Asymptotics for graphically divergent series: dense digraphs and 2-SAT formulae

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Abstract

We propose a new method for obtaining complete asymptotic expansions in a systematic manner, which is suitable for counting sequences of various graph families in dense regime. The core idea is to encode the two-dimensional array of expansion coefficients into a special bivariate generating function, which we call a *coefficient generating function*. We show that coefficient generating functions possess certain general properties that make it possible to express asymptotics in a short closed form. Also, in most scenarios, we indicate a combinatorial meaning of the involved coefficients. Applications of our method include asymptotics of connected graphs, irreducible tournaments, strongly connected digraphs, 2-SAT formulae and contradictory strongly connected implication digraphs. Moreover, due to its flexibility, the method allows to treat a wide range of structural variations, including fixing the numbers of connected, irreducible, strongly connected and contradictory components, as well as source-like, sink-like and isolated ones, or adding weights and marking variables.

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1 Introduction

1.1 Motivation and historical context

Asymptotic methods are a powerful tool widely used in enumerative combinatorics. Typically, they help to determine how fast different counting sequences grow and how to compare their growth. This quantitative information allows scientists to predict the properties of large combinatorial objects and understand their structure. There is extensive literature on this account, for instance, the surveys of Bender [4], Odlyzko [29] and books of De Bruijn [9], Flajolet and Sedgewick [15].

While, in combinatorics, the main concern is often to get the dominant term of the asymptotics, there are certain reasons to go further. First of them is rather obvious: the more terms in the asymptotic expansion, the more accurate the estimate of the behavior of a counting sequence. Ideally, we would like to have a complete asymptotic expansion that allows us to obtain estimates of any predetermined accuracy.

Another reason to look for complete asymptotic expansions is that they may possess certain structure themselves. In other words, coefficients in these expansions often have combinatorial meanings on their own. For instance, it follows from results of Dixon [13] and Cori [8] that, for any $r \geqslant 1$, the probability t_n that a uniform random square-tiled surface is connected, satisfies

$$t_n = 1 - \sum_{k=1}^{r-1} \frac{\mathfrak{ip}_k}{n^{\underline{k}}} + O\left(\frac{1}{n^r}\right),$$

where the sequence $(\mathfrak{ip}_k)_{k=1}^{\infty}$ counts indecomposable permutations and, for any positive integer k, the value $n^{\underline{k}} = n(n-1)\dots(n-k+1)$ is the falling factorial (see also [24]). The same way, the probability p_n that a uniform random graph is connected is equal to

$$p_n = 1 - \sum_{k=1}^{r-1} \mathfrak{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{kn}} + O\left(\frac{n^r}{2^{nr}}\right),$$

where $\mathfrak{it}_k$ is the number of irreducible tournaments of size k, see [23]. In some cases, coefficients are rather linear combinations of certain counting sequences. Thus, the asymptotic expansion of the number of permutations of size n is given by

$$n! = \frac{n^n}{e^n}\sqrt{2\pi n}\left(1 + \sum_{q=1}^{r-1} \frac{c_q}{n^q} + O\left(\frac{1}{n^r}\right)\right),$$

where

$$c_q = \sum_{k=1}^{2q} \frac{(-1)^k h_{2q+2k,k}}{2^{q+k}(q+k)!}$$

and $h_{m,k}$ is the number of permutations of size m having k cycles, all of length at least three, see [15, Proposition B.1]. The reader can find more examples of that types in [24] and [25].

Comprehension of this "second level" structure leads to better understanding of the structure of initial combinatorial objects and can be potentially used, for instance, to guess new recurrences and bijections. Let us illustrate this idea by the following example. Consider a uniform random graph with n vertices and $m = \frac{1}{2}n(1 + \mu n^{-1/3})$ edges, as $\mu \to -\infty$, while $|\mu| = o(n^{1/3})$. The first two terms of the asymptotic expansion of the probability that this graph is a union of trees and unicycles are $1 - \frac{5}{24}|\mu|^{-3} + \mathcal{O}\left(|\mu|^{-6}\right)$. On the one hand, this can be proved with the help of analytical tools (see Equation (10.3) in [19, Lemma 3] with $y = 1/2$). In this case, the rational constant $\frac{5}{24}$ appears formally as a term in the asymptotic expansion of a complex contour integral. On the other hand, the coefficient $\frac{5}{24} = \frac{1}{2!}(\frac{1}{4} + \frac{1}{6})$ can be interpreted as the total weight of bicyclic cubic cores, *i.e.* connected multigraphs with two vertices of degree 3. The corresponding term is negative, since these cores are nor trees neither unicycles; they form the skeletons of the majority of objects excluded from a typical random graph in the given range. Further coefficients of this asymptotics can also be expressed via combinations of cubic cores weights. The presence of these weights in the asymptotic expansion means that the corresponding cores are excluded from the graph: in the absence of bicyclic cores, the graph is almost surely a set of trees and unicycles.

Apparently, this phenomenon must be universal within various graph structures. Thus, the probability that the strongly connected components of a uniform random digraph with n vertices and $m = n(1 + \mu n^{-1/3})$ edges are only isolated vertices or cycles behaves as $1 - \frac{1}{2}|\mu|^{-3} + \mathcal{O}\left(|\mu|^{-6}\right)$, as $\mu \to -\infty$, see [11]. Again, the factor $\frac{1}{2}$ has an interpretation in terms of the weight of excluded structures. From similar point of view, the combinatorial interpretation of the coefficients of complete asymptotic expansions is discussed in [24], [25] and [26] (see also the PhD dissertation [28], which covers most of the above-mentioned papers).

The tool that serves to establish the asymptotic expansions of [13], [23] and [28] in a form suitable for combinatorial interpretation is Bender's theorem [5]. Given an analytic function $F(x, y)$ and a formal power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$, typically with coefficients that grow factorially or superfactorially, Bender's theorem provides the complete asymptotic expansion of the composition $B(z) = F\big(A(z), z\big)$ in the form

$$b_n = \sum_{k=0}^{r-1} c_k a_{n-k} + \mathcal{O}(a_{n-r}),$$

(for a simplified version of Bender's theorem that we use in this paper, see Theorem 23). It is necessary to emphasize here that Bender's theorem itself does not provide a combinatorial meaning for the coefficients. There is always additional work that depends on the initial data. The advantage of this theorem is that the coefficients c_k are calculated simultaneously: the corresponding formal power series $C(z) = \sum_{n=0}^{\infty} c_n z^n$ is expressed in a closed form via the input, that is, $F(x, y)$ and $A(z)$. This helps, in certain cases, to give the coefficients c_k a combinatorial interpretation.

Inspired by the work of Bender [5], Borinsky studied the asymptotic behavior of *fac-*

torially divergent series [7], *i.e.* series whose coefficients admit an expansion of the form

$$a_n \sim \alpha^{n+\beta}\Gamma(n+\beta)\left(d_0 + \frac{d_1}{\alpha(n+\beta-1)} + \frac{d_1}{\alpha^2(n+\beta-1)(n+\beta-2)} + \dots\right),$$

where $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$ are fixed parameters. He proved that, for the set $\mathbb{R}[[x]]^\alpha_\beta$ of series of that type, the following theorem holds.

Theorem 1. *For any $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$, the subspace $\mathbb{R}[[x]]^\alpha_\beta$ is a subring of $\mathbb{R}[[x]]$. Moreover, if $f, g \in \mathbb{R}[[x]]^\alpha_\beta$, then $f \cdot g \in \mathbb{R}[[x]]^\alpha_\beta$ and, with the additional conditions $g_0 = 0$ and $g_1 = 1$, we also have $f \circ g, g^{-1} \in \mathbb{R}[[x]]^\alpha_\beta$. The* transfer map

$$\left(\sum_{n=0}^{\infty} a_n z^n\right) \xmapsto{\mathcal{A}^\alpha_\beta} \left(\sum_{k=0}^{\infty} d_k z^k\right)$$

is a derivation*, i.e. this map obeys a* Leibniz rule

$$\big(\mathcal{A}^\alpha_\beta(f \cdot g)\big)(z) = f(z)\big(\mathcal{A}^\alpha_\beta g\big)(z) + g(z)\big(\mathcal{A}^\alpha_\beta f\big)(z)$$

and a chain rule

$$\big(\mathcal{A}^\alpha_\beta(f \circ g)\big)(z) = f'\big(g(z)\big)\big(\mathcal{A}^\alpha_\beta g\big)(z) + \left(\frac{z}{g(z)}\right)^\beta \exp\left(\frac{g(z)-z}{\alpha z g(z)}\right)\big(\mathcal{A}^\alpha_\beta f\big)\big(g(z)\big). \qquad (1)$$

The formalism of Borinsky makes it possible to derive the asymptotic expansions of certain implicitly defined power series. However, the peculiar form of the correction term in chain rule (1) makes it difficult to interpret the result combinatorially.

1.2 Notations

Before moving on to a description of our work, let us introduce some notations that will be useful for this purpose and will be used throughout the paper.

Sets and expressions

The sets of integers and real numbers are designated by their usual notations, $\mathbb{Z}$ and $\mathbb{R}$, respectively. To represent their subsets, we employ subscripts. For instance, we write $\mathbb{R}_{>1}$ and $\mathbb{Z}_{>0}$ for the sets $\{x \in \mathbb{R} \mid x > 1\}$ and $\{n \in \mathbb{Z} \mid n > 0\}$, respectively. We also use indicator functions in one or several variables. Thus,

$$\mathbf{1}_{n=0} = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \mathbf{1}_{n=k} = \begin{cases} 1 & n = k \\ 0 & \text{otherwise} \end{cases}$$

are functions in the variable $n \in \mathbb{Z}$ and variables $n, k \in \mathbb{Z}$, respectively. The domain of any such function will be clear from the context.

We use Knuth's notation $n^{\underline{k}}$ for falling factorials [18]:

$$n^{\underline{k}} := n(n-1)\dots(n-k+1),$$

where n and k are non-negative integers.

Typically, to designate a family of graphs or 2-SAT-formulae, we use the first letters of its name. To represent its generating function and the corresponding counting sequence, we employ these letters written in uppercase serifs and lowercase Gothic fonts, respectively. For instance, the exponential generating function of irreducible tournaments is denoted by $\mathsf{IT}(z)$, while $\mathfrak{it}_n$ means the number of irreducible tournaments of size n. All the families that will be used in our paper are introduced in Section 2.

Sequences

For a sequence $(a_n)_{n=0}^{\infty}$ and an integer M, not necessarily positive, we write

$$a_n \approx \sum_{k \geqslant M} f_k(n), \tag{2}$$

if for all integers $r \geqslant M+1$, as $n \to \infty$, one has an asymptotic expansion

$$a_n = \sum_{k=M}^{r-1} f_k(n) + \mathcal{O}\big(f_r(n)\big),$$

where the sequence $\big(f_k\big)_{n=M}^{\infty}$ satisfies $f_{k+1}(n) = o\big(f_k(n)\big)$ for each $k < r$. In this case, we also write

$$a_n \sim f_M(n)$$

to identify the leading term of the asymptotics. Note that the sum in the right-hand side of (2) is formal and does not necessarily have to converge.

Formal power series

We use an operator $[z^n]$ to extract nth coefficient of formal power series in z:

$$\text{if} \quad A(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad \text{then} \quad [z^n]A(z) := a_n.$$

The *exponential Hadamard product* of formal power series $A(z)$ and $B(z)$ is designated by $A(z) \odot B(z)$. In other words, if

$$A(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \qquad \text{and} \qquad B(z) = \sum_{n=0}^{\infty} b_n \frac{z^n}{n!} \ ,$$

then

$$A(z) \odot B(z) := \sum_{n=0}^{\infty} a_n b_n \frac{z^n}{n!} \ .$$

In the case of several arguments, we use the notation $\odot_z$ to emphasize that the exponential Hadamard product is taken with respect to the argument z.

For different needs, we use several *types* of generating functions (GFs). To highlight the type of GF used, we capitalize them. Overall, we employ the following four types: *Exponential* GF, *Graphic* GF, *Implication* GF and *Coefficient* GF. The details are forthcoming in the next sections (see formulae (11), (16), (29) and (5), respectively).

1.3 Our contribution

Our paper is devoted to the asymptotic study of exponential generating series of a certain kind: the corresponding counting sequences diverge as powers of quadratic functions. For this purpose, we introduce a new type of generating series and an asymptotic transfer.

Definition 2. Let $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>0}$.

1. By $\mathfrak{G}_\alpha^\beta$ (the Gothic "G" for "graphic") we denote the set of formal power series

$$A(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \tag{3}$$

whose coefficients a_n satisfy an asymptotic expansion

$$a_n \approx \alpha^{\beta\binom{n}{2}} \left[\sum_{m \geqslant M} \alpha^{-mn} \sum_{\ell=0}^{\infty} n^{\underline{\ell}}\, a^\circ_{m,\ell} \right] \tag{4}$$

for some integer M with the additional assumption that, for each $m \in \mathbb{Z}_{\geqslant M}$, the support of the sequence $(a^\circ_{m,\ell})_{\ell=0}^\infty$ is finite. Such series will be called *graphically divergent*.

2. The *Coefficient GF of type* (α, β) associated with Exponential GF (3) whose coefficients $(a_n)_{n=0}^\infty$ admit an asymptotic expansion of the form (4) is the bivariate formal power series

$$A^\circ(z, w) := \sum_{m=M}^{\infty} \sum_{\ell=0}^{\infty} a^\circ_{m,\ell} \frac{z^m}{\alpha^{\frac{1}{\beta}\binom{m}{2}}} w^\ell. \tag{5}$$

We denote by $\mathfrak{C}_\alpha^\beta$ (the Gothic "C" for "coefficient") the set of Coefficient GF corresponding to $\mathfrak{G}_\alpha^\beta$. [1]

3. The *asymptotic transfer* $\mathcal{Q}_\alpha^\beta \colon \mathfrak{G}_\alpha^\beta \to \mathfrak{C}_\alpha^\beta$ is the linear operator that transfers a formal power series (an Exponential GF) to the respective bivariate Coefficient GFs, *i.e.* for $A \in \mathfrak{G}_\alpha^\beta$, we have

$$\mathcal{Q}_\beta^\alpha A = A^\circ. \tag{6}$$

In other words, if $(a_n)_{n=0}^\infty$ and $(a^\circ_{m,\ell})_{m=M,\ell=0}^\infty$ satisfy (3) and (4), then

$$\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \quad \xmapsto{\mathcal{Q}_\alpha^\beta} \quad \sum_{m=M}^{\infty} \sum_{\ell=0}^{\infty} a^\circ_{m,\ell} \frac{z^m}{\alpha^{\frac{1}{\beta}\binom{m}{2}}} w^\ell.$$

Here we assume that $\binom{m}{2} = \frac{m(m-1)}{2}$ for any $m \in \mathbb{Z}$.

[1] In addition to the sets $\mathfrak{G}_\alpha^\beta$ and $\mathfrak{C}_\alpha^\beta$, our paper contains two more objects that depend on parameters α and β: namely the operators Δ_α and $\Phi_\alpha^{\beta_1,\beta_2}$; see Theorem 30 and Theorem 33, respectively. Unlike Borinsky, we decided to set α as the subscript (and β as the superscript), in order to write powers of the operator Δ_α as Δ_α^m and avoid confusing powers and indices.

We will see in Section 3.3 that, for a fixed parameter α, the sets $\mathfrak{G}_\alpha^\beta$ are subsequently embedded one into another:

$$\mathfrak{G}_\alpha^1 \subset \mathfrak{G}_\alpha^2 \subset \mathfrak{G}_\alpha^3 \subset \mathfrak{G}_\alpha^4 \subset \dots$$

On the contrary, from the formal point of view, linear spaces $\mathfrak{C}_\alpha^\beta$ are the same for different values of α and β:

$$\mathfrak{C}_\alpha^\beta = \mathbb{R}[w][[z]].$$

The choice of parameters α and β corresponds to the choice of basis in this space.

For fixed parameters α and β, we show that the set $\mathfrak{G}_\alpha^\beta$ admit a ring structure, and that the asymptotic transfers are consistent with the ring operations. More precisely, the following holds.

Proposition 3. *For any fixed $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>0}$, the set $\mathfrak{G}_\alpha^\beta$ form a ring. For each pair $A, B \in \mathfrak{G}_\alpha^\beta$, the operations of addition and multiplication satisfy*

$$\big(\mathcal{Q}_\alpha^\beta(A+B)\big)(z,w) = (\mathcal{Q}_\alpha^\beta A)(z,w) + (\mathcal{Q}_\alpha^\beta B)(z,w) \tag{7}$$

and

$$\big(\mathcal{Q}_\alpha^\beta(A\cdot B)\big)(z,w) = A\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big)\cdot(\mathcal{Q}_\alpha^\beta B)(z,w) + B\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big)\cdot(\mathcal{Q}_\alpha^\beta A)(z,w). \tag{8}$$

Proposition 4. *Let $\alpha \in \mathbb{R}_{>1}$, $\beta \in \mathbb{Z}_{>0}$ and $A \in \mathfrak{G}_\alpha^\beta$ with $a_0 = 0$. If F is a function analytic in a neighbourhood of the origin, and $H(z) = \partial_x F(x)|_{x=A(z)}$, then $F \circ A \in \mathfrak{G}_\alpha^\beta$ and*

$$\big(\mathcal{Q}_\alpha^\beta(F\circ A)\big)(z,w) = H\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big)\cdot(\mathcal{Q}_\alpha^\beta A)(z,w). \tag{9}$$

Theorem 3 and Theorem 4, along with their multivariate versions Theorem 38 and Theorem 39, are our key tools for establishing asymptotic expansions of graphically divergent series. As in the case of Theorem 1, the proof is based on a careful estimation of the growth rate of the middle terms. However, compared to Borinsky's paper, we can see several differences. First, we deal with series whose growth rate is higher: roughly speaking, it is α^{n^2} versus $\alpha^n \cdot n!$. Second, we also keep track of polynomial "fluctuations" by means of two-dimensional arrays of coefficients. This allows us to get more information versus one-dimensional case of Borinsky, but the exact form of our coefficient generating functions needs to be carefully designed. We pay for the additional information by complexifying the function A°, which is reflected in the term $\alpha^{\frac{\beta+1}{2}} z^\beta w$ inserted into the Leibniz and chain rules. It is worth mentioning that the main advantage of Borinsky's approach is the possibility of obtaining compositions and inverses within the ring under consideration. In our case, only compositions with analytic functions are allowed. In principle, this means that all series we deal with can be treated with the help of Bender's theorem. Thus, our principal contribution is not to compute asymptotic expansions, but to present them in a nice looking, concise form, easy to understand, convenient to interpret and adapted to use.

We present several applications of this technique. First, we revisit in a simple manner the complete asymptotic expansions for connected digraphs and irreducible tournaments (the asymptotic coefficients were obtained by Wright [33] and [34]; a combinatorial interpretation of these coefficients was given by Monteil and Nurligareev [23]). Second, we obtain a complete asymptotic expansion of strongly connected digraphs (here, the asymptotic coefficients were known [35], but no interpretation was given). We also establish a complete asymptotic expansion for digraphs with a fixed number of strongly connected components (this result is new). In particular, we provide a refined version of this result, for the case when the number of source-like, sink-like and isolated components are given. Finally, we establish complete asymptotic expansions for satisfiable and contradictory strongly connected 2-CNF formulae (these results are also new). We also discuss the case where the number of components are fixed. The method can be potentially refined even further to restrain the strongly connected components.

Note that the coefficients involved into the asymptotic expansions under discussion are virtually always expressed in a relatively simple way via enumerating sequences of other combinatorial families. As a consequence, somehow as a byproduct of our method, they have combinatorial meanings on their own.

1.4 Structure of the paper

The paper consists of six sections, the first of which is the present introduction. Section 2 can be considered as a brief listing of all necessary prerequisites. We introduce different generating functions, as well as recall various graph families and 2-SAT model, and provide results related to their enumeration. The section is divided into two parts, but this division is rather arbitrary, since the parts share common ideas. The described material is largely not new and can be covered by [10, 12] and, for example, [15]. However, we encourage the reader to have a look at the presentation in order to get familiar with the differences (for instance, we prefer operators Δ_α and $\Phi_\alpha^{\beta_1,\beta_2}$ rather than exponential Hadamard product).

Section 3 is devoted to our method of the asymptotic transfer. This is the core part of the paper. We prove that graphically divergent series form a ring, and study properties of the asymptotic transfer. In Section 4, we consider applications of our method to asymptotics of undirected and directed graphs, while Section 5 is devoted to applications to 2-SAT formulae. Numerical values corresponding to the results of Section 4 and Section 5 are presented in Section A. Finally, in Section 6, we discuss possible extensions of the method not covered by the current paper: the behavior of expansions when the edge probability tends to zero, the enumeration of 2-connected graphs and blocks, and possible extensions to enumeration of the k-SAT formulae.

2 Generating functions and enumeration

2.1 Digraphs

In this section, we recall the symbolic method applied for enumeration of various (undirected or directed) graph families. We start with observing different graph structures

that are employed in our investigation. Note that all objects we work with are *labeled*; this is assumed throughout, and the word "labeled" will be omitted below. Next, we describe two types of generating functions, exponential and graphic, that serve for graph enumeration purposes, and explain relations between them. In particular, we discuss multivariate generating functions that are useful for marking patterns and parameters. The presentation of the topic is completed by enumeration results for diverse digraph classes, including close formulae for generating functions of acyclic, strongly connected and semi-strong digraphs. Our exposition is mainly based on the book [15] and the papers [10, 11], to which we refer the reader for further details.

2.1.1 Graph families

Let us recall various graph families that will be used throughout the paper.

A *graph* is a pair (V, E), where V is a finite set of *vertices*, typically represented by an interval $[n] := \{1, \dots, n\}$, and $E \subset \big\{\{x, y\} \mid x, y \in V,\, x \neq y\big\}$ is the set of *edges*. In particular, loops and multiple edges are forbidden in this model. In contrast to directed graphs, which are discussed below, these graphs are referred to as *undirected*.

A graph is *connected* if any pair of its vertices is joined by a path. In other words, for any pair $x, y \in V$, there exists a sequence of vertices

$$x = v_0,\, v_1,\, \dots,\, v_{m-1},\, v_m = y, \tag{10}$$

such that $\{v_{i-1}, v_i\} \in E$ for all $i = 1, \dots, m$. Every graph can be uniquely represented as a disjoint union of its connected components.

A *directed graph* (or, simply, a *digraph*) is a pair (V, E), where V is a finite set of vertices, and $E \subset \big\{(x, y) \mid x, y \in V,\, x \neq y\big\}$ is the set of edges. Contrary to undirected graphs, the order of vertices in an edge is important, so these edges are referred to as *directed*.

A digraph is *strongly connected* if any pair $x, y \in V$ is joined by a directed path, meaning that there exists a sequence of vertices (10) such that $(v_{i-1}, v_i) \in E$ for all $i = 1, \dots, m$. Every graph consists of several strongly connected components. In contrast to the undirected case, two components can be joined by directed edges. However, all the edges that join two distinct components must have the same direction.

Depending on its nature, a strongly connected component of a digraph can be

1. *source-like*, if it does not have incoming edges from other components;
2. *sink-like*, if it does not have outgoing edges towards other components;
3. *isolated*, if it is source-like and sink-like at the same time;
4. *purely source-like*, if it is source-like and not isolated;
5. *purely sink-like*, if it is sink-like and not isolated.

A digraph is *semi-strong*, if all its strongly connected components are isolated.

A *tournament* is a digraph such that each pair of its vertices $x, y \in V$ is joined by exactly one of two directed edges: either (x, y) or (y, x). A tournament is *reducible*, if there exists a partition of its set of vertices into two nonempty subsets A and B such that any pair of vertices $(a, b) \in A \times B$ are joined by the edge (a, b). Otherwise, the tournament is *irreducible*. Equivalently, a tournament is irreducible if and only if it is strongly connected [30].

Finally, a digraph is *acyclic*, if it does not contain directed cycles. Usually, the corresponding subclass of digraphs is referred to as *directed acyclic graphs*.

2.1.2 Exponential generating functions

Recall that $(a_n)_{n=0}^{\infty}$ is a *counting sequence* of a family $\mathcal{A}$ of (undirected or directed) graphs, if a_n denotes the number of graphs from $\mathcal{A}$ with n vertices. The *Exponential GF* of $\mathcal{A}$ is

$$A(z) := \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}. \tag{11}$$

Exponential GFs are commonly used for enumerating labeled combinatorial classes, since their behavior is consistent with the labeled product. More precisely, if $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are the counting sequences of classes $\mathcal{A}$ and $\mathcal{B}$, respectively, then the counting sequence $(c_n)_{n=0}^{\infty}$ of the labeled product $\mathcal{C} = \mathcal{A} \star \mathcal{B}$ obeys the binomial convolution rule,

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k},$$

which corresponds to the relation $C(z) = A(z)B(z)$ of the Exponential GFs. For further details, see [15].

