

Tightness of a MaxCut Lower Bound via Vector Chromatic Number

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Submitted: Jun 1, 2026; Accepted: Jul 16, 2026; Published: Sep 11, 2026

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Abstract

Recently, Balla, Janzer, and Sudakov showed a lower bound on the MaxCut in terms of the vector chromatic number, recovering known results on the MaxCut of H -free graphs. In this paper, we show that their bound is tight, providing a construction that achieves a value arbitrarily close to the optimal constant. This answers a question raised by Elphick. Our construction is a modification of the spherical random geometric graph used by Feige and Schechtman to establish the integrality gap for the Goemans–Williamson semidefinite relaxation of the MaxCut.

Mathematics Subject Classifications: 05C35, 05C15, 90C27

1 Introduction

The *MaxCut* of a graph $G = (V, E)$, denoted by $\text{mc}(G)$, is the size of the largest bipartite subgraph of G . This parameter has been extensively studied over the past 50 years. From a computational perspective, computing the MaxCut of a graph appeared as one of Karp’s original 21 NP-complete problems [9]. Moreover, it is the first optimization problem for which semidefinite programming was used to obtain a constant-factor approximation guarantee, established in the landmark work of Goemans and Williamson [6].

A standard probabilistic argument shows that any graph G with m edges satisfies $\text{mc}(G) \geq m/2$. This motivates the study of the *surplus* of a graph, defined as $\text{sp}(G) = \text{mc}(G) - m/2$. A classical result of Edwards establishes that $\text{sp}(G) \geq \frac{\sqrt{8m+1}-1}{8}$ for any graph on m edges, a bound that is tight for complete graphs of odd order. To improve this lower bound for graphs that are “far from complete”, Erdős and Lovász [3] initiated the study of the surplus in H -free graphs in the 1970s. A central conjecture in this area, posed by Alon, Bollobás, Krivelevich, and Sudakov [1], asserts that for any fixed graph H , every H -free graph G with m edges has a surplus of at least $m^{3/4+\varepsilon_H}$ for some $\varepsilon_H > 0$. Recently, a major breakthrough by Jin, Milojević, Tomon, and Zhang [7] made significant progress

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toward this conjecture by combining novel linear-algebraic techniques with methods from spectral graph theory and semidefinite programming.

The *vector chromatic number* of a graph G , denoted $\chi_{\text{vec}}(G)$, is the minimum real number $\kappa \geq 2$ for which there exists an assignment of unit vectors $\mathbf{x}_i$ to each vertex $i \in V$ such that $\langle \mathbf{x}_i, \mathbf{x}_j \rangle \leq -(\kappa - 1)^{-1}$ for all edges $ij \in E$. This formulation was introduced by Karger, Motwani, and Sudan [8] to design a semidefinite programming-based approximation algorithm for the graph coloring problem, showing that graphs that are vector κ -colorable can be colored by roughly $\Delta^{1-2/\kappa}$ colors, where Δ is the maximum degree in the graph and is assumed to be polynomial in the number of vertices. Feige, Langberg, and Schechtman [4] showed that their result is essentially optimal, by constructing a κ -vector colorable spherical random geometric graph with chromatic number at least $\Delta^{1-2/\kappa-\varepsilon}$ for every constant $\kappa > 2$ and every $\varepsilon > 0$. It is worth mentioning that the vector chromatic number is also known as the *Schrijver theta number* of the complement graph, denoted $\vartheta^-(\overline{G})$, a semidefinite programming formulation that first appeared 20 years before the vector chromatic number in a paper by McEliece, Rodemich, Rumsey, and Welch [10] and was independently introduced by Schrijver [11].

Connecting these concepts, a recent work by Balla, Janzer, and Sudakov [2] bounded the surplus of a graph in terms of its vector chromatic number, giving a unified approach and short proofs that recover known results on the MaxCut of H -free graphs.

