Planar ternary graphs, flag spheres, and Delannoy polynomials

Margaret Bayer[a] Richard Danner[b] Thiago Holleben[c]
Marie Kramer[d] Yirong Yang[e]

Submitted: Oct 31, 2025; Accepted: Jul 1, 2026; Published: Sep 11, 2026

Abstract

In 2022 Kim showed that when a graph G is ternary (without induced cycles of length divisible by three), its independence complex $\mathrm{Ind}(G)$ is either contractible or homotopy equivalent to a sphere. In this paper, we show that when $\mathrm{Ind}(G)$ is homotopy equivalent to a sphere of dimension $\dim \mathrm{Ind}(G)$, the complex is Gorenstein. Equivalently, G is a 1-well-covered graph. This answers a question by Faridi and Holleben.

We then focus on the independence complexes of Gorenstein planar ternary graphs. We prove that they are boundaries of vertex decomposable simplicial polytopes. We show that the transformations among these flag spheres using edge subdivisions and contractions can be modeled by the Hasse diagram of the partition refinement poset. In addition, their h-polynomials are products of Delannoy polynomials and thus real-rooted. Finally, we demonstrate a way to construct nonplanar Gorenstein (1-well-covered) ternary graphs from planar ones.

Mathematics Subject Classifications: 05C10, 05C69, 55P15, 05E45

1 Introduction

In this paper, we explore the algebraic, combinatorial, geometric, and topological properties of the independence complexes of *ternary* graphs, graphs that do not contain induced cycles of length divisible by three. Independence complexes of graphs are *flag simplicial complexes*. One way to study the topological properties of these simplicial complexes is

[a]Department of Mathematics, University of Kansas, Lawrence, KS, U.S.A. (`bayer@ku.edu`).
[b]Department of Mathematics, University of Vermont, Burlington, VT, U.S.A. (`rsdanner@uvm.edu`).
[c]Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany (`holleben@mis.mpg.de`).
[d]Department of Mathematics & Statistics, Youngstown State University, Youngstown, OH, U.S.A. (`mkramer02@ysu.edu`).
[e]School of Mathematics, Georgia Institute of Technology, Atlanta, GA, U.S.A. (`yyang917@gatech.edu`).

to use the translation into algebraic properties of the associated Stanley–Reisner ring. For example, Stanley [20] proved that the Stanley–Reisner ring of a homology sphere is *Gorenstein*.

It turns out that the Gorenstein property for independence complexes is strongly related to properties of the graphs. Staples [21] introduced a class of graphs called *1-well-covered*, denoted W_2; these are defined by a property of the independent sets of the graph. This class of graphs was further studied by Pinter [17], and plays an important role in our work. Hoang and Trung [9] proved that for triangle-free graphs without isolated vertices, a graph G is in W_2 if and only if the independence complex of G is Gorenstein. Trung [22] determined the complete list of connected planar graphs in W_2 (and hence with Gorenstein independence complexes); the list contains exactly one graph for each independence number. These graphs (with one exception) are in fact ternary.

Kim [11] proved that a graph is ternary if and only if the independence complex of every induced subgraph is contractible or homotopy equivalent to a sphere. (This resolved a conjecture by Engström [5].) This led Faridi and Holleben [6] to ask the question of the connection between homotopy type and homology of these independence complexes. For our purposes we phrase the question as follows.

Question 1. Let G be a ternary graph without isolated vertices, $\mathrm{Ind}(G)$ its independence complex, and $d-1$ the dimension of $\mathrm{Ind}(G)$. Are the following two statements equivalent?

1. $\mathrm{Ind}(G)$ is homotopy equivalent to a $(d-1)$-dimensional sphere.
2. $\mathrm{Ind}(G)$ is a homology sphere.

In this paper, we give an affirmative answer to this question.

Theorem A. *Let G be a ternary graph without isolated vertices. Then $\mathrm{Ind}(G)$ is a homology sphere if and only if $\mathrm{Ind}(G)$ is homotopy equivalent to S^d, where $d = \dim \mathrm{Ind}(G)$.*

Moreover, using the characterization of planar graphs with independence complexes that are homology spheres ([22]), we are able to provide further details on the independence complex of planar ternary graphs.

In general, for $d \geqslant 4$, deciding if a simplicial $(d-1)$-sphere can be realized as the boundary of a d-polytope is a very hard problem. Our result below shows that being planar and ternary imposes not only strong topological restrictions on the independence complexes but also strong geometric ones.

Theorem B. *Let G be a planar ternary graph such that $\mathrm{Ind}(G)$ is a homology sphere. Then $\mathrm{Ind}(G)$ is combinatorially equivalent to the boundary of a simplicial polytope and is vertex decomposable.*

Remark 2. Theorems A and B together imply that if G is a planar ternary graph whose independence complex is $(d-1)$-dimensional, then $\mathrm{Ind}(G)$ is homotopy equivalent to a $(d-1)$-dimensional sphere if and only if $\mathrm{Ind}(G)$ is the boundary of a vertex decomposable simplicial polytope. Moreover, the only nonternary planar graph G for which $\mathrm{Ind}(G)$ is a homology sphere is the graph of the triangular prism, for which $\mathrm{Ind}(G)$ is the hexagon.

We study how the independence complexes of planar ternary graphs are related by edge subdivisions. Lutz and Nevo [14] showed that any two PL homeomorphic flag simplicial complexes can be connected by a sequence of edge subdivisions. For d-dimensional flag PL spheres we thus have a connected graph we call T_d: the graph has a vertex for each flag PL d-sphere and an edge from one sphere to another that is obtained by an edge subdivision.

In this paper, we describe the restriction H_d of T_d to independence complexes of planar ternary graphs. In the statement below, let $\mathcal{P}_n$ denote the graph of the Hasse diagram of the partition refinement poset of the partitions of n.

Theorem C. *The graph H_{n-1} is isomorphic to $\mathcal{P}_n$. In particular, $\mathcal{P}_n$ is an induced subgraph of T_{n-1} for every n.*

We then focus on combinatorial properties of the flag spheres in Theorem B. We begin by giving a combinatorial interpretation for their h-vectors, and as a consequence, using results from [23, 15] we prove the following.

Theorem D. *Let G be a planar graph such that $\mathrm{Ind}(G)$ is a homology sphere. Then the h-polynomial $h(\mathrm{Ind}(G), t)$ of $\mathrm{Ind}(G)$ is real-rooted. If additionally G is ternary and connected, then $h(\mathrm{Ind}(G), t)$ is a Delannoy polynomial.*

The paper is structured as follows. In Section 2 we recall important definitions and related results. In Section 3, we investigate ternary graphs and their independence complexes, proving Theorem A. We introduce and delve into the properties of the graph family $\{G_m\}$, which gives all triangle-free connected planar Gorenstein graphs without isolated vertices. The section ends with a proof of Theorem B. In Section 4, we further examine the effect of edge subdivisions and contractions on the flag spheres arising from G_m and prove Theorem C. Section 5 is dedicated to proving Theorem D. Along the way, we give an explicit formula for the h-polynomial of $\mathrm{Ind}(G_m)$ and find a connection between the h-vector and the Delannoy numbers. We conclude this paper by posing some open questions and giving future research directions that might be of interest in Section 6.

2 Background

2.1 Simplicial complexes

A *simplicial complex* Δ on a vertex set V is a collection of subsets of V such that $\tau \subset \sigma \in \Delta$ implies $\tau \in \Delta$. Elements of Δ are called *faces*, and maximal faces (with respect to inclusion) are called *facets*. The number $\dim \sigma = |\sigma| - 1$ is called the *dimension* of σ, and the dimension of Δ is the maximum dimension of its faces. If every facet of Δ has the same dimension, we say Δ is *pure*. Moreover, 0-dimensional faces are called vertices, and 1-dimensional faces are called edges. For 0- and 1-dimensional faces, we often omit the usual set notation if it does not cause ambiguity. That is, if $\{x\}$ is a face of Δ, we denote it by x, and if $\{x, y\}$ is a face of Δ, we denote it by xy.

2.2 Graphs and independence complexes

1-dimensional simplicial complexes are exactly *simple graphs* (or just "graphs" from now on). Given a graph G, an *independent set* of G is a collection of vertices S of G such that there is no edge in G between elements in S. The *independence number*, denoted $\alpha(G)$, is the size of the largest independent set of G. The collection of independent sets of G forms a simplicial complex called the *independence complex* of G, which we denote by $\mathrm{Ind}(G)$. Simplicial complexes that arise as independence complexes of graphs are *flag complexes*, complexes Δ where the minimal sets (with respect to inclusion) not in Δ have size two; in other words, every minimal nonface is a missing edge. Figure 1 illustrates two graphs and their independence complexes.

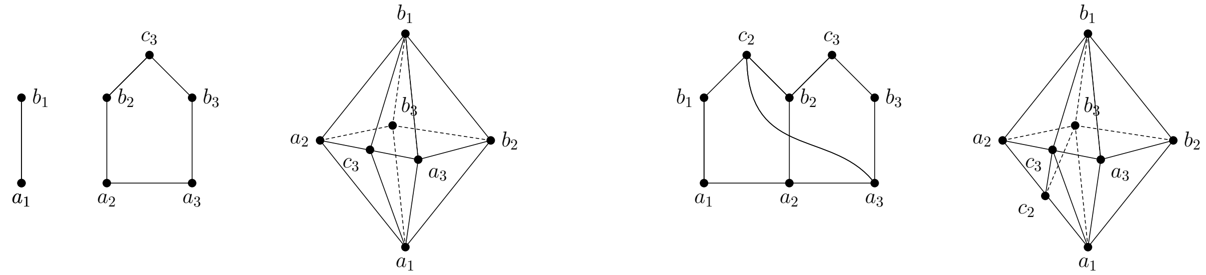


Figure 1: Two graphs whose independence complexes are spheres

The *induced subcomplex* of Δ on vertex set A is the complex $\Delta[A]$ on vertex set A with faces $\{\sigma \in \Delta : \sigma \subseteq A\}$. In the case of a graph G with vertex set V, we denote the induced subgraph of G on the set $V \setminus A$ as $G - A$.