Generating functions serve to express structural relationships between different classes of combinatorial objects in the language of algebra and vice versa. Thus, the Exponential GFs $\mathsf{G}(z)$ and $\mathsf{CG}(z)$ of graphs and connected graphs, respectively, satisfy the so-called *exponential formula* (see, for example, [32, Example 5.2.1]):

$$\mathsf{G}(z) = e^{\mathsf{CG}(z)}. \tag{12}$$

A similar formula gives a link between the Exponential GFs $\mathsf{SSD}(z)$ and $\mathsf{SCD}(z)$ of semi-strong digraphs and strongly connected digraphs, respectively:

$$\mathsf{SSD}(z) = e^{\mathsf{SCD}(z)}. \tag{13}$$

Another relation following from [27, formula (1)] links together the Exponential GFs $\mathsf{T}(z)$ and $\mathsf{IT}(z)$ of tournaments and irreducible tournaments, respectively:

$$\mathsf{T}(z) = \frac{1}{1 - \mathsf{IT}(z)}. \tag{14}$$

Note that the Exponential GFs of graphs and tournaments are the same:

$$\mathsf{G}(z) = \mathsf{T}(z) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{z^n}{n!}. \tag{15}$$

2.1.3 Graphic generating functions

For some families of directed graphs, it is more convenient to use *Graphic GFs*:

$$\widehat{A}(z) := \sum_{n=0}^{\infty} a_n \frac{z^n}{2^{\binom{n}{2}} n!}. \tag{16}$$

This is the case where, instead of the labeled product, we employ a so-called *arrow product*: given two families of directed graphs $\mathcal{A}$ and $\mathcal{B}$, we consider a new class $\mathcal{C}$ of pairs (a, b), $a \in \mathcal{A}$, $b \in \mathcal{B}$, equipped with additional edges directed from vertices of a to vertices of b. Indeed, the convolution rule corresponding to the arrow product is

$$c_n = \sum_{k=0}^{n} \binom{n}{k} 2^{k(n-k)} a_k b_{n-k}.$$

Hence, their Graphic GFs satisfy $\widehat{C}(z) = \widehat{A}(z)\widehat{B}(z)$. For details, see [10].

There are certain bridges between Exponential GFs and Graphic GFs. For instance, the Exponential GF $\mathsf{D}(z)$ and the Graphic GF $\widehat{\mathsf{D}}(z)$ of directed graphs are:

$$\mathsf{D}(z) = \sum_{n=0}^{\infty} 2^{n(n-1)} \frac{z^n}{n!} \qquad \text{and} \qquad \widehat{\mathsf{D}}(z) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{z^n}{n!} = \mathsf{G}(z),$$

respectively. To proceed from the Exponential GF of a family $\mathcal{A}$ to its Graphic GF, we use the linear operator Δ_2 first defined by Robinson [31]:

$$\Delta_2 A(z) := \widehat{A}(z). \tag{17}$$

This operator divides z^n by $2^{\binom{n}{2}}$ and can be expressed in terms of the exponential Hadamard product. Namely, the conversion between the Exponential GF $A(z)$ and the corresponding Graphic GF $\widehat{A}(z)$ is done according to the formula

$$\widehat{A}(z) = \Delta_2 A(z) = A(z) \odot \widehat{\mathsf{Set}}(z) \qquad \text{and} \qquad A(z) = \Delta_2^{-1} A(z) = \widehat{A}(z) \odot \mathsf{G}(z), \tag{18}$$

where, in notations of de Panafieu and Dovgal [10],

$$\widehat{\mathsf{Set}}(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^{\binom{n}{2}} n!}$$

is the Graphic GF of *digraphs without edges* (or, equivalently, *sets of isolated vertices*).

2.1.4 Marking variables

It is said that u is a *marking variable* for the number of occurrences of a pattern π (or, simply, that u *marks* the number of π) in a generating function

$$F(z, u) = \sum_{m=0}^{\infty} F_m(z) u^m,$$

if $F_m(z)$ is a generating function for objects having exactly m occurrences of π for every $m \geqslant 0$. This concept can be recursively extended to an arbitrary number of marking variables.

As an example of particular importance, consider undirected graphs. Introducing a marking variable w for the number of edges, we get the corresponding Exponential GF to be

$$\mathsf{G}(z,w) = \sum_{n=0}^{\infty} (1+w)^{\binom{n}{2}} \frac{z^n}{n!} \ . \tag{19}$$

In this case, the definition of a Graphic GF should be modified to

$$\widehat{A}(z,w) := \sum_{n=0}^{\infty} a_n(w) \frac{z^n}{(1+w)^{\binom{n}{2}} n!}, \tag{20}$$

so that conversions (18) hold. In particular,

$$\widehat{\mathsf{Set}}(z,w) = \sum_{n=0}^{\infty} \frac{z^n}{(1+w)^{\binom{n}{2}} n!}. \tag{21}$$

Another option is to consider a marking variable t for the number of connected components. Doing that, we obtain the following generalization of formula (12):

$$\mathsf{G}(z;t) = e^{t \cdot \mathsf{CG}(z)}. \tag{22}$$

Studying directed graphs, the reader may need several marking variables. First of all, it is reasonable to introduce a marking variable t for the number of strongly connected components. For instance, the bivariate Exponential GFs of semi-strong digraphs and tournaments satisfy, respectively,

$$\mathsf{SSD}(z;t) = e^{t \cdot \mathsf{SCD}(z)} \qquad \text{and} \qquad \mathsf{T}(z;t) = \frac{1}{1 - t \cdot \mathsf{IT}(z)}, \tag{23}$$

which generalize relations (13) and (14). We could be interested in source-like components too. In the next section, we also introduce marking variables for the numbers of purely source-like, purely sink-like and isolated components of digraphs.

Remark 5. Introducing a marking variable for the number of edges in a graph is closely related the Erdős-Rényi model $\mathbb{G}(n,p)$ [14, 17] and the similar model $\mathbb{D}(n,p)$. Recall that, according to these models, each edge of a (undirected or directed) graph with n vertices appears independently with a fixed positive probability p. In particular, in $\mathbb{G}(n,p)$ a graph with m edges appears with the probability of

$$p^m (1-p)^{\binom{n}{2}-m} = \left(\frac{p}{1-p}\right)^m (1-p)^{\binom{n}{2}},$$

while in $\mathbb{D}(n,p)$ the probability to obtain a fixed digraph with m directed edges is

$$p^m (1-p)^{2\binom{n}{2}-m} = \left(\frac{p}{1-p}\right)^m (1-p)^{2\binom{n}{2}}.$$

It can be shown that if $\mathcal{F}$ is a undirected (resp. directed) graph family whose Exponential GF (resp. Graphic GF) is $F(z,w)$, then the probability that a randomly generated graph from $\mathbb{G}(n,p)$ (resp. $\mathbb{D}(n,p)$) belongs to $\mathcal{F}$ is exactly

$$\mathbb{P}_{\mathcal{F}}(n,p) = (1-p)^{\binom{n}{2}} n! [z^n] F\left(z, \frac{p}{1-p}\right)$$

(see [11, Lemma 2.8]). Putting the weight of a graph to be $(\frac{p}{1-p})^m$, we get the total weight of all graphs equal to $(1-p)^{-\binom{n}{2}}$, and the probability of a specific family can be obtained by dividing its weight by the total weight. Note, that the case $p = \frac{1}{2}$ corresponds to enumeration of graphs, since $w = \frac{p}{1-p} = 1$ in this case.

2.1.5 Enumeration of digraphs with marking variables

Let us now proceed to general enumerative results that provide generating functions for digraphs with various marking variables. They are based on a technique developed by Gessel [16] for counting acyclic digraphs by sources and sinks. Some of results presented here seem to be new, others can be found in the works [31, 10, 11].

Proposition 6 ([10, Theorem 3.4])**.** *Let $\mathcal{A}$ be a family of strongly connected digraphs and $A(z)$ be its Exponential GF. If s marks the number of source-like components, then the bivariate Graphic GF $\widehat{D}_{\mathcal{A}}(z;s)$ of digraphs whose strongly connected components belong to $\mathcal{A}$ is given by*

$$\widehat{\mathsf{D}}_{\mathcal{A}}(z;s) = \frac{\Delta_2\big(e^{(s-1)A(z)}\big)}{\Delta_2\big(e^{-A(z)}\big)}.$$

Corollary 7. *If s marks the number of source-like components and t marks the total number of strongly connected components, then the Graphic GF $\widehat{\mathsf{D}}(z;s,t)$ of digraphs satisfies*

$$\widehat{\mathsf{D}}(z;s,t) = \frac{\Delta_2\big(e^{(s-1)t\cdot\mathsf{SCD}(z)}\big)}{\Delta_2\big(e^{-t\cdot\mathsf{SCD}(z)}\big)} = \frac{\Delta_2\big(\mathsf{SSD}(z;(s-1)t)\big)}{\Delta_2\big(\mathsf{SSD}(z;-t)\big)}. \tag{24}$$

In particular, we have the following expression for the bivariate Graphic GF $\widehat{\mathsf{D}}(z;t)$ of digraphs:

$$\widehat{\mathsf{D}}(z;t) = \frac{1}{\Delta_2\big(e^{-t\cdot\mathsf{SCD}(z)}\big)} = \frac{1}{\Delta_2\big(\mathsf{SSD}(z;-t)\big)}. \tag{25}$$

Proof. Let us apply Theorem 6 to the family $\mathcal{A} = \mathcal{S}_t$ consisting of all strongly connected digraphs taken with a weight t each. The Exponential GF of this family is $t\cdot\mathsf{SCD}(z)$, while $\widehat{D}_{\mathcal{S}_t}(z;s)$ is the Graphic GF of weighted digraphs, where a digraph with k strongly connected components has the weight t^k and s marks the number of source-like components. Together with (23), this gives us (24), and putting $s=1$, we obtain (25). □

Proposition 8. *If u, v and y mark, respectively, the numbers of purely source-like, purely sink-like and isolated components, while t marks the total number of strongly connected components, then the multivariate Exponential GF $\mathsf{D}(z;u,v,y,t)$ of digraphs is given by*

$$\mathsf{D}(z;u,v,y,t) = e^{(y-u-v+1)t\cdot\mathsf{SCD}(z)} \cdot \Delta_2^{-1}\left(\frac{\Delta_2\big(e^{(u-1)t\cdot\mathsf{SCD}(z)}\big)\cdot\Delta_2\big(e^{(v-1)t\cdot\mathsf{SCD}(z)}\big)}{\Delta_2\big(e^{-t\cdot\mathsf{SCD}(z)}\big)}\right). \tag{26}$$

Proof. Consider the family of all digraphs with a distinguished subset of purely source-like, purely sink-like, and isolated components. Let each purely source-like component be marked with a variable $\hat{u}$, each purely sink-like component be marked with a variable $\hat{v}$, and each isolated component be marked with one of the variables $\hat{u}$, $\hat{v}$, or $\hat{y}$. If, additionally, t marks the total number of connected components, then the Graphic GF of such digraphs is $\widehat{\mathsf{D}}(z;1+\hat{u},1+\hat{v},1+\hat{u}+\hat{v}+\hat{y},t)$.

The constructed family can be decomposed into the labeled product of the class of isolated components marked by $\hat{y}$ and another digraph family. The latter, in its turn, is the arrow product of the following three digraph families: the class of source-like components marked by $\hat{u}$ (some of which may be isolated), the class of arbitrary digraphs, and the class of sink-like components marked by $\hat{v}$ (again, some of them may be isolated). For example, Figure 1 shows the following components of a digraph from the family, from left to right: the one marked with $\hat{u}$ (violet), the non-distinguished one (blue), the one marked with $\hat{v}$ (green), and the one marked with $\hat{y}$ (orange). Note that each component is marked by t as well.

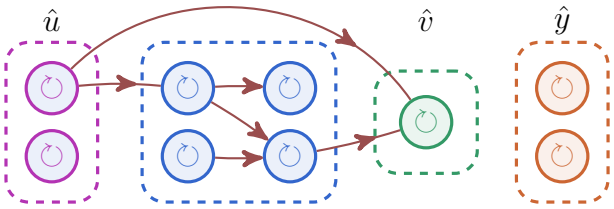

Figure 1: Decomposition with marked source-like, sink-like and isolated components.

The above decomposition can be turned into a functional equation with several intermediate conversions between Exponential and Graphic GFs:

$$\mathsf{D}(z;1+\hat{u},1+\hat{v},1+\hat{u}+\hat{v}+\hat{y},t) = e^{\hat{y}t\cdot\mathsf{SCD}(z)}\cdot\Delta_2^{-1}\Big(\Delta_2\big(e^{\hat{u}t\cdot\mathsf{SCD}(z)}\big)\cdot\widehat{\mathsf{D}}(z;t)\cdot\Delta_2\big(e^{\hat{v}t\cdot\mathsf{SCD}(z)}\big)\Big),$$

where $\widehat{\mathsf{D}}(z;t) = \widehat{\mathsf{D}}(z;1,1,1,t)$, is the Graphic GF for digraphs with marking variable t for the total number of strongly connected components. The above equation can be solved by putting $\hat{u}=u-1$, $\hat{v}=v-1$, $\hat{y}=y-u-v+1$ and substituting $\widehat{\mathsf{D}}(z;t)$ from Theorem 7, which completes the proof. □

Remark 9. Following de Panafieu and Dovgal [10], relations (24)-(26) can be rewritten with the help of the exponential Hadamard product in the following way:

$$\widehat{\mathsf{D}}(z;s,t)=\frac{e^{(s-1)t\cdot\mathsf{SCD}(z)}\odot_z\widehat{\mathsf{Set}}(z)}{e^{-t\cdot\mathsf{SCD}(z)}\odot_z\widehat{\mathsf{Set}}(z)},\qquad\qquad\widehat{\mathsf{D}}(z;t)=\frac{1}{e^{-t\cdot\mathsf{SCD}(z)}\odot_z\widehat{\mathsf{Set}}(z)}$$

and

$$\mathsf{D}(z;u,v,y,t)=\\ e^{(y-u-v+1)t\cdot\mathsf{SCD}(z)}\left(\left[\frac{\left(e^{(u-1)t\cdot\mathsf{SCD}(z)}\odot_z\widehat{\mathsf{Set}}(z)\right)\left(e^{(v-1)t\cdot\mathsf{SCD}(z)}\odot_z\widehat{\mathsf{Set}}(z)\right)}{e^{-t\cdot\mathsf{SCD}(z)}\odot_z\widehat{\mathsf{Set}}(z)}\right]\odot_z\mathsf{G}(z)\right).$$

2.1.6 Enumeration of acyclic, strongly connected and semi-strong digraphs

Here, we recall exact enumeration results for directed acyclic graphs, strongly connected digraphs and semi-strong digraphs. They appeared for the first time in the paper of Robinson [31], and later were rewritten in terms of the exponential Hadamard product by de Panafieu and Dovgal [10].

Proposition 10 ([31, Corollary 1])**.** *The Graphic GF* $\widehat{\mathsf{DAG}}(z)$ *of acyclic digraphs is given by*

$$\widehat{\mathsf{DAG}}(z)=\frac{1}{\Delta_2(e^{-z})}.$$

Proof. Apply Proposition 6 to the family $\mathcal{A}$ consisting of a single vertex and put $s=1$. □

Proposition 11 ([10, Corollary 3.5])**.** *The Exponential GF* $\mathsf{SCD}(z)$ *of strongly connected digraphs is given by*

$$\mathsf{SCD}(z)=-\log\left(\mathsf{G}(z)\odot\frac{1}{\mathsf{G}(z)}\right).$$

Corollary 12. *The Exponential GFs* $\mathsf{SSD}(z)$ *and* $\mathsf{SCD}(z)$ *of semi-strong digraphs and strongly connected digraphs, respectively, are given by*

$$\mathsf{SSD}(z)=\frac{1}{1-\mathsf{IT}(z)\odot\mathsf{G}(z)}\qquad\textit{and}\qquad\mathsf{SCD}(z)=\log\frac{1}{1-\mathsf{IT}(z)\odot\mathsf{G}(z)}.\tag{27}$$

Proof. It is sufficient to apply Proposition 11 and formulae (13-15). □

Notation 13. We designate by $\mathfrak{dag}_n$, $\mathfrak{scd}_n$ and $\mathfrak{ssd}_n$, respectively, the numbers of acyclic, strongly connected and semi-strong digraphs on n vertices, so that

$$\widehat{\mathsf{DAG}}(z)=\sum_{n=0}^{\infty}\mathfrak{dag}_n\frac{z^n}{2^{\binom{n}{2}}n!},\qquad\mathsf{SCD}(z)=\sum_{n=0}^{\infty}\mathfrak{scd}_n\frac{z^n}{n!},\qquad\mathsf{SSD}(z)=\sum_{n=0}^{\infty}\mathfrak{ssd}_n\frac{z^n}{n!}.$$

Remark 14. While Theorem 11 also holds in the case of marked edges, a proper generalization of Theorem 12 is more subtle. Behind the proof of Theorem 12, we can see an additional combinatorial meaning of the function $\big(\mathsf{G}(z)\big)^{-1}$, namely,

$$\frac{1}{\mathsf{G}(z)} = 1 - \mathsf{IT}(z) \tag{28}$$

(the proof of (28) follows immediately from relations (14) and (15)). While formula (12) is directly generalized to

$$\mathsf{G}(z, w) = e^{\mathsf{CG}(z,w)}$$

by marking edges, the way to generalize relation (14) is hidden. The natural analogue for tournaments would be marking *descents*, that is, directed edges (s, t) whose labels satisfy $s < t$. However, as it was shown in [2], this approach fails, since relation (14) remains true only if we replace Exponential GFs $\mathsf{T}(z)$ and $\mathsf{IT}(z)$ with the so-called *Eulerian GFs*.

Nevertheless, it turns out that a required generalization is possible [26]. The key idea is to pass to another tournament analogue of the Erdős-Rényi model $\mathbb{G}(n, p)$ for random graphs discussed in Theorem 5. In order to do so, for $p \in [1/2, 1]$ and $q = 1 - p$, we introduce a *tournament with ties* of size n as a directed graph with the set of vertices $[n]$. Each pair of vertices x and y in this graph, independently of other pairs, is joined by the edge (x, y), the edge (y, x), and by both of them with probability $1-p$, $1-p$, and $2p-1$, respectively. Putting the weight of such a tournament to be

$$(w-1)^m = \left(\frac{2p-1}{1-p}\right)^m,$$

where m is the total number of *ties* in the tournament (that is, pairs of vertices joined by two directed edges) and $w = \frac{p}{1-p}$, we have the total weight of all tournaments with ties of size n equal to $(1-p)^{-\binom{n}{2}}$, and

$$\mathsf{T}(z; w) = \frac{1}{1 - \mathsf{IT}(z; w)}.$$

However, it is important to emphasize that it is not the variable w that counts ties now, but $u = w - 1$.

2.2 2-SAT formulae

In this section, we shortly describe the symbolic method for enumeration of 2-SAT formulae and implication digraphs recently developed in [12]. We start by recalling the model and related notions, such as implication digraphs and contradictory components. Next, we discuss the Implication GFs employed for enumeration purposes and their connections to other types of generating functions. Finally, we give exact expressions for generating functions of satisfiable formulae, contradictory strongly connected implication digraphs and implication digraphs with marking variables for contradictory and ordinary strongly connected components.

2.2.1 Definitions and basic properties

A *k-conjunctive normal form* (k-CNF) formula with n Boolean variables $\{x_1, \ldots, x_n\}$ and m clauses is a conjunction of the form

$$\bigwedge_{i=1}^{m} (c_{i1} \vee \ldots \vee c_{ik}),$$

where each of the *literals* c_{ij} belongs to the set $\{x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n\}$. These formulae are also called k-SAT formulae. A formula is *satisfiable* if it takes a True value under at least one variable assignment.

In the case $k = 2$, a 2-CNF can be mapped to a so-called *implication digraph*. The vertices of this digraph are the literals $\{x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n\}$. Each clause $u \vee v$ corresponds to two edges $\overline{u} \to v$ and $\overline{v} \to u$ in the implication digraph. The meaning of each edge is a logical implication under satisfiability. We assume that there is no clause of types $(x \vee x)$ and $(x \vee \overline{x})$ in our model. Also, we suppose that each clause in a 2-CNF can occur at most once. As a consequence, the corresponding implication digraph has no loops and no multiple edges.

It is now a well-known property of the 2-SAT problem (see [3]) that a formula is not satisfiable if and only if there is a *contradictory variable*, *i.e.* a pair of literals x and $\overline{x}$ such that there exists a directed path from x to $\overline{x}$ and from $\overline{x}$ to x. It can be shown that if there is at least one contradictory variable inside a strongly connected component of the implication digraph, then all the variables included into this component are contradictory. We say that a *component* is *contradictory* if it contains at least one contradictory variable, and *ordinary* otherwise.

2.2.2 Implication generating function

Let $(a_n)_{n=0}^{\infty}$ be the counting sequence of a class $\mathcal{A}$ of 2-SAT formulae, *i.e.* a_n denotes the number of Boolean 2-SAT formulae from $\mathcal{A}$ with n variables. The *Implication GF* of $\mathcal{A}$ is

$$\ddot{A}(z) := \sum_{n=0}^{\infty} a_n \frac{z^n}{2^{n^2} n!}. \tag{29}$$

For example, the Implication GF of all 2-SAT formulae is

$$\ddot{\mathsf{CNF}}(z) = \sum_{n=0}^{\infty} 2^{n(n-1)} \frac{z^n}{2^n n!} = D(z/2). \tag{30}$$

Clearly, the Exponential GF and Implication GF of the same class can be expressed in terms of each other:

$$A(z) = \Delta_2^{-2}\Big(\ddot{A}(2z)\Big) = \ddot{A}(z) \odot \mathsf{D}(2z) \tag{31}$$

and

$$\ddot{A}(z) = \Delta_2^{2}\Big(A\left(z/2\right)\Big) = A(z) \odot \ddot{\mathsf{Set}}(z), \tag{32}$$

where, in notations of de Panafieu, Dovgal and Ravelomanana [12],

$$\ddot{\mathsf{Set}}(z) := \sum_{n=0}^{\infty} \frac{z^n}{2^{n^2} n!}$$

is the Implication GF of digraphs without vertices.

Similarly to Exponential GFs and Graphic GF, the design of Implication GFs comes from enumerative needs. We will not discuss this issue here. For more detailed information, we refer the reader to [12], in particular to Proposition 3.11.

If we introduce a marking variable w for the total number of clauses, then the bivariate Implication GF takes the following form:

$$\ddot{A}(z,w) := \sum_{n=0}^{\infty} a_n(w) \frac{z^n}{(1+w)^{n(n-1)} 2^n n!}.$$

In particular,

$$\ddot{\mathsf{Set}}(z,w) := \sum_{n=0}^{\infty} \frac{z^n}{(1+w)^{n(n-1)} 2^n n!},$$

which is used for the bivariate analogue of (32).

2.2.3 Enumeration of 2-SAT

Proposition 15 ([12, Proposition 4.5])**.** *Let $\mathcal{S}$ and $\mathcal{C}$ be two families of strongly connected digraphs whose Exponential GFs are $S(z)$ and $C(z)$, respectively. Then the Implication GF $\ddot{\mathsf{CNF}}_{\mathcal{S},\mathcal{C}}(z)$ of implication digraphs whose ordinary and contradictory strongly connected components belong to families $\mathcal{S}$ and $\mathcal{C}$, respectively, is given by*

$$\ddot{\mathsf{CNF}}_{\mathcal{S},\mathcal{C}}(z) = \frac{\Delta_2^2\big(e^{C(z/2)-S(z)/2}\big)}{\Delta_2\big(e^{-S(z)}\big)}.$$

Corollary 16 ([12, Theorem 4.6])**.** *The Implication GF $\ddot{\mathsf{SAT}}(z)$ of satisfiable 2-CNFs is given by*

$$\ddot{\mathsf{SAT}}(z) = \mathsf{G}(z) \cdot \Delta_2^2\left(e^{-\frac{1}{2}\mathsf{SCD}(z)}\right). \tag{33}$$

Proof. Apply Theorem 15 with $\mathcal{C} = \emptyset$ and $\mathcal{S}$ consisting of all strongly connected digraphs and take into account that $\Delta_2\big(e^{-\mathsf{SCD}(z)}\big) = G^{-1}(z)$ by Theorem 11. □

Corollary 17 ([12, Theorem 4.8])**.** *The Exponential GF $\mathsf{CSCC}(z)$ of contradictory strongly connected implication digraphs is given by*

$$\mathsf{CSCC}(z) = \frac{1}{2}\mathsf{SCD}(2z) + \log\left(\Delta_2^{-2}\Big(\mathsf{D}(z)\big(1-\mathsf{IT}(2z)\big)\Big)\right). \tag{34}$$

Proof. Apply Theorem 15 with $\mathcal{C}$ and $\mathcal{S}$ consisting of all contradictory strongly connected and all strongly connected digraphs, respectively, and use Theorem 12 and relation (30). □

Corollary 18. *If s marks the number of contradictory strongly connected components and t marks the number of pairs of ordinary strongly connected components in the corresponding implication digraph, then the multivariate Implication GF $\ddot{\mathsf{CNF}}(z;s,t)$ of 2-CNFs is given by*

$$\ddot{\mathsf{CNF}}(z;s,t) = \frac{\Delta_2^2\left(e^{s\cdot \mathsf{CSCC}(z/2)-t/2\cdot \mathsf{SCD}(z)}\right)}{\Delta_2\left(e^{-t\cdot \mathsf{SCD}(z)}\right)}. \tag{35}$$

Proof. The statement is obtained by equipping the Exponential GFs of the strongly connected components from Theorem 15 with marking variables for their weights. In other words, we put $C(z) = s \cdot \mathsf{CSCC}(z)$ and $S(z) = t \cdot \mathsf{SCD}(z)$. □

Remark 19. Initially, in [12], relations (33)-(35) were stated in terms of the exponential Hadamard product, respectively, as

$$\ddot{\mathsf{SAT}}(z) = \mathsf{G}(z) \cdot \left(e^{-\frac{1}{2}\mathsf{SCD}(z)} \odot \ddot{\mathsf{Set}}(2z)\right),$$

$$\mathsf{CSCC}(z) = \frac{1}{2}\mathsf{SCD}(2z) + \log\left(\mathsf{D}(z) \odot \frac{\mathsf{D}(z)}{\mathsf{G}(2z)}\right)$$

and

$$\ddot{\mathsf{CNF}}(z;s,t) = \frac{e^{s\cdot \mathsf{CSCC}(z)-t/2\cdot \mathsf{SCD}(2z)} \odot_z \ddot{\mathsf{Set}}(z)}{e^{-t\cdot \mathsf{SCD}(z)} \odot_z \widehat{\mathsf{Set}}(z)}.$$

3 Asymptotic transfers for graphically divergent series

This section is devoted to the study of graphically divergent series and the corresponding Coefficient GFs. We show that the graphically divergent series admit a ring structure, and that the asymptotic transfers are consistent with ring operations and compositions with analytic functions, as well as with several other operations such as differentiation and integration. Moreover, the concept of asymptotic transfer can be extended to the case of marking variables.