Theorem 1 ([2, Theorem 1.7]). *Let G be a graph which has m edges. Then*

$$\text{sp}(G) \geq \frac{m}{\pi(\chi_{\text{vec}}(G) - 1)}.$$

Theorem 1 is proved by assigning an optimal vector coloring to the vertices of G and applying the classic random hyperplane rounding technique of Goemans and Williamson [6]. Specifically, given a unit vector embedding where $\langle \mathbf{x}_i, \mathbf{x}_j \rangle \leq -(\chi_{\text{vec}}(G) - 1)^{-1}$ for all edges $ij \in E$, a cut is formed by choosing a uniform random hyperplane through the origin. The lower bound then follows by computing the expected number of edges crossing this hyperplane.

So far, no graph is known to satisfy the inequality in Theorem 1 with the correct constant factor. For this reason, Elphick (personal communication with I. Balla) questioned whether the leading constant π in the denominator was tight. In this paper, we answer this question by constructing a family of graphs showing that the constant π is optimal. Consequently, our result implies that for certain graphs, a random hyperplane cut is essentially the best possible cut achievable from this vector embedding.

Theorem 2. *For every $\delta \in (0, 1)$, there is a graph G with $m \geq 1$ edges for which*

$$\text{sp}(G) \leq \frac{m}{(\pi - \delta)(\chi_{\text{vec}}(G) - 1)}.$$

To prove Theorem 2, we construct a geometric graph whose vertex set consists of points on the $(d - 1)$ -dimensional unit sphere $\mathcal{S}^{d-1} \subset \mathbb{R}^d$ and two vertices are adjacent if the angle between them is at least θ , for an appropriate choice of θ . This angular threshold

condition provides an immediate upper bound on the vector chromatic number. Moreover, in this graph the size of any hyperplane cut is very close to that of the maximum cut.

Our construction is directly inspired by the seminal work of Feige and Schechtman [5], who established the optimal integrality gap $\alpha_{\text{GW}} = \min_{\theta \in [\pi/2, \pi)} \frac{2\theta}{\pi(1 - \cos \theta)}$ for the Goemans–Williamson semidefinite relaxation of MaxCut. They use a spherical random geometric graph construction in which vertices are adjacent if the angle between them lies within a small interval around the angle θ^* which is the solution to the minimization problem of α_{GW} . This restriction was necessary to bound the objective value of the semidefinite program relaxation. Since we are only interested in bounding the surplus relative to the vector chromatic number, we simplify both the graph structure and its subsequent analysis.

2 Construction

Let $\mathcal{S}^{d-1}$ denote the unit sphere in $\mathbb{R}^d$. We measure distances between points on $\mathcal{S}^{d-1}$ by the angle between their respective unit vectors; that is, $d(\mathbf{x}, \mathbf{y}) = \arccos(\mathbf{x}^\top \mathbf{y})$. The parameters of our construction are chosen as follows. Fix $\theta \in (\pi/2, \pi)$ and let $\varepsilon \in (0, 1)$. The dimension d is chosen as the smallest integer such that the $(d-1)$ -dimensional volume of a spherical cap of radius $\pi - \theta - \varepsilon$ is an ε -fraction of the volume of a cap of radius $\pi - \theta$, which is achieved by taking $d \approx \log(1/\varepsilon)/(\varepsilon(\theta - \pi/2))$.

2.1 Continuous version

We define a continuous graph $G_c = G_c(\theta, \varepsilon)$ as follows. The vertices of G_c are all points of $\mathcal{S}^{d-1}$. The edges of G_c are pairs of vertices at distance at least θ . Denote by μ the normalized $(d-1)$ -dimensional natural measure on $\mathcal{S}^{d-1}$ and by μ^2 the induced product measure $\mu \times \mu$ on $\mathcal{S}^{d-1} \times \mathcal{S}^{d-1}$. Given a (measurable) subset A of $\mathcal{S}^{d-1}$ and θ between 0 and π , we also denote

$$\mu_\theta(A) = \mu^2(\{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in A, \mathbf{y} \notin A, d(\mathbf{x}, \mathbf{y}) \geq \theta\}).$$

The following lemma of Feige and Schechtman states which set A maximizes the measure of edges cut.