Remark 3 (*Well-covered graphs and pure flag complexes*). A graph G is *well-covered* if all its maximal independent sets have the same cardinality. Since the maximal independent sets of G correspond to the facets of $\mathrm{Ind}(G)$, this is equivalent to saying $\mathrm{Ind}(G)$ is pure of dimension $\alpha(G) - 1$. A well-covered graph is called 1-*well-covered* if the deletion of any vertex leaves a graph that is also well-covered. In 1979, Staples [21] introduced the class of W_n graphs, which consists of graphs where every n disjoint independent sets are contained in n disjoint maximum independent sets. Levit and Mandrescu [12, Theorem 2.2] then showed that 1-well-covered graphs without isolated vertices are exactly the graphs in the class W_2.

2.3 Links, deletions, stars and joins

If Δ is a simplicial complex and v is a vertex of Δ, we denote by $\mathrm{lk}_\Delta(v) = \{\tau \in \Delta \colon v \notin \tau \quad \text{and} \quad \tau \cup v \in \Delta\}$ the *link* of v in Δ, $\mathrm{del}_\Delta(v) = \{\tau \in \Delta \colon v \notin \tau\}$ the *deletion* of v in Δ, and $\mathrm{st}_\Delta(v) = \{\tau \in \Delta \colon \tau \cup v \in \Delta\}$ the *star* of v in Δ. If there exists a vertex v of Δ such that $\mathrm{st}_\Delta(v) = \Delta$, then Δ is a *cone with apex* v, and if Δ is a cone with apex v for every $v \in \sigma$ for some $\sigma \in \Delta$, we say Δ is a *cone with* σ. Note that $\mathrm{lk}_\Delta(v) = \mathrm{del}_\Delta(v) \cap \mathrm{st}_\Delta(v)$ and $\Delta = \mathrm{del}_\Delta(v) \cup \mathrm{st}_\Delta(v)$. More generally, the link of a face σ of Δ is the complex $\mathrm{lk}_\Delta(\sigma) = \{\tau \in \Delta \colon \tau \cup \sigma \in \Delta \quad \text{and} \quad \tau \cap \sigma = \emptyset\}$. The coning operation is a special case of the joining operation; given two simplicial complexes Δ and Γ on disjoint vertex sets, the *join* of Δ and Γ is $\Delta * \Gamma = \{\tau \sqcup \sigma : \tau \in \Delta, \sigma \in \Gamma\}$.

For a vertex v of a graph G with vertex set V and edge set E, we denote by $N(v) = \{u \in V : \{u, v\} \in E\}$ the *neighborhood of* v and by $N[v] = \{v\} \cup N(v)$ the *closed neighborhood of* v. We will often use the following observations:

$$\mathrm{lk}_{\mathrm{Ind}(G)}(v) = \mathrm{Ind}(G - N[v]), \quad \mathrm{del}_{\mathrm{Ind}(G)}(v) = \mathrm{Ind}(G - v) \text{ and } \mathrm{st}_{\mathrm{Ind}(G)}(v) = \mathrm{Ind}(G - N(v)).$$

Moreover, if G and H are disjoint graphs, then one can check by the definition of independent set that

$$\mathrm{Ind}(G \sqcup H) = \mathrm{Ind}(G) * \mathrm{Ind}(H).$$

2.4 Cohen–Macaulay complexes, Gorenstein graphs, homology spheres and pseudomanifolds

A simplicial complex Δ is *Cohen–Macaulay* over a field $\mathbb{K}$ if for every $\sigma \in \Delta$

$$\tilde{H}_i(\mathrm{lk}_\Delta(\sigma); \mathbb{K}) = 0 \quad \text{for} \quad i < \dim \mathrm{lk}_\Delta(\sigma),$$

where $\tilde{H}_i(\Gamma; \mathbb{K})$ denotes the reduced i-th simplicial homology group of Γ over $\mathbb{K}$. If moreover Δ satisfies $\tilde{H}_{\dim \mathrm{lk}_\Delta(\sigma)}(\mathrm{lk}_\Delta(\sigma); \mathbb{K}) \cong \mathbb{K}$ for every $\sigma \in \Delta$, then we say Δ is a $\mathbb{K}$-*homology sphere*. Finally, if Δ is a homology sphere, or is a cone with σ and $\mathrm{lk}_\Delta(\sigma)$ is a homology sphere, then we say Δ is *Gorenstein*. A graph G for which $\mathrm{Ind}(G)$ is Gorenstein is called a *Gorenstein graph* (for more details see [20, II Theorem 5.1]).

A $(d-1)$-dimensional simplicial complex Δ is a *pseudomanifold* if the following 3 conditions are satisfied:

1. Δ is pure;

2. Given any two facets F_1, F_2 of Δ, there exists a sequence of facets $G_0, \ldots, G_s$ such that $G_0 = F_1$, $G_s = F_2$ and $|G_i \cap G_{i+1}| = d - 1$. In other words, Δ is *strongly connected*; and

3. Every $(d-2)$-dimensional face of Δ is contained in at most two facets of Δ.

The *boundary* of a $(d-1)$-dimensional pseudomanifold Δ is the collection of $(d-2)$-faces of Δ contained in exactly one facet. If the boundary of a pseudomanifold is empty, we say it is a pseudomanifold without boundary.

It is well known that a Cohen–Macaulay complex Δ is pure and strongly connected, and hence satisfies properties (1) and (2). In particular, a Cohen–Macaulay complex Δ can only fail to be a pseudomanifold if Δ has a $(d-2)$-face contained in more than two facets of Δ.

Remark 4. It turns out that the class of W_2 graphs (see Remark 3) plays a key role in the study of flag homology spheres. It was shown in [8, Proposition 3.7] that a triangle-free graph G without isolated vertices has a Gorenstein independence complex if and only if G is a W_2 graph.

2.5 Vertex-decomposability, simplicial spheres, and polytopes

A pure simplicial complex Δ is *vertex decomposable* if it is a simplex (including $\emptyset$) or there is a vertex $v \in \Delta$ such that the link $\mathrm{lk}_\Delta(v)$ and deletion $\mathrm{del}_\Delta(v)$ are vertex decomposable simplicial complexes.

A *polytope* P is the convex hull of finitely many points in a Euclidean space. Its *dimension* is given by the dimension of the affine hull of P. For a d-polytope $P \subset \mathbb{R}^d$, the proper *faces* of P are the intersections of P with its *supporting hyperplanes*. The polytope P is *simplicial* if all of its $(d-1)$-dimensional faces are simplices. The *boundary complex* of a simplicial d-polytope is the simplicial complex consisting of the vertex sets of all faces of dimension less than d. We refer the reader to [24] for more exposition on polytopes.

A $(d-1)$-dimensional simplicial complex is a *simplicial* $(d-1)$*-sphere* if its geometric realization in Euclidean space is homeomorphic to a $(d-1)$-sphere. Simplicial spheres are examples of pseudomanifolds and are necessarily homology spheres. The Poincaré homology sphere is a homology sphere but not a simplicial sphere.

The boundary complex of any simplicial polytope is a simplicial sphere. A simplicial $(d-1)$-sphere is said to be *polytopal* if it can be realized as the boundary complex of a simplicial d-polytope. An example of a nonpolytopal simplicial 3-sphere is the Barnette sphere [2].

2.6 PL spheres and stellar moves.

A simplicial $(d-1)$-sphere is *piecewise linear (PL)* if it is PL homeomorphic to the boundary of a d-simplex. Let Δ be a simplicial complex and σ be a face of Δ. Then the *stellar subdivision* of Δ at σ is given by

1. first, introducing a new vertex a in the interior of σ, and
2. next, replacing $\mathrm{st}_\Delta(\sigma)$ in Δ with $a * \partial\sigma * \mathrm{lk}_\Delta(\sigma)$.

The inverse operation is called a *stellar weld*. When σ is an edge of Δ, these operations are also called *edge subdivision* (see also Theorem 16) and *edge contraction*.

A classical result by Alexander [1] states that two simplicial complexes are PL homeomorphic if and only if they can be connected by a sequence of stellar moves. More recently, Lutz and Nevo [14] showed that two flag simplicial spheres are PL homeomorphic if and only if they can be connected by a sequence of edge subdivisions and contractions such that all the complexes in the sequence are flag. The barycentric subdivision of the double suspension of the Poincaré homology sphere is an example of a non-PL flag sphere.

2.7 f-, h-, γ-vectors and γ-positivity

For a simplicial complex Δ of dimension $d-1$, we denote by $f_i(\Delta)$ the number of i-dimensional faces of Δ. The *f-vector of* Δ is the sequence $f(\Delta) = (f_{-1}(\Delta), \ldots, f_{d-1}(\Delta))$, and the f-polynomial is the generating function of $f(\Delta)$. The *h-vector of* Δ is the sequence

$h(\Delta) = (h_0(\Delta), \ldots, h_d(\Delta))$ obtained from $f(\Delta)$ via the polynomial relations

$$\sum_{i=0}^{d} h_i(\Delta)t^{d-i} = \sum_{i=0}^{d} f_{i-1}(\Delta)(t-1)^{d-i}.$$

The *f-polynomial* and *h-polynomial* of Δ are respectively $f(\Delta, t) := \sum_{i=0}^{d} f_{i-1}(\Delta)t^i$ and $h(\Delta, t) := \sum_{i=0}^{d} h_i(\Delta)t^i$. It is well-known that

$$f(\Delta * \Gamma, t) = f(\Delta, t) \cdot f(\Gamma, t) \quad \text{and} \quad h(\Delta * \Gamma, t) = h(\Delta, t) \cdot h(\Gamma, t).$$

If it is clear from the context, we may omit Δ in the previous definitions.

When Δ is a $(d-1)$-dimensional homology sphere, the h-vector of Δ satisfies the Dehn–Sommerville relations: $h_i = h_{d-i}$ for $0 \leqslant i \leqslant d$. It is then possible to express $h(\Delta, t)$ as $h(\Delta, t) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i (1+t)^{d-2i}$. The *$\gamma$-vector* of Δ is the sequence $\gamma(\Delta) = (\gamma_0, \ldots, \gamma_{\lfloor d/2 \rfloor})$. If $\gamma_i \geqslant 0$ for every i, we say $h(\Delta, t)$ is *γ-positive* (or simply Δ is γ-positive). In 2005, Gal [7] conjectured that every flag homology sphere is γ-positive. As is pointed out in [7, Remark 3.1.1] (see also [4, Lemma 4.1]), if every root of the polynomial $h(\Delta, t)$ is real, then $h(\Delta, t)$ is γ-positive.

3 Flag spheres arising from ternary graphs

In 1978, Lovász [13] proved Kneser's conjecture by showing connections between topological invariants of certain spaces arising from a graph G and the chromatic number of G. Since then, many topological obstructions to graph coloring have been shown. One particular example can be found in the class of *ternary graphs*, that is, graphs without induced cycles of length divisible by three. Two conjectures of Kalai and Meshulam [10, Conjectures 1, 2] would imply there exists a constant C, such that the chromatic number of every ternary graph G is bounded above by C. The bound on chromatic numbers of ternary graphs was proven by Bonamy, Charbit, and Thomassé in [3], while the following topological characterization was proven by Kim.

Theorem 5 ([11, Theorem 1.1])**.** *A graph G is ternary if and only if the independence complex of every induced subgraph of G is either contractible or homotopy equivalent to a sphere.*

In this section, we take a closer look at ternary graphs and their independence complexes. We prove Theorem A characterizing when the independence complex of a ternary graph is a homology sphere. Then we turn our attention to a family of graphs $\{G_m\}$, which together with the graph R_3 give all planar Gorenstein graphs [22]. We show that each G_m is ternary and provide some additional observations about their structure. Finally, we prove that the independence complex of each G_m can be realized as the boundary of a polytope.

For a ternary graph G and a sequence of vertices $v_1, \ldots, v_s$ of G, a direct consequence of Theorem 5 is that by recursively taking links (or deletions) of the v_i, the resulting complex is either contractible or homotopy equivalent to a sphere. This property should be contrasted with the definition of homology spheres. On the one hand, subcomplexes of homology spheres obtained by taking links always have the homology of a sphere of the highest possible dimension. On the other hand, at the homology level, the condition from Theorem 5 includes more subcomplexes than just the ones obtained from links, but it does not require the spheres to be of the highest possible dimension. In [6] the authors introduce the class of *spherical complexes* as the class of complexes satisfying the topological condition arising from Theorem 5. The algebraic perspective from [6] led the authors to conjecture a characterization of Gorenstein spherical complexes, which we state for ternary graphs. We give an affirmative answer to [6, Question 6.5] below.

Lemma 6. *Let G be a ternary graph and $d = \dim \operatorname{Ind}(G)$. If $\operatorname{Ind}(G)$ is homotopy equivalent to a d-dimensional sphere, then $\operatorname{Ind}(G)$ is Cohen–Macaulay.*

Proof. Let $\Delta = \operatorname{Ind}(G)$. By [6, Theorem 3.6], we know that for any $\sigma \in \Delta$ either $\operatorname{lk}_\Delta(\sigma)$ has trivial homology or $\operatorname{lk}_\Delta(\sigma)$ has the homology of a d'-dimensional sphere, where

$$d - |\sigma| \leqslant d' \leqslant d.$$

In particular, $\tilde{H}_i(\operatorname{lk}_\Delta(\sigma); \mathbb{K}) = 0$ for $i < d - |\sigma|$, and since $\dim \operatorname{lk}_\Delta(\sigma) \leqslant d - |\sigma|$, we conclude Δ is Cohen–Macaulay. $\square$

Lemma 7. *Let G be a ternary graph. If $\operatorname{Ind}(G)$ is Cohen–Macaulay, then $\operatorname{Ind}(G)$ is a pseudomanifold.*

Proof. Since $\operatorname{Ind}(G)$ is Cohen–Macaulay, we conclude it is pure and strongly connected. Let σ be a $(d-1)$-dimensional face of $\operatorname{Ind}(G)$. Then $\operatorname{lk}_{\operatorname{Ind}(G)}(\sigma) = \operatorname{Ind}(G - N[\sigma])$, and since $G - N[\sigma]$ is also a ternary graph, we conclude the 0-dimensional complex $\operatorname{lk}_{\operatorname{Ind}(G)}(\sigma)$ is either contractible or homotopy equivalent to S^0. More specifically, $\operatorname{lk}_{\operatorname{Ind}(G)}(\sigma)$ is either one point or two points. In other words, there are at most two vertices of $\operatorname{Ind}(G)$ such that $v \notin \sigma$ and $\sigma \cup v \in \operatorname{Ind}(G)$. This property, together with the fact that $\operatorname{Ind}(G)$ is pure and strongly connected, implies $\operatorname{Ind}(G)$ is a (Cohen–Macaulay) pseudomanifold. $\square$

Theorem A. *Let G be a ternary graph without isolated vertices. Then $\operatorname{Ind}(G)$ is a homology sphere if and only if $\operatorname{Ind}(G)$ is homotopy equivalent to S^d, where $d = \dim \operatorname{Ind}(G)$.*

Proof. Let G be a ternary graph without isolated vertices. Suppose $\operatorname{Ind}(G)$ is a homology sphere. Then $\operatorname{Ind}(G)$ has the same homology as S^d by [20, Theorem II.5.1]. Theorem 5 then implies $\operatorname{Ind}(G)$ is homotopy equivalent to S^d.

Suppose now that $\operatorname{Ind}(G)$ is homotopy equivalent to S^d, where $d = \dim \operatorname{Ind}(G)$. By Theorem 6 we know $\operatorname{Ind}(G)$ is Cohen–Macaulay, hence by Theorem 7 we conclude $\operatorname{Ind}(G)$ is a Cohen–Macaulay pseudomanifold.

We now show that $\operatorname{Ind}(G)$ is a Cohen–Macaulay pseudomanifold without boundary. Since $\operatorname{Ind}(G)$ is Cohen–Macaulay and homotopy equivalent to S^d, $\tilde{H}_d(\operatorname{Ind}(G); \mathbb{Z}_2) = \mathbb{Z}_2$. Consider the corresponding chain complex

$$0 \longrightarrow C_d \xrightarrow{\partial_d} C_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \longrightarrow 0.$$

Since $\tilde{H}_d$ is nontrivial, there exists a nonempty set $\{F_i\}$ of facets of $\mathrm{Ind}(G)$ such that $\partial_d(\sum F_i) = 0$. Let Δ be the subcomplex of $\mathrm{Ind}(G)$ generated by $\{F_i\}$. Since $\partial_d(\sum F_i) = 0$, every ridge of Δ is contained in an even number of facets, and thus it must be contained in exactly two since $\mathrm{Ind}(G)$ is a pseudomanifold. Thus, Δ is a d-dimensional pseudomanifold without boundary, forcing $\Delta = \mathrm{Ind}(G)$.

Finally, since $\mathrm{Ind}(G)$ is a Cohen–Macaulay orientable pseudomanifold, [20, Theorem II.5.1] directly implies $\mathrm{Ind}(G)$ is Gorenstein. □

Remark 8. Although we state Theorem A for ternary graphs, as it fits the context of this paper, the same proof works for arbitrary spherical complexes (see [6, Definition 2.3]).

In [17] a recursive construction of planar 1-well-covered graphs, G_m, $m \in \mathbb{N}$, of girth four is given.

Definition 9. Let G_1 be the single edge a_1b_1, and G_2 be the 5-cycle $(a_1, b_1, c_2, b_2, a_2, a_1)$. For $m \geqslant 3$, define G_m to be the graph with vertex and edge sets V_m and E_m, where

$$V_m = V(G_m) = V(G_{m-1}) \cup \{a_m, b_m, c_m\} \quad \text{and}$$
$$E_m = E(G_m) = E(G_{m-1}) \cup \{a_{m-1}a_m, b_{m-1}c_m, c_{m-1}a_m, a_mb_m, b_mc_m\}.$$

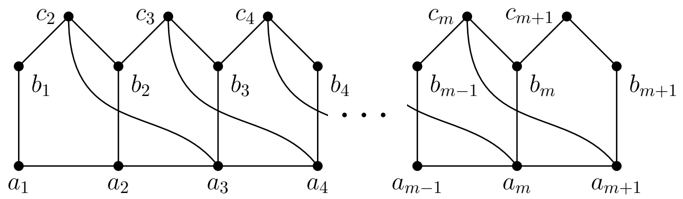


Figure 2: G_m

See Figures 2 and 3 for representations of G_m and G_3 respectively. In [9] this construction is shown to produce Gorenstein graphs, and in [22] it is shown that these are all the triangle-free, connected, planar (see Figure 3), Gorenstein graphs without isolated vertices.

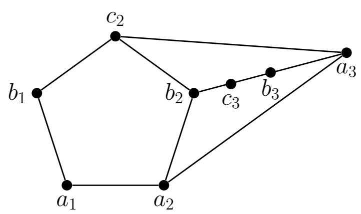


Figure 3: G_3 is planar

More generally, the following holds. Here R_3 is the graph of a triangular prism (see Figure 4). The independence complex of R_3 is the complement of R_3, which is a 6-cycle.

Theorem 10 ([22, Theorem 3.7]). *If a planar graph G is Gorenstein, then G is a disjoint union of graphs of the form G_m and R_3.*

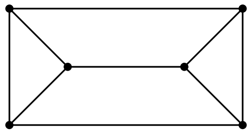
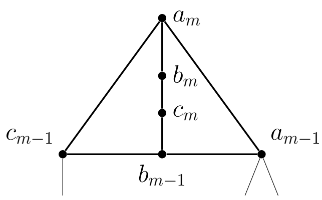


Figure 4: The graph R_3

We now establish some more properties of G_m for any $m \in \mathbb{N}$, unless otherwise stated. Recall that it was proven in [22] that G_m is triangle-free. In fact, we can show the following stronger statement.

Lemma 11. *G_m is ternary for all m.*

Proof. We proceed by induction on m. Recall that G_1 is isomorphic to a single edge K_2, and G_2 is isomorphic to a 5-cycle C_5. Thus, we immediately see that G_1 and G_2 are ternary. Now assume that G_{m-1} is ternary for some $m \geqslant 3$. Recall that $V(G_m) = V(G_{m-1}) \cup \{a_m, b_m, c_m\}$ and $E(G_m) = E(G_{m-1}) \cup \{a_{m-1}a_m, b_{m-1}c_m, c_{m-1}a_m, a_m b_m, b_m c_m\}$.



Note that all induced cycles in G_{m-1} are induced cycles in G_m. As G_{m-1} is ternary, none of these induced cycles have length $3k$ for any $k \in \mathbb{N}$. Additionally, G_m has induced 5-cycles $(a_{m-1}, b_{m-1}, c_m, b_m, a_m, a_{m-1})$ and $(b_{m-1}, c_{m-1}, a_m, b_m, c_m, b_{m-1})$, and an induced 4-cycle $(a_{m-1}, b_{m-1}, c_{m-1}, a_m, a_{m-1})$. Finally, we can build induced cycles in G_m from induced cycles in G_{m-1} as follows. Observe that as $\deg_{G_{m-1}}(b_{m-1}) = 2$, a cycle in G_{m-1} contains the edge $a_{m-1}b_{m-1}$ if and only if it contains the edge $b_{m-1}c_{m-1}$. Let C be such a cycle in G_{m-1}, in particular, $\operatorname{len}(C) \neq 3k$ for any $k \in \mathbb{N}$. Replacing the path $(a_{m-1}, b_{m-1}, c_{m-1})$ in C by the path (a_{m-1}, a_m, c_{m-1}) results in an induced cycle C' in G_m of length $\operatorname{len}(C') = \operatorname{len}(C) \neq 3k$. We can now conclude that none of the induced cycles in G_m have length $3k$ for any $k \in \mathbb{N}$, and thus G_m is ternary as desired. $\square$

Remark 12. In view of Theorems 10 and 11, we will often state our results for planar ternary graphs. For results we state with this assumption, there is a less clean version including the graph R_3, which is the only nonternary connected planar Gorenstein graph. For the sake of brevity, we omit these modified versions.

Having excluded the existence of induced cycles of length divisible by three, we now investigate the existence and interactions of other cycles in G_m.

Proposition 13. *Let $m \geqslant 1$. Then G_m has the following properties:*

(a) Any two distinct 4-cycles in G_m are edge-disjoint.

(b) G_m has no 6-cycles.

(c) Any pair consisting of a 4-cycle and a 5-cycle in G_m is either edge-disjoint or intersects in a 2-path.

Proof.

(a) By construction of G_m, all of its 4-cycles are of the form $(a_{t-1}, a_t, c_{t-1}, b_{t-1}, a_{t-1})$ for some $t \geqslant 3$. Thus, any two distinct 4-cycles are edge-disjoint.

(b) Suppose G_m had a 6-cycle, say $(x_1, x_2, \ldots, x_6, x_1)$. As G_m is ternary, there exists an edge $x_i x_{i+3}$ (indices mod 6) for some $i \in [6]$. However, this gives two distinct 4-cycles sharing one edge, which do not exist by part (a). Hence, G_m has no 6-cycles.

(c) Recall that every 4-cycle in G_m is of the form $(a_{t-1}, a_t, c_{t-1}, b_{t-1}, a_{t-1})$ for some $t \geqslant 3$. By construction of G_m, every 5-cycle is of the form $(a_{t-1}, a_t, b_t, c_t, b_{t-1}, a_{t-1})$ for some $t \geqslant 2$, $(a_{t-2}, a_{t-1}, a_t, c_{t-1}, b_{t-2}, a_{t-2})$ for some $t \geqslant 3$, $(a_t, b_t, c_t, b_{t-1}, c_{t-1}, a_t)$ for some $t \geqslant 3$, or $(a_{t-1}, a_t, c_{t-1}, b_{t-2}, c_{t-2}, a_{t-1})$ for some $t \geqslant 4$. If a 4-cycle and a 5-cycle are not edge-disjoint, it is now straightforward to see that the intersection is a 2-path. $\square$

We now turn our attention to induced paths in G_m. We find that unless two vertices are close to each other, there exist multiple induced paths of different lengths between them.

Lemma 14. *Let $x, y \in V(G_m)$. If $\operatorname{dist}(x, y) \geqslant 3$, then there exist induced paths of length $0, 1,$ and $2 \mod 3$ between x and y in G_m.*

Proof. In the following, all paths are induced paths unless otherwise stated. We start by finding paths of different lengths between a_k and a_{k+l}, and between b_k and b_{k+l}. These paths are a crucial step in finding the desired paths claimed in the statement of the lemma. Let $k \in [m]$ and $l \geqslant 0$ such that $k + l \leqslant m$. A *type–$(1, a)$–path* is a path of length l of the form $(a_k, a_{k+1}, \ldots, a_{k+l})$, where $l \geqslant 1$. A *type–$(2, a)$–path* is a path of length $l + 1$ of the form $(a_k, b_k, c_{k+1}, a_{k+2}, a_{k+3}, \ldots, a_{k+l})$, where $l \geqslant 2$. A *type–$(3, a)$–path* is a path of length $l + 2$ of the form $(a_k, b_k, c_{k+1}, b_{k+1}, c_{k+2}, a_{k+3}, a_{k+4}, \ldots, a_{k+l})$, where $l \geqslant 3$. (See Figure 5.) Observe that these three paths immediately give the desired paths if $x, y \in \{a_i\}_{i \in [m]}$. Moreover, if $x = a_i$ and $y = b_j$, we obtain the desired paths as follows. Note that since $\operatorname{dist}(x, y) \geqslant 3$, we may disregard $j \in \{i-2, i-1, i, i+1\}$. If $j \geqslant i+3$, concatenating a type–(t, a)–path for $t \in [3]$ with the edge $a_j b_j$ gives the desired paths. Similarly, if $j \leqslant i-4$, concatenating the path (b_j, a_j, a_{j+1}) with a type–(t, a)–path for $t \in [3]$ gives the desired paths. For the remaining two cases, we refer to Table 1.

If $x = a_i$ and $y = c_j$, we obtain the desired paths as follows. Again, since $\operatorname{dist}(x, y) \geqslant 3$, we may disregard $j \in \{i-2, i-1, i, i+1\}$. If $j \geqslant i+3$, concatenating a type–(t, a)–path for $t \in [3]$ with the path (a_j, b_j, c_j) gives the desired paths. Similarly, if $j \leqslant i-4$, concatenating the edge $c_j a_{j+1}$ with a type–(t, a)–path for $t \in [3]$ gives the desired paths. For the remaining two cases, we refer to Table 1. $\square$

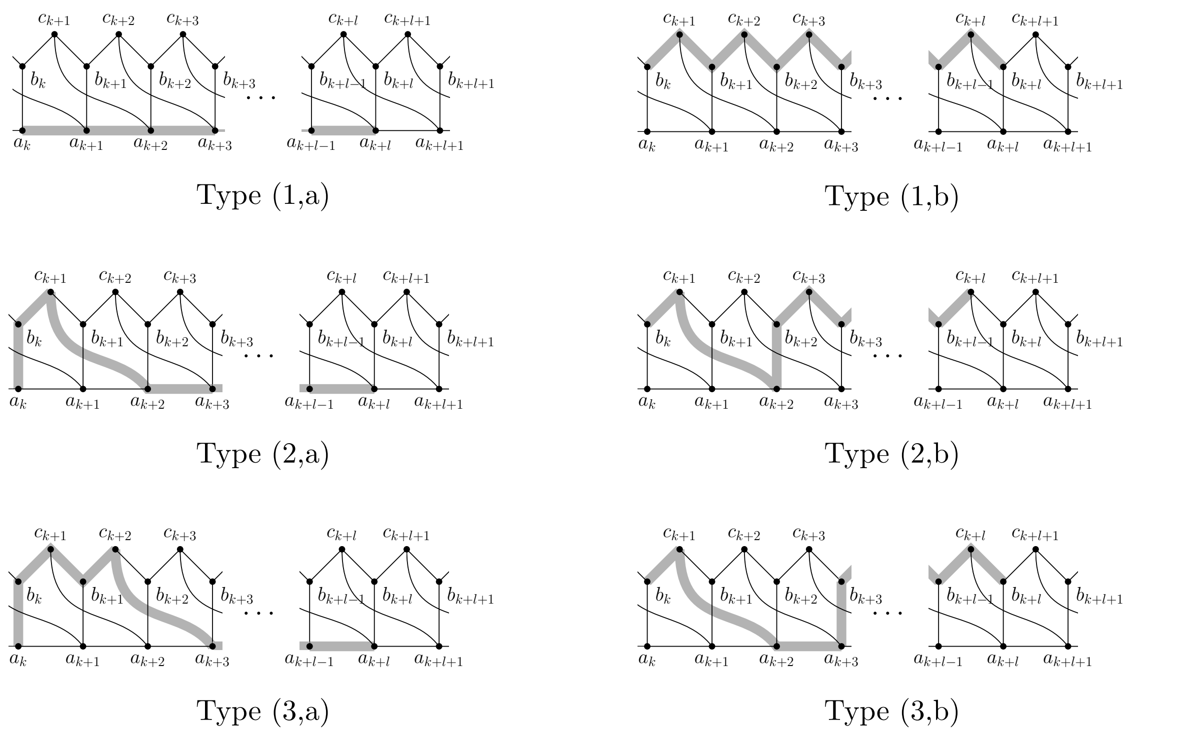


Figure 5: Induced paths of length $0, 1$, and $2 \mod 3$ in G_m.

x	y	$0 \mod 3$	$1 \mod 3$	$2 \mod 3$
a_i	b_{i+2}	$(a_i, a_{i+1}, a_{i+2}, b_{i+2})$	$(a_i, b_i, c_{i+1}, a_{i+2}, b_{i+2})$	$(a_i, b_i, c_{i+1}, b_{i+1}, c_{i+2}, b_{i+2})$
a_i	b_{i-3}	$(b_{i-3}, c_{i-2}, a_{i-1}, a_i)$	$(b_{i-3}, a_{i-3}, a_{i-2}, a_{i-1}, a_i)$	$(b_{i-3}, a_{i-3}, a_{i-2}, b_{i-2}, c_{i-1}, a_i)$
a_i	c_{i+2}	$(a_i, a_{i+1}, b_{i+1}, c_{i+2})$	$(a_i, b_i, c_{i+1}, b_{i+1}, c_{i+2})$	$(a_i, b_i, c_{i+1}, a_{i+2}, b_{i+2}, c_{i+2})$
a_i	c_{i-3}	$(c_{i-3}, a_{i-2}, a_{i-1}, a_i)$	$(c_{i-3}, b_{i-3}, c_{i-2}, a_{i-1}, a_i)$	$(c_{i-3}, b_{i-3}, a_{i-3}, a_{i-2}, a_{i-1}, a_i)$

Table 1: Short paths: a_i

Similarly, we can define a *type–$(1,b)$–path* as a path of length $2l$ of the form $(b_k, c_{k+1}, b_{k+1}, \ldots, c_{k+l}, b_{k+l})$, where $l \geqslant 1$. A *type–$(2,b)$–path* is a path of length $2l-1$ of the form $(b_k, c_{k+1}, a_{k+2}, b_{k+2}, c_{k+3}, b_{k+3}, \ldots, c_{k+l}, b_{k+l})$, where $l \geqslant 2$. A *type–$(3,b)$–path* is a path of length $2l-2$ of the form $(b_k, c_{k+1}, a_{k+2}, a_{k+3}, b_{k+3}, c_{k+4}, b_{k+4}, \ldots, c_{k+l}, b_{k+l})$, where $l \geqslant 3$. (See Figure 5.) These immediately give the desired paths if $x = b_i, y = b_j$, and $|i-j| \geqslant 3$. As $\operatorname{dist}(x,y) \geqslant 3$, we may disregard pairs $i, j \in [m]$ with $|i-j| \leqslant 1$. If $|i-j| = 2$, assume without loss of generality that $i < j$, and we refer to Table 2.

If $x = b_i$ and $y = c_j$, we obtain the desired paths as follows. Note that since $\operatorname{dist}(x,y) \geqslant 3$, we may disregard $j \in \{i-1, i, i+1\}$. If $j \geqslant i+4$, concatenating a type–(t,b)–path for $t \in [3]$ with the edge b_jc_j gives the desired paths. If $j \leqslant i-3$, concatenating the edge c_jb_j with a type–(t,b)–path for $t \in [3]$ gives the desired paths. For the remaining three cases, we refer to Table 2.

x	y	0 mod 3	1 mod 3	2 mod 3
b_i	b_{i+2}	$(b_i, c_{i+1}, a_{i+2}, b_{i+2})$	$(b_i, c_{i+1}, b_{i+1}, c_{i+2}, b_{i+2})$	$(b_i, a_i, a_{i+1}, b_{i+1}, c_{i+2}, b_{i+2})$
b_i	c_{i+3}	$(b_i, a_i, a_{i+1}, a_{i+2}, a_{i+3}, b_{i+3}, c_{i+3})$	$(b_i, c_{i+1}, a_{i+2}, b_{i+2}, c_{i+3})$	$(b_i, c_{i+1}, b_{i+1}, c_{i+2}, b_{i+2}, c_{i+3})$
b_i	c_{i+2}	$(b_i, c_{i+1}, b_{i+2}, c_{i+2})$	$(b_i, a_i, a_{i+1}, b_{i+1}, c_{i+2})$	$(b_i, a_i, a_{i+1}, a_{i+2}, b_{i+2}, c_{i+2})$
b_i	c_{i-2}	$(c_{i-2}, a_{i-1}, a_i, b_i)$	$(c_{i-2}, b_{i-2}, c_{i-1}, a_i, b_i)$	$(c_{i-2}, b_{i-2}, c_{i-1}, b_{i-1}, c_i, b_i)$

Table 2: Short paths: b_i

Finally, if $x = c_i, y = c_j$, assume without loss of generality that $i < j$. As $\operatorname{dist}(x, y) \geqslant 3$, we may disregard pairs $i, j \in [m]$ with $j \leqslant i + 1$. If $j \geqslant i + 4$, concatenating the edge $c_i b_i$, a type–(t, b)–path for $t \in [3]$, and the edge $b_{j-1} c_j$ gives the desired paths. For the remaining two cases, we refer to Table 3.

x	y	0 mod 3	1 mod 3	2 mod 3
c_i	c_{i+3}	$(c_i, b_i, c_{i+1}, b_{i+1}, c_{i+2}, b_{i+2}, c_{i+3})$	$(c_i, a_{i+1}, a_{i+2}, b_{i+2}, c_{i+3})$	$(c_i, a_{i+1}, a_{i+2}, a_{i+3}, b_{i+3}, c_{i+3})$
c_i	c_{i+2}	$(c_i, a_{i+1}, b_{i+1}, c_{i+2})$	$(c_i, b_i, c_{i+1}, b_{i+1}, c_{i+2})$	$(c_i, b_i, c_{i+1}, a_{i+2}, b_{i+2}, c_{i+2})$

Table 3: Short paths: c_i

The recursive construction of G_m allows us to recursively describe the independent sets of G_m. We can moreover specify when such independent sets are maximal.

Proposition 15. *Every independent set A of G_m has one of the following forms:*

1. *$A \cap \{a_m, b_m, c_m\} = \emptyset$ and A is an independent set in G_k for some $k < m$.*

2. *$A \cap \{a_m, b_m, c_m\} = \{b_m\}$ and $A \setminus \{b_m\}$ is an independent set in G_k for some $k < m$.*

3. *$A \cap \{a_m, b_m, c_m\} = \{a_m, c_m\}$ and $A \setminus \{a_m, c_m\}$ is an independent set in G_k for some $k \leqslant m - 2$. In particular, $A \cap \{a_{m-1}, b_{m-1}, c_{m-1}\} = \emptyset$.*

4. *$A \cap \{a_m, b_m, c_m\} = \{a_m\}$ and $A \setminus \{a_m\}$ is an independent set in G_k for some $k < m$ disjoint from $\{a_{m-1}, c_{m-1}\}$.*

5. *$A \cap \{a_m, b_m, c_m\} = \{c_m\}$ and $A \setminus \{c_m\}$ is an independent set in G_k for some $k < m$ disjoint from $\{b_{m-1}\}$.*

Moreover, the independent sets from (2), (4), and (5) are maximal if $A \setminus \{b_m\}$, $A \setminus \{a_m\}$, and $A \setminus \{c_m\}$ are maximal independent sets in G_{m-1}, respectively. Finally, the independent set from (3) is maximal if $A \setminus \{a_m, c_m\}$ is a maximal independent set in G_{m-2}.

Proof. The independent sets (1) - (5) immediately follow from the construction of G_m. To obtain the maximality statements, recall that a maximal independent set in G_m has cardinality m. $\square$

Definition 16. Let Δ be a simplicial complex and $\{x, y\}$ an edge of Δ. The *edge subdivision* of Δ at $\{x, y\}$ is the simplicial complex Δ' with vertex set $V(\Delta) \sqcup \{a\}$ and faces:

- σ, for $\sigma \in \Delta$ such that $\{x, y\} \not\subset \sigma$,

- $\sigma \setminus \{x\} \cup \{a\}$, for $\sigma \in \Delta$ such that $\{x, y\} \subset \sigma$,
- $\sigma \setminus \{y\} \cup \{a\}$, for $\sigma \in \Delta$ such that $\{x, y\} \subset \sigma$.

Remark 17. Edge subdivisions preserve flagness of the complex ([14, Lemma 5.2]). We will focus on performing edge subdivisions on Δ when $\Delta = \mathrm{Ind}(G)$ for some graph G. We see how an edge subdivision of Δ comes from an operation on G. Observe that G is the complement of the 1-skeleton of Δ. Therefore, using the notation from the definition above, $\Delta' = \mathrm{Ind}(G')$ where G' is a graph with vertex set $V(G) \sqcup \{a\}$ and edges

- xy,
- e, for every $e \in E(G)$,
- az, for every $z \in N_G(x) \cup N_G(y)$.

In Figure 1, the right-hand-side sphere is obtained by performing an edge subdivision at the edge a_1a_2 on the left-hand-side sphere with a new vertex c_2. On the left-hand-side graph, $N(a_1) \cup N(a_2) = \{b_1, b_2, a_3\}$. Therefore, the edge subdivision is reflected on the graph as adding a vertex c_3 and edges $a_1a_2, c_2b_1, c_2b_2, c_2a_3$.

A special polytope that will play an important role in the following results in this section is the m-dimensional *crosspolytope*, the convex hull of $\{\vec{e}_i, -\vec{e}_i : i = 1, \ldots, m\}$ in $\mathbb{R}^m$. Moreover, any polytope combinatorially equivalent to a crosspolytope is called a crosspolytope. The independence complex of the graph of m disjoint edges (mK_2) is the boundary of a crosspolytope.

Theorem 18. *For each $m \geqslant 1$ the simplicial complex $\mathrm{Ind}(G_m)$ is obtained by a sequence of $m-1$ edge subdivisions of the boundary of the m-dimensional crosspolytope.*

Proof. Fix m and let Q_1^m be the boundary of the m-dimensional crosspolytope. Q_1^m is the independence complex of mK_2, which is mG_1; call this graph S_1. Let the vertex set of S_1 be $\{b_i : 1 \leqslant i \leqslant m\} \cup \{c_i : 1 \leqslant i \leqslant m\}$, with edge set $\{b_ic_i : 1 \leqslant i \leqslant m\}$. Now we perform a sequence of edge subdivisions on Q_1^m, producing a sequence of flag complexes, Q_i^m, and of graphs S_i, the complements of the 1-skeletons. (These are the graphs for which Q_i^m are the independence complexes.) Subdivide edge b_1c_2 of Q_1^m with new vertex a_2. Call the new complex Q_2^m, and the complement of its 1-skeleton S_2. The graph S_2 has edges b_ic_i and b_1c_2, a_2c_1, and a_2b_2. Thus S_2 is a pentagon along with $m-2$ disjoint edges, that is, S_2 is the disjoint union of G_2 and $m-2$ copies of G_1. Suppose we have formed Q_i^m and S_i, $2 \leqslant i \leqslant m-2$, and we know that S_i is the disjoint union of G_i and $m-i$ copies of G_1. Then subdivide the edge b_ic_{i+1} with new vertex a_{i+1}. Call the new complex Q_{i+1}^m and the complement of its 1-skeleton S_{i+1}. S_{i+1} has the edges of S_i (including $b_{i+1}c_{i+1}$), and edges b_ic_{i+1} and $a_{i+1}c_i$, $a_{i+1}b_{i+1}$, and a_ia_{i+1}. Thus S_{i+1} is the disjoint union of G_{i+1} and $m-i-1$ copies of G_1. Finally, when $i = m-1$, $S_m = G_m$. So G_m is the complement of the 1-skeleton of Q_m^m. Thus, Q_m^m, the simplicial complex resulting from subdividing $m-1$ edges of the boundary of the m-dimensional crosspolytope, is the independence complex of G_m. □

In order to prove Theorem B, we establish the two corollaries below, which follow from [7, Proposition 2.4.2] and [19, Theorem 2.7], respectively.

Corollary 19. *For $m \geqslant 1$, $\mathrm{Ind}(G_m)$ is the boundary of an m-dimensional simplicial polytope.*

Corollary 20. *For $m \geqslant 1$, $\mathrm{Ind}(G_m)$ is vertex decomposable.*

Theorem B now quickly follows from Theorem 10 and the two previous corollaries.

Theorem B. *Let G be a planar ternary graph such that $\mathrm{Ind}(G)$ is a homology sphere. Then $\mathrm{Ind}(G)$ is combinatorially equivalent to the boundary of a simplicial polytope and is vertex decomposable.*

Proof. Let G be a planar ternary graph such that $\mathrm{Ind}(G)$ is a homology sphere. Then G is Gorenstein, so by Theorem 10, G is the disjoint union of G_m graphs. Thus $\mathrm{Ind}(G)$ is the join of the independence complexes $\mathrm{Ind}(G_m)$, each of which is the boundary of a simplicial polytope and is vertex decomposable. These properties are preserved under the join operation, so $\mathrm{Ind}(G)$ is the boundary of a simplicial polytope and is vertex decomposable. $\square$

4 An induced subgraph of the Lutz–Nevo graph

Theorem 18 demonstrates how to obtain $\mathrm{Ind}(G_m)$ from the boundary of the crosspolytope by edge subdivisions. As a result, independence complexes of disjoint unions of G_m's are polytopal simplicial spheres. Given two such spheres $\mathrm{Ind}(G)$ and $\mathrm{Ind}(G')$ of the same dimension, we now show how to obtain one from the other by edge subdivisions and contractions. For example, Figure 1 illustrates an edge subdivision that turns $\mathrm{Ind}(G_1 \sqcup G_2)$ into $\mathrm{Ind}(G_3)$. These observations allow us to show that the subgraph of the Lutz–Nevo graph induced on the set of vertices representing these spheres is isomorphic to the partition refinement graph.

Theorem 21 ([14, Theorem 1.2])**.** *Two flag simplicial complexes are PL homeomorphic if and only if they can be connected by a sequence of edge contractions and subdivisions such that every complex in the sequence is flag.*

In view of Theorem 21 the goal of this section is to study the following graph.

Definition 22. Let V_d be the set of all isomorphism classes of d-dimensional flag PL spheres for some fixed d, and E_d be the collection of pairs $\{\Delta_1, \Delta_2\}$ of elements of V_d such that $\{\Delta_1, \Delta_2\} \in E_d$ if and only if Δ_2 can be obtained from Δ_1 by either contracting an edge, or subdividing an edge. For an integer $d \geqslant 0$, the *Lutz–Nevo graph* T_d is the graph with vertex set V_d and edge set E_d.

Lemma 23 ([14])**.** *Let Δ be a flag complex. The complex Δ' obtained by contracting an edge e of Δ is flag if and only if e is not part of an induced square of Δ.*

We are interested in the subgraph associated with the planar ternary graphs.

Definition 24. Denote by H_{n-1} the induced subgraph of T_{n-1} where the vertices are given by the (isomorphism types) of independence complexes of planar ternary graphs with independence number n.

We will show this subgraph is isomorphic to a graph arising from partitions.

A *partition* $\lambda = (m_1, \ldots, m_s)$ of n is an ordered nonincreasing list of integers such that $m_1 + \cdots + m_s = n$.

Definition 25 (Partition refinement graph). A partition λ of n *refines* a partition μ of n if each part of μ is the sum of parts of λ. The *partition refinement graph*, $\mathcal{P}_n$, is the graph of the Hasse diagram of the refinement poset on the integer partitions of n.

We will prove that $\mathcal{P}_n$ is isomorphic to H_{n-1}.

Lemma 26. *There is a bijection between $V(H_{n-1})$ and $V(\mathcal{P}_n)$.*

Proof. If the independence complex of a planar ternary graph G is an $(n-1)$-dimensional sphere, then by [22]

$$G = G_{m_1} \sqcup G_{m_2} \sqcup \cdots \sqcup G_{m_s}$$

where $m_1 + \cdots + m_s = n$. Reordering the m_i does not change the isomorphism class. Hence the map from $V(H_{n-1})$ to $V(\mathcal{P}_n)$ given by

$$\mathrm{Ind}(G_{m_1} \sqcup G_{m_2} \sqcup \cdots \sqcup G_{m_s}) \mapsto (m_1, \ldots, m_s)$$

is a bijection. □

Lemma 27. *Let $\mathrm{Ind}(G)$ be a vertex of H_{n-1}, and xy an edge of $\mathrm{Ind}(G)$ where x and y are vertices of the same connected component in G. Then the sphere given by subdividing the edge xy is not a vertex of H_{n-1}.*

Proof. Subdividing the edge xy of $\mathrm{Ind}(G_m)$ adds the edge xy to G_m (among other edges / vertices). There are two cases:

1. If the distance between x and y is two, then the new graph has a triangle, and hence is not ternary.
2. If the distance between x and y is at least three, by Theorem 14 there exists an induced path between x and y of length 2 mod 3. In particular, this path together with the new edge xy gives us an induced cycle of length $3k$ and hence the graph is not ternary. □

Lemma 28. *Let $G = G_{m_1} \sqcup G_{m_2} \sqcup \cdots \sqcup G_{m_s}$ and $\Delta = \mathrm{Ind}(G)$. Suppose xy is an edge of Δ, where x is a vertex of G_{m_i} and y is a vertex of G_{m_j} $(i \neq j)$. Let Δ' be obtained from Δ by an edge subdivision of xy, and let G' be the graph with $\mathrm{Ind}(G') = \Delta'$. Then G' has a connected component with subgraphs G_{m_i} and G_{m_j}, and this component has independence number $m_i + m_j$.*

Proof. Let Δ' be as given in the statement of the lemma, with new vertex a created by the edge subdivision. Δ' is a flag complex, so it is the independence complex of some graph G'. The faces of Δ' are those faces of Δ not containing both x and y, and the faces $\sigma \setminus \{x\} \cup \{a\}$ and $\sigma \setminus \{y\} \cup \{a\}$ for each $\sigma \in \Delta$ containing $\{x, y\}$. The edges of G' are the nonedges of Δ'; these are the nonedges of Δ, the pair xy and some edges containing a.

Note that if u is a vertex of G not in $V(G_{m_i} \cup G_{m_j})$, then $ux \in \Delta$, so $ua \in \Delta'$, so ua is not an edge of G'. Thus, the subgraph G'' induced by the vertex set $V(G_{m_i} \cup G_{m_j}) \cup \{a\}$ is a connected component of G' containing as subgraphs G_{m_i} and G_{m_j}. If A_i is a maximal independent set of G_{m_i} containing x, and A_j is a maximal independent set of G_{m_j} containing y, then $A_i \cup A_j \setminus \{\mathrm{x}\} \cup \{a\}$ is an independent set of size $m_i + m_j$ in G'. Moreover, this set must be a maximal independent set in G'', because every other vertex of $G_{m_i} \cup G_{m_j}$ is adjacent to some vertex of $A_i \cup A_j$. $\square$

Theorem C. *The graph H_{n-1} is isomorphic to $\mathcal{P}_n$. In particular, $\mathcal{P}_n$ is an induced subgraph of T_{n-1} for every n.*

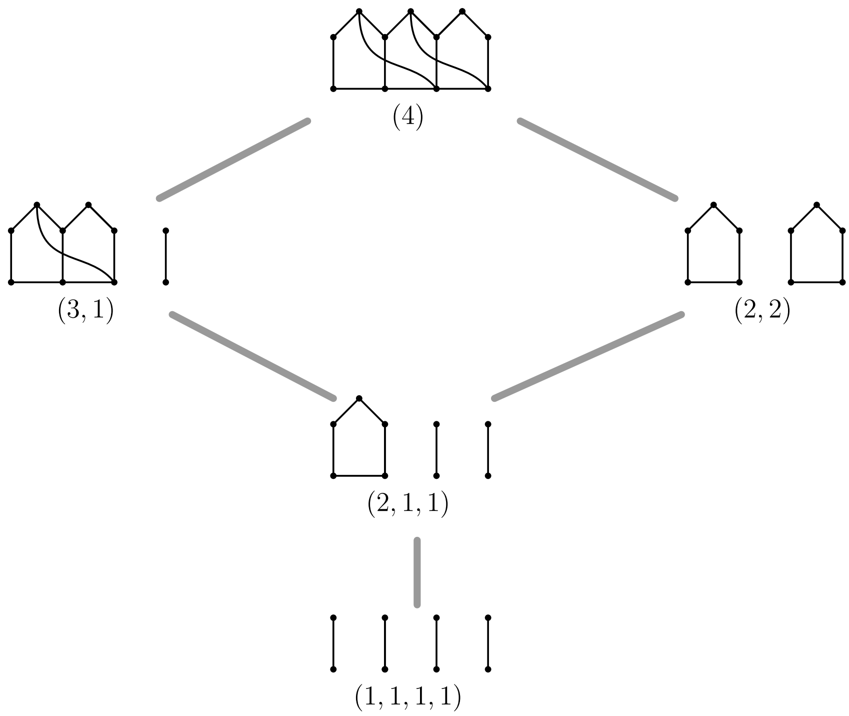


Figure 6: A visualization of Theorem C for $n = 4$

Proof. We know there is a bijection between the vertices of H_{n-1} and $\mathcal{P}_n$; hence we just need to show that this bijection preserves edges. Let $\mathrm{Ind}(G)$ and $\mathrm{Ind}(G')$ be neighbors in H_{n-1}, and assume without loss of generality that $\mathrm{Ind}(G')$ is obtained by subdividing (instead of contracting) an edge xy of $\mathrm{Ind}(G)$. Furthermore, assume $G = G_{m_1} \sqcup G_{m_2} \sqcup \cdots \sqcup G_{m_s}$. By Theorem 27, x and y must belong to two different components of G, say

G_{m_1} and G_{m_2} without loss of generality. By Theorem 28 and the fact that G' is planar and ternary, we conclude that $G' \cong G_{m_1+m_2} \sqcup G_{m_3} \sqcup \cdots \sqcup G_{m_s}$. Therefore, the edge between $\mathrm{Ind}(G)$ and $\mathrm{Ind}(G')$ in H_{n-1} gets mapped to the edge between $(m_1, \ldots, m_s)$ and $(m_1 + m_2, m_3, \ldots, m_s)$ in $\mathcal{P}_n$.

Conversely, given the edge between $(m_1, \ldots, m_s)$ and $(m_1 + m_2, m_3, \ldots, m_s)$ in $\mathcal{P}_n$, we will show that there exists an edge between their corresponding vertices $\mathrm{Ind}(G_{m_1} \sqcup G_{m_2} \sqcup \cdots \sqcup G_{m_s})$ and $\mathrm{Ind}(G_{m_1+m_2} \sqcup G_{m_3} \sqcup \cdots \sqcup G_{m_s})$ in H_{n-1}. Indeed, one can go from the first complex to the second by subdividing any xy where $x \in G_{m_1}$ and $y \in G_{m_2}$ both having degree two. See Figure 7 for an illustration. $\square$

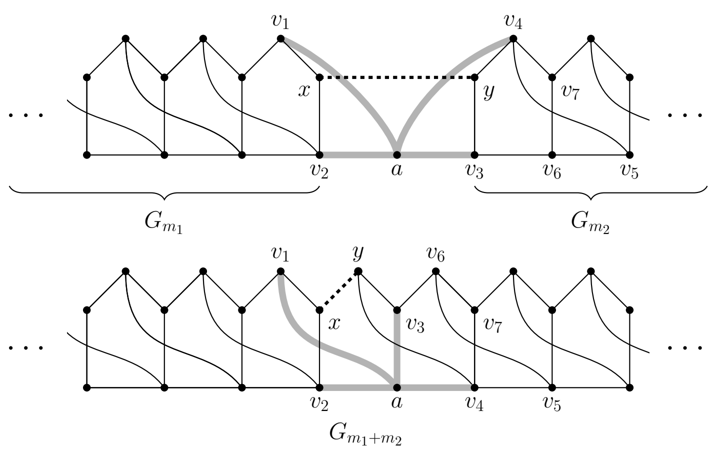


Figure 7: Isomorphism between $G_{m_1+m_2}$ and graph obtained after subdividing the edge xy (connecting two disconnected components G_{m_1} and G_{m_2}). The dotted edge is being subdivided, and the gray edges are the new ones in the new graph.

5 The f, h and γ-vectors of planar flag spheres

Recall that Gal's conjecture [7] posits that all flag homology spheres are γ-positive. Moreover, a real-rooted h-polynomial implies γ-positivity. Gal's conjecture was proposed after he found counterexamples to the stronger conjecture that the h-polynomials of flag spheres were real rooted. By [7, Section 2.4], flag simplicial spheres obtained by successive edge subdivisions from the crosspolytope are γ-positive. Therefore, Theorem 18 implies that $\mathrm{Ind}(G_m)$ is γ-positive. In this section, we investigate further the f-vector, h-vector and γ-vector of $\mathrm{Ind}(G_m)$, and prove the stronger result that a flag homology sphere whose 1-skeleton is complementary to a planar graph has a real-rooted h-polynomial.

For $m \geqslant 1$ and $0 \leqslant i \leqslant m-1$, let $f_i(m) := f_i(\mathrm{Ind}(G_m))$. When needed, we use $f_m(m) = 0$.

Proposition 29. *For each $m \geqslant 1$, $f_0(m) = 3m - 1$. For each $m \geqslant 2$, $f_1(m) = (9m^2 - 19m + 12)/2$. For $m \geqslant 3$ and $2 \leqslant i \leqslant m-1$,*

$$f_i(m) = 2f_{i-1}(m-1) + f_i(m-1) + f_{i-2}(m-2) + f_{i-1}(m-2).$$

Proof. First note that $G_1 = K_2$, and $\mathrm{Ind}(G_1) = 2K_1$, so $f_0(1) = 2$. Also, $f_1(m)$ is the number of edges of $\mathrm{Ind}(G_m)$, that is, the number of pairs of vertices in G_m that do not form an edge. So for $m \geqslant 2$, $f_1(m) = \binom{3m-1}{2} - 5(m-1) = (9m^2 - 19m + 12)/2$. Now assume $m \geqslant 3$ and $2 \leqslant i \leqslant m-1$.

Refer to the recursive construction of G_m for the vertices and edges of G_m not in G_{m-1}. Note that $G_m - (N[a_m] \cup N[c_m]) = G_m - \{a_{m-1}, b_{m-1}, c_{m-1}, a_m, b_m, c_m\} = G_{m-2}$.

Using the characterization of independent sets of G_m from Proposition 15, we count the number of independent sets of size $i+1$ for each of the five cases: Case (1) gives exactly $f_i(m-1)$ subsets, Case (2) gives $f_{i-1}(m-1)$ subsets, and Case (3) gives exactly $f_{i-2}(m-2)$.

For Cases (4) and (5), note that every independent set of size i of G_{m-1} intersects $\{a_{m-1}, b_{m-1}, c_{m-1}\}$ in either $a_{m-1}, b_{m-1}, c_{m-1}, \{a_{m-1}, c_{m-1}\}$ or $\emptyset$. We now show that the number of these intersections is $f_{i-1}(m-1) + f_{i-1}(m-2)$. To see this, first consider independent sets J of size i (dimension $i-1$) in G_{m-1}. If $J \cap \{a_{m-1}, b_{m-1}, c_{m-1}\} = \emptyset$, extend J to a set of case 4 by adding a_m. If $J \cap \{a_{m-1}, b_{m-1}, c_{m-1}\} \neq \emptyset$, then J can be extended to a unique set of case 4 or 5 by adding a_m (if $b_{m-1} \in J$) or c_m (if a_{m-1} and/or c_{m-1} is in J). This gives all independent sets of Case (4) or (5) except the sets A for which $A \cap \{a_{m-1}, b_{m-1}, c_{m-1}, a_m, b_m, c_m\} = \{c_m\}$. The number of these is $f_{i-1}(m-2)$.

Counting the elements in each partition we have

$$\begin{aligned} f_i(m) &= f_i(m-1) + f_{i-1}(m-1) + f_{i-2}(m-2) + f_{i-1}(m-1) + f_{i-1}(m-2) \\ &= 2f_{i-1}(m-1) + f_i(m-1) + f_{i-2}(m-2) + f_{i-1}(m-2). \end{aligned}$$ ☐

We now define the Delannoy polynomials mentioned in Theorem D.

Definition 30. *The Delannoy number $D(m,k)$ is the number of lattice paths from $(0,0)$ to (m,k) using steps $(1,0)$, $(0,1)$, and $(1,1)$.*

Following [23], let $d(m,k) = D(m-k,k)$, and for fixed m, let $d_m(t)$ be the generating function $d_m(t) = \sum_{k=0}^{m} d(m,k)t^k$. It is known (see for example [16, A008288] and [23, Section 2]) that $d(m,k)$ satisfies the recursion $d(m,0) = d(m,m) = 1$ for $m \geqslant 0$ and, for $m > 1$ and $0 < k < m$,

$$d(m,k) = d(m-1,k) + d(m-1,k-1) + d(m-2,k-1).$$

In 2019, Wang, Zheng and Chen [23] showed that Delannoy polynomials are real-rooted.

Theorem 31 ([23, Theorem 2.1])**.** *The zeros of the Delannoy polynomial $\sum_{k=0}^{m} d(m,k)t^k$ are negative real numbers.*

We are now ready to tackle Theorem D. The first part of the theorem is stated for all planar graphs whose independence complexes are homology spheres. Let G be such a graph and assume further it is connected. Then recall from Theorem 12 that G is either a G_m or R_3. The case of $G = G_m$ is stated as the second part of Theorem D and proved as Theorem 32. When $G = R_3$, the h-polynomial of $\mathrm{Ind}(G) = C_6$ is $t^2 + 4t + 1$, which is real-rooted.

Theorem 32. *Let G be a planar ternary connected graph such that* $\operatorname{Ind}(G)$ *is an* $(m-1)$-*dimensional homology sphere. Then* $h_k(\operatorname{Ind}(G)) = d(m,k)$.

Proof. We may assume $G = G_m$ by the previous discussion. Recall $f_i(m) = f_i(\operatorname{Ind}(G_m))$ and, similarly, write $h_k(m) = h_k(\operatorname{Ind}(G_m))$ and

$$h(m,t) = \sum_{k=0}^{m} h_k(m)t^{m-k} = \sum_{i=0}^{m} f_{i-1}(m)(t-1)^{m-i}.$$

Proposition 29 gives

$$f_{i-1}(m) = 2f_{i-2}(m-1) + f_{i-1}(m-1) + f_{i-3}(m-2) + f_{i-2}(m-2).$$

So

$$\begin{aligned}
h(m,t) &= \sum_{i=0}^{m} f_{i-1}(m)(t-1)^{m-i} \\
&= 2\sum_{i=0}^{m} f_{i-2}(m-1)(t-1)^{m-i} + \sum_{i=0}^{m} f_{i-1}(m-1)(t-1)^{m-i} \\
&\quad + \sum_{i=0}^{m} f_{i-3}(m-2)(t-1)^{m-i} + \sum_{i=0}^{m} f_{i-2}(m-2)(t-1)^{m-i} \\
&= 2\sum_{i=0}^{m-1} f_{i-1}(m-1)(t-1)^{m-1-i} + (t-1)\sum_{i=0}^{m-1} f_{i-1}(m-1)(t-1)^{m-1-i} \\
&\quad + \sum_{i=0}^{m-2} f_{i-1}(m-2)(t-1)^{m-2-i} + (t-1)\sum_{i=0}^{m-2} f_{i-1}(m-2)(t-1)^{m-2-i} \\
&= (t+1)h(m-1,t) + th(m-2,t).
\end{aligned}$$

Equating coefficients of t^{m-k}, $h_k(m) = h_k(m-1) + h_{k-1}(m-1) + h_{k-1}(m-2)$. This is the same recursion as for $d(m,k)$. Since $(h_0(1), h_1(1)) = (1,1) = (d(1,0), d(1,1))$ and $(h_0(2), h_1(2), h_2(2)) = (1,3,1) = (d(2,0), d(2,1), d(2,2))$, $h_k(m) = d(m,k)$ for all $m \geqslant 1$, $0 \leqslant k \leqslant m$. ☐

We immediately conclude the following from Theorems 31 and 32.

Corollary 33. *The zeros of the h-polynomial of* $\operatorname{Ind}(G_m)$ *are negative real numbers.*

To remove the requirement of connectedness, recall from Sections 2.3 and 2.7 that for any pair of graphs G and H,

$$h(\operatorname{Ind}(G \sqcup H), t) = h(\operatorname{Ind}(G) * \operatorname{Ind}(H), t) = h(\operatorname{Ind}(G), t) \cdot h(\operatorname{Ind}(H), t).$$

Corollary 34. *If G is a planar graph such that* $\operatorname{Ind}(G)$ *is a homology sphere, then the zeros of* $h(\operatorname{Ind}(G), t)$ *are negative real numbers.*

Theorem D then follows directly from Theorem 32 and the corollaries above.

6 Future work

Recall that Trung characterized Gorenstein planar graphs [22], all of which are ternary except the graph R_3. Pinter constructed Gorenstein nonplanar graphs in [18], but the ternary property of such graphs was not explored. In particular, many of the graphs introduced in [18] contain an induced subgraph isomorphic to G_m for some m, see for example [18, Figure 4, Figure 10]. This naturally leads to the question of finding nonplanar ternary graphs that are Gorenstein.

We now provide one such example below. Start with a disconnected graph $G^0 = H_1 \sqcup H_2 \sqcup H_3$ with $H_i \cong C_5$ for $i \in [3]$. Consider vertices $1 \in V(H_1)$ and $6 \in V(H_2)$. We obtain a graph G^1 by subdividing the edge $\{1, 6\}$ in $\mathrm{Ind}(G^0)$. Next, consider vertices $7 \in N_{H_2}(6)$ and $11 \in V(H_3)$. We obtain the graph G^2 pictured below by subdividing the edge $\{7, 11\}$ in $\mathrm{Ind}(G^1)$.

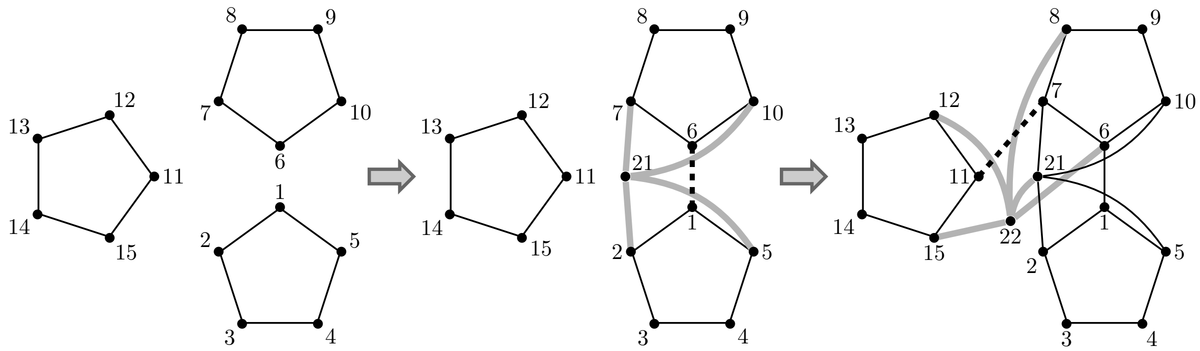


Figure 8: Constructing a nonplanar ternary Gorenstein graph

It is quick to check that all induced cycles in G^2 have length four or five. G^2 is nonplanar as the induced subgraph of G^2 on vertices $\{6, 7, 8, 9, 10, 21, 22\}$ is homeomorphic to $K_{3,3}$. Lastly, to see that G^2 is a Gorenstein graph, notice that $\mathrm{Ind}(G^0)$ is homeomorphic to $\mathrm{Ind}(G^2)$ since we obtain $\mathrm{Ind}(G^2)$ from $\mathrm{Ind}(G^0)$ by subdividing edges, and by Theorem A $\mathrm{Ind}(G^0)$ is homotopy equivalent to $S^{\dim \mathrm{Ind}(G^0)}$. The result then follows since $\dim \mathrm{Ind}(G^0) = \dim \mathrm{Ind}(G^2)$.

We can use a generalization of this construction to obtain Gorenstein graphs, and we suspect that many of these are ternary. Additionally, under certain conditions we can guarantee the output of this construction to be nonplanar. The general construction looks as follows.

1. Set $k = 0$.
2. Let $G^k \cong \bigsqcup_{i=1}^{n} H_i$ with $H_i \cong G_{m_i}$ for all i and $n \geqslant 2$.
3. Pick two vertices $v_i \in V(H_i)$ and $v_j \in V(H_j)$ such that v_i, v_j are in different components of G^k.
4. Subdivide the edge $v_i v_j$ in $\mathrm{Ind}(G^k)$ and call the resulting graph G^{k+1}.

5. If G^{k+1} is connected, stop the procedure. Else, either stop the procedure or set $k = k + 1$ and return to (3).

Observe that we used a special case of this in the proof of Theorem C. In particular, as long as we choose the vertices v_i and v_j such that $\deg v_i = \deg v_j = 2$, the graph G^k is isomorphic to a disjoint union of G_m, that is, it is a planar ternary Gorenstein graph.

We can actually show that G^k is nonplanar if and only if we at least once pick some vertex v_i such that $\deg v_i \geqslant 3$: Note that the forward direction is given by the above observation. For the other direction, let k be the largest such that at each prior step $\deg v_i = \deg v_j = 2$. Hence, we currently have $G^k \cong \bigsqcup_{i=1}^{n} H_i$ with $H_i \cong G_{m_i}$ for all i. Now assume $\deg v_i \geqslant 3$ and let $N(v_i) \supseteq \{x_1, x_2, x_3\}$. Calling the new vertex in G^{k+1} y, the induced subgraph on vertices $\{y, x_1, x_2, x_3, v_i\}$ is isomorphic to $K_{2,3}$. By the structure of G_m we can additionally find a vertex $z \in V(H_i)$ such that for each $\ell \in [3]$ there exist pairwise disjoint paths from x_ℓ to z in H_i avoiding v_i. Hence, G^{k+1} contains a subgraph homeomorphic to $K_{3,3}$ and is therefore nonplanar.

Question 35. If G^k is nonplanar, under which conditions is it also ternary?

Remark 36. Suppose G^0 has N connected components, and the construction above produces a graph G. Form a graph W on N vertices, each corresponding to a connected component of G^0, with two vertices adjacent if at some point the construction uses vertices from the corresponding components. We believe that if W is a tree, then the graph G is ternary.

Question 37. How do the conditions for being nonplanar or ternary change if we just require v_i and v_j to be in different components of G^k, but not necessarily vertices of H_i?

Acknowledgements

This project started during the Graduate Research Workshop in Combinatorics 2024, which was hosted by the University of Wisconsin–Milwaukee and supported in part by NSF Grant DMS-195344. The authors would like to thank Aram Bingham, Jeremy Martin and Shira Zerbib for their early contributions to this project. The authors also thank the anonymous referee for several helpful suggestions. Richard Danner's research was partially supported by NSF Grant DMS-2246967 and Simons Collaboration Gift #854037. Yirong Yang's research was partially supported by a Graduate Fellowship from NSF Grant DMS-2246399.

References

[1] J. W. Alexander. The combinatorial theory of complexes. *Ann. of Math. (2)*, 31:292–320, 1930.

[2] D. Barnette. The triangulations of the 3-sphere with up to 8 vertices. *J. Combin. Theory, Ser. A*, 14(1):37–52, 1973.

[3] M. Bonamy, P. Charbit, and S. Thomassé. Graphs with large chromatic number induce $3k$-cycles. arXiv:1408.2172, 2014.

[4] P. Brändén. Sign-graded posets, unimodality of W-polynomials and the Charney–Davis conjecture. *Electron. J. Combin.*, 11(2): #R9, 15, 2004/06.

[5] A. Engström. On the topological Kalai–Meshulam conjecture. arXiv:2009.11077, 2020.

[6] S. Faridi and T. Holleben. Spherical complexes. arXiv:2311.07727, 2023.

[7] S. R. Gal. Real root conjecture fails for five- and higher-dimensional spheres. *Discrete Comput. Geom.*, 34(2):269–284, 2005.

[8] D. T. Hoang, N. C. Minh, and T. N. Trung. Cohen–Macaulay graphs with large girth. *J. Algebra Appl.*, 14:1550112, 16, 2015.

[9] D. T. Hoang and T. N. Trung. A characterization of triangle-free Gorenstein graphs and Cohen–Macaulayness of second powers of edge ideals. *J. Algebraic Combin.*, 43:325–338, 2016.

[10] G. Kalai. When do a few colors suffice? https://gilkalai.wordpress.com/2014/12/19/when-a-few-colors-suffice/. Accessed 09-11-2025.

[11] J. Kim. The homotopy type of the independence complex of graphs with no induced cycles of length divisible by 3. *European J. Combin.*, 104:Paper No. 103534, 9, 2022.

[12] V. E. Levit and E. Mandrescu. 1-well-covered graphs revisited. *European J. Combin.*, 80:261–272, 2019.

[13] L. Lovász. Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A*, 25(3):319–324, 1978.

[14] F. H. Lutz and E. Nevo. Stellar theory for flag complexes. *Math. Scand.*, 118(1):70–82, 2016.

[15] E. Munarini. Pell graphs. *Discrete Math.*, 342(8):2415–2428, 2019.

[16] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, 2025. Published electronically at http://oeis.org.

[17] M. R. Pinter. A class of planar well-covered graphs with girth four. *J. Graph Theory*, 19(1):69–81, 1995.

[18] M. R. Pinter. A class of well-covered graphs with girth four. *Ars Combin.*, 45:241–255, 1997.

[19] J. S. Provan and L. J. Billera. Decompositions of simplicial complexes related to diameters of convex polyhedra. *Math. Oper. Res.*, 5(4):576–594, 1980.

[20] R. P. Stanley. *Combinatorics and commutative algebra*, volume 41 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 1996.

[21] J. A. W. Staples. On some subclasses of well-covered graphs. *J. Graph Theory*, 3(2):197–204, 1979.

[22] T. N. Trung. A characterization of Gorenstein planar graphs. In *The 50th Anniversary of Gröbner Bases*, volume 77 of *Adv. Stud. Pure Math.*, pages 399–410. Math. Soc. Japan, Tokyo, 2018.

[23] Y. Wang, S.-N. Zheng, and X. Chen. Analytic aspects of Delannoy numbers. *Discrete Math.*, 342(8):2270–2277, 2019.

[24] G. M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer New York, 1995.