3.1 Proofs of the main asymptotic transfer properties

The goal of this section is to provide proofs of Theorem 3 and Theorem 4. In the case of compositions with analytic functions, a common tool for establishing asymptotic expansions is Bender's theorem [5] (see Theorem 23). To simplify the presentation, we introduce *gargantuan sequences*, which are implicitly employed in its statement as one of its hypotheses. We show that all counting sequences under consideration are gargantuan, and therefore, Bender's theorem is applicable. The concept of gargantuan sequences is also useful for the product of two series, although in this case the proof is rather straightforward.

Definition 20. We will call a sequence (a_n) *gargantuan*, if for any positive integer R the following two conditions hold, as $n \to \infty$:

$$\text{(i)} \quad \frac{a_{n-1}}{a_n} \to 0, \qquad\qquad \text{(ii)} \quad \sum_{k=R}^{n-R} |a_k a_{n-k}| = \mathcal{O}\big(a_{n-R}\big).$$

Lemma 21. *Let $\alpha \in \mathbb{R}_{>1}$, $\beta \in \mathbb{Z}_{>0}$ and $m, \ell \in \mathbb{Z}_{\geqslant 0}$ be fixed numbers. Suppose that*

$$c_n \sim \frac{\alpha^{\beta\binom{n}{2}} \alpha^{-mn} n^\ell}{n!},$$

as $n \to \infty$. Then the sequence (c_n) is gargantuan.

Proof. Let d_n denote $\alpha^{\beta\binom{n}{2}-mn} n^\ell / n!$. Since $c_n \sim d_n$ as $n \to \infty$, we have $c_n \leqslant 2d_n$ for large enough n. Furthermore, $d_n = o(d_{n+1})$ as $n \to \infty$, and hence, $c_n = o(c_{n+1})$ as $n \to \infty$. Thus, condition (i) of Theorem 20 holds. In particular, $d_n \leqslant d_{n+1}$ for large enough n.

In order to verify condition (ii), we prove that the maximum of the product $d_k d_{n-k}$ over $k \in [R, n-R]$ is attained on the boundary of the interval $[R, n-R]$, as n is sufficiently large. For this aim, consider the function

$$d\colon \mathbb{R}_{>0} \to \mathbb{R}, \qquad d(x) = \frac{\alpha^{\frac{\beta x(x-1)}{2} - mx} x^\ell}{\Gamma(x+1)},$$

coinciding with $(d_n)_{n=0}^\infty$ at positive integers. It is sufficient to show that $\log d(x)$ is a convex function for large enough argument x. Indeed, according to [1, (6.4.12)],

$$\frac{\partial^2}{\partial x^2}(\log \Gamma(x)) \sim \frac{1}{x}, \quad \text{and therefore,} \quad \frac{\partial^2}{\partial x^2} \log d(x) = \beta \log \alpha + \mathcal{O}\left(\frac{1}{x}\right),$$

as $x \to \infty$. The latter quantity is positive for large x, since $\alpha > 1$ and $\beta > 0$.

Now, let N be an integer, large enough, but fixed, such that the inequalities $c_k \leqslant 2d_k$ and $d_k \leqslant d_{k+1}$ hold for all $k \geqslant N$. Assume further that $N \geqslant R+1$. Then, as $n \to \infty$, we have

$$\begin{aligned}
\sum_{k=R}^{n-R} |c_k c_{n-k}| &= 2\sum_{k=R}^{N-1} |c_k c_{n-k}| + \sum_{k=N}^{n-N} |c_k c_{n-k}| \\
&\leqslant 4(N-R) \max_{k \in [R,N]} |c_k| \cdot d_{n-R} + 4 \sum_{k=N}^{n-N} d_k d_{n-k} \\
&\leqslant \mathcal{O}(d_{n-R}) + 4(n-2N+1) d_N d_{n-N} \\
&= \mathcal{O}(d_{n-R}) = \mathcal{O}(c_{n-R}).
\end{aligned}$$

□

Lemma 22. *If a sequence (a_n) is gargantuan and a sequence (b_n) satisfies $b_n = \mathcal{O}(a_n)$, as $n \to \infty$, then, for any positive integer R,*

$$\sum_{k=R}^{n-R} |b_k a_{n-k}| = \mathcal{O}(a_{n-R}),$$

as $n \to \infty$.

Proof. The proof of this statement is direct. Since $b_n = \mathcal{O}(a_n)$, there exist constants $C \in \mathbb{R}_{>0}$ and $N \in \mathbb{Z}_{>0}$ such that, for $k \geqslant N$, we have $|b_k| \leqslant C|a_k|$. Hence, the sum can be split into two parts:

$$\begin{aligned}\sum_{k=R}^{n-R} |b_k a_{n-k}| &\leqslant \sum_{k=R}^{N-1} |b_k a_{n-k}| + C \sum_{k=N}^{n-R} |a_k a_{n-k}| \\ &\leqslant |b_R a_{n-R}| + \max_{k\in[R,N-1]} |b_k| \cdot (N-R-1) \cdot o(a_{n-R}) + C \sum_{k=R}^{n-R} |a_k a_{n-k}| \\ &= \mathcal{O}(a_{n-R}).\end{aligned}$$

□

Proof of Theorem 3. Let $A, B \in \mathfrak{G}_\alpha^\beta$. In this case, formula (7) holds, since

$$a_n + b_n \approx \alpha^{\beta\binom{n}{2}} \left[\sum_{m\geqslant M} \alpha^{-mn} \sum_{\ell=0}^{\infty} n^{\underline{\ell}} (a^\circ_{m,\ell} + b^\circ_{m,\ell}) \right],$$

where M is the minimum of the constants corresponding to expansions of a_n and b_n.

To get (8), note that, according to Theorem 21, the sequences $(a_n/n!)$ and $(b_n/n!)$ are gargantuan. Fixing a positive integer R, we have

$$n![z^n]A(z)B(z) = \sum_{k=0}^{R-1} \binom{n}{k} \big(a_k b_{n-k} + b_k a_{n-k}\big) + \sum_{k=R}^{n-R} \binom{n}{k} b_k a_{n-k}.$$

Due to Theorem 22, the second sum is negligible. Thus, it is sufficient to rewrite the first sum in asymptotic form (4) and verify that its coefficients coincide with those of the right-hand side of (8). Given a fixed integer k, the asymptotics of the shifted sequence (a_{n-k}), as $n \to \infty$, is

$$\begin{aligned}\binom{n}{k} b_k a_{n-k} &\approx \frac{b_k}{k!} \alpha^{\beta\binom{n-k}{2}} \sum_{m\geqslant M} \alpha^{-m(n-k)} \sum_{\ell\geqslant 0} n^{\underline{k}} (n-k)^{\underline{\ell}} a^\circ_{m,\ell} \\ &\approx \frac{b_k}{k!} \alpha^{\beta\binom{n}{2}+\beta\binom{k}{2}+\beta k} \sum_{m\geqslant M} \alpha^{-mn-\beta kn} \sum_{\ell\geqslant 0} n^{\underline{\ell+k}} a^\circ_{m,\ell} \alpha^{mk} \\ &\approx \frac{b_k}{k!} \alpha^{\beta\binom{n}{2}+\beta\binom{k}{2}+\beta k} \sum_{m\geqslant M+\beta k} \alpha^{-mn} \sum_{\ell\geqslant k} n^{\underline{\ell}} a^\circ_{m-\beta k,\ell-k} \alpha^{(m-\beta k)k} \\ &\approx \alpha^{\beta\binom{n}{2}} \sum_{m\geqslant M+\beta k} \alpha^{-mn} \sum_{\ell\geqslant k} n^{\underline{\ell}} a^\circ_{m-\beta k,\ell-k} \frac{b_k \alpha^{km}}{k! \alpha^{\beta\binom{k}{2}}}.\end{aligned}$$

At the same time, by expanding a product of the form $B\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big) \cdot (\mathcal{Q}_\alpha^\beta A)(z,w)$, we

obtain:

$$\begin{aligned}
&\alpha^{\frac{1}{\beta}\binom{m}{2}}[z^m w^\ell]\left(\sum_{k=0}^{\infty} b_k \frac{\left(\alpha^{\frac{\beta+1}{2}} z^\beta w\right)^k}{k!}\right)\left(\sum_{r=M}^{\infty}\sum_{s=0}^{\infty} a^\circ_{r,s}\frac{z^r}{\alpha^{\frac{1}{\beta}\binom{r}{2}}}w^s\right)\\
&=\sum_{k\geqslant 0}\frac{b_k}{k!}a^\circ_{m-\beta k,\ell-k}\alpha^{\frac{(\beta+1)k}{2}+\frac{1}{\beta}\binom{m}{2}-\frac{1}{\beta}\binom{m-\beta k}{2}}\\
&=\sum_{k\geqslant 0}\frac{b_k\alpha^{km}}{k!\alpha^{\beta\binom{k}{2}}}a^\circ_{m-\beta k,\ell-k}.
\end{aligned}$$

That is why, comparing this expression with the previous one summed up over k such that $k\leqslant \ell$ and $M+\beta k\leqslant m$, we conclude that formula (8) is valid. □

To prove Theorem 4, we use Bender's theorem [5] that commonly serves to provide asymptotic expansions for divergent formal power series. We cite here an adaptation of his theorem originally presented in a more general form.

Proposition 23 ([5, Theorem 2])**.** *Consider a formal power series*

$$A(z)=\sum_{n=1}^{\infty} a_n z^n$$

and a function $F(x)$, which is analytic in some neighborhood of origin. Define

$$B(z)=\sum_{n=0}^{\infty} b_n z^n = F\big(A(z)\big) \qquad \text{and} \qquad C(z)=\sum_{n=0}^{\infty} c_n z^n = \frac{\partial}{\partial x}F(x)\bigg|_{x=A(z)}.$$

Assume that the sequence (a_n) is gargantuan and $a_n\neq 0$ for any positive integer $n>0$. Then

$$b_n \approx \sum_{k\geqslant 0} c_k a_{n-k}$$

and the sequence (b_n) is gargantuan.

Proof of Theorem 4. Let $A\in\mathfrak{G}_\alpha^\beta$ and F be a function analytic in a neighbourhood of the origin. According to Theorem 23 (which is applicable due to Theorem 21),

$$n![z^n](F\circ A)(z)\approx\sum_{k\geqslant 0}\binom{n}{k}\eta_k a_{n-k}, \tag{36}$$

where η_k are the coefficients of the $H(z)=\partial_x F(x)|_{x=A(z)}$,

$$H(z)=\sum_{k=0}^{\infty}\eta_k\frac{z^k}{k!}.$$

Now we follow the scheme of the proof of Theorem 3. Namely, we rewrite (36) in asymptotic form (4) and verify that its coefficients are equivalent to those of $H\big(\alpha^{\frac{\beta+1}{2}}z^\beta w\big)\cdot(\mathcal{Q}_\alpha^\beta A)(z,w)$. Indeed, for fixed $k\in\mathbb{Z}_{>0}$ and $n\to\infty$, we have

$$\binom{n}{k}\eta_k a_{n-k}\approx\alpha^{\beta\binom{n}{2}}\sum_{m\geqslant M+\beta k}\alpha^{-mn}\sum_{\ell\geqslant k}n^{\underline{\ell}}a^\circ_{m-\beta k,\ell-k}\frac{\eta_k\alpha^{km}}{k!\alpha^{\beta\binom{k}{2}}}$$

and

$$\alpha^{\frac{1}{\beta}\binom{m}{2}}[z^m w^\ell]H\big(\alpha^{\frac{\beta+1}{2}}z^\beta w\big)(\mathcal{Q}_\alpha^\beta A)(z,w)=\sum_{k\geqslant 0}\frac{\eta_k\alpha^{km}}{k!\alpha^{\beta\binom{k}{2}}}a^\circ_{m-\beta k,\ell-k}.$$

Hence, formula (9) is valid. □

3.2 Other transfer properties

In this section, we discuss four more properties of the asymptotic transfer $\mathcal{Q}_\alpha^\beta$. First, we express explicitly the behavior of $\mathcal{Q}_\alpha^\beta$ with respect to powers of a series $A\in\mathfrak{G}_\alpha^\beta$. Next, we study its relations with linear change of variable $z\mapsto\alpha z$. Finally, we discuss how the asymptotic transfer $\mathcal{Q}_\alpha^\beta$ can be applied together with the operations of differentiation and integration.

Lemma 24. *If $\alpha\in\mathbb{R}_{>1}$, $\beta\in\mathbb{Z}_{>0}$ and $A\in\mathfrak{G}_\alpha^\beta$, then $A^m\in\mathfrak{G}_\alpha^\beta$ for each $m\in\mathbb{Z}_{\geqslant 0}$ with*

$$\big(\mathcal{Q}_\alpha^\beta A^m\big)(z,w)=m\cdot A^{m-1}\big(\alpha^{\frac{\beta+1}{2}}z^\beta w\big)\cdot(\mathcal{Q}_\alpha^\beta A)(z,w).\tag{37}$$

If, additionally, $a_0=1$, then (37) holds for any $m\in\mathbb{Q}$.

Proof. For non-negative integer m, this follows from (8) by induction. In the other cases, we apply Theorem 4 to the series $\big(A(z)-1\big)$ and function $F(x)=(1+x)^m$. □

Lemma 25. *Let $\alpha\in\mathbb{R}_{>1}$, $\beta\in\mathbb{Z}_{>0}$, and $A,B\in\mathfrak{G}_\alpha^\beta$. If $B(z)=A(\alpha^d z)$ for some $d\in\mathbb{Z}$, then*

$$(\mathcal{Q}_\alpha^\beta B)(z,w)=\frac{(\mathcal{Q}_\alpha^\beta A)(\alpha^{d/\beta}z,w)}{\alpha^{\frac{1}{\beta}\binom{d+1}{2}}z^d}.$$

Proof. If (4) holds for the coefficients $(a_n)_{n=0}^\infty$ of $A(z)$, then the coefficients of $B(z)$ satisfy

$$b_n:=\alpha^{dn}a_n\approx\alpha^{\beta\binom{n}{2}}\left[\sum_{m\geqslant M-d}\alpha^{-mn}\sum_{\ell=0}^{\infty}n^{\underline{\ell}}a^\circ_{m+d,\ell}\right].$$

Hence, using Theorem 2 and the relation $\binom{m-d}{2}=\binom{m}{2}-md+\binom{d+1}{2}$, we obtain

$$(\mathcal{Q}_\alpha^\beta B)(z,w)=\frac{z^{-d}}{\alpha^{\frac{1}{\beta}\binom{d+1}{2}}}\sum_{m=M}^{\infty}\sum_{\ell=0}^{\infty}a^\circ_{m,\ell}\frac{(\alpha^d z)^m}{\alpha^{\frac{1}{\beta}\binom{m}{2}}}w^\ell,$$

which implies the statement of the lemma. □

Corollary 26. *Let $\alpha \in \mathbb{R}_{>1}$, $\beta \in \mathbb{Z}_{>0}$, and $A, B, C \in \mathfrak{G}_\alpha^\beta$. If $B(z) = A(\alpha z)$ and $C(z) = A(z/\alpha)$, then*

$$(\mathcal{Q}_\alpha^\beta B)(z,w) = \alpha^{-1/\beta} z^{-1} \cdot (\mathcal{Q}_\alpha^\beta A)(\alpha^{1/\beta} z, w) \quad \text{and} \quad (\mathcal{Q}_\alpha^\beta C)(z,w) = z \cdot (\mathcal{Q}_\alpha^\beta A)(\alpha^{-1/\beta} z, w).$$

Proof. Apply Theorem 25 for $d = 1$ and $d = -1$, respectively. □

Proposition 27. *If $\alpha \in \mathbb{R}_{>1}$, $\beta \in \mathbb{Z}_{>0}$ and $A \in \mathfrak{G}_\alpha^\beta$, then*

$$(\mathcal{Q}_\alpha^\beta A')(z,w) = \frac{(\mathcal{Q}_\alpha^\beta A)(z,w) + \frac{\partial}{\partial w}(\mathcal{Q}_\alpha^\beta A)(z,w)}{\alpha^{\frac{\beta+1}{2}} z^\beta}.$$

and

$$\left(\mathcal{Q}_\alpha^\beta \int A\right)(z,w) = \alpha^{\frac{\beta+1}{2}} z^\beta \sum_{k=0}^{\infty} (-1)^k \frac{\partial^k}{\partial w^k} \big(\mathcal{Q}_\alpha^\beta A\big)(z,w).$$

Proof. Similarly to the proof of Theorem 25, this follows from Theorem 2 with the help of direct calculations. Given coefficients $(a_n)_{n=0}^\infty$ satisfying (4), the idea is to express asymptotics expansions of a_{n+1} and a_{n-1} in the same form as the initial sequence. That can be done due to relations

$$(n+1)^{\underline{\ell}} = n^{\underline{\ell}} + \ell n^{\underline{\ell-1}}$$

and

$$(n-1)^{\underline{\ell}} = \sum_{k=0}^{\ell} (-1)^{\ell-k} \ell^{\underline{\ell-k}} n^{\underline{k}}.$$

Since the calculations are straightforward, we allow ourselves to omit the details. □

3.3 Relations between different rings of graphically divergent series

The goal of this section is to study relations between rings $\mathfrak{G}_\alpha^\beta$ for a fixed parameter α and different values of β. First, we show that there is a natural inclusion

$$\mathfrak{G}_\alpha^1 \subset \mathfrak{G}_\alpha^2 \subset \mathfrak{G}_\alpha^3 \subset \mathfrak{G}_\alpha^4 \subset \dots$$

and that each ring $\mathfrak{G}_\alpha^\beta$ in this row belongs to the "kernel" of the next one, meaning that asymptotic coefficients of the elements of $\mathfrak{G}_\alpha^\beta$ are zeroes with respect to $\mathfrak{G}_\alpha^{\beta+1}$. To "compare" elements of $\mathfrak{G}_\alpha^\beta$ and $\mathfrak{G}_\alpha^{\beta+1}$, we introduce a linear operator Δ_α that changes the growth rate of a formal power series, leaving the asymptotic coefficients the same. The operator Δ_α comes together with another family of operators, denoted by $\Phi_\alpha^{\beta_1,\beta_2}$, that change the type of a Coefficient GF, leaving its coefficients unchanged. The connections between these operators and asymptotic transfers are reflected by the commutative diagram described in Theorem 35.

Lemma 28. *If $\alpha \in \mathbb{R}_{>1}$ and $\beta_1, \beta_2 \in \mathbb{Z}_{>0}$, such that $\beta_1 < \beta_2$, then*

$$\mathfrak{G}_\alpha^{\beta_1} \subset \mathfrak{G}_\alpha^{\beta_2}.$$

Moreover, for any $A \in \mathfrak{G}_\alpha^{\beta_1}$, we have

$$(\mathcal{Q}_\alpha^{\beta_2} A)(z,w) = 0.$$

Proof. Since, for some integers m and ℓ and constant C,

$$n![z^n] \sim C\alpha^{\beta_1\binom{n}{2}}\alpha^{-mn}n^{\ell},$$

all the coefficients of expansion (4) are zeroes if $\alpha^{\beta_2\binom{n}{2}}$ is chosen as the main term. □

Remark 29. Power series whose coefficients grow exponentially or factorially also take part of the ring $\mathfrak{G}_\alpha^\beta$ for any $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>0}$. So are power series with non-zero radius of convergence. Indeed, if A is a series of one of the mentioned kind, then its expansion coefficients $a^\circ_{m,\ell}$ are all zeroes. In particular, $(\mathcal{Q}_\alpha^\beta A)(z,w) = 0$.

Definition 30. Let $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>1}$. The linear operator $\Delta_\alpha \colon \mathfrak{G}_\alpha^\beta \to \mathfrak{G}_\alpha^{\beta-1}$ is defined by

$$\Delta_\alpha\left(\sum_{n=0}^{\infty} a_n\frac{z^n}{n!}\right) := \sum_{n=0}^{\infty}\frac{a_n}{\alpha^{\binom{n}{2}}}\frac{z^n}{n!}, \tag{38}$$

where $(a_n)_{n=0}^\infty$ satisfy (3) and (4).

Remark 31. Taking into account Theorem 29, we can consider Δ_α as an operator $\mathfrak{G}_\alpha^\beta \to \mathfrak{G}_\alpha^\beta$ for any positive integer β, including $\beta = 1$. This allows us to make sense of the power $\Delta_\alpha^m A$ for any $A \in \mathfrak{G}_\alpha^\beta$ and $m \in \mathbb{Z}$. The operator Δ_2 that we have seen in (17) is the particular case of the above operator Δ_α with $\alpha = 2$. As well as for Δ_2, the action of Δ_α can be expressed in terms of the exponential Hadamard product. To this end, in relations (18) we need to replace Exponential GFs $\mathsf{G}(z)$ and $\widehat{\mathsf{Set}}(z)$ by their bivariate analogues (19) and (21) taken at $w = \alpha - 1$:

$$\Delta_\alpha A(z) = A(z) \odot \widehat{\mathsf{Set}}(z, \alpha-1) \qquad \text{and} \qquad \Delta_\alpha^{-1} A(z) = A(z) \odot \mathsf{G}(z, \alpha-1).$$

More generally, for any non-zero integer m, we have

$$\Delta_\alpha^m A(z) = A(z) \odot \mathsf{G}(z, \alpha^{-m}-1). \tag{39}$$

This observation is particularly useful for numerical calculations.

Lemma 32. *If $\alpha \in \mathbb{R}_{>1}$, $\beta \in \mathbb{Z}_{>0}$ and $A \in \mathfrak{G}_\alpha^\beta$, then*

$$\big(\mathcal{Q}_\alpha^\beta(\Delta_\alpha A)\big)(z,w) = 0.$$

Proof. This follows directly from Theorem 28 and Theorem 29. □

Definition 33. Let $\alpha \in \mathbb{R}_{>1}$ and $\beta_1, \beta_2 \in \mathbb{Z}_{>0}$. The linear operator $\Phi_\alpha^{\beta_1,\beta_2} \colon \mathfrak{C}_\alpha^{\beta_1} \to \mathfrak{C}_\alpha^{\beta_2}$ is the mapping that transfers a Coefficient GF of type (α, β_1) to the Coefficient GF of type (α, β_2) with the same coefficients. In other words,

$$\Phi_\alpha^{\beta_1,\beta_2}\left(\sum_{m=M}^{\infty}\sum_{\ell=0}^{\infty} a^\circ_{m,\ell}\frac{z^m}{\alpha^{\frac{1}{\beta_1}\binom{m}{2}}}w^\ell\right) := \sum_{m=M}^{\infty}\sum_{\ell=0}^{\infty} a^\circ_{m,\ell}\frac{z^m}{\alpha^{\frac{1}{\beta_2}\binom{m}{2}}}w^\ell, \tag{40}$$

where $M \in \mathbb{Z}$ and the support of the sequence $(a^\circ_{m,\ell})_{\ell=0}^\infty$ is finite for any $m \in \mathbb{Z}_{m\geqslant M}$.

Remark 34. We have already mentioned that the sets $\mathfrak{C}_\alpha^\beta$ represent different choices of bases in the same linear space $\mathbb{R}[w][[z]]$. In its turn, the operator $\Phi_\alpha^{\beta_1,\beta_2}$ represents a change of basis in this space. This corresponds to the change of the Coefficient GF associated with a series, when we pass from $A \in \mathfrak{G}_\alpha^{\beta_1}$ to $B = \Delta_\alpha^{\beta_1-\beta_2} A \in \mathfrak{G}_\alpha^{\beta_2}$ whose coefficients are defined by $b_n = \alpha^{\binom{n}{2}(\beta_2-\beta_1)} a_n$. As well as Δ_α, the operator $\Phi_\alpha^{\beta_1,\beta_2}$ can be expressed in terms of the exponential Hadamard product:

$$\Phi_\alpha^{\beta_1,\beta_2} A^\circ(z,w) = A^\circ(z,w) \odot_z \mathsf{G}(z, \alpha^{\frac{1}{\beta_1}-\frac{1}{\beta_2}} - 1).$$

For calculations, the following identity, understood in terms of the Hadamard product, could be useful:

$$\Phi_\alpha^{\beta_1,\beta_2} = \Delta_\alpha^{\frac{1}{\beta_2}-\frac{1}{\beta_1}}.$$

Lemma 35. *If $\alpha \in \mathbb{R}_{>1}$ and $\beta_1, \beta_2 \in \mathbb{Z}_{>0}$, then, for any $A \in \mathfrak{G}_\alpha^{\beta_1}$,*

$$\big(\mathcal{Q}_\alpha^{\beta_2}(\Delta_\alpha^{\beta_1-\beta_2} A)\big)(z,w) = \Phi_\alpha^{\beta_1,\beta_2}\Big((\mathcal{Q}_\alpha^{\beta_1} A)(z,w)\Big).$$

In other words, the following diagram is commutative.

$$\begin{array}{ccc} \mathfrak{G}_\alpha^{\beta_1} & \xrightarrow{\mathcal{Q}_\alpha^{\beta_1}} & \mathfrak{C}_\alpha^{\beta_1} \\ {\scriptstyle \Delta_\alpha^{\beta_1-\beta_2}}\downarrow & & \downarrow{\scriptstyle \Phi_\alpha^{\beta_1,\beta_2}} \\ \mathfrak{G}_\alpha^{\beta_2} & \xrightarrow{\mathcal{Q}_\alpha^{\beta_2}} & \mathfrak{C}_\alpha^{\beta_2} \end{array}$$

Proof. This follows directly from Theorem 2, Theorem 30 and Theorem 33. □

3.4 Transfers and marking variables

The theory developed in the previous sections can be naturally extended for the case of marking variables. The aim of this section is to convince the reader that the results we have seen above are still valid for this extension. For simplicity, we consider only one marking variable u. The reader will see that the statements we formulate are fulfilled in the case of several variables as well.