Lemma 3 ([5, Corollary 6]). *Fix θ between 0 and π . Then the maximum of $\mu_\theta(A)$, where A ranges over all (measurable) subsets of $\mathcal{S}^{d-1}$, is attained for any cap of measure $1/2$ (namely, a hemisphere).*

Therefore, we just need to estimate the measure of edges cut by a hyperplane. Since each hyperplane cuts the same measure of edges, for edges with endpoints at distance θ apart, a fraction of θ/π of them will be cut by any hyperplane. The following result can be extracted from [5, Section 3.1], but we give the full proof for completeness.

Lemma 4. *Let $G_c = G_c(\theta, \varepsilon)$ be defined as above. Denote by opt_c the measure of edges cut by a random hyperplane, divided by the measure of edges in G_c . Then,*

$$\text{opt}_c \leq \theta/\pi + 2\varepsilon.$$

Proof. Let E_1 be the set of edges of G_c whose endpoints have distance at most $\theta + \varepsilon$, and let E_2 be the remaining edges. Since the edges in E_1 make angles in the range $[\theta, \theta + \varepsilon]$, the probability that a random hyperplane cuts an edge in E_1 is at most $(\theta + \varepsilon)/\pi$. By the choice of d , the total measure of E_2 is at most an ε -fraction of the total measure of $E(G_c)$ and every edge in E_2 is cut with probability at most 1. Hence,

$$\text{opt}_c \leq \frac{\mu^2(E_1)}{\mu^2(E(G_c))} \left(\frac{\theta + \varepsilon}{\pi} \right) + \frac{\mu^2(E_2)}{\mu^2(E(G_c))} \leq \frac{\theta}{\pi} + 2\varepsilon. \quad \square$$

2.2 Discrete version

We choose a sufficiently small parameter $\gamma > 0$ (e.g., $\gamma < \varepsilon/(d|\cot(\theta)|) \approx \varepsilon/d$) such that the volume of a spherical cap of radius $\pi - \theta - \gamma$ is at least a $(1 - \varepsilon)$ -fraction of the volume of a cap of radius $\pi - \theta + \gamma$. The discrete graph $G = G(\theta, \varepsilon)$ is then constructed by partitioning $\mathcal{S}^{d-1}$ into $n = (O(1)/\gamma)^d$ equal-volume cells of diameter at most γ , following the approach in [5, Lemma 21]. The vertex set of G is formed by selecting an arbitrary point within each cell, and two vertices are connected by an edge if the distance between them is at least θ .

Lemma 5. *For every $\theta \in (\pi/2, \pi)$ and for every $\varepsilon \in (0, 1)$, the graph $G = G(\theta, \varepsilon)$ satisfies*

$$\chi_{\text{vec}}(G) \leq 1 - \frac{1}{\cos(\theta)}.$$

Proof. Recall that $\chi_{\text{vec}}(G)$ is the minimum $\kappa \geq 2$ for which there exists an assignment of unit vectors $\mathbf{y}_i$ to the vertices $i \in V(G)$ satisfying $\mathbf{y}_i^\top \mathbf{y}_j \leq -(\kappa - 1)^{-1}$ for all edges $ij \in E(G)$. By construction, each vertex $i \in V(G)$ corresponds to a point $\mathbf{x}_i \in \mathcal{S}^{d-1}$, and every edge $ij \in E(G)$ satisfies $\mathbf{x}_i^\top \mathbf{x}_j \leq \cos(\theta)$. By considering the identity embedding $\mathbf{y}_i = \mathbf{x}_i$ for all $i \in V(G)$ and setting $\kappa = 1 - 1/\cos(\theta)$, we obtain $\cos(\theta) = -(\kappa - 1)^{-1}$. It follows immediately that $\chi_{\text{vec}}(G) \leq \kappa$. $\square$

Lemma 6. *For every $\theta \in (\pi/2, \pi)$ and for every $\varepsilon \in (0, 1)$, the graph $G = G(\theta, \varepsilon)$ satisfies*

$$\text{mc}(G) \leq \left(\frac{\theta}{\pi} + 5\varepsilon \right) e(G).$$

Proof. Let $C \subseteq V$ be a set such that $e(C, V \setminus C) = \text{mc}(G)$. Color each vertex of C in red and each vertex in $V \setminus C$ in blue and color each of the n cells of the sphere $\mathcal{S}^{d-1}$ with the same color as the vertex on it (like a disco ball). Let R be the set of red points in $\mathcal{S}^{d-1}$.

Denote by μ_θ the measure of all edges in G_c and by $\mu_\theta(A, B)$ the measure of edges between cells A and B . We give an upper bound on $\text{mc}(G)/e(G)$ from the ratio $\mu_\theta(R)/\mu_\theta$.

Let $\mathcal{P}$ be the set of all pairs of cells (A, B) . We classify the pairs in $\mathcal{P}$ based on the distances between their points. Let $\mathcal{P}_1$ be the set of *near pairs*, where $d(x, y) < \theta$ for all $x \in A$ and $y \in B$. Let $\mathcal{P}_2$ be the set of *contributing pairs*, where $d(x, y) \geq \theta$ for all $x \in A$ and $y \in B$. Because every pair of points between these cells forms an edge, and each cell has a volume of $1/n$, every pair in $\mathcal{P}_2$ contributes exactly $1/n^2$ to the measure of edges.

Finally, let $\mathcal{P}_3$ be the set of *mixed pairs*, which contain some points at distance less than θ and others at distance at least θ .

A cell participates in mixed pairs only with cells entirely contained in the difference of two caps, one of radius $\pi - \theta + \gamma$ and the other of radius $\pi - \theta - \gamma$. As the volume of this region is an ε -fraction of the volume of a cap of radius $\pi - \theta$, the influence of mixed pairs on the total measure of edges is bounded by

$$\sum_{(A,B) \in \mathcal{P}_3} \mu_\theta(A, B) \leq \frac{|\mathcal{P}_3|}{n^2} \leq \varepsilon \mu_\theta.$$

Now we are ready to bound μ_θ and $\mu_\theta(R)$. Since the near pairs do not contribute to the measure of edges, we can upper bound μ_θ by

$$\mu_\theta = \sum_{(A,B) \in \mathcal{P}_2} \mu_\theta(A, B) + \sum_{(A,B) \in \mathcal{P}_3} \mu_\theta(A, B) \leq \sum_{(A,B) \in \mathcal{P}_2} \frac{1}{n^2} + \varepsilon \mu_\theta \leq \frac{e(G)}{n^2} + \varepsilon \mu_\theta.$$

Let $\mathcal{P}'_i \subseteq \mathcal{P}_i$ denote the set of pairs $(A, B) \in \mathcal{P}_i$ such that A and B have different colors. Even if every pair in $\mathcal{P}'_3$ contributes to the cut while having zero measure, we can lower bound $\mu_\theta(R)$ by

$$\mu_\theta(R) = \sum_{(A,B) \in \mathcal{P}'_2} \mu_\theta(A, B) + \sum_{(A,B) \in \mathcal{P}'_3} \mu_\theta(A, B) \geq \frac{|\mathcal{P}'_2|}{n^2} \geq \frac{\text{mc}(G) - |\mathcal{P}'_3|}{n^2} \geq \frac{\text{mc}(G)}{n^2} - \varepsilon \mu_\theta.$$

We conclude the result by applying Lemma 4:

$$\frac{\text{mc}(G)}{e(G)} \leq \frac{\mu_\theta(R) + \varepsilon \mu_\theta}{\mu_\theta - \varepsilon \mu_\theta} \leq \frac{\mu_\theta(R)}{\mu_\theta} + 3\varepsilon \leq \text{opt}_c + 3\varepsilon \leq \frac{\theta}{\pi} + 5\varepsilon. \quad \square$$

3 Proof of Theorem 2

Before proving Theorem 2, we need the following trigonometric estimate.

Lemma 7. *For every $\delta \in (0, 1)$, there exist constants $\theta \in (\pi/2, \pi)$ and $\varepsilon \in (0, 1)$ such that*

$$\frac{-\arcsin(\cos(\theta))}{\pi} + \varepsilon \leq \frac{-\cos(\theta)}{\pi - \delta}.$$

Proof. Let θ be such that $-\cos(\theta) = x = \delta/\pi$ and let $\varepsilon = \delta x/(2\pi^2)$. From the Taylor expansion of $\arcsin$, since $x > 0$, it follows that:

$$x \leq \arcsin(x) = x + \frac{x^3}{6} + \dots \leq x + x^3.$$

The result then follows by a straightforward computation. By substituting this bound and dropping the strictly negative terms, we have:

$$\left(1 - \frac{\delta}{\pi}\right) \arcsin(x) + (\pi - \delta)\varepsilon \leq x + x^3 - \frac{\delta}{2\pi}x = x + \frac{\delta^3}{\pi^3} - \frac{\delta^2}{2\pi^2} \leq x.$$

Dividing this inequality by $\pi - \delta$ and recalling that $\arcsin(x) = -\arcsin(\cos(\theta))$ gives the result. $\square$

Now, we are ready to show our main result.

Proof of Theorem 2. Let θ and ε be chosen as in Lemma 7, and let $G = G(\theta, \varepsilon/5)$. Applying Lemma 6 followed by the identity $\arccos(x) + \arcsin(x) = \pi/2$, we obtain the initial equality below. The subsequent bounds follow directly from Lemma 7 and Lemma 5, respectively:

$$\frac{\text{mc}(G)}{m} - \frac{1}{2} \leq \frac{\theta}{\pi} - \frac{1}{2} + \varepsilon = \frac{-\arcsin(\cos(\theta))}{\pi} + \varepsilon \leq \frac{-\cos(\theta)}{\pi - \delta} \leq \frac{1}{(\pi - \delta)(\chi_{\text{vec}}(G) - 1)}. \quad \square$$

4 Concluding Remarks

In this paper we show that for every $\delta \in (0, 1)$, there exists a graph G with m edges for which

$$\text{sp}(G) \leq \frac{m}{(\pi - \delta)(\chi_{\text{vec}}(G) - 1)}.$$

The Lovász theta function of the complement graph $\vartheta(\overline{G})$, also known as the strict vector chromatic number of G , is the minimum real number $\kappa \geq 2$ for which there exists a unit vector $\mathbf{x}_i$ for each vertex $i \in V$ such that $\mathbf{x}_i^\top \mathbf{x}_j = -(\kappa - 1)^{-1}$ holds whenever i and j are distinct vertices in G and $ij \in E$. From this definition, it follows that $\vartheta(\overline{G}) \geq \chi_{\text{vec}}(G)$. By combining this inequality with Theorem 1, Balla, Janzer, and Sudakov deduced the following theorem:

Theorem 8 ([2, Theorem 1.5]). *Let G be a graph with m edges and let $\vartheta = \vartheta(\overline{G})$ be the Lovász theta function of the complement graph $\overline{G}$. Then,*

$$\text{sp}(G) \geq \frac{m}{\pi(\vartheta - 1)}.$$

A natural open question is whether the graph constructed in Section 2.2 also shows that the constant π in Theorem 8 is tight. This would follow immediately if $\vartheta(\overline{G})$ and $\chi_{\text{vec}}(G)$ were arbitrarily close for $G(\theta, \varepsilon)$. A related question was raised by Feige, Langberg, and Schechtman [4], who asked whether their results for the vector chromatic number could be extended to the stronger relaxation given by the Lovász theta function. Showing that $\chi_{\text{vec}}(G)$ and $\vartheta(\overline{G})$ are close for their spherical random geometric graph construction would provide such an extension. We expect this to be the case as very few graphs are known for which these two parameters differ [11, 2].

Acknowledgements

I am grateful to my advisor, Gabriel Coutinho, for his insightful feedback and mentorship. I would like to thank Igor Balla for the fun and engaging discