theorem 5 $(5, 2, 1; 3; 4) \sim (5, 1, 2; 3; 4)$.

So far, we have 6 Wilf-equivalent classes. It is easy to check that the remaining POPs are trivially related to POPs mentioned so far. Therefore, there are 6 Wilf-equivalence classes of type II. A detailed classification of type II POPs is provided in Table 5.

No.	POP of Type II (p)	$\lvert\mathfrak{S}_n(p)\rvert$ $(n = 1, 2, \ldots, 8, \ldots)$	Ref
1	$(1, 2, 3; 4; 5)$ $(1, 3, 2; 4; 5)$ $(3, 1, 2; 4; 5)$	$1, 2, 6, 24, 100, 420, 1764, 7392, \ldots$	Theorem 32
2	$(1, 3, 4; 2; 5)$ $(1, 4, 3; 2; 5)$ $(4, 1, 3; 2; 5)$ $(1, 2, 4; 3; 5)$ $(1, 4, 2; 3, 5)$ $(4, 1, 2; 3; 5)$	$1, 2, 6, 24, 100, 426, 1848, 8120, \ldots$	Theorem 32 Section 7.1
3	$(1, 3, 5; 2; 4)$	$1, 2, 6, 24, 100, 434, 1934, 8828, \ldots$	
4	$(1, 5, 3; 2; 4)$	$1, 2, 6, 24, 100, 430, 1889, 8494, \ldots$	
5	$(1, 2, 5; 3; 4)$ $(5, 1, 2; 3; 4)$	$1, 2, 6, 24, 100, 426, 1875, 8482, \ldots$	Theorem 5
6	$(1, 5, 2; 3; 4)$	$1, 2, 6, 24, 100, 426, 1855, 8278, \ldots$	

Table 5: Classification of Type II POPs of size 5

8.3 Type III

Let the POP 

be denoted by $(a, b, c; d, e)$. We have 120 POPs of type III. The complement and reverse of $(a, b, c; d, e)$ are $(c, b, a; e, d)$ and $(6-a, 6-b, 6-c; 6-d, 6-e)$ respectively. Therefore, the POPs with $(d, e) = (5, 4), (5, 3), (5, 2), (5, 1), (4, 3), (4, 2)$ are the reverse of the POPs with $(d, e) = (1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4)$ respectively. Therefore, it is enough to consider the POPs with $(d, e) = (1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4)$.

When $\{a, b, c\} = \{1, 2, 3\}$ and $\{d, e\} = \{4, 5\}$ or $\{a, b, c\} = \{3, 4, 5\}$ and $\{d, e\} = \{1, 2\}$, all POPs are Wilf-equivalent by Theorems 3 and 4. Under trivial bijections, the following POPs are in 18 (6 from each) different equivalence classes.

1. $\{a, b, c\} = \{1, 2, 4\}$ and $\{d, e\} = \{3, 5\}$ or $\{a, b, c\} = \{2, 4, 5\}$ and $\{d, e\} = \{1, 3\}$.
2. $\{a, b, c\} = \{1, 3, 4\}$ and $\{d, e\} = \{2, 5\}$ or $\{a, b, c\} = \{2, 3, 5\}$ and $\{d, e\} = \{1, 4\}$.
3. $\{a, b, c\} = \{1, 2, 5\}$ and $\{d, e\} = \{3, 4\}$ or $\{a, b, c\} = \{1, 4, 5\}$ and $\{d, e\} = \{2, 3\}$.

Similarly, under trivial bijections, the following POPs are in 8 (4 from each) different Wilf-equivalence classes.

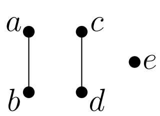
1. $\{a, b, c\} = \{1, 3, 5\}$ and $\{d, e\} = \{2, 4\}$.
2. $\{a, b, c\} = \{2, 3, 4\}$ and $\{d, e\} = \{1, 5\}$.

Therefore, there are 27 Wilf-equivalence classes of type III. A detailed classification of type III POPs is provided in Table 6.

No.	**POP of Type III** (p)	$\lvert\mathfrak{S}_n(p)\rvert$ $(n = 1, 2, \ldots, 8, \ldots)$	**Ref**
1	$(3, 4, 5; 1, 2)$ $(3, 5, 4; 1, 2)$ $(5, 3, 4; 1, 2)$	$1, 2, 6, 24, 110, 530, 2597, 12796, \ldots$	Theorem 3, 4
2	$(2, 4, 5; 1, 3)$	$1, 2, 6, 24, 110, 532, 2629, 13135, \ldots$	
3	$(2, 5, 4; 1, 3)$	$1, 2, 6, 24, 110, 532, 2632, 13188, \ldots$	
4	$(4, 2, 5; 1, 3)$	$1, 2, 6, 24, 110, 532, 2628, 13095, \ldots$	
5	$(5, 4, 2; 1, 3)$	$1, 2, 6, 24, 110, 533, 2658, 13527, \ldots$	
6	$(4, 5, 2; 1, 3)$	$1, 2, 6, 24, 110, 532, 2638, 13329, \ldots$	
7	$(5, 2, 4; 1, 3)$	$1, 2, 6, 24, 110, 533, 2640, 13195, \ldots$	
8	$(2, 3, 5; 1, 4)$	$1, 2, 6, 24, 110, 535, 2679, 13632, \ldots$	
9	$(2, 5, 3; 1, 4)$	$1, 2, 6, 24, 110, 535, 2690, 13836, \ldots$	
10	$(3, 2, 5; 1, 4)$	$1, 2, 6, 24, 110, 531, 2613, 12974, \ldots$	
11	$(5, 3, 2; 1, 4)$	$1, 2, 6, 24, 110, 531, 2601, 12817, \ldots$	
12	$(3, 5, 2; 1, 4)$	$1, 2, 6, 24, 110, 531, 2626, 13192, \ldots$	
13	$(5, 2, 3; 1, 4)$	$1, 2, 6, 24, 110, 534, 2666, 13534, \ldots$	
14	$(2, 3, 4; 1, 5)$	$1, 2, 6, 24, 110, 536, 2690, 13711, \ldots$	
15	$(2, 4, 3; 1, 5)$	$1, 2, 6, 24, 110, 530, 2595, 12759, \ldots$	
16	$(4, 3, 2; 1, 5)$	$1, 2, 6, 24, 110, 530, 2564, 12190, \ldots$	
17	$(3, 4, 2; 1, 5)$	$1, 2, 6, 24, 110, 530, 2575, 12407, \ldots$	
18	$(1, 4, 5; 2, 3)$	$1, 2, 6, 24, 110, 533, 2663, 13637, \ldots$	
19	$(1, 5, 4; 2, 3)$	$1, 2, 6, 24, 110, 534, 2678, 13748, \ldots$	
20	$(4, 1, 5; 2, 3)$	$1, 2, 6, 24, 110, 530, 2607, 12997, \ldots$	
21	$(5, 4, 1; 2, 3)$	$1, 2, 6, 24, 110, 530, 2605, 12996, \ldots$	
22	$(4, 5, 1; 2, 3)$	$1, 2, 6, 24, 110, 530, 2617, 13202, \ldots$	
23	$(5, 1, 4; 2, 3)$	$1, 2, 6, 24, 110, 533, 2633, 13156, \ldots$	
24	$(1, 3, 5; 2, 4)$	$1, 2, 6, 24, 110, 537, 2727, 14261, \ldots$	
25	$(1, 5, 3; 2, 4)$	$1, 2, 6, 24, 110, 533, 2673, 13757, \ldots$	
26	$(5, 3, 1; 2, 4)$	$1, 2, 6, 24, 110, 533, 2644, 13319, \ldots$	
27	$(3, 5, 1; 2, 4)$	$1, 2, 6, 24, 110, 532, 2628, 13175, \ldots$	

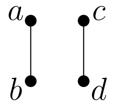
Table 6: Classification of Type III POPs of size 5

8.4 Type IV

Let the POP 

be denoted by $(a, b; c, d; e)$. We clearly have 120 POPs of type IV. The complement and reverse of $(a, b; c, d; e)$ are $(b, a; d, c; e)$ and $(6-a, 6-b; 6-c, 6-$

$d; 6-e)$ respectively. Therefore, POPs with $e = 1, 2$ are the reverse of POPs with $e = 5, 4$ respectively. As $(a, b; c, d; e) = (c, d; a, b; e)$, it is enough to consider POPs with $e = 3, 4, 5$ and with (a, b) begin one of $(1, 2), (1, 3), (1, 4), (1, 5)$.

In Section 7.2, we proved that there are 5 Wilf-equivalence classes of the shape . By Theorem 32, there are 5 Wilf-equivalence classes of POPs $(a, b; c, d; e)$ where $e = 1$ or 5. Theorem 3 implies $(1, 2; 3, 4; 5) \sim (1, 2; 4, 3; 5)$. By Theorem 4, we see that$(1, 2; 3, 4; 5) \sim (1, 2; 4, 5; 3)$ and $(1, 2; 4, 3; 5) \sim (1, 2; 5, 4; 3)$. By Theorem 3, we see that $(1, 2; 3, 5; 4) \sim (1, 2; 5, 3; 4)$. The POPs $(1, 4; 2, 5; 3), (1, 4; 5, 2; 3)$, $(1, 5; 2, 4; 3)$, $(1, 5; 4, 2; 3)$, $(1, 3; 2, 5; 4)$, $(1, 3; 5, 2; 4)$, $(1, 5; 2, 3; 4)$, $(1, 5; 3, 2; 4)$ are in 8 different Wilf-equivalence classes.

So far, we have 14 Wilf-equivalence classes. It is easy to check that the remaining POPs are trivially related to the POPs mentioned so far, giving us 14 Wilf-equivalence classes of type IV. A detailed classification of type IV POPs is provided in Table 7.

No.	**POP of Type IV** (p)	$\lvert\mathfrak{S}_n(p)\rvert$ $(n = 1, 2, \ldots, 8, \ldots)$	**Ref**
1	$(1, 2; 3, 4; 5)$ $(1, 2; 4, 3; 5)$ $(1, 2; 4, 5; 3)$ $(1, 2; 5, 4; 3)$	$1, 2, 6, 24, 90, 300, 910, 2576, \ldots$	Theorem 3, 4
2	$(1, 3; 4, 2; 5)$	$1, 2, 6, 24, 90, 312, 1064, 3552, \ldots$	
3	$(1, 3; 2, 4; 5)$	$1, 2, 6, 24, 90, 312, 1029, 3304, \ldots$	
4	$(1, 4; 3, 2; 5)$	$1, 2, 6, 24, 90, 300, 938, 2864, \ldots$	
5	$(1, 4; 2, 3; 5)$	$1, 2, 6, 24, 90, 318, 1092, 3680, \ldots$	
6	$(1, 2; 3, 5; 4)$ $(1, 2; 5, 3; 4)$	$1, 2, 6, 24, 90, 315, 1043, 3318, \ldots$	Theorem 3
7	$(1, 3; 2, 5; 4)$	$1, 2, 6, 24, 90, 311, 1034, 3401, \ldots$	
8	$(1, 3; 5, 2; 4)$	$1, 2, 6, 24, 90, 323, 1129, 3951, \ldots$	
9	$(1, 5; 2, 3; 4)$	$1, 2, 6, 24, 90, 310, 1034, 3458, \ldots$	
10	$(1, 5; 3, 2; 4)$	$1, 2, 6, 24, 90, 301, 914, 2768, \ldots$	
11	$(1, 4; 2, 5; 3)$	$1, 2, 6, 24, 90, 329, 1192, 4302, \ldots$	
12	$(1, 4; 5, 2; 3)$	$1, 2, 6, 24, 90, 310, 1088, 3888, \ldots$	
13	$(1, 5; 2, 4; 3)$	$1, 2, 6, 24, 90, 334, 1235, 4567, \ldots$	
14	$(1, 5; 4, 2; 3)$	$1, 2, 6, 24, 90, 316, 1009, 3210, \ldots$	

Table 7: Classification of Type IV POPs of size 5

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