Given $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>0}$, let $\mathfrak{G}_\alpha^\beta(u)$ be the set of formal power series $A = A(z;u)$ of form (3) whose coefficients $a_n = a_n(u)$ satisfy (4) with

$$a_{m,\ell}^\circ = a_{m,\ell}^\circ(u) = \sum_{k=0}^{\infty} a_{m,\ell;k}^\circ u^k.$$

Here we suppose that the support of the two-dimensional array $(a_{m,\ell;k}^\circ)_{\ell,k=0}^\infty$ is finite for each $m \in \mathbb{Z}_{\geqslant M}$. In particular, $a_{m,\ell}^\circ(u)$ are polynomials in u, and relation (4) can be rewritten as

$$a_n(u) \approx \alpha^{\beta\binom{n}{2}} \left[\sum_{m \geqslant M} \alpha^{-mn} \sum_{\ell=0}^{L_m} n^{\underline{\ell}} \sum_{k=0}^{K_m} a_{m,\ell;k}^\circ u^k \right]$$

for some constants L_m and K_m. In this case, similarly to Theorem 2, the Coefficient GF of type (α, β) associated with $A(z;u)$ is the formal power series $A^\circ = A^\circ(z,w;u)$ defined by (5), and the set of Coefficient GFs is denoted by $\mathfrak{C}_\alpha^\beta(u)$. The asymptotic transfer $\mathcal{Q}_\alpha^\beta \colon \mathfrak{G}_\alpha^\beta(u) \to \mathfrak{C}_\alpha^\beta(u)$ is defined as before, so that $\mathcal{Q}_\alpha^\beta A = A^\circ$.

Lemma 36. *If $\alpha \in \mathbb{R}_{>1}$, $\beta \in \mathbb{Z}_{>0}$ and $A \in \mathfrak{G}_\alpha^\beta(u)$, then for any $\kappa \in \mathbb{Z}_{\geqslant 0}$*

$$[u^\kappa]\,(\mathcal{Q}_\alpha^\beta A)(z,w;u) = \big(\mathcal{Q}_\alpha^\beta\,[u^\kappa]A\big)(z,w).$$

Proof. Straightforward calculations show that both expressions are equal to

$$\sum_{m=M}^{\infty}\sum_{\ell=0}^{\infty} a^\circ_{m,\ell;\kappa}\frac{z^m}{\alpha^{\frac{1}{\beta}\binom{m}{2}}}w^\ell.$$

□

Notation 37. For a fixed $\kappa \in \mathbb{Z}_{\geqslant 0}$, denote

$$A_\kappa(z) := [u^\kappa]\,A(z;u).$$

It follows from the above that if $A(z;u) \in \mathfrak{G}_\alpha^\beta(u)$, then $A_\kappa(z) \in \mathfrak{G}_\alpha^\beta$.

Proposition 38. *For any fixed $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>0}$, the set $\mathfrak{G}_\alpha^\beta(u)$ form a ring. For each pair $A, B \in \mathfrak{G}_\alpha^\beta(u)$, the operations of addition and multiplication satisfy*

$$\big(\mathcal{Q}_\alpha^\beta(A+B)\big)(z,w;u) = (\mathcal{Q}_\alpha^\beta A)(z,w;u) + (\mathcal{Q}_\alpha^\beta B)(z,w;u) \tag{41}$$

and

$$\big(\mathcal{Q}_\alpha^\beta(A\cdot B)\big)(z,w;u) = A\big(\alpha^{\frac{\beta+1}{2}}z^\beta w;u\big)\cdot(\mathcal{Q}_\alpha^\beta B)(z,w;u) + B\big(\alpha^{\frac{\beta+1}{2}}z^\beta w;u\big)\cdot(\mathcal{Q}_\alpha^\beta A)(z,w;u). \tag{42}$$

Proof. Relation (41) comes directly from the definitions. In order to verify formula (42), we prove that the corresponding coefficients in u are the same. Indeed, according to Theorem 36 and Theorem 3, we have

$$\begin{aligned}[u^\kappa]\,(\mathcal{Q}_\alpha^\beta AB)(z,w;u) = \big(\mathcal{Q}_\alpha^\beta\,[u^\kappa]AB\big)(z,w) &= \left(\mathcal{Q}_\alpha^\beta\sum_{s=0}^{\kappa}A_sB_{\kappa-s}\right)(z,w)\\ &= \sum_{s=0}^{\kappa}\Big(A_s\big(\alpha^{\frac{\beta+1}{2}}z^\beta w\big)\cdot(\mathcal{Q}_\alpha^\beta B_{\kappa-s})(z,w) + B_{\kappa-s}\big(\alpha^{\frac{\beta+1}{2}}z^\beta w\big)\cdot(\mathcal{Q}_\alpha^\beta A_s)(z,w)\Big).\end{aligned}$$

On the other hand, the same tools give us

$$\begin{aligned}[u^\kappa]\left(A\big(\alpha^{\frac{\beta+1}{2}}z^\beta w;u\big)\cdot(\mathcal{Q}_\alpha^\beta B)(z,w;u)\right) &= \sum_{s=0}^{\kappa}A_s\big(\alpha^{\frac{\beta+1}{2}}z^\beta w\big)\cdot[u^{\kappa-s}]\,\big(\mathcal{Q}_\alpha^\beta B\big)(z,w;u)\\ &= \sum_{s=0}^{\kappa}A_s\big(\alpha^{\frac{\beta+1}{2}}z^\beta w\big)\cdot(\mathcal{Q}_\alpha^\beta B_{\kappa-s})(z,w)\end{aligned}$$

for the first summand of the right-hand side of (42), and a similar relation holds for the second summand. Comparing the obtained expressions, we conclude that relation (42) holds. □

Proposition 39. *Let $\alpha \in \mathbb{R}_{>1}$, $\beta \in \mathbb{Z}_{>0}$ and $A \in \mathfrak{G}_\alpha^\beta$ with $a_0 = 0$. If we have $F(x;u)$ to be a function analytic in a neighbourhood of the origin, $G(z;u) = F\big(A(z);u\big)$, and $H(z;u) = \partial_x F(x;u)|_{x=A(z)}$, then $G \in \mathfrak{G}_\alpha^\beta(u)$ and*

$$\big(\mathcal{Q}_\alpha^\beta G\big)(z,w;u) = H\big(\alpha^{\frac{\beta+1}{2}} z^\beta w;u\big) \cdot (\mathcal{Q}_\alpha^\beta A)(z,w), \tag{43}$$

Proof. Similarly to the proof of Theorem 38, we verify that the coefficients in u are the same for both expressions. For this purpose, for any $\kappa \in \mathbb{Z}_{\geqslant 0}$, let us introduce

$$F_\kappa(x) := [u^\kappa]\, F(x;u) \qquad \text{and} \qquad H_\kappa(z) := [u^\kappa]\, H(z;u).$$

Since $[u^\kappa]G(z;u) = [u^\kappa]F\big(A(z);u\big) = F_\kappa\big(A(z)\big)$ and $H_\kappa(z) = \partial_x F_\kappa(x)|_{x=A(z)}$, due to Theorem 36 and Theorem 4 we have

$$\begin{aligned}
[u^\kappa]\big(\mathcal{Q}_\alpha^\beta G\big)(z,w;u) &= \big(\mathcal{Q}_\alpha^\beta (F_\kappa \circ A)\big)(z,w) \\
&= H_\kappa\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big) \cdot (\mathcal{Q}_\alpha^\beta A)(z,w) \\
&= [u^\kappa] H\big(\alpha^{\frac{\beta+1}{2}} z^\beta w;u\big) \cdot (\mathcal{Q}_\alpha^\beta A)(z,w). \qquad \square
\end{aligned}$$

Proposition 40. *If $\alpha \in \mathbb{R}_{>1}$, $\beta \in \mathbb{Z}_{>0}$ and $A \in \mathfrak{G}_\alpha^\beta(u)$, then $A^m \in \mathfrak{G}_\alpha^\beta(u)$ for each $m \in \mathbb{Z}_{\geqslant 0}$ with*

$$\big(\mathcal{Q}_\alpha^\beta A^m\big)(z,w;u) = m \cdot A^{m-1}\big(\alpha^{\frac{\beta+1}{2}} z^\beta w;u\big) \cdot (\mathcal{Q}_\alpha^\beta A)(z,w;u). \tag{44}$$

If, additionally, $[u^0]A(z;u) = 1$, then (37) holds for any $m \in \mathbb{Z}$.

Proof. The essential part of the proof concerns the case when $m = -1$, since the rest follows from Theorem 38. Let us check that, for each $\kappa \in \mathbb{Z}_{>0}$, extracting κth coefficients from both sides of (44) leads to the same result. To do this, first notice that $A_+(z;u) = A(z;u) - 1$ is divisible by u. Hence, $\big(A_+(z;u)\big)^k$ is divisible by $u^{\kappa+1}$ for any $k > \kappa$ and

$$[u^\kappa]\big(A(z;u)\big)^{-1} = \sum_{s=0}^{\kappa} (-1)^s\, [u^\kappa]\big(A_+(z;u)\big)^s.$$

As a consequence, Theorem 36 and Theorem 3 imply that

$$[u^\kappa]\big(\mathcal{Q}_\alpha^\beta A^{-1}\big)(z,w;u) = [u^\kappa]\left[\sum_{s=1}^{\kappa} (-1)^s s \cdot A_+^{s-1}\big(\alpha^{\frac{\beta+1}{2}} z^\beta w;u\big) \cdot (\mathcal{Q}_\alpha^\beta A_+)(z,w;u)\right].$$

On the other hand,

$$\begin{aligned}
-\,[u^\kappa]\left[A^{-2}\big(\alpha^{\frac{\beta+1}{2}} z^\beta w;u\big) \cdot (\mathcal{Q}_\alpha^\beta A)(z,w;u)\right] = \\
-\,[u^\kappa]\left[\left(\sum_{s=0}^{\infty} (-1)^s A_+^s\big(\alpha^{\frac{\beta+1}{2}} z^\beta w;u\big)\right)^2 (\mathcal{Q}_\alpha^\beta A)(z,w;u)\right],
\end{aligned}$$

and, taking into account that $(\mathcal{Q}_\alpha^\beta A)(z,w;u) = (\mathcal{Q}_\alpha^\beta A_+)(z,w;u)$ is divisible by u, this gives us the same expression. $\square$

Similarly to what has been done in Section 3.3, we define $\Delta_\alpha\colon \mathfrak{G}_\alpha^\beta(u) \to \mathfrak{G}_\alpha^{\beta-1}(u)$ and $\Phi_\alpha^{\beta_1,\beta_2}\colon \mathfrak{C}_\alpha^{\beta_1}(u) \to \mathfrak{C}_\alpha^{\beta_2}(u)$, so that these linear operators satisfy relations (38) and (40), respectively. In this case, the following generalization of Theorem 35 holds.

Lemma 41. *If $\alpha \in \mathbb{R}_{>1}$ and $\beta_1, \beta_2 \in \mathbb{Z}_{>0}$, then, for any $A \in \mathfrak{G}_\alpha^{\beta_1}(u)$,*

$$\big(\mathcal{Q}_\alpha^{\beta_2}(\Delta_\alpha^{\beta_1-\beta_2}A)\big)(z,w;u) = \big(\Phi_\alpha^{\beta_1,\beta_2}(\mathcal{Q}_\alpha^{\beta_1}A)\big)(z,w;u).$$

Proof. This follows directly from definitions. □

4 Digraphs

4.1 Asymptotics for graphs and tournaments

In this section, as a first example of the application of our method, we discuss asymptotics of undirected graphs and tournaments. Both asymptotics are known: they were established for the first time in 1970 by Wright [33, 34] and were combinatorially interpreted in 2021 by Monteil and Nurligareev [23]. Here we, first, revisit these results in terms of Coefficient GFs by applying asymptotic transfer, and second, employ the obtained results to get more general asymptotics. Namely, we obtain the Coefficient GFs of graphs and tournaments with a marking variables for the number of connected graphs and irreducible tournaments, respectively. The latter result appears to be novel in its general form, although essential information related to the asymptotics under consideration (including the interpretation of the coefficients) can be obtained in another way; see [25] and [26].

Lemma 42. *The Exponential GFs $\mathsf{G}(z)$, $\mathsf{T}(z)$, and $\mathsf{D}(z)$ of undirected graphs, tournaments, and directed graphs, respectively, belong to the rings $\mathfrak{G}_2^1$, $\mathfrak{G}_2^1$, and $\mathfrak{G}_2^2$. Their Coefficient GFs of type $(2,1)$, $(2,1)$, and $(2,2)$, respectively, satisfy*

$$(\mathcal{Q}_2^1\,\mathsf{G})(z,w) = (\mathcal{Q}_2^1\,\mathsf{T})(z,w) = 1 \qquad \textit{and} \qquad (\mathcal{Q}_2^2\,\mathsf{D})(z,w) = 1.$$

Proof. This follows immediately from Theorem 2, since

$$\mathsf{G}(z) = \mathsf{T}(z) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}}\frac{z^n}{n!} \qquad \text{and} \qquad \mathsf{D}(z) = \sum_{n=0}^{\infty} 2^{2\binom{n}{2}}\frac{z^n}{n!}.$$ □

Theorem 43. *The Exponential GF $\mathsf{CG}(z)$ of connected graphs belongs to the ring $\mathfrak{G}_2^1$ and its Coefficient GF of type $(2,1)$ satisfies*

$$(\mathcal{Q}_2^1\,\mathsf{CG}) = 1 - \mathsf{IT}(2zw). \tag{45}$$

Proof. As we have seen in Theorem 42, the Exponential GF $\mathsf{G}(z)$ of graphs belongs to $\mathfrak{G}_2^1$ with

$$(\mathcal{Q}_2^1\,\mathsf{G})(z,w) = 1.$$

Since the Exponential GF of connected graphs satisfy the exponential formula

$$\mathsf{CG}(z) = \log\big(\mathsf{G}(z)\big),$$

we can apply Theorem 4 to $A(z) = \mathsf{G}(z) - 1$ and $F(x) = \log(1+x)$ with $\alpha = 2$ and $\beta = 1$. Taking into account formulae (14) and (15), we have

$$H(z) = \frac{1}{\mathsf{G}(z)} = 1 - \mathsf{IT}(z),$$

which implies target relation (45). □

Corollary 44. *The bivariate Exponential GF $G(z;t)$ of graphs with a variable t that marks the number of connected components belongs to the ring $\mathfrak{G}_2^1(t)$ and its Coefficient GF of type $(2,1)$ satisfies*

$$(\mathcal{Q}_2^1\,\mathsf{G})(z,w;t) = t \cdot \mathsf{G}(2zw;t) \cdot \big(1 - \mathsf{IT}(2zw)\big).$$

In particular, for any $m \in \mathbb{Z}_{\geqslant 0}$, the asymptotics of graphs with $(m+1)$ connected components is given by

$$[t^{m+1}](\mathcal{Q}_2^1\,\mathsf{G})(z,w;t) = \frac{1}{m!}\mathsf{CG}^m(2zw) \cdot \big(1 - \mathsf{IT}(2zw)\big). \tag{46}$$

Proof. Taking into account relation (22), it is sufficient to apply Theorem 39 to the series $A(z) = \mathsf{CG}(z)$ and the function $F(x;t) = e^{tx}$ with $\alpha = 2$ and $\beta = 1$. □

Remark 45. Relation (45) corresponds to the asymptotic expansion

$$\mathfrak{cg}_n \approx 2^{\binom{n}{2}} \left(1 - \sum_{k\geqslant 1} \mathfrak{it}_k \binom{n}{k} \frac{2^{\binom{k+1}{2}}}{2^{kn}}\right)$$

established in [23], where $\mathfrak{cg}_n$ and $\mathfrak{it}_n$ are the numbers of connected graphs and irreducible tournaments of size n, respectively. Formula (46) reflects the asymptotic expansion from [28, Theorem 7.3.1], see also [26].

Theorem 46. *The Exponential GF $\mathsf{IT}(z)$ of irreducible tournaments belongs to the ring $\mathfrak{G}_2^1$ and its Coefficient GF of type $(2,1)$ satisfies*

$$(\mathcal{Q}_2^1\,\mathsf{IT})(z,w) = \big(1 - \mathsf{IT}(2zw)\big)^2. \tag{47}$$

Proof. Due to Theorem 42, the Exponential GF of tournaments belongs to $\mathfrak{G}_2^1$ and

$$(\mathcal{Q}_2^1\,\mathsf{T})(z,w) = 1.$$

According to (14), the Exponential GF of irreducible tournaments satisfy

$$\mathsf{IT}(z) = 1 - \frac{1}{\mathsf{T}(z)}.$$

Hence, to get relation (47), it is sufficient to apply Theorem 4 to $A(z) = \mathsf{T}(z) - 1$ and $F(x) = 1 - (1+x)^{-1}$ with $\alpha = 2$ and $\beta = 1$. □

Corollary 47. *The bivariate Exponential GF $T(z;t)$ of tournaments with a variable t that marks the number of irreducible parts belongs to the ring $\mathfrak{G}_2^1(t)$ and its Coefficient GF of type $(2,1)$ satisfies*

$$(\mathcal{Q}_2^1\,\mathsf{T})(z,w;t) = t\cdot\big(\mathsf{T}(2zw;t)\cdot\big(1-\mathsf{IT}(2zw)\big)\big)^2.$$

In particular, for any $m\in\mathbb{Z}_{\geqslant 0}$, the asymptotics of tournaments with $(m+1)$ irreducible parts is given by

$$[t^{m+1}](\mathcal{Q}_2^1\,\mathsf{T})(z,w;t) = (m+1)\cdot\mathsf{IT}^m(2zw)\cdot\big(1-\mathsf{IT}(2zw)\big)^2. \tag{48}$$

Proof. Due to the second of relations (23), it is sufficient to apply Theorem 39 to $A(z)=\mathsf{IT}(z)$ and $F(x;t)=(1-tx)^{-1}$ with $\alpha=2$ and $\beta=1$. □

Remark 48. Formula (47) reflects the asymptotic expansion

$$\mathfrak{it}_n \approx 2^{\binom{n}{2}}\left(1-\sum_{k\geqslant 1}\big(2\mathfrak{it}_k-\mathfrak{it}_k^{(2)}\big)\binom{n}{k}\frac{2^{\binom{k+1}{2}}}{2^{kn}}\right)$$

proved in [23], where $\mathfrak{it}_n$ and $\mathfrak{it}_n^{(2)}$ are the number of irreducible tournaments and tournaments consisting of two irreducible parts (both of size n). Formula (48) is consistent with the asymptotics established in [25] and [28, Theorem 5.3.1], see also [26].

Remark 49. Taking into account Theorem 5, we can establish the asymptotic behavior of graphs within the Erdős-Rényi model. Indeed, denoting $\alpha=(1-p)^{-1}$, we get the Exponential GF of graphs expressed as

$$\mathsf{G}(z)=\sum_{n=0}^{\infty}\alpha^{\binom{n}{2}}\frac{z^n}{n!}.$$

Relations (12) and (22) remain valid, which lead us to

$$(\mathcal{Q}_\alpha^1\mathsf{CG})(z,w)=\frac{1}{\mathsf{G}(\alpha zw)}=e^{-\mathsf{CG}(\alpha zw)}$$

and

$$(\mathcal{Q}_\alpha^1\mathsf{G})(z,w;t)=t\cdot\frac{\mathsf{G}(\alpha zw;t)}{\mathsf{G}(\alpha zw)}=t\cdot e^{(t-1)\cdot\mathsf{CG}(\alpha zw)}.$$

However, there is no combinatorial interpretation in terms of irreducible tournaments anymore, see Theorem 14.

4.2 Asymptotics for strongly connected digraphs

In this section, we explore the asymptotic behavior of strongly connected digraphs. A classical enumeration result related to this combinatorial class was first obtained by Wright [35] in 1971 and reproved by Bender [5] several years later (see also the papers

of Liskovets [20, 21]). Here, we establish the corresponding Coefficient GF and the exact form of its coefficients in terms of semi-strong digraphs and tournaments. This allows us to rewrite the previously known asymptotics in a compact form and provide a combinatorial meaning to the involved coefficients.

Theorem 50. *The Exponential GF* $\mathsf{SCD}(z)$ *of strongly connected digraphs belongs to the ring* $\mathfrak{G}_2^2$ *and its Coefficient GF of type* $(2,2)$ *satisfies*

$$(\mathcal{Q}_2^2\,\mathsf{SCD})(z,w) = \mathsf{SSD}(2^{3/2}z^2w)\cdot\Phi_2^{1,2}\big(1-\mathsf{IT}(2zw)\big)^2. \tag{49}$$

Proof. Recall that, according to Theorem 12, the Exponential GF of strongly connected graphs satisfy

$$\mathsf{SCD}(z) = \log\frac{1}{1-\mathsf{IT}(z)\odot\mathsf{G}(z)}.$$

As we have seen in Theorem 46, the Exponential GF of irreducible tournaments belongs to $\mathfrak{G}_2^1$ with

$$(\mathcal{Q}_2^1\,\mathsf{IT})(z,w) = \big(1-\mathsf{IT}(2zw)\big)^2.$$

Hence, $\big(\Delta_2^{-1}\mathsf{IT}\big)(z) = \mathsf{IT}(z)\odot\mathsf{G}(z)$ belongs to $\mathfrak{G}_2^2$, and to obtain $(\mathcal{Q}_2^2\,\mathsf{SCD})(z,w)$ we can apply Theorem 4 to

$$A(z) = \mathsf{IT}(z)\odot\mathsf{G}(z) \qquad \text{and} \qquad F(x) = -\log(1-x)$$

with $\alpha=\beta=2$. Taking into account relations (27), in the case in hand we have

$$\left.\frac{\partial F}{\partial x}\right|_{x=A(z)} = \frac{1}{1-\mathsf{IT}(z)\odot\mathsf{G}(z)} = \mathsf{SSD}(z).$$

To finish the proof, we use Theorem 35 and relation (47):

$$\big(\mathcal{Q}_2^2(\Delta_2^{-1}\mathsf{IT})\big)(z,w) = \big(\Phi_2^{1,2}(\mathcal{Q}_2^1\,\mathsf{IT})\big)(z,w) = \Phi_2^{1,2}\big(1-\mathsf{IT}(2zw)\big)^2. \qquad \square$$

Corollary 51. *The probability* p_n *that a uniform random digraph with* n *vertices is strongly connected satisfies*

$$p_n \approx \sum_{m\geqslant 0}\frac{1}{2^{nm}}\sum_{\ell=\lceil m/2\rceil}^{m} n^{\underline{\ell}}\,\mathfrak{scd}^{\circ}_{m,\ell}, \tag{50}$$

where

$$\mathfrak{scd}^{\circ}_{m,\ell} = 2^{m(m+1)/2+\ell(\ell-m)}\frac{\mathfrak{ssd}_{m-\ell}}{(m-\ell)!}\frac{\mathbf{1}_{m=2\ell}-2\mathfrak{it}_{2\ell-m}+\mathfrak{it}^{(2)}_{2\ell-m}}{(2\ell-m)!},$$

and $\mathfrak{ssd}_k$, $\mathfrak{it}_k$ *and* $\mathfrak{it}^{(2)}_k$ *denote the numbers of semi-strong digraphs, irreducible tournaments and tournaments with two irreducible components, respectively (all of them are of size* k*).*

Proof. By definition, we have the relation $p_n = \mathfrak{scd}_n / 2^{2\binom{n}{2}}$. According to Theorem 50, the Exponential GF $\mathsf{SCD}(z)$ belongs to $\mathfrak{G}_2^2$. Hence, it follows from (4) and (49) that

$$p_n \approx \sum_{m\geqslant 0} \frac{1}{2^{nm}} \sum_{\ell=0}^{\infty} n^{\underline{\ell}}\, \mathfrak{scd}^{\circ}_{m,\ell}.$$

To establish the limits of summation, let us denote, for any $n, k \in \mathbb{Z}_{\geqslant 0}$,

$$b_n := n![z^n]\big(1-\mathsf{IT}(z)\big)^2 = \mathbf{1}_{n=0} - 2\mathfrak{it}_n + \mathfrak{it}^{(2)}_n \tag{51}$$

and

$$b_{n,k} := n![z^n w^k]\big(1-\mathsf{IT}(zw)\big)^2 = b_n \cdot \mathbf{1}_{n=k}.$$

Since the coefficients $b_{n,k}$ are non-zero when $n = k$ only, by direct calculations we obtain

$$\begin{aligned}
\mathfrak{scd}^{\circ}_{m,\ell} &= 2^{\frac{1}{2}\binom{m}{2}}[z^m w^\ell](\mathcal{Q}_2^2\,\mathsf{SCD})(z,w)\\
&= \sum_{k=0}^{\infty} 2^{3k/2}\frac{\mathfrak{sSD}_k}{k!}\cdot 2^{\frac{1}{2}\binom{m-2k}{2}}[z^{m-2k}w^{\ell-k}]\big(1-\mathsf{IT}(2zw)\big)^2\\
&= \sum_{k=0}^{\infty} 2^{m(m+1)/2+k(k-m)}\frac{\mathfrak{sSD}_k}{k!}\,\frac{b_{m-2k,\ell-k}}{(m-2k)!}\\
&= 2^{m(m+1)/2+\ell(\ell-m)}\,\frac{\mathfrak{sSD}_{m-\ell}}{(m-\ell)!}\,\frac{b_{2\ell-m}}{(2\ell-m)!},
\end{aligned}$$

which is non-zero for $\ell \leqslant m \leqslant 2\ell$ only. □

Remark 52. Relation (49) corresponds to the asymptotic expansion of the form

$$\mathfrak{scd}_n \approx 2^{n^2-n}\sum_{m\geqslant 0}\frac{w_m(n)}{2^{nm}}$$

investigated by Wright [35] who established a recursive method of computing the polynomials $w_m(n)$. The latter asymptotics was also studied by Bender [5] who proposed a direct way of computing these polynomials based on Theorem 23. For numerical values of $\mathfrak{scd}^{\circ}_{m,\ell}$ and $w_m(n)$, as well as for more details, see Section A.4.

4.3 Fixed number of strongly connected components

This section is devoted to the asymptotics of digraphs with a marking variable for the number of strongly connected components. We start with establishing the Coefficient GF of semi-strong digraphs. Next, we proceed to the Coefficient GF of all digraphs. Finally, we provide the leading term of the probability that a random digraph has a fixed number of strongly connected components and indicate the combinatorial meaning of this term, which involves directed acyclic graphs. All the results discussed here appear to be novel.

Theorem 53. *The bivariate Exponential GF* $\mathsf{SSD}(z;t)$ *of semi-strong digraphs with the marking variable* t *for the number of strongly connected components belongs to the ring* $\mathfrak{G}_2^2(t)$ *and the corresponding Coefficient GF of type* $(2,2)$ *satisfies*

$$(\mathcal{Q}_2^2\,\mathsf{SSD})(z,w;t) = t\cdot \mathsf{SSD}(2^{3/2}z^2w;1+t)\cdot \Phi_2^{1,2}\big(1-\mathsf{IT}(2zw)\big)^2. \tag{52}$$

In particular, for any $m\in\mathbb{Z}_{\geqslant 0}$*, the asymptotics of semi-strong digraphs with* $(m+1)$ *strongly connected components is given by*

$$[t^{m+1}](\mathcal{Q}_2^2\,\mathsf{SSD})(z,w;t) = \frac{1}{m!}\mathsf{SCD}^m(2^{3/2}z^2w)\cdot(\mathcal{Q}_2^2\,\mathsf{SCD})(z,w). \tag{53}$$

Proof. Taking into account relation (23), it is sufficient to apply Theorem 39 to the series $A(z)=\mathsf{SCD}(z)$ and function $F(x;t)=e^{tx}$ with $\alpha=\beta=2$. To complete the proof, we use Theorem 50. □

Theorem 54. *The Graphic GF* $\widehat{\mathsf{D}}(z;t)$ *of digraphs with the marking variable* t *for the number of strongly connected components belongs to the ring* $\mathfrak{G}_2^1(t)$ *and the corresponding Coefficient GF of type* $(2,1)$ *is given by*

$$(\mathcal{Q}_2^1\,\widehat{\mathsf{D}})(z,w;t) = \Big(\widehat{\mathsf{D}}(2zw;t)\Big)^2\cdot\Phi_2^{2,1}\Big((\mathcal{Q}_2^2\,\mathsf{SSD})(z,w;-t)\Big). \tag{54}$$

Proof. It is sufficient to apply Theorem 40, Theorem 41 and Theorem 39 to the Graphic GF $\widehat{\mathsf{D}}(z;t)$ of digraphs written in the form

$$\widehat{\mathsf{D}}(z;t) = \frac{1}{\Delta_2\big(e^{-t\cdot\mathsf{SCD}(z)}\big)}.$$ □

Corollary 55. *The probability* $p_{n,m+1}$ *that a uniform random digraph on* n *vertices has exactly* $(m+1)$ *strongly connected components satisfies the following asymptotic behavior, as* $n\to\infty$*:*

$$p_{n,m+1}\sim\binom{n}{m}\frac{2^m\mathfrak{dag}_m^{(\vec{2})}}{2^{mn}}, \tag{55}$$

where $\mathfrak{dag}_m^{(\vec{2})}$ *are the coefficients of the Graphic GF* $\widehat{\mathsf{DAG}}^2(z)$*.*

Proof. In order to obtain the dominant term of the probability $p_{n,m+1}$, we need to trace the term with the smallest power of z in $[t^{m+1}](\mathcal{Q}_2^1\,\widehat{\mathsf{D}})(z,w;t)$. In order to do that, rewrite (54) as

$$(\mathcal{Q}_2^1\,\widehat{\mathsf{D}})(z,w;t) = t\cdot\frac{\Phi_2^{2,1}\left(e^{(1-t)\cdot\mathsf{SCD}(2^{3/2}z^2w)}\cdot\Phi_2^{1,2}\big(1-\mathsf{IT}(2zw)\big)^2\right)}{\Big(\Delta_2\big(e^{-t\cdot\mathsf{SCD}(2zw)}\big)\Big)^2}. \tag{56}$$

Note that the (only) smallest strongly connected digraph has one vertex, hence, $\mathsf{SCD}(z)$ starts with z. As a consequence, for every t^m, the smallest exponent in z in the numerator of (56) is equal to $2m$. At the same time, the smallest exponent in z in the denominator

is m, which is smaller for every $m > 0$. Since the numerator starts with 1, the whole fraction can be simplified in the following way:

$$[t^{m+1}](\mathcal{Q}_2^1\,\widehat{\mathsf{D}})(z,w;t) = [t^m]\frac{1}{\left(e^{-2tzw}\odot_z \widehat{\mathsf{Set}}(z)\right)^2} = 2^m z^m w^m [t^m]\left(\frac{1}{\widehat{\mathsf{Set}}(-t)}\right)^2.$$

Theorem 10 implies that

$$\left(\frac{1}{\widehat{\mathsf{Set}}(-z)}\right)^2 = \left(\frac{1}{\Delta_2(e^{-z})}\right)^2 = \widehat{\mathsf{DAG}}^2(z).$$

Thus, the expansion coefficient at $2^{-mn}n^{\underline{m}}$ is equal to

$$2^{\binom{m}{2}}[z^m w^m][t^{m+1}](\mathcal{Q}_2^1\,\widehat{\mathsf{D}})(z,w;t) = 2^m\frac{\mathfrak{dag}_m^{(\vec{2})}}{m!}.$$

The proof is completed by noting that $\binom{n}{m} = n^{\underline{m}}/m!$. □

Remark 56. There is also a straightforward combinatorial way to prove asymptotics (55), which is based on the structural analysis of involved digraphs. This method works for more general scenarios as well, even when generating functions are not available, but obtaining secondary terms in this case may become too tedious. The key idea is that the main contribution into the asymptotics of digraphs with $(m+1)$ components is given by digraphs whose m components contain one vertex each, and the remaining component contains $(n-m)$ vertices. We will call the one-vertex components the *small* ones, and the component with $(n-m)$ vertices the *large* one. As $n \to \infty$, with high probability each of the small components is connected to the large component by at least one edge. Note that, for any small component, the connections can only be in one direction, otherwise the small component would be merged into the large one.

This observation allows us to repartition the m one-vertex components into the group of k components having the edges *towards* the large component, and the group of $(m-k)$ components that have the edges *from* the large component (see Figure 2). The vertices within each of the above two groups form a directed acyclic graph structure, with k and $(m-k)$ vertices, respectively. Furthermore, there can be additional edges from the first group to the second one. Hence, the number of arrangements of m vertices beyond the large component is

$$\sum_{k=0}^{m}\binom{m}{k}2^{k(m-k)}\mathfrak{dag}_k\mathfrak{dag}_{m-k} = \mathfrak{dag}_n^{(\vec{2})}$$

and the total number of digraphs on n vertices defined in such a way is asymptotically equal to

$$\binom{n}{m}2^{(n-m)(n-m-1)}2^{m(n-m)}\mathfrak{dag}_m^{(\vec{2})}.$$

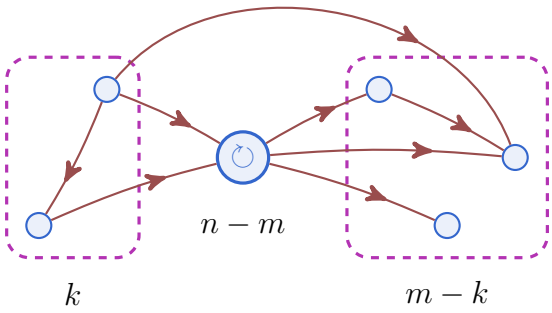

Figure 2:

Thus, the dominant term of the probability that a random digraph contains $(m+1)$ strongly connected components is

$$\frac{1}{2^{n(n-1)}}\binom{n}{m}2^{(n-m)(n-m-1)}2^{m(n-m)}\mathfrak{dag}_m^{(\vec{2})}=\binom{n}{m}\frac{2^m\mathfrak{dag}_m^{(\vec{2})}}{2^{mn}}.$$

4.4 Different kinds of strongly connected components

In this section, we provide complete asymptotic expansions for two more multivariate versions of digraphs. The advantage of the corresponding Coefficient GFs lies in possessing information related to the asymptotic behavior of digraphs with an arbitrary number of components of arbitrary types. At the same time, the presented new results are a straightforward application of the asymptotic transfer to the relations discussed in Section 2.1.5. That is why we omit tedious details of their proofs.

Theorem 57. *The Graphic GF $\widehat{\mathsf{D}}(z;s,t)$ of digraphs with marking variables s and t for the numbers of source-like and all strongly connected components, respectively, belongs to the ring $\mathfrak{G}_2^1(s,t)$ and the corresponding Coefficient GF of type $(2,1)$ is given by*

$$(\mathcal{Q}_2^1\,\widehat{\mathsf{D}})(z,w;s,t)=$$
$$\widehat{\mathsf{D}}(2zw;t)\left[\Phi_2^{2,1}\Big((\mathcal{Q}_2^2\,\mathsf{SSD})\big(z,w;(s-1)t\big)\Big)+\widehat{\mathsf{D}}(2zw;s,t)\cdot\Phi_2^{2,1}\Big((\mathcal{Q}_2^2\,\mathsf{SSD})(z,w;-t)\Big)\right].$$

Proof. It is sufficient to apply Theorem 38 to relation (24) in the form

$$\widehat{\mathsf{D}}(z;s,t)=\Delta_2\Big(\mathsf{SSD}\big(z;(s-1)t\big)\Big)\cdot\widehat{\mathsf{D}}(z;t).$$

The first summand of the result comes directly from the chain rule. To get the second summand, we additionally apply Theorem 54 and use relation (24) again, but in the opposite direction. □

Theorem 58. *The Exponential GF* $\mathsf{D}(z;u,v,y,t)$ *of digraphs with marking variables* u,v,y *and* t *for the numbers of purely source-like, purely sink-like, isolated and all strongly connected components, respectively, belongs to the ring* $\mathfrak{G}_2^2(u,v,y,t)$ *and the corresponding Coefficient GF of type* $(2,2)$ *is given by*

$$(\mathcal{Q}_2^2\,\mathsf{D})(z,w;u,v,y,t)=\mathsf{D}_1^{\circ}+\mathsf{D}_{20}^{\circ}\cdot\Phi_2^{1,2}\Big(\mathsf{D}_{21}^{\circ}+\mathsf{D}_{22}^{\circ}+\mathsf{D}_{23}^{\circ}\Big),\tag{57}$$

where

$$\begin{aligned}
\mathsf{D}_1^{\circ}(z,w;u,v,y,t)&=(y-u-v+1)t\cdot\mathsf{D}(2^{3/2}z^2w;u,v,y,t)\cdot(\mathcal{Q}_2^2\,\mathsf{SCD})(z,w)\,;\\
\mathsf{D}_{20}^{\circ}(z,w;u,v,y,t)&=\mathsf{SSD}\big(2^{3/2}z^2w;(y-u-v+1)t\big)\,;\\
\mathsf{D}_{21}^{\circ}(z,w;u,v,y,t)&=\widehat{\mathsf{D}}(2zw;u,t)\cdot\Phi_2^{2,1}\Big((\mathcal{Q}_2^2\,\mathsf{SSD})\big(z,w;(v-1)t\big)\Big)\,;\\
\mathsf{D}_{22}^{\circ}(z,w;u,v,y,t)&=\widehat{\mathsf{D}}(2zw;v,t)\cdot\Phi_2^{2,1}\Big((\mathcal{Q}_2^2\,\mathsf{SSD})\big(z,w;(u-1)t\big)\Big)\,;\\
\mathsf{D}_{23}^{\circ}(z,w;u,v,y,t)&=\widehat{\mathsf{D}}(2zw;u,t)\cdot\widehat{\mathsf{D}}(2zw;v,t)\cdot\Phi_2^{2,1}\Big((\mathcal{Q}_2^2\,\mathsf{SSD})(z,w;-t)\Big).
\end{aligned}$$

Proof. The proof is a straightforward application of Theorem 38 and Theorem 41 to relation (26) with further simplifications done by Theorem 7, Theorem 53 and Theorem 54. □

Remark 59. Similarly to Theorem 55, we can prove that the probability that a uniform random digraph on n vertices has $(m+1)$ strongly connected components, from which $i>1$ are purely source-like, $j>1$ are purely sink-like and $\ell\geqslant 0$ are isolated, is asymptotically

$$2^{-n(m+\ell)}\binom{n}{m}\binom{m}{\ell}2^{m(\ell+1)}\sum_k\binom{m-\ell}{k}2^{k(m-\ell-k)}\mathfrak{dag}_{k,i}\mathfrak{dag}_{m-\ell-k,j},$$

where $\mathfrak{dag}_{m,j}$ denotes the number of directed acyclic graphs with m vertices, from which j are source-like. The idea is to trace the term with the smallest exponent with respect to z in $[t^{m+1}u^iv^jy^\ell](\mathcal{Q}_2^2\,\mathsf{D})(z,w;u,v,y,t)$. The easiest way is to start with extracting $[y^\ell]$ in the right-hand side of expression (57), and then extract other variables. At some point, extraction of the dominant term can be obtained by expressing all parts in terms of SCD and replacing all $\mathsf{SCD}(z)$ with z.

Again, one can obtain the dominant term in the asymptotics combinatorially, by considering the structure of involved digraphs. Indeed, if a digraph has $(m+1)$ strongly connected components, then, with high probability, all the components except one consist of a single vertex. Moreover, the large component containing the rest $(n-m)$ vertices cannot be isolated, source-like or sink-like, with an exception of the two cases: either when all the components are isolated, or when there is only one source-like or sink-like component. Once the large component is marked, the digraph splits into several parts, and computations similar to those carried out in Theorem 56 yield the above result.

Remark 60. Similarly to the undirected case discussed in Theorem 49, we can establish asymptotics of digraphs within the $\mathbb{D}(n,p)$ model. Again, we put $\alpha = (1-p)^{-1}$, so that the Exponential GF of digraphs is equal to

$$\mathsf{D}(z) = \sum_{n=0}^{\infty} \alpha^{2\binom{n}{2}} \frac{z^n}{n!}.$$

As we have observed in Theorem 5, all enumerative results, with the exception of Theorem 12 and relation (28), remain the same. Therefore, to obtain the correct statements that generalize the ones we have seen in Section 4, we need to replace 2 by α and $1-\mathsf{IT}(z)$ by $\mathsf{G}^{-1}(z)$. In particular, the asymptotics of strongly connected digraphs is described by

$$(\mathcal{Q}_\alpha^2 \mathsf{SCD})(z,w) = \mathsf{SSD}(\alpha^{3/2} z^2 w) \cdot \Phi_\alpha^{1,2}\Big(\mathsf{G}^{-2}(\alpha z w)\Big),$$

the generalizations of formulae (52) and (54) are, respectively,

$$(\mathcal{Q}_\alpha^2 \mathsf{SSD})(z,w;t) = t \cdot \mathsf{SSD}(\alpha^{3/2} z^2 w; 1+t) \cdot \Phi_\alpha^{1,2}\Big(\mathsf{G}^{-2}(\alpha z w)\Big)$$

and

$$(\mathcal{Q}_\alpha^1 \widehat{\mathsf{D}})(z,w;t) = \Big(\widehat{\mathsf{D}}(\alpha z w; t)\Big)^2 \cdot \Phi_\alpha^{2,1}\Big((\mathcal{Q}_\alpha^2 \mathsf{SSD})(z,w;-t)\Big),$$

and so on.

5 2-SAT formulae

5.1 Asymptotics for satisfiable 2-CNFs

In this section, we provide the complete asymptotic expansion of satisfiable 2-CNFs, both in the form of the Coefficient GF and as a series. Additionally, we give a combinatorial interpretation of the leading term of the asymptotics of the number of satisfiable 2-CNF formulae. All the results presented in this section are novel.

Theorem 61. *The Implication GF* $\ddot{\mathsf{SAT}}(z)$ *of satisfiable 2-CNF formulae belongs to the ring* $\mathfrak{G}_2^1$ *and its Coefficient GF of type* $(2,1)$ *is given by*

$$(\mathcal{Q}_2^1 \ddot{\mathsf{SAT}})(z,w) = \ddot{\mathsf{SAT}}(2zw)\big(1 - \mathsf{IT}(2zw)\big).$$

Proof. This follows directly from Theorem 3 applied to relation (33),

$$\ddot{\mathsf{SAT}}(z) = \mathsf{G}(z) \cdot \Delta_2^2\left(e^{-\frac{1}{2}\mathsf{SCD}(z)}\right).$$

Indeed, as we have seen in Theorem 42, $\mathsf{G}(z)$ belongs to $\mathfrak{G}_2^1$ and $(\mathcal{Q}_2^1 \mathsf{G})(z,w) = 1$. On the other hand, according to Theorem 53, the Exponential GF $\mathsf{SSD}(z)$ belongs to the ring $\mathfrak{G}_2^2$,

and hence, by Theorem 24, so does $e^{-\frac{1}{2}\mathsf{SCD}(z)} = \big(\mathsf{SSD}(z)\big)^{-1/2}$. As a consequence, due to Theorem 32,

$$\Delta_2^2\left(e^{-\frac{1}{2}\mathsf{SCD}(z)}\right) \xrightarrow{\mathcal{Q}_2^1} 0.$$

Thus,

$$(\mathcal{Q}_2^1\,\mathsf{S\ddot{A}T})(z,w) = (\mathcal{Q}_2^1\,\mathsf{G})(z,w)\cdot\Delta_2^2\left(e^{-\frac{1}{2}\mathsf{SCD}(2zw)}\right) = \frac{\mathsf{S\ddot{A}T}(z)}{\mathsf{G}(2zw)}$$

and the relation $\big(\mathsf{G}(2zw)\big)^{-1} = 1-\mathsf{IT}(2zw)$ completes the proof. □

Corollary 62. *The number $\mathfrak{sat}_n$ of satisfiable 2-CNF formulae with n Boolean variables satisfies*

$$\mathfrak{sat}_n \approx 2^{3\binom{n}{2}+n}\left(1-\sum_{m\geqslant 1} s_m^{\circ}\binom{n}{m}\frac{2^{\binom{m+1}{2}}}{2^{mn}}\right),$$

where

$$s_m^{\circ} = \left(\sum_{k=0}^{m-1}\binom{m}{k}\frac{\mathfrak{sat}_k\mathfrak{it}_{m-k}}{2^{k^2}}\right) - \frac{\mathfrak{sat}_m}{2^{m^2}},$$

and $\mathfrak{it}_k$ denotes the number of irreducible tournaments with k vertices.

Proof. This follows from Theorem 61 with the help of the definitions of the Implication GF and the Coefficient GFs by extracting the required coefficient. □

Remark 63. The fact that $\mathfrak{sat}_n \sim 2^{3\binom{n}{2}+n}$ has a simple combinatorial explanation. With high probability, the implication digraph of a typical satisfiable formula consists of one pair of ordinary strongly connected components, supplied with additional edges going from one of these components to another. There are 2^n ways to choose which literals go into each of the components. Furthermore, there are $2^{n(n-1)}$ ways to choose directed edges within the first of these components (the edges of the second component are uniquely defined by this choice). Finally, there are $2^{\binom{n}{2}}$ ways to draw directed edges between the components, *i.e.* to choose pairs of edges $x\to\overline{y}$ and $y\to\overline{x}$ such that vertices x,y belong to the first component, and $\overline{x},\overline{y}$ belong to the second one.

5.2 Further asymptotics

This section contains two more new asymptotic results. First, we establish the Coefficient GF of contradictory strongly connected implication digraphs. The second result is more general. Namely, we obtain the Coefficient GF of implication digraphs with marking variables for the numbers of contradictory and ordinary strongly connected components. We also briefly discuss the combinatorial interpretation of its leading term.

Theorem 64. *The Exponential GF $\mathsf{CSCC}(z)$ of contradictory strongly connected implication digraphs belongs to the ring $\mathfrak{G}_2^4$, and its Coefficient GF of type $(2,4)$ is given by*

$$(\mathcal{Q}_2^4\,\mathsf{CSCC})(z,w) = \exp\left(\frac{1}{2}\mathsf{SCD}(2^{7/2}z^4w) - \mathsf{CSCC}(2^{5/2}z^4w)\right)\cdot\Phi_2^{2,4}\Big(1-\mathsf{IT}(2^{5/2}z^2w)\Big).$$

Proof. The proof is rather straightforward. Recall that, according to Theorem 17,

$$\mathsf{CSCC}(z) = \frac{1}{2}\mathsf{SCD}(2z) + \log\Big(\Delta_2^{-2}\Big(\mathsf{D}(z)\big(1-\mathsf{IT}(2z)\big)\Big)\Big).$$

First of all, $\mathsf{SCD}(2z) \in \mathfrak{G}_2^2$ and $\big(1-\mathsf{IT}(2z)\big) \in \mathfrak{G}_2^1$ by Theorem 50 and Theorem 46, respectively. Hence, due to Theorem 28,

$$\mathsf{SCD}(2z) \xmapsto{\mathcal{Q}_2^4} 0 \qquad \text{and} \qquad \big(1-\mathsf{IT}(2z)\big) \xmapsto{\mathcal{Q}_2^2} 0.$$

This implies that

$$A(z) = \Big(\Delta_2^{-2}\Big(\mathsf{D}(z)\big(1-\mathsf{IT}(2z)\big)\Big) - 1\Big) \in \mathfrak{G}_2^4$$

and, according to Theorem 35 and Theorem 3, $(\mathcal{Q}_2^4 A)(z,w) = \Phi_2^{2,4}\Big(1-\mathsf{IT}(2^{5/2}z^2w)\Big)$. Now, applying Theorem 4 to the series $A(z)$ and function $F(x) = \log(x+1)$ with $\alpha = 2$ and $\beta = 4$, we have

$$H(z) = \frac{1}{A(z)+1} = \exp\left(\frac{1}{2}\mathsf{SCD}(2z) - \mathsf{CSCC}(z)\right)$$

and $(\mathcal{Q}_2^4\,\mathsf{CSCC})(z,w) = H(2^{5/2}z^4w)\cdot(\mathcal{Q}_2^4 A)(z,w)$. □

Theorem 65. *The Implication GF* $\ddot{\mathsf{CNF}}(z;s,t)$ *of 2-CNF formulae with variables* s *and* t *that mark, respectively, the numbers of contradictory strongly connected components and pairs of ordinary strongly connected components in the corresponding implication digraph, belongs to the ring* $\mathfrak{G}_2^2(s,t)$ *and the corresponding Coefficient GF of type* $(2,2)$ *is given by*

$$(\mathcal{Q}_2^2\,\ddot{\mathsf{CNF}})(z,w;s,t) = s\cdot\widehat{\mathsf{D}}(2^{3/2}z^2w;t)\cdot$$
$$\Phi_2^{4,2}\left[z\cdot\exp\left(\big(s-1\big)\,\mathsf{CSCC}(2^{3/2}z^4w) + \frac{(1-t)}{2}\,\mathsf{SCD}(2^{5/2}z^4w)\right)\cdot\Phi_2^{2,4}\Big(1-\mathsf{IT}(4z^2w)\Big)\right].$$

Proof. The main idea of the proof is to apply Theorem 38 and Theorem 41 to relation (35) written in the form

$$\ddot{\mathsf{CNF}}(z;s,t) = \widehat{\mathsf{D}}(z;t)\cdot\Delta_2^2\left(e^{s\cdot\mathsf{CSCC}(z/2)-t/2\cdot\mathsf{SCD}(z)}\right).$$

Due to Theorem 28,

$$\widehat{\mathsf{D}}(z;t) \xmapsto{\mathcal{Q}_2^2} 0 \qquad \text{and} \qquad e^{t/2\cdot\mathsf{SCD}(z)} \xmapsto{\mathcal{Q}_2^4} 0.$$

Therefore, the essential part of the proof comes from Theorem 39 applied to the series $A(z) = \mathsf{CSCC}(z/2)$ and function $F(x;s) = e^{sx}$ with $\alpha = 2$ and $\beta = 4$:

$$(\mathcal{Q}_2^2\,\ddot{\mathsf{CNF}})(z,w;s,t) = s\cdot\widehat{\mathsf{D}}(2^{3/2}z^2w;t)\cdot\Phi_2^{4,2}\left(e^{s\cdot\mathsf{CSCC}(2^{3/2}z^4w)-t/2\cdot\mathsf{SCD}(2^{5/2}z^4w)}\cdot(\mathcal{Q}_2^4 A)(z,w)\right).$$

To complete the proof, use Theorem 26, so that $(\mathcal{Q}_2^4 A)(z,w) = z\cdot(\mathcal{Q}_2^4\,\mathsf{CSCC})\big(2^{-1/4}z,w\big)$, and finally, Theorem 64. □

Remark 66. By analysing the leading term of the Coefficient GF in Theorem 65, we can obtain the structure of 2-CNF formulae with given constraints. Thus, the structure of a typical 2-CNF implication digraph with $(m+1)$ contradictory components is as follows. One large component contains almost all the variables, and the remaining m components contain 2 Boolean variables each. Note that this implication digraph, with high probability, consists of isolated contradictory components only.

Next, if the numbers of contradictory components and ordinary components are fixed, then, with high probability, all the ordinary components contain one vertex each, and they are connected by an edge with the large contradictory component. More precisely, one-node components are partitioned into two copies of directed acyclic graphs: one of them points towards the large contradictory component, and the other has edges directed from that component. There can be, in addition, an arbitrary subset of directed edges from the first of these directed acyclic graphs towards the small 2-variable contradictory components (as well as the complementary subset of edges directed from small contradictory components toward the second directed acyclic graph).

Remark 67. Similarly to the Erdős-Rényi model, let us define $\mathbb{F}(n,p)$ to be the random 2-SAT model with n Boolean variables, so that each of the $n(n-1)$ possible clauses appears independently with a fixed positive probability p. In this case, the probability that a randomly generated 2-CNF formula from $\mathbb{F}(n,p)$ belongs to a family $\mathcal{F}$ whose Implication GF is $\ddot{F}(z,w)$ is equal to

$$\mathbb{P}_{\mathcal{F}}(n,p) = (1-p)^{n(n-1)} 2^n n! [z^n] \ddot{F}\left(z, \frac{p}{1-p}\right)$$

(*c.f.* [12, Proposition 3.3], compare with Theorem 5). It is natural, as we have it done for undirected and directed graphs, to introduce $\alpha = (1-p)^{-1}$. However, this leads to

$$\ddot{\mathsf{CNF}}(z) = \sum_{n=0}^{\infty} \alpha^{n(n-1)} \frac{z^n}{2^n n!}$$

and the complete asymptotic expansion of the counting sequence corresponding to any nontrivial 2-SAT family is not homogeneous anymore. Now, different powers of 2 and α are interlacing, which complicates the order of terms in the full asymptotic expansion. Indeed, if

$$f_n \approx \alpha^{\beta\binom{n}{2}} \sum_{m \geqslant M} \alpha^{-mn} \sum_{\ell=0}^{\infty} n^{\underline{\ell}} f^{\circ}_{m,\ell},$$

then, dividing the argument by 2, we have

$$f_n 2^{-n} \approx \alpha^{\beta\binom{n}{2}} \sum_{m \geqslant M} \alpha^{-mn} 2^{-n} \sum_{\ell=0}^{\infty} n^{\underline{\ell}} f^{\circ}_{m,\ell}.$$

When the corresponding generating function is multiplied by another function or participates in a functional composition, the order of the first few dominant terms of the

expansion depends on the value of α. Furthermore, the sequence of expansion coefficients can no longer be captured by a conventional Coefficient GF, unless when $\log_2 \alpha$ is rational. All these observations show that the presented method is not applicable to the $\mathbb{F}(n,p)$ model in full generality.

6 Discussion and open problems

In this section, we discuss open problems related to the asymptotic transfer method presented in this paper and its applications. We have seen that this method works well for various dense graph families and 2-SAT formulae. It can also be used to study the Erdős-Rényi model where edges of a random graph are drawn independently with a constant probability $p \in (0,1)$, see Theorem 49 and Theorem 60. Apparently, as $p \to 0$, the limit where the method ceases to be applicable is at $p = \Theta(\log n/n)$, since the divergence of a quadratic term $(1-p)^{-\beta\binom{n}{2}}$ should be faster than the divergence of a factorial. At this threshold, the terms of the asymptotic expansion may start having a comparable order. This phenomenon has a heuristic combinatorial explanation: in random Erdős-Rényi graphs, $p \sim \log n/n$ indicates the connectivity threshold where all but one component consists of a single vertex. A similar phenomenon must take place for random digraphs and 2-CNF formulae. Our method could be potentially applied to the case where other components have finite sizes by summing all the contributions and using a delicate generating function argument. Below the threshold $p = \Theta(\log n/n)$, a different approach is clearly required. The development of the mentioned technique is one of the directions for a future research. Note that, in the case of 2-SAT, this might be tricky because α and 2 are on average algebraically independent, see Theorem 67.

It is of interest whether our method can be extended to the case of a functional composition of two graphically divergent generating functions, as it happens in the case of factorially divergent series (see the Borinsky's paper [7]). The positive answer to this question would potentially unlock refined asymptotic enumeration of *2-vertex-connected graphs* (also known as *nonseparable graphs* or *blocks*, *i.e.* connected graphs without cut-points) and *2-edge-connected graphs* (connected graphs without bridges), whose respective Exponential GFs $B(x)$ and $H(x)$ satisfy functional equations

$$\mathsf{CG}'(x) = e^{B'(x\mathsf{CG}'(x))} \qquad \text{and} \qquad \mathsf{CG}'(x) = H'(xe^{x\mathsf{CG}'(x)}).$$

Another possible way to get this enumeration would be to restate Lagrange inversion of appropriate generating series in terms of Coefficient GFs (see the paper [6] of Bender and Richmond who established the first of these asymptotics by developing the method for treating inverses and functional compositions with analytic functions).

The next set of questions are related to the k-SAT problem. As we have seen in Theorem 63, the fact that the number of satisfiable 2-SAT formulae grows asymptotically as $2^{n+3\binom{n}{2}}$ has a simple combinatorial interpretation in terms of implication digraphs. Would there be an equivalently simple heuristic explanation for the asymptotic number of satisfiable k-CNF formulae on n Boolean variables for any $k > 2$? Furthermore, the Coefficient GF of the satisfiable 2-SAT formulae has a remarkably simple form

$(\mathcal{Q}_2^1 \ddot{\mathsf{SAT}})(z,w) = \ddot{\mathsf{SAT}}(2zw)\big(1 - \mathsf{IT}(2zw)\big)$. Could there be an equally simple expression for a sort of Coefficient GF for k-SAT with $k > 2$, even if their exact enumeration is elusive?

It is worth mentioning that, for problems like k-SAT, the logarithm of the total number of objects is growing faster than a quadratic function. This may suggest that the required analogue of the Coefficient GF must have more than two dimensions, which leads to another question. Namely, could the asymptotic transfer method be meaningfully generalized to higher dimensions as well? The latter would be useful, for instance, to count families of hypergraphs and directed hypergraphs.

We conclude our review of open problems with a particular question related to the enumeration of digraphs. Curiously, the statement of Theorem 12 suggest that there might be a combinatorial explanation of expressions (27). The first of them could follow from the fact that the family of semi-strong digraphs would be in a natural one-to-one correspondence with sequences of irreducible tournaments decorated with an arbitrary subset of $\binom{n}{2}$ edges of additional color. The corresponding counting sequence $(\eta_n)_{n=1}^{\infty}$ first appeared in the paper of Wright [35] who obtained the following recurrence for the counting sequence $(\mathfrak{scd}_n)$ of strongly connected digraphs:

$$\mathfrak{scd}_n = \eta_n + \sum_{t=1}^{n-1} \binom{n-1}{t-1} \mathfrak{scd}_t \eta_{n-t}.$$

Liskovets later discovered in [22] that $\eta_n 2^{-\binom{n}{2}}$ indeed enumerates irreducible tournaments and even extended this enumeration result to the case of unlabelled structures. Recently, Archer, Gessel, Graves and Liang [2], among other results, revealed some fine enumerative properties of the combinatorial class corresponding to η_n. They also noted that there might be a natural bijection between strong digraphs and cycles of irreducible decorated tournaments, but could not identify such a bijection (which would correspond to the second expression of (27)). Despite our attempts, we have not been able to find any of such bijections either and, to our best knowledge, they still remain an open problem.

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A Numerical values of Coefficient GFs

A.1 Connected graphs, counting connected components

According to Theorem 43, the Coefficient GF of type $(2,1)$ of connected graphs satisfies

$$(\mathcal{Q}_2^1\,\mathsf{CG})(z,w) = 1 - \mathsf{IT}(2zw).$$

As a consequence, the corresponding asymptotic coefficients $\mathfrak{cg}^{\circ}_{m,\ell}$ are of form

$$\mathfrak{cg}^{\circ}_{m,\ell} = \mathbf{1}_{m=\ell=0} - \mathbf{1}_{m=\ell>0} \cdot \mathfrak{it}_m \cdot \frac{2^{\binom{m+1}{2}}}{m!}.$$

The sequence $(\mathfrak{it}_m)_{m=1}^{\infty}$ counts irreducible tournaments and is given by https://oeis.org/A054946 from the OEIS:

$$(\mathfrak{it}_m) = 1,\ 0,\ 2,\ 24,\ 544,\ 22\,320,\ 1\,677\,488,\ 236\,522\,496,\ 64\,026\,088\,576,\ \ldots$$

Thus, the sequence $\big(\mathfrak{cg}^{\circ}_{m,m}\big)_{m=0}^{\infty}$ starts by

$$\big(\mathfrak{cg}^{\circ}_{m,m}\big) = 1,\ -2,\ 0,\ -\frac{64}{3},\ -1024,\ -\frac{2\,228\,224}{15},\ -65\,011\,712,\ -\frac{28\,143\,578\,513\,408}{315},\ \ldots$$

According to Theorem 44, the Coefficient GF of type $(2,1)$ of graphs, enriched with a marking variable t for counting connected components, satisfies

$$(\mathcal{Q}_2^1\,\mathsf{G})(z,w;t) = t \cdot \mathsf{G}(2zw;t) \cdot \big(1 - \mathsf{IT}(2zw)\big).$$

This series starts with the following terms:

$$(\mathcal{Q}_2^1\,\mathsf{G})(z,w;t) = t + 2(t^2-t)zw + 2(t^3-t^2)z^2w^2 + \frac{4}{3}(t^4+t^2-2t)z^3w^3 + \ldots$$

The corresponding asymptotic coefficients $\mathfrak{g}^{\circ}_{m,\ell}(t)$ can be written as

$$\mathfrak{g}^{\circ}_{m,\ell}(t) = \mathbf{1}_{m=\ell=0} - \mathbf{1}_{m=\ell>0} \cdot \big(\bar{\mathfrak{g}}_m^{(m+1)} t^{m+1} + \ldots + \bar{\mathfrak{g}}_m^{(1)} t\big) \cdot \frac{2^{\binom{m+1}{2}}}{m!},$$

where $\bar{\mathfrak{g}}_m^{(k)}$ are integers listed in Table 1.

For a fixed positive integer k, the sequence $\big(\bar{\mathfrak{g}}_m^{(k)}\big)_{m=0}^{\infty}$ appears in the asymptotics of graphs with k connected components, which is given by $[t^k](\mathcal{Q}_2^1\,\mathsf{G})(z,w;t)$, see relation (46). In particular, the case $k=1$ corresponds to the asymptotics of connected graphs discussed in Section A.1, and

$$\bar{\mathfrak{g}}_m^{(1)} = -\mathfrak{it}_m.$$

Table 1: Values of the coefficients $\bar{\mathfrak{g}}_m^{(k)}$ for $m, k \leqslant 8$.

m	0	1	2	3	4	5	6	7	8
$\big(\bar{\mathfrak{g}}_m^{(1)}\big)$	1	−1	0	−2	−24	−544	−22 320	−1 677 488	−236 522 496
$\big(\bar{\mathfrak{g}}_m^{(2)}\big)$	0	1	−1	1	14	398	18 552	1 505 644	222 306 448
$\big(\bar{\mathfrak{g}}_m^{(3)}\big)$	0	0	1	0	7	115	3 238	156 576	13 457 052
$\big(\bar{\mathfrak{g}}_m^{(4)}\big)$	0	0	0	1	2	25	455	13 783	711 788
$\big(\bar{\mathfrak{g}}_m^{(5)}\big)$	0	0	0	0	1	5	65	1 330	43 673
$\big(\bar{\mathfrak{g}}_m^{(6)}\big)$	0	0	0	0	0	1	9	140	3 248
$\big(\bar{\mathfrak{g}}_m^{(7)}\big)$	0	0	0	0	0	0	1	14	266
$\big(\bar{\mathfrak{g}}_m^{(8)}\big)$	0	0	0	0	0	0	0	1	20

A.2 Irreducible tournaments, counting irreducible parts

According to Theorem 46, the Coefficient GF of irreducible tournaments of type $(2,1)$ satisfies

$$(\mathcal{Q}_2^1\,\mathsf{IT})(z,w) = \big(1-\mathsf{IT}(2zw)\big)^2.$$

This gives us the asymptotic coefficients $\mathfrak{it}^\circ_{m,\ell}$ that turn out to be

$$\mathfrak{it}^\circ_{m,\ell} = \mathbf{1}_{m=\ell=0} - \mathbf{1}_{m=\ell>0}\cdot\big(2\mathfrak{it}_m - \mathfrak{it}_m^{(2)}\big)\cdot\frac{2^{\binom{m+1}{2}}}{m!},$$

where the counting sequence $(\mathfrak{it}_m)_{m=1}^\infty$ of irreducible tournaments is described in Section A.1, and the sequence $\big(\mathfrak{it}_m^{(2)}\big)_{m=1}^\infty$ of tournaments with exactly two irreducible parts is given by

$$\big(\mathfrak{it}_m^{(2)}\big) = 0,\ 2,\ 0,\ 16,\ 240,\ 6\,608,\ 315\,840,\ 27\,001\,984,\ 4\,268\,194\,560,\ 1\,281\,626\,527\,232,\ \ldots$$

Thus, we have the following starting values of $\big(\mathfrak{it}^\circ_{m,m}\big)_{m=0}^\infty$:

$$\big(\mathfrak{it}^\circ_{m,m}\big) = 1,\ -4,\ 8,\ -\frac{128}{3},\ -\frac{4\,096}{3},\ -\frac{3\,473\,408}{15},\ -\frac{4\,984\,930\,304}{45},\ -\frac{50\,988\,241\,125\,376}{315},\ \ldots$$

According to Theorem 47, the Coefficient GF of type $(2,1)$ of tournaments, enriched with a marking variable t for counting irreducible parts, satisfies

$$(\mathcal{Q}_2^1\,\mathsf{T})(z,w;t) = t\cdot\big(\mathsf{T}(2zw;t)\cdot\big(1-\mathsf{IT}(2zw)\big)\big)^2.$$

The first several terms of this series are

$$(\mathcal{Q}_2^1\,\mathsf{T})(z,w;t) = t + 4(t^2-t)zw + 4(3t^3-4t^2+t)z^2w^2 + \frac{16}{3}(6t^4-9t^3+4t^2-t)z^3w^3 + \ldots$$

The corresponding asymptotic coefficients $\mathfrak{t}^\circ_{m,\ell}(t)$ can be written as

$$\mathfrak{t}^\circ_{m,\ell}(t) = \mathbf{1}_{m=\ell=0} - \mathbf{1}_{m=\ell>0}\cdot\big(\bar{\mathfrak{t}}_m^{(m+1)}t^{m+1} + \ldots + \bar{\mathfrak{t}}_m^{(1)}t\big)\cdot\frac{2^{\binom{m+1}{2}}}{m!},$$

Table 2: Values of the coefficients $\bar{\mathfrak{t}}_m^{(k)}$ for $m, k \leqslant 8$.

m	0	1	2	3	4	5	6	7	8
$(\bar{\mathfrak{t}}_m^{(1)})$	1	−2	2	−4	−32	−848	−38 032	−3 039 136	−446 043 008
$(\bar{\mathfrak{t}}_m^{(2)})$	0	2	−8	16	−16	368	22 528	2 232 064	372 697 856
$(\bar{\mathfrak{t}}_m^{(3)})$	0	0	6	−36	120	0	9 744	586 656	60 297 600
$(\bar{\mathfrak{t}}_m^{(4)})$	0	0	0	24	−192	960	960	153 216	10 063 872
$(\bar{\mathfrak{t}}_m^{(5)})$	0	0	0	0	120	−1 200	8 400	16 800	2 177 280
$(\bar{\mathfrak{t}}_m^{(6)})$	0	0	0	0	0	720	−8 640	80 640	241 920
$(\bar{\mathfrak{t}}_m^{(7)})$	0	0	0	0	0	0	5 040	−70 560	846 720
$(\bar{\mathfrak{t}}_m^{(8)})$	0	0	0	0	0	0	0	40 320	−645 120

where $\bar{\mathfrak{t}}_m^{(k)}$ are integers listed in Table 2.

For a fixed positive integer k, the sequence $\big(\bar{\mathfrak{t}}_m^{(k)}\big)_{m=0}^{\infty}$ appears in the asymptotics of tournaments with k irreducible parts, which is given by $[t^k](\mathcal{Q}_2^1\,\mathsf{T})(z,w;t)$, see relation (48). In particular,

$$\bar{\mathfrak{t}}_m^{(k)} = k \cdot \Big(\mathfrak{it}_m^{(k-1)} - 2\mathfrak{it}_m^{(k)} + \mathfrak{it}_m^{(k+1)} \Big),$$

where $\mathfrak{it}_m^{(k)}$ is the number of tournaments of size m with k irreducible parts, supplemented by the convention $\mathfrak{it}_m^{(0)} = \mathbf{1}_{m=0}$ that leads us to the asymptotics of irreducible tournaments discussed in Section A.2.

A.3 The Erdős-Rényi model

As it was mentioned in Theorem 49, the Coefficient GF of type $(\alpha, 1)$ of graphs within the Erdős-Rényi model $\mathbb{G}(n,p)$ satisfies

$$(\mathcal{Q}_\alpha^1 \mathsf{G})(z,w;t) = t \cdot \frac{\mathsf{G}(\alpha z w; t)}{\mathsf{G}(\alpha z w)} = t \cdot e^{(t-1)\cdot \mathsf{CG}(\alpha z w)},$$

where $\alpha = (1-p)^{-1}$ and t is the marking variable for connected components. The corresponding asymptotic coefficients $\mathfrak{g}^{\circ}_{m,\ell}(\alpha, t)$ can be written as

$$\mathfrak{g}^{\circ}_{m,\ell}(\alpha,t) = \mathbf{1}_{m=\ell=0} - \mathbf{1}_{m=\ell>0} \cdot \big(\bar{\mathfrak{g}}_m^{(m+1)}(\alpha)\, t^{m+1} + \ldots + \bar{\mathfrak{g}}_m^{(1)}(\alpha)\, t \big) \cdot \frac{\alpha^{\binom{m+1}{2}}}{m!},$$

where $\bar{\mathfrak{g}}_m^{(k)}(\alpha)$ are polynomials in α listed in Table 3.

We observe that the column sums in Table 3 equal zero, except for the column that corresponds to $m = 0$. Clearly, this can be explained by the fact that the total weight of all graphs within the Erdős-Rényi model is $\alpha^{\binom{n}{2}}$. Another observation is that the case where $p = 1/2$, *i.e.* $\alpha = 2$, corresponds to the situation discussed in Section A.1. In other words, substituting the value $\alpha = 2$ into Table 3, we obtain Table 1.

Table 3: Values of the coefficients $\bar{\mathfrak{g}}_m^{(k)}(\alpha)$ for $m, k \leqslant 4$.

m	0	1	2	3	4
$\big(\bar{\mathfrak{g}}_m^{(1)}(\alpha)\big)$	1	-1	$-\alpha+2$	$-\alpha^3+6\alpha-6$	$-\alpha^6+8\alpha^3+6\alpha^2-36\alpha+24$
$\big(\bar{\mathfrak{g}}_m^{(2)}(\alpha)\big)$	0	1	$\alpha-3$	$\alpha^3-9\alpha+11$	$\alpha^6-12\alpha^3-9\alpha^2+66\alpha-50$
$\big(\bar{\mathfrak{g}}_m^{(3)}(\alpha)\big)$	0	0	1	$3\alpha-6$	$4\alpha^3+3\alpha^2-36\alpha+35$
$\big(\bar{\mathfrak{g}}_m^{(4)}(\alpha)\big)$	0	0	0	1	$6\alpha-10$

For a fixed non-negative integer k, the sequence $\big(\bar{\mathfrak{g}}_m^{(k+1)}(\alpha)\big)_{m=0}^{\infty}$ appears in the asymptotics of graphs with exactly $(k+1)$ connected components, which is given by the relation

$$[t^{k+1}](\mathcal{Q}_\alpha^1\,\mathsf{G})(z,w;t) = \frac{1}{k!}\cdot\frac{\mathsf{CG}^k(\alpha zw)}{\mathsf{G}(\alpha zw)}.$$

In particular, the case $k=0$ corresponds to the asymptotics of connected graphs:

$$(\mathcal{Q}_\alpha^1\mathsf{CG})(z,w) = \frac{1}{\mathsf{G}(\alpha zw)} = e^{-\mathsf{CG}(\alpha zw)}.$$

The last relation can be interpreted in the following way: the probability p_n that a random graph of size n within the Erdős-Rényi model is connected satisfies

$$p_n \approx 1 - \sum_{m\geqslant 1} P_m(\alpha)\binom{n}{m}\frac{\alpha^{\binom{m+1}{2}}}{\alpha^{mn}},$$

where $P_m(\alpha) = -\bar{\mathfrak{g}}_m^{(1)}(\alpha)$. The first six polynomials $P_m(\alpha)$ are

$$\begin{aligned}
&P_1(\alpha) = 1,\\
&P_2(\alpha) = \alpha-2,\\
&P_3(\alpha) = \alpha^3-6\alpha+6,\\
&P_4(\alpha) = \alpha^6-8\alpha^3-6\alpha^2+36\alpha-24,\\
&P_5(\alpha) = \alpha^{10}-10\alpha^6-20\alpha^4+60\alpha^3+90\alpha^2-240\alpha+120,\\
&P_6(\alpha) = \alpha^{15}-12\alpha^{10}-30\alpha^7+70\alpha^6+360\alpha^4-390\alpha^3-1080\alpha^2+1800\alpha-720.
\end{aligned}$$

A.4 Strongly connected digraphs

According to Theorem 50, the Coefficient GF of type $(2,2)$ of strongly connected digraphs satisfies

$$(\mathcal{Q}_2^2\,\mathsf{SCD})(z,w) = \mathsf{SSD}(2^{3/2}z^2w)\cdot\Phi_2^{1,2}\big(1-\mathsf{IT}(2zw)\big)^2,$$

or, in terms of the exponential Hadamard product,

$$(\mathcal{Q}_2^2\,\mathsf{SCD})(z,w) = \mathsf{SSD}(2^{3/2}z^2w)\cdot\Big(\big(1-\mathsf{IT}(2zw)\big)^2\odot_z\mathsf{G}(z,\sqrt{2}-1)\Big).$$

Due to Theorem 51, the corresponding coefficients $\mathfrak{scd}^\circ_{m,\ell}$ are

$$\mathfrak{scd}^\circ_{m,\ell} = 2^{m(m+1)/2+\ell(\ell-m)} \frac{\mathfrak{ssd}_{m-\ell}}{(m-\ell)!} \frac{\mathbf{1}_{m=2\ell} - 2\mathrm{it}_{2\ell-m} + \mathrm{it}^{(2)}_{2\ell-m}}{(2\ell-m)!}.$$

The sequence $(\mathfrak{ssd}_k)_{k=0}^\infty$ counts semi-strong digraphs and is given by https://oeis.org/A054948 from the OEIS:

$$(\mathfrak{ssd}_k) = 1,\ 1,\ 2,\ 22,\ 1\,688,\ 573\,496,\ 738\,218\,192,\ 3\,528\,260\,038\,192,\ \ldots$$

Together with the values of $(\mathrm{it}_k)_{k=1}^\infty$ and $\left(\mathrm{it}^{(2)}_k\right)_{k=1}^\infty$ indicated in Section A.1 and Section A.2, respectively, this gives us numerical values of $\mathfrak{scd}^\circ_{m,\ell}$ indicated in Table 4.

Table 4: Values of the coefficients $\mathfrak{scd}^\circ_{m,\ell}$ for $m, \ell \leqslant 7$.

ℓ	0	1	2	3	4	5	6	7
$\mathfrak{scd}^\circ_{0,\ell}$	1	0	0	0	0	0	0	0
$\mathfrak{scd}^\circ_{1,\ell}$	0	-4	0	0	0	0	0	0
$\mathfrak{scd}^\circ_{2,\ell}$	0	4	8	0	0	0	0	0
$\mathfrak{scd}^\circ_{3,\ell}$	0	0	-32	$-\frac{128}{3}$	0	0	0	0
$\mathfrak{scd}^\circ_{4,\ell}$	0	0	64	128	$-\frac{4\,096}{3}$	0	0	0
$\mathfrak{scd}^\circ_{5,\ell}$	0	0	0	$-1\,024$	$-\frac{4\,096}{3}$	$-\frac{3\,473\,408}{15}$	0	0
$\mathfrak{scd}^\circ_{6,\ell}$	0	0	0	$\frac{45\,056}{3}$	8 192	$-\frac{262\,144}{3}$	$-\frac{4\,984\,930\,304}{45}$	0
$\mathfrak{scd}^\circ_{7,\ell}$	0	0	0	0	$-\frac{1\,441\,792}{3}$	$-\frac{524\,288}{3}$	$-\frac{444\,596\,224}{15}$	$-\frac{50\,988\,241\,125\,376}{315}$

If we rewrite the above relation as

$$\mathfrak{scd}_n \approx 2^{2\binom{n}{2}} \sum_{m\geqslant 0} \frac{w_m(n)}{2^{nm}}, \qquad \text{where} \quad w_m(n) = \sum_{\ell=\lceil m/2\rceil}^{m} n^{\underline{\ell}}\, \mathfrak{scd}^\circ_{m,\ell},$$

then, for $m \leqslant 6$, the polynomials $w_m(n)$ have the following explicit form:

$$\begin{aligned}
w_0(n) &= 1,\\
w_1(n) &= -4n,\\
w_2(n) &= 4n(2n-1),\\
w_3(n) &= -\frac{32}{3}n(n-1)(4n-5),\\
w_4(n) &= -\frac{64}{3}n(n-1)(64n^2-326n+393),\\
w_5(n) &= -\frac{1\,024}{15}n(n-1)(n-2)(3\,392n^2-23\,724n+40\,659),\\
w_6(n) &= -\frac{4\,096}{45}n(n-1)(n-2)(1\,217\,024n^3-14\,603\,328n^2+57\,193\,318n-73\,009\,815).
\end{aligned}$$

This corresponds to the asymptotic expansion of strongly connected digraphs established by Wright [35] and Bender [5]. In their papers, a different notations were used: the asymptotics was expressed via three sequences $(\eta_k)_{k=0}^{\infty}$, $(\gamma_k)_{k=0}^{\infty}$ and $(\xi_k)_{k=0}^{\infty}$ determined by certain recurrences. It follows from Theorem 50 and Theorem 51 that these sequences satisfy the following relations:

$$\eta_k = 2^{\binom{k}{2}}\mathfrak{it}_k, \qquad \gamma_k = \frac{\mathfrak{ssd}_k}{k!}, \qquad \xi_k = \frac{b_k}{k!},$$

where the sequence $(b_k)_{k=0}^{\infty}$ is defined by (51).

Furthermore, Wright explicitly computed the polynomials of $w_m(n)$ for $m \leqslant 5$. We use the occasion to fix a typo in his expression for $w_5(n)$. The corrected value is indicated above, while Wright mistakenly omitted the last digit in the number 23 724.

A.5 Semi-strong digraphs, counting strongly connected components

According to Theorem 53, the Coefficient GF of type $(2,2)$ of semi-strong digraphs satisfies

$$(\mathcal{Q}_2^2\,\mathsf{SSD})(z,w;t) = t\cdot \mathsf{SSD}(2^{3/2}z^2w; 1+t)\cdot \Phi_2^{1,2}\big(1-\mathsf{IT}(2zw)\big)^2,$$

where t is the marking variable for strongly connected components. In terms of the exponential Hadamard product, we can rewrite this formula in the following way:

$$(\mathcal{Q}_2^2\,\mathsf{SSD})(z,w;t) = t\cdot e^{(t+1)\cdot \mathsf{SCD}(2^{3/2}z^2w)}\cdot \Big(\big(1-\mathsf{IT}(2zw)\big)^2 \odot_z \mathsf{G}(z,\sqrt{2}-1)\Big).$$

The corresponding coefficients $\mathfrak{ssd}^{\circ}_{m,\ell}(t)$ are polynomials in t listed in Table 5.

Table 5: Values of the coefficients $\mathfrak{ssd}^{\circ}_{m,\ell}(t)$ for $m,\ell \leqslant 5$.

ℓ	0	1	2	3	4	5
$\mathfrak{ssd}^{\circ}_{0,\ell}(t)$	t	0	0	0	0	0
$\mathfrak{ssd}^{\circ}_{1,\ell}(t)$	0	$-4t$	0	0	0	0
$\mathfrak{ssd}^{\circ}_{2,\ell}(t)$	0	$4(t^2+t)$	$8t$	0	0	0
$\mathfrak{ssd}^{\circ}_{3,\ell}(t)$	0	0	$-32(t^2+t)$	$-\frac{128}{3}t$	0	0
$\mathfrak{ssd}^{\circ}_{4,\ell}(t)$	0	0	$32(t^3+3t^2+2t)$	$128(t^2+t)$	$-\frac{4\,096}{3}t$	0
$\mathfrak{ssd}^{\circ}_{5,\ell}(t)$	0	0	0	$-512(t^3+3t^2+2t)$	$-\frac{4\,096}{3}(t^2+t)$	$-\frac{3\,473\,408}{15}t$

Putting $t=1$, we obtain the asymptotic coefficients of semi-strong digraphs without reference to strongly connected components, see Table 6.

Another option is to extract kth coefficient of $(\mathcal{Q}_2^2\,\mathsf{SSD})(z,w;t)$ in t, which leads to the asymptotics of semi-strong digraphs with k strongly connected components (see relation (53)). For $k=1$, these digraphs are strongly connected, and their asymptotics was discussed in Section A.5, see Table 4. For the cases $k=2$ and $k=3$, we present the asymptotic coefficients in Table 7 and Table 8, respectively.

Table 6: Values of the coefficients $\mathfrak{ssd}^{\circ}_{m,\ell}$ for $m, \ell \leqslant 7$.

ℓ	0	1	2	3	4	5	6	7
$\mathfrak{ssd}^{\circ}_{0,\ell}$	1	0	0	0	0	0	0	0
$\mathfrak{ssd}^{\circ}_{1,\ell}$	0	-4	0	0	0	0	0	0
$\mathfrak{ssd}^{\circ}_{2,\ell}$	0	8	8	0	0	0	0	0
$\mathfrak{ssd}^{\circ}_{3,\ell}$	0	0	-64	$-\frac{128}{3}$	0	0	0	0
$\mathfrak{ssd}^{\circ}_{4,\ell}$	0	0	192	256	$-\frac{4\,096}{3}$	0	0	0
$\mathfrak{ssd}^{\circ}_{5,\ell}$	0	0	0	$-3\,072$	$-\frac{8\,192}{3}$	$-\frac{3\,473\,408}{15}$	0	0
$\mathfrak{ssd}^{\circ}_{6,\ell}$	0	0	0	$\frac{114\,688}{3}$	24 576	$-\frac{524\,288}{3}$	$-\frac{4\,984\,930\,304}{45}$	0
$\mathfrak{ssd}^{\circ}_{7,\ell}$	0	0	0	0	$-\frac{3\,670\,016}{3}$	$-524\,288$	$-\frac{889\,192\,448}{15}$	$-\frac{50\,988\,241\,125\,376}{315}$

Table 7: Values of the coefficients $[t^2]\mathfrak{ssd}^{\circ}_{m,\ell}(t)$ for $m \leqslant 8$ and $\ell \leqslant 7$.

ℓ	0	1	2	3	4	5	6	7
$[t^2]\,\mathfrak{ssd}^{\circ}_{0,\ell}(t)$	0	0	0	0	0	0	0	0
$[t^2]\,\mathfrak{ssd}^{\circ}_{1,\ell}(t)$	0	0	0	0	0	0	0	0
$[t^2]\,\mathfrak{ssd}^{\circ}_{2,\ell}(t)$	0	4	0	0	0	0	0	0
$[t^2]\,\mathfrak{ssd}^{\circ}_{3,\ell}(t)$	0	0	-32	0	0	0	0	0
$[t^2]\,\mathfrak{ssd}^{\circ}_{4,\ell}(t)$	0	0	96	128	0	0	0	0
$[t^2]\,\mathfrak{ssd}^{\circ}_{5,\ell}(t)$	0	0	0	$-1\,536$	$-\frac{4\,096}{3}$	0	0	0
$[t^2]\,\mathfrak{ssd}^{\circ}_{6,\ell}(t)$	0	0	0	18 432	12 288	$-\frac{262\,144}{3}$	0	0
$[t^2]\,\mathfrak{ssd}^{\circ}_{7,\ell}(t)$	0	0	0	0	$-589\,824$	262 144	$-\frac{444\,596\,224}{15}$	0
$[t^2]\,\mathfrak{ssd}^{\circ}_{8,\ell}(t)$	0	0	0	0	$\frac{233\,046\,016}{3}$	9 437 184	$-33\,554\,432$	$-\frac{1\,276\,142\,157\,824}{45}$

Table 8: Values of the coefficients $[t^3]\mathfrak{ssd}^{\circ}_{m,\ell}(t)$ for $m \leqslant 9$ and $\ell \leqslant 7$.

ℓ	0	1	2	3	4	5	6	7
$[t^3]\,\mathfrak{ssd}^{\circ}_{0,\ell}(t)$	0	0	0	0	0	0	0	0
$[t^3]\,\mathfrak{ssd}^{\circ}_{1,\ell}(t)$	0	0	0	0	0	0	0	0
$[t^3]\,\mathfrak{ssd}^{\circ}_{2,\ell}(t)$	0	0	0	0	0	0	0	0
$[t^3]\,\mathfrak{ssd}^{\circ}_{3,\ell}(t)$	0	0	0	0	0	0	0	0
$[t^3]\,\mathfrak{ssd}^{\circ}_{4,\ell}(t)$	0	0	32	0	0	0	0	0
$[t^3]\,\mathfrak{ssd}^{\circ}_{5,\ell}(t)$	0	0	0	-512	0	0	0	0
$[t^3]\,\mathfrak{ssd}^{\circ}_{6,\ell}(t)$	0	0	0	4 096	4 096	0	0	0
$[t^3]\,\mathfrak{ssd}^{\circ}_{7,\ell}(t)$	0	0	0	0	$-131\,072$	$-\frac{262\,144}{3}$	0	0
$[t^3]\,\mathfrak{ssd}^{\circ}_{8,\ell}(t)$	0	0	0	0	4 325 376	2 097 152	$-\frac{33\,554\,432}{3}$	0
$[t^3]\,\mathfrak{ssd}^{\circ}_{9,\ell}(t)$	0	0	0	0	0	$-276\,824\,064$	$-\frac{268\,435\,456}{3}$	$-\frac{113\,816\,633\,344}{15}$

A.6 Digraphs, counting strongly connected components, part 1

According to Theorem 54, the Coefficient GF of type $(2,1)$ of digraphs with the marking variable t for the number of strongly connected components satisfies

$$(\mathcal{Q}_2^1\,\widehat{\mathsf{D}})(z,w;t) = \Big(\widehat{\mathsf{D}}(2zw;t)\Big)^2 \cdot \Phi_2^{2,1}\Big((\mathcal{Q}_2^2\,\mathsf{SSD})(z,w;-t)\Big),$$

or, in terms of the exponential Hadamard product,

$$(\mathcal{Q}_2^1\widehat{\mathsf{D}})(z,w;t) = \\ t\cdot\frac{\Big(e^{(1-t)\cdot\mathsf{SCD}(2^{3/2}z^2w)}\cdot\Big(\big(1-\mathsf{IT}(2zw)\big)^2\odot_z\mathsf{G}(z,\sqrt{2}-1)\Big)\Big)\odot_z\mathsf{G}(z,1/\sqrt{2}-1)}{\big(\Delta_2\big(e^{-t\cdot\mathsf{SCD}(2zw)}\big)\big)^2}.$$

The corresponding coefficients $\hat{\mathfrak{d}}^{\circ}_{m,\ell}(t)$ are polynomials in t. For small values of the parameters m and ℓ, they are given in Table 9.

Table 9: Values of the coefficients $\hat{\mathfrak{d}}^{\circ}_{m,\ell}(t)$ for $m,\ell\leqslant 3$.

ℓ	0	1	2	3
$\hat{\mathfrak{d}}^{\circ}_{0,\ell}(t)$	t	0	0	0
$\hat{\mathfrak{d}}^{\circ}_{1,\ell}(t)$	0	$4(t^2-t)$	0	0
$\hat{\mathfrak{d}}^{\circ}_{2,\ell}(t)$	0	$-4(t^2-t)$	$4(5t^3-7t^2+2t)$	0
$\hat{\mathfrak{d}}^{\circ}_{3,\ell}(t)$	0	0	$-32(2t^3-3t^2+t)$	$8\big(\frac{61}{3}t^4-29t^3+14t^2-\frac{16}{3}t\big)$

Extracting kth coefficient of $(\mathcal{Q}_2^1\,\widehat{\mathsf{D}})(z,w;t)$ in t, we obtain asymptotics of digraphs with k strongly connected components whose dominant term is described by Theorem 55. For $k=1$, these digraphs are strongly connected, and their asymptotics was discussed in Section A.5, see Table 4. For the cases $k=2$ and $k=3$, we present the asymptotic coefficients in Table 10 and Table 11, respectively.

Table 10: Values of the coefficients $[t^2]\hat{\mathfrak{d}}^{\circ}_{m,\ell}(t)$ for $m,\ell\leqslant 7$.

ℓ	0	1	2	3	4	5	6	7
$[t^2]\,\hat{\mathfrak{d}}^{\circ}_{0,\ell}(t)$	0	0	0	0	0	0	0	0
$[t^2]\,\hat{\mathfrak{d}}^{\circ}_{1,\ell}(t)$	0	4	0	0	0	0	0	0
$[t^2]\,\hat{\mathfrak{d}}^{\circ}_{2,\ell}(t)$	0	-4	-28	0	0	0	0	0
$[t^2]\,\hat{\mathfrak{d}}^{\circ}_{3,\ell}(t)$	0	0	96	112	0	0	0	0
$[t^2]\,\hat{\mathfrak{d}}^{\circ}_{4,\ell}(t)$	0	0	-96	-896	-248	0	0	0
$[t^2]\,\hat{\mathfrak{d}}^{\circ}_{5,\ell}(t)$	0	0	0	5 632	$\frac{40\,960}{3}$	$\frac{271\,808}{3}$	0	0
$[t^2]\,\hat{\mathfrak{d}}^{\circ}_{6,\ell}(t)$	0	0	0	$-18\,432$	$-77\,824$	1 449 984	$\frac{2\,895\,728\,576}{45}$	0
$[t^2]\,\hat{\mathfrak{d}}^{\circ}_{7,\ell}(t)$	0	0	0	0	$\frac{13\,303\,808}{3}$	10 747 904	$\frac{14\,709\,161\,984}{15}$	$\frac{5\,311\,318\,221\,568}{45}$

Table 11: Values of the coefficients $[t^3]\hat{\mathfrak{d}}^{\circ}_{m,\ell}(t)$ for $m, \ell \leqslant 7$.

ℓ	0	1	2	3	4	5	6	7
$[t^3]\,\hat{\mathfrak{d}}^{\circ}_{0,\ell}(t)$	0	0	0	0	0	0	0	0
$[t^3]\,\hat{\mathfrak{d}}^{\circ}_{1,\ell}(t)$	0	0	0	0	0	0	0	0
$[t^3]\,\hat{\mathfrak{d}}^{\circ}_{2,\ell}(t)$	0	0	20	0	0	0	0	0
$[t^3]\,\hat{\mathfrak{d}}^{\circ}_{3,\ell}(t)$	0	0	-64	-232	0	0	0	0
$[t^3]\,\hat{\mathfrak{d}}^{\circ}_{4,\ell}(t)$	0	0	32	2 048	2 140	0	0	0
$[t^3]\,\hat{\mathfrak{d}}^{\circ}_{5,\ell}(t)$	0	0	0	$-6\,656$	$-30\,720$	56 720	0	0
$[t^3]\,\hat{\mathfrak{d}}^{\circ}_{6,\ell}(t)$	0	0	0	4 096	430 080	$\frac{913\,408}{3}$	$\frac{87\,938\,960}{3}$	0
$[t^3]\,\hat{\mathfrak{d}}^{\circ}_{7,\ell}(t)$	0	0	0	0	$-4\,849\,664$	$-\frac{41\,156\,608}{3}$	$-347\,275\,264$	$\frac{1\,476\,072\,351\,296}{45}$

A.7 Digraphs, counting strongly connected components, part 2

According to Theorem 57, the Coefficient GF of type $(2,1)$ of digraphs with the marking variables s and t for the number of source-like and all strongly connected components, respectively, satisfies

$$(\mathcal{Q}_2^1\,\widehat{\mathsf{D}})(z,w;s,t) = \\ \widehat{\mathsf{D}}(2zw;t)\left[\Phi_2^{2,1}\Big((\mathcal{Q}_2^2\,\mathsf{SSD})\big(z,w;(s-1)t\big)\Big) + \widehat{\mathsf{D}}(2zw;s,t)\cdot\Phi_2^{2,1}\Big((\mathcal{Q}_2^2\,\mathsf{SSD})(z,w;-t)\Big)\right].$$

The corresponding coefficients $\hat{\mathfrak{d}}^{\circ}_{m,\ell}(s,t)$ are polynomials in s and t. For $m, \ell \leqslant 3$, the non-zero values of the coefficients $\hat{\mathfrak{d}}^{\circ}_{m,\ell}(s,t)$ are the following:

$$\begin{aligned}
\hat{\mathfrak{d}}^{\circ}_{0,0}(s,t) &= st,\\
\hat{\mathfrak{d}}^{\circ}_{1,1}(s,t) &= 4st^2 - 4st,\\
\hat{\mathfrak{d}}^{\circ}_{2,1}(s,t) &= 4(s^2-2s)t^2 + 4st,\\
\hat{\mathfrak{d}}^{\circ}_{2,2}(s,t) &= 2(s^2+9s)t^3 - 28st^2 + 8st,\\
\hat{\mathfrak{d}}^{\circ}_{3,2}(s,t) &= 32(s^2-3s)t^3 - 32(s^2-4s)t^2 - 32st,\\
\hat{\mathfrak{d}}^{\circ}_{3,3}(s,t) &= \frac{4}{3}(s^3+21s^2+100s)t^4 - 4(7s^2+51s)t^3 + 112st^2 - \frac{128}{3}st.
\end{aligned}$$

Clearly, putting $s = 1$, on can get the asymptotics discussed in Section A.6. On the other hand, putting $t = 1$, we obtain asymptotics of digraphs with respect to source-like components. We provide the values of $\hat{\mathfrak{d}}^{\circ}_{m,\ell}(s,1)$ in Table 12.

Finally, extracting coefficients leads to asymptotics of digraphs with prescribed number of strongly connected components or source-like strongly connected components. Thus, if we extract kth coefficient of $(\mathcal{Q}_2^1\,\widehat{\mathsf{D}})(z,w;s,t)$ in t, we obtain asymptotics of digraphs with k strongly connected components with respect to source-like strongly connected components. For $k = 1$, these digraphs are strongly connected, and the result is given by Table 4 whose entries are multiplied by s. For the cases $k = 2$ and $k = 3$, the asymptotic coefficients are presented in Table 13 and Table 14, respectively. On the other hand,

Table 12: Values of the coefficients $\hat{\mathfrak{d}}^{\circ}_{m,\ell}(s,1)$ for $m,\ell \leqslant 4$.

ℓ	0	1	2	3	4
$\hat{\mathfrak{d}}^{\circ}_{0,\ell}(s,1)$	s	0	0	0	0
$\hat{\mathfrak{d}}^{\circ}_{1,\ell}(s,1)$	0	0	0	0	0
$\hat{\mathfrak{d}}^{\circ}_{2,\ell}(s,1)$	0	$4(s^2-s)$	$2(s^2-s)$	0	0
$\hat{\mathfrak{d}}^{\circ}_{3,\ell}(s,1)$	0	0	0	$\frac{4}{3}(s^3-s)$	0
$\hat{\mathfrak{d}}^{\circ}_{4,\ell}(s,1)$	0	0	$32(s^3-s)$	$128(s^2-s)$	$\frac{2}{3}s^4+\frac{4}{3}s^3+42s^2-44s$

if we extract kth coefficient of $(\mathcal{Q}_2^1\,\widehat{\mathsf{D}})(z,w;s,t)$ in s, we obtain asymptotics of digraphs with k source-like strongly connected components with respect to all strongly connected components. The corresponding asymptotics for the cases $k=1,2,3$ are summarized in Table 15, Table 16 and Table 17.

Table 13: Values of the coefficients $[t^2]\hat{\mathfrak{d}}^{\circ}_{m,\ell}(s,t)$ for $m,\ell \leqslant 5$.

ℓ	0	1	2	3	4	5
$[t^2]\,\hat{\mathfrak{d}}^{\circ}_{0,\ell}(s,t)$	0	0	0	0	0	0
$[t^2]\,\hat{\mathfrak{d}}^{\circ}_{1,\ell}(s,t)$	0	$4s$	0	0	0	0
$[t^2]\,\hat{\mathfrak{d}}^{\circ}_{2,\ell}(s,t)$	0	$4(s^2-2s)$	$-28s$	0	0	0
$[t^2]\,\hat{\mathfrak{d}}^{\circ}_{3,\ell}(s,t)$	0	0	$-32(s^2-4s)$	$112s$	0	0
$[t^2]\,\hat{\mathfrak{d}}^{\circ}_{4,\ell}(s,t)$	0	0	$96(s^2-2s)$	$128(s^2-8s)$	$-248s$	0
$[t^2]\,\hat{\mathfrak{d}}^{\circ}_{5,\ell}(s,t)$	0	0	0	$-512(3s^2-14s)$	$-\frac{4\,096}{3}(s^2-11s)$	$\frac{271\,808}{3}s$

Table 14: Values of the coefficients $[t^3]\hat{\mathfrak{d}}^{\circ}_{m,\ell}(s,t)$ for $m,\ell \leqslant 4$.

ℓ	0	1	2	3	4
$[t^3]\,\hat{\mathfrak{d}}^{\circ}_{0,\ell}(s,t)$	0	0	0	0	0
$[t^3]\,\hat{\mathfrak{d}}^{\circ}_{1,\ell}(s,t)$	0	0	0	0	0
$[t^3]\,\hat{\mathfrak{d}}^{\circ}_{2,\ell}(s,t)$	0	0	$2(s^2+9s)$	0	0
$[t^3]\,\hat{\mathfrak{d}}^{\circ}_{3,\ell}(s,t)$	0	0	$32(s^2-3s)$	$-4(7s^2+51s)$	0
$[t^3]\,\hat{\mathfrak{d}}^{\circ}_{4,\ell}(s,t)$	0	0	$32(s^3-3s^2+3s)$	$-256(s^2-9s)$	$2(89s^2+981s)$

Table 15: Values of the coefficients $[s]\hat{\mathfrak{d}}^{\circ}_{m,\ell}(s,t)$ for $m,\ell \leqslant 3$.

ℓ	0	1	2	3
$[s]\,\hat{\mathfrak{d}}^{\circ}_{0,\ell}(s,t)$	t	0	0	0
$[s]\,\hat{\mathfrak{d}}^{\circ}_{1,\ell}(s,t)$	0	$4(t^2-t)$	0	0
$[s]\,\hat{\mathfrak{d}}^{\circ}_{2,\ell}(s,t)$	0	$-4(2t^2-t)$	$2(9t^3-14t^2+4t)$	0
$[s]\,\hat{\mathfrak{d}}^{\circ}_{3,\ell}(s,t)$	0	0	$-32(3t^3-4t^2+t)$	$4\big(\frac{100}{3}t^4-51t^3+28t^2-\frac{32}{3}t\big)$

Table 16: Values of the coefficients $[s^2]\hat{\mathfrak{d}}^{\circ}_{m,\ell}(s,t)$ for $m,\ell \leqslant 4$.

ℓ	0	1	2	3	4
$[s^2]\,\hat{\mathfrak{d}}^{\circ}_{0,\ell}(s,t)$	0	0	0	0	0
$[s^2]\,\hat{\mathfrak{d}}^{\circ}_{1,\ell}(s,t)$	0	0	0	0	0
$[s^2]\,\hat{\mathfrak{d}}^{\circ}_{2,\ell}(s,t)$	0	$4t^2$	$2t^3$	0	0
$[s^2]\,\hat{\mathfrak{d}}^{\circ}_{3,\ell}(s,t)$	0	0	$32(t^3-t^2)$	$28(t^4-t^3)$	0
$[s^2]\,\hat{\mathfrak{d}}^{\circ}_{4,\ell}(s,t)$	0	0	$-96(t^3-t^2)$	$128(2t^4-2t^3+t^2)$	$2(258t^5-326t^4+89t^3)$

Table 17: Values of the coefficients $[s^3]\hat{\mathfrak{d}}^{\circ}_{m,\ell}(s,t)$ for $m,\ell \leqslant 5$.

ℓ	0	1	2	3	4	5
$[s^3]\,\hat{\mathfrak{d}}^{\circ}_{0,\ell}(s,t)$	0	0	0	0	0	0
$[s^3]\,\hat{\mathfrak{d}}^{\circ}_{1,\ell}(s,t)$	0	0	0	0	0	0
$[s^3]\,\hat{\mathfrak{d}}^{\circ}_{2,\ell}(s,t)$	0	0	0	0	0	0
$[s^3]\,\hat{\mathfrak{d}}^{\circ}_{3,\ell}(s,t)$	0	0	0	$\frac{4}{3}t^4$	0	0
$[s^3]\,\hat{\mathfrak{d}}^{\circ}_{4,\ell}(s,t)$	0	0	$32t^3$	0	$4(10t^5-\frac{29}{3}t^4)$	0
$[s^3]\,\hat{\mathfrak{d}}^{\circ}_{5,\ell}(s,t)$	0	0	0	$512(2t^4-t^3)$	$-\frac{1024}{3}(t^5-t^4)$	$\frac{4}{3}(1\,154t^6-1\,480t^5+359t^4)$

A.8 Digraphs, counting strongly connected components, part 3

According to Theorem 58, the Coefficient GF of type $(2,2)$ of digraphs with the marking variables u,v,y and t for the numbers of purely source-like, purely sink-like, isolated and all strongly connected components, respectively, is given by

$$(\mathcal{Q}_2^2\,\mathsf{D})(z,w;u,v,y,t) = \mathsf{D}^{\circ}_1 + \mathsf{D}^{\circ}_{20}\cdot\Phi_2^{1,2}\Big(\mathsf{D}^{\circ}_{21}+\mathsf{D}^{\circ}_{22}+\mathsf{D}^{\circ}_{23}\Big),$$

where

$$\begin{aligned}
\mathsf{D}^{\circ}_1(z,w;u,v,y,t) &= (y-u-v+1)t\cdot\mathsf{D}(2^{3/2}z^2w;u,v,y,t)\cdot(\mathcal{Q}_2^2\,\mathsf{SCD})(z,w)\,;\\
\mathsf{D}^{\circ}_{20}(z,w;u,v,y,t) &= \mathsf{SSD}\big(2^{3/2}z^2w;(y-u-v+1)t\big)\,;\\
\mathsf{D}^{\circ}_{21}(z,w;u,v,y,t) &= \widehat{\mathsf{D}}(2zw;u,t)\cdot\Phi_2^{2,1}\Big((\mathcal{Q}_2^2\,\mathsf{SSD})\big(z,w;(v-1)t\big)\Big)\,;\\
\mathsf{D}^{\circ}_{22}(z,w;u,v,y,t) &= \widehat{\mathsf{D}}(2zw;v,t)\cdot\Phi_2^{2,1}\Big((\mathcal{Q}_2^2\,\mathsf{SSD})\big(z,w;(u-1)t\big)\Big)\,;\\
\mathsf{D}^{\circ}_{23}(z,w;u,v,y,t) &= \widehat{\mathsf{D}}(2zw;u,t)\cdot\widehat{\mathsf{D}}(2zw;v,t)\cdot\Phi_2^{2,1}\Big((\mathcal{Q}_2^2\,\mathsf{SSD})(z,w;-t)\Big).
\end{aligned}$$

The first non-zero values of the corresponding coefficients $\mathfrak{d}^{\circ}_{m,\ell}(u,v,y,t)$, which are polynomials in u,v,y and t, are listed below.

$$\begin{aligned}
&\mathfrak{d}^{\circ}_{0,0}(u,v,y,t) = yt,\\
&\mathfrak{d}^{\circ}_{1,1}(u,v,y,t) = 4uvt^2-4yt,\\
&\mathfrak{d}^{\circ}_{2,1}(u,v,y,t) = 4(y^2-2uv)t^2+4yt,\\
&\mathfrak{d}^{\circ}_{2,2}(u,v,y,t) = 2uv(u+v+8)t^3-28uvt^2+8yt,
\end{aligned}$$

$$\mathfrak{d}^{\circ}_{3,2}(u,v,y,t) = 32uv(y-3)t^3 + 32(4uv-y^2)t^2 - 32yt,$$
$$\mathfrak{d}^{\circ}_{3,3}(u,v,y,t) = \frac{4}{3}uv\big(u^2+v^2+21(u+v)+78\big)t^4 - 4uv\big(7(u+v)+44\big)t^3 + 112uvt^2 - \frac{128}{3}yt.$$

Note that, after the substitution $u = s$, $v = 1$, $y = s$, we get the coefficients $\hat{\mathfrak{d}}^{\circ}_{m,\ell}(s,t)$ seen in Section A.7. In other words,

$$\mathfrak{d}^{\circ}_{m,\ell}(s,1,s,t) = \hat{\mathfrak{d}}^{\circ}_{m,\ell}(s,1).$$

Many other possible substitutions can be considered. For instance, to obtain the asymptotics of sink-like strongly connected components, one may put $u = 1$, $v = s$, $y = s$, which leads to exactly the same result for symmetry reasons. Another example is the asymptotics of strongly connected components that are neither source-like, nor sink-like. This asymptotics can be reached by the substitution $u = \bar{s}^{-1}$, $v = \bar{s}^{-1}$, $y = \bar{s}^{-1}$, $t = \bar{s}\bar{t}$, where $\bar{s}$ is the marking variable for the target components (see the list of first non-zero values below).

$$\begin{aligned}
\mathfrak{d}^{\circ}_{0,0}(\bar{s}^{-1},\bar{s}^{-1},\bar{s}^{-1},\bar{s}\bar{t}) &= \bar{t},\\
\mathfrak{d}^{\circ}_{1,1}(\bar{s}^{-1},\bar{s}^{-1},\bar{s}^{-1},\bar{s}\bar{t}) &= 4\bar{t}^2 - 4\bar{t},\\
\mathfrak{d}^{\circ}_{2,1}(\bar{s}^{-1},\bar{s}^{-1},\bar{s}^{-1},\bar{s}\bar{t}) &= -4\bar{t}^2 + 4\bar{t},\\
\mathfrak{d}^{\circ}_{2,2}(\bar{s}^{-1},\bar{s}^{-1},\bar{s}^{-1},\bar{s}\bar{t}) &= 4\big(1+4\bar{s}\big)\bar{t}^3 - 28\bar{t}^2 + 8\bar{t},\\
\mathfrak{d}^{\circ}_{3,2}(\bar{s}^{-1},\bar{s}^{-1},\bar{s}^{-1},\bar{s}\bar{t}) &= 32\big(1-3\bar{s}\big)\bar{t}^3 - 96\bar{t}^2 - 32\bar{t},\\
\mathfrak{d}^{\circ}_{3,3}(\bar{s}^{-1},\bar{s}^{-1},\bar{s}^{-1},\bar{s}\bar{t}) &= \frac{8}{3}\big(1+21\bar{s}+39\bar{s}^2\big)\bar{t}^4 - 8\big(7+22\bar{s}\big)\bar{t}^3 + 112\bar{t}^2 - \frac{128}{3}\bar{t}.
\end{aligned}$$

As usually, extracting coefficients leads to asymptotics of digraphs with prescribed number of strongly connected components. We will not give an exhaustive list of all possibilities and will limit ourselves to a few remarks. First of all, the expression

$$[u^p v^q y^r t^k](\mathcal{Q}_2^2\,\mathsf{D})(z,w;u,v,y,t)$$

represents the asymptotics of digraphs with k strongly connected components such that p of them are strongly source-like, q of them are strongly sink-like, and r of them are isolated. As an example, we provide asymptotic coefficients for the digraphs with one purely source-like component, one purely sink-like component, one isolated component and one component of the general type (neither source-like, nor sink-like). This corresponds to the case where $u = v = y = 1$ and $t = 4$, see Table 18.

If we extract kth coefficient of $(\mathcal{Q}_2^2\,\mathsf{D})(z,w;u,v,y,t)$ in t, we obtain the asymptotics of digraphs with k strongly connected components with respect to strongly connected components of different types. In particular, for $k = 1$, we revisit the asymptotics of strongly connected digraphs, and the result is given by Table 4 whose entries are multiplied by y. For the cases $k = 2$ and $k = 3$, the asymptotic coefficients are presented in Table 19 and Table 20, respectively.

Similarly, extracting pth coefficient in u, we obtain the asymptotics of digraphs with p purely source-like strongly connected components. Thus, Table 21 shows this asymptotics

Table 18: Values of the coefficients $\alpha_{m,\ell} = [uvyt^4]\mathfrak{d}^{\circ}_{m,\ell}(u,v,y,t)$ for $m,\ell \leqslant 8$.

ℓ	0	1	2	3	4	5	6	7	8
$\alpha_{0,\ell}$	0	0	0	0	0	0	0	0	0
$\alpha_{1,\ell}$	0	0	0	0	0	0	0	0	0
$\alpha_{2,\ell}$	0	0	0	0	0	0	0	0	0
$\alpha_{3,\ell}$	0	0	0	0	0	0	0	0	0
$\alpha_{4,\ell}$	0	0	0	256	0	0	0	0	0
$\alpha_{5,\ell}$	0	0	0	−3 072	−5 632	0	0	0	0
$\alpha_{6,\ell}$	0	0	0	16 384	143 360	114 176	0	0	0
$\alpha_{7,\ell}$	0	0	0	0	−1 966 080	−4 620 288	8 392 704	0	0
$\alpha_{8,\ell}$	0	0	0	0	31 457 280	139 460 608	$-\frac{859\,635\,712}{3}$	$-\frac{23\,526\,842\,368}{3}$	0

Table 19: Values of the coefficients $\beta_{m,\ell} = [t^2]\mathfrak{d}^{\circ}_{m,\ell}(u,v,y,t)$ for $m,\ell \leqslant 5$.

ℓ	0	1	2	3	4	5
$\beta_{0,\ell}$	0	0	0	0	0	0
$\beta_{1,\ell}$	0	$4uv$	0	0	0	0
$\beta_{2,\ell}$	0	$4(y^2-2uv)$	$-28uv$	0	0	0
$\beta_{3,\ell}$	0	0	$-32(y^2-4uv)$	$112uv$	0	0
$\beta_{4,\ell}$	0	0	$96(y^2-2uv)$	$128(y^2-8uv)$	$-248uv$	0
$\beta_{5,\ell}$	0	0	0	$-512(3y^2-14uv)$	$-\frac{4\,096}{3}(y^2-11uv)$	$\frac{271\,808}{3}uv$

Table 20: Values of the coefficients $\gamma_{m,\ell} = [t^3]\mathfrak{d}^{\circ}_{m,\ell}(u,v,y,t)$ for $m,\ell \leqslant 4$.

ℓ	0	1	2	3	4
$\gamma_{0,\ell}$	0	0	0	0	0
$\gamma_{1,\ell}$	0	0	0	0	0
$\gamma_{2,\ell}$	0	0	$2uv(u+v+8)$	0	0
$\gamma_{3,\ell}$	0	0	$32uv(y-3)$	$-4uv\big(7(u+v)+44\big)$	0
$\gamma_{4,\ell}$	0	0	$32\big(y^3-uv(2y+u+v-4)\big)$	$-64uv\big(7y-3(u+v+11)\big)$	$2uv\big(89(u+v)+892\big)$

Table 21: Values of the coefficients $\eta_{m,\ell} = [u]\mathfrak{d}^{\circ}_{m,\ell}(u,1,1,t)$ for $m,\ell \leqslant 4$.

ℓ	0	1	2	3	4
$\eta_{0,\ell}$	0	0	0	0	0
$\eta_{1,\ell}$	0	$4t^2$	0	0	0
$\eta_{2,\ell}$	0	$-8t^2$	$2(9t^3-14t^2)$	0	0
$\eta_{3,\ell}$	0	0	$-64(t^3-2t^2)$	$4\big(\frac{100}{3}t^4-51t^3+28t^2\big)$	0
$\eta_{4,\ell}$	0	0	$32(t^3-6t^2)$	$-32(39t^4-58t^3+32t^2)$	$2\big(905t^5-\frac{3\,304}{3}t^4+981t^3-124t^2\big)$

for the case $p=1$ with respect to the number of all strongly connected components. Here, to avoid keeping references to other types of strongly connected components, we also make the substitution $v=1$ and $y=1$. In the same way, to get the asymptotics of digraphs with $r=1$ isolated components that keeps track of the total number of connected components,

Table 22: Values of the coefficients $\theta_{m,\ell} = [y]\mathfrak{d}^{\circ}_{m,\ell}(1,1,y,t)$ for $m, \ell \leqslant 5$.

ℓ	0	1	2	3	4	5
$\theta_{0,\ell}$	t	0	0	0	0	0
$\theta_{1,\ell}$	0	$-4t$	0	0	0	0
$\theta_{2,\ell}$	0	$4t$	$8t$	0	0	0
$\theta_{3,\ell}$	0	0	$32(t^3-t)$	$\frac{128}{3}t$	0	0
$\theta_{4,\ell}$	0	0	$-64(t^3-t)$	$64(5t^4-7t^3+2t)$	$-\frac{4\,096}{3}t$	0
$\theta_{5,\ell}$	0	0	0	$-512(6t^4-7t^3+2t)$	$256\big(\frac{61}{3}t^5-29t^4+14t^3-\frac{16}{3}t\big)$	$-\frac{3\,473\,408}{15}t$

we extract rth coefficient in y and substitute $u = 1$ and $v = 1$, see Table 22. In particular, we can see from these tables that purely source-like and isolated components behave differently.

A.9 Satisfiable 2-CNFs

According to Theorem 61, the Coefficient GF of type $(2,1)$ of satisfiable 2-CNF formulae satisfies

$$(\mathcal{Q}_2^1 \ddot{\mathsf{SAT}})(z,w) = \ddot{\mathsf{SAT}}(2zw)\big(1 - \mathsf{IT}(2zw)\big).$$

As a consequence, the corresponding asymptotic coefficients $\ddot{\mathfrak{sat}}^{\circ}_{m,\ell}$ are of form

$$\ddot{\mathfrak{sat}}^{\circ}_{m,\ell} = \mathbf{1}_{m=\ell=0} - \mathbf{1}_{m=\ell>0} \cdot \frac{\bar{s}_m}{2^{\binom{m-1}{2}} \cdot m!},$$

where (compare to Theorem 62)

$$\bar{s}_m = \left(\sum_{k=0}^{m-1} \binom{m}{k} \cdot 2^{m^2-k^2} \cdot \mathfrak{sat}_k \cdot \mathfrak{it}_{m-k} \right) - \mathfrak{sat}_m.$$

Here, $(\mathfrak{sat}_m)_{m=0}^{\infty}$ is the counting sequence of satisfiable 2-CNF formulae,

$$(\mathfrak{sat}_m) = 1,\ 1,\ 15,\ 2\,397,\ 3\,049\,713,\ 28\,694\,311\,447,\ 2\,034\,602\,766\,692\,687,\ \ldots\ ;$$

and $(\mathfrak{it}_m)_{m=1}^{\infty}$ is the counting sequence of irreducible tournaments; see Section A.1. Thus, the sequence $(\bar{s}_m)_{m=1}^{\infty}$ starts with

$$(\bar{s}_m) = 1,\ 1,\ 67,\ 12\,559,\ 8\,976\,361,\ 23\,458\,307\,761,\ 225\,313\,054\,216\,027,\ \ldots$$

and

$$\big(\ddot{\mathfrak{sat}}^{\circ}_{m,m}\big) = 1,\ -1,\ -\frac{1}{4},\ -\frac{67}{48},\ -\frac{12\,559}{1\,536},\ -\frac{8\,976\,361}{122\,880},\ -\frac{23\,458\,307\,761}{23\,592\,960},\ \ldots$$

A.10 Contradictory strongly connected implication digraphs

Due to Theorem 64, the Coefficient GF of type $(2,4)$ of contradictory strongly connected implication digraphs is given by

$$(\mathcal{Q}_2^4 \mathsf{CSCC})(z,w) = \exp\left(\frac{1}{2}\mathsf{SCD}(2^{7/2}z^4w) - \mathsf{CSCC}(2^{5/2}z^4w)\right) \cdot \Phi_2^{2,4}\Big(1 - \mathsf{IT}(2^{5/2}z^2w)\Big).$$

From this relation, it is cleat that the corresponding asymptotic coefficients $\mathfrak{cscc}^{\circ}_{m,\ell}$ are zeroes for all the odd values of m. Another necessary condition for these coefficients to be non-zero is $2\ell \leqslant m \leqslant 4\ell$. We provide the numerical values of $\mathfrak{cscc}^{\circ}_{2m,\ell}$ for $m, \ell \leqslant 6$ in Table 23.

Table 23: Values of the coefficients $\mathfrak{cscc}^{\circ}_{2m,\ell}$ for $m, \ell \leqslant 6$.

ℓ	0	1	2	3	4	5	6
$\mathfrak{cscc}^{\circ}_{0,\ell}$	1	0	0	0	0	0	0
$\mathfrak{cscc}^{\circ}_{2,\ell}$	0	-8	0	0	0	0	0
$\mathfrak{cscc}^{\circ}_{4,\ell}$	0	16	0	0	0	0	0
$\mathfrak{cscc}^{\circ}_{6,\ell}$	0	0	-512	$-\frac{32\,768}{3}$	0	0	0
$\mathfrak{cscc}^{\circ}_{8,\ell}$	0	0	4 096	0	$-16\,777\,216$	0	0
$\mathfrak{cscc}^{\circ}_{10,\ell}$	0	0	0	$-524\,288$	$-\frac{33\,554\,432}{3}$	$-\frac{2\,336\,462\,209\,024}{15}$	0
$\mathfrak{cscc}^{\circ}_{12,\ell}$	0	0	0	$-4\,278\,190\,080$	0	$-68\,719\,476\,736$	$-8\,725\,724\,278\,030\,336$

As seen from Table 23, the counting sequence $(\mathfrak{cscc}_n)_{n=0}^{\infty}$ of contradictory strongly connected implication digraphs,

$$(\mathfrak{cscc}_n) = 0,\ 0,\ 1,\ 1\,606,\ 12\,864\,042,\ 1\,035\,697\,286\,504,\ 1\,137\,724\,245\,192\,445\,576,\ \ldots$$

behaves as $2^{4\binom{n}{2}}$, as $n \to \infty$. Similarly to the case of strongly connected digraphs, we can establish its asymptotic behavior more precisely, namely,

$$\mathfrak{cscc}_n \approx 2^{4\binom{n}{2}} \sum_{m \geqslant 0} \frac{w_m(n)}{4^{nm}}, \qquad \text{where} \quad w_m(n) = \sum_{\ell=\lceil m/2 \rceil}^{m} n^{\underline{\ell}}\, \mathfrak{cscc}^{\circ}_{2m,\ell}.$$

For $m \leqslant 6$, the polynomials $w_m(n)$ are the following:

$$\begin{aligned}
w_0(n) &= 1,\\
w_1(n) &= -8n,\\
w_2(n) &= 16n,\\
w_3(n) &= -\frac{512}{3} n(n-1)(64n-125),\\
w_4(n) &= -4\,096 n(n-1)(4\,096n^2 - 20\,480n + 24\,575),\\
w_5(n) &= -\frac{524\,288}{15} n(n-1)(n-2)(4\,456\,448n^2 - 31\,194\,816n + 53\,476\,431),\\
w_6(n) &= -16\,777\,216 n(n-1)(n-2)(520\,097\,792n^2 + 6\,241\,153\,024n - 14\,042\,579\,199).
\end{aligned}$$

A.11 2-CNFs, counting strongly connected components

According to Theorem 65, the Coefficient GF of type $(2,2)$ of 2-CNF formulae is given by

$$(\mathcal{Q}_2^2\, \ddot{\mathsf{CNF}})(z,w;s,t) = s \cdot \widehat{\mathsf{D}}(2^{3/2}z^2w;t) \cdot \Phi_2^{4,2}\left[z \cdot \exp\left(\big(s-1\big)\, \mathsf{CSCC}(2^{3/2}z^4w) + \frac{(1-t)}{2}\, \mathsf{SCD}(2^{5/2}z^4w) \right) \cdot \Phi_2^{2,4}\Big(1 - \mathsf{IT}(4z^2w)\Big) \right].$$

Here, the variables s and t mark, respectively, the numbers of contradictory strongly connected components and pairs of ordinary strongly connected components in the corresponding implication digraph. The asymptotic coefficients $\mathfrak{c\ddot{n}f}_{m,\ell}(s,t)$ are polynomials in s and t. It is clear from the above relation that these polynomials are zeroes for even values of m. For small odd values of m, the polynomials $\mathfrak{c\ddot{n}f}_{m,\ell}(s,t)$ are shown in Table 24.

Table 24: Values of the coefficients $\mathfrak{c\ddot{n}f}_{2m+1,\ell}(s,t)$ for $m,\ell \leqslant 3$.

ℓ	0	1	2	3
$\mathfrak{c\ddot{n}f}_{1,\ell}(s,t)$	s	0	0	0
$\mathfrak{c\ddot{n}f}_{3,\ell}(s,t)$	0	$8s(t-1)$	0	0
$\mathfrak{c\ddot{n}f}_{5,\ell}(s,t)$	0	$-16s(t-1)$	$192s(t^2-t)$	0
$\mathfrak{c\ddot{n}f}_{7,\ell}(s,t)$	0	0	$-512s(4t^2-5t+1)$	$2\,048s\left(\frac{25}{3}t^3-5t^2+2t-\frac{16}{3}\right)$

Substituting $t=0$, we obtain the asymptotic coefficients of implication digraphs that do not contain ordinary strongly connected components, see Table 25. In other words, all strongly connected components of such graphs are contradictory.

Table 25: Values of the coefficients $\mathfrak{c\ddot{n}f}_{2m+1,\ell}(s,0)$ for $m,\ell \leqslant 5$.

ℓ	0	1	2	3	4	5
$\mathfrak{c\ddot{n}f}_{1,\ell}(s,0)$	s	0	0	0	0	0
$\mathfrak{c\ddot{n}f}_{3,\ell}(s,0)$	0	$-8s$	0	0	0	0
$\mathfrak{c\ddot{n}f}_{5,\ell}(s,0)$	0	$16s$	0	0	0	0
$\mathfrak{c\ddot{n}f}_{7,\ell}(s,0)$	0	0	$-512s$	$-\frac{32\,768}{3}s$	0	0
$\mathfrak{c\ddot{n}f}_{9,\ell}(s,0)$	0	0	$2\,048s(s+2)$	0	$-16\,777\,216s$	0
$\mathfrak{c\ddot{n}f}_{11,\ell}(s,0)$	0	0	0	$-262\,144s(s+2)$	$-\frac{33\,554\,432}{3}s$	$-\frac{2\,336\,462\,209\,024}{15}s$

Extracting kth coefficient in s from $\mathfrak{c\ddot{n}f}_{m,\ell}(s,0)$, we get the asymptotics of implication digraphs with exactly k contradictory strongly connected components. In particular, taking $[s]\mathfrak{c\ddot{n}f}_{2m+1,\ell}(s,0)$ leads us to the asymptotics of contradictory strongly connected implication digraphs whose coefficients are described by Table 23, while $[s^2]\mathfrak{c\ddot{n}f}_{2m+1,\ell}(s,0)$ corresponds to implication digraphs with two contradictory strongly connected components Table 26

Similarly, it is possible to consider implication digraphs with a given number k of pairs of ordinary strongly connected components. To obtain the corresponding asymptotics, it is sufficient to extract kth coefficient in t from $\mathfrak{c\ddot{n}f}_{m,\ell}(s,t)$. For $k=1,2,3$, the reader can find the result of calculations in Table 27, Table 28 and Table 29.

Table 26: Values of the coefficients $[s^2]\mathfrak{cnf}_{2m+1,\ell}(s,0)$ for $m,\ell \leqslant 7$.

ℓ	0	1	2	3	4	5	6	7
$[s^2]\,\mathfrak{cnf}_{1,\ell}(s,0)$	0	0	0	0	0	0	0	0
$[s^2]\,\mathfrak{cnf}_{3,\ell}(s,0)$	0	0	0	0	0	0	0	0
$[s^2]\,\mathfrak{cnf}_{5,\ell}(s,0)$	0	0	0	0	0	0	0	0
$[s^2]\,\mathfrak{cnf}_{7,\ell}(s,0)$	0	0	0	0	0	0	0	0
$[s^2]\,\mathfrak{cnf}_{9,\ell}(s,0)$	0	0	2 048	0	0	0	0	0
$[s^2]\,\mathfrak{cnf}_{11,\ell}(s,0)$	0	0	0	$-262\,144$	0	0	0	0
$[s^2]\,\mathfrak{cnf}_{13,\ell}(s,0)$	0	0	0	$\frac{13\,497\,270\,272}{3}$	0	0	0	0
$[s^2]\,\mathfrak{cnf}_{15,\ell}(s,0)$	0	0	0	0	$-\frac{6\,910\,602\,379\,264}{3}$	$-\frac{274\,877\,906\,944}{3}$	0	0

Table 27: Values of the coefficients $[t]\mathfrak{cnf}_{2m+1,\ell}(s,t)$ for $m,\ell \leqslant 5$.

ℓ	0	1	2	3	4	5
$[t]\,\mathfrak{cnf}_{1,\ell}(s,t)$	0	0	0	0	0	0
$[t]\,\mathfrak{cnf}_{3,\ell}(s,t)$	0	$8s$	0	0	0	0
$[t]\,\mathfrak{cnf}_{5,\ell}(s,t)$	0	$-16s$	$-192s$	0	0	0
$[t]\,\mathfrak{cnf}_{7,\ell}(s,t)$	0	0	$2\,560s$	$4\,096s$	0	0
$[t]\,\mathfrak{cnf}_{9,\ell}(s,t)$	0	0	$-8\,192s$	0	$\frac{16\,973\,824}{3}s$	0
$[t]\,\mathfrak{cnf}_{11,\ell}(s,t)$	0	0	0	$1\,048\,576s(4s+9)$	$\frac{2\,046\,820\,352}{3}s$	$84\,859\,158\,528s$

Table 28: Values of the coefficients $[t^2]\mathfrak{cnf}_{2m+1,\ell}(s,t)$ for $m,\ell \leqslant 5$.

ℓ	0	1	2	3	4	5
$[t^2]\,\mathfrak{cnf}_{1,\ell}(s,t)$	0	0	0	0	0	0
$[t^2]\,\mathfrak{cnf}_{3,\ell}(s,t)$	0	0	0	0	0	0
$[t^2]\,\mathfrak{cnf}_{5,\ell}(s,t)$	0	0	$192s$	0	0	0
$[t^2]\,\mathfrak{cnf}_{7,\ell}(s,t)$	0	0	$-2\,048s$	$-10\,240s$	0	0
$[t^2]\,\mathfrak{cnf}_{9,\ell}(s,t)$	0	0	$2\,048s$	$786\,432s$	$5\,472\,256s$	0
$[t^2]\,\mathfrak{cnf}_{11,\ell}(s,t)$	0	0	0	$-17\,039\,360s$	$-134\,217\,728s$	$\frac{123\,505\,475\,584}{3}s$

Table 29: Values of the coefficients $[t^3]\mathfrak{cnf}_{2m+1,\ell}(s,t)$ for $m,\ell \leqslant 6$.

ℓ	0	1	2	3	4	5	6
$[t^3]\,\mathfrak{cnf}_{1,\ell}(s,t)$	0	0	0	0	0	0	0
$[t^3]\,\mathfrak{cnf}_{3,\ell}(s,t)$	0	0	0	0	0	0	0
$[t^3]\,\mathfrak{cnf}_{5,\ell}(s,t)$	0	0	0	0	0	0	0
$[t^3]\,\mathfrak{cnf}_{7,\ell}(s,t)$	0	0	0	$\frac{51\,200}{3}s$	0	0	0
$[t^3]\,\mathfrak{cnf}_{9,\ell}(s,t)$	0	0	0	$-786\,432s$	$-\frac{851\,968}{3}s$	0	0
$[t^3]\,\mathfrak{cnf}_{11,\ell}(s,t)$	0	0	0	$4\,194\,304s$	$\frac{1\,744\,830\,464}{3}s$	$16\,965\,959\,680s$	0
$[t^3]\,\mathfrak{cnf}_{13,\ell}(s,t)$	0	0	0	$-\frac{8\,388\,608}{3}$	$-96\,636\,764\,160s$	$-\frac{14\,809\,047\,236\,608}{3}s$	$\frac{4\,602\,751\,631\,753\,216}{9}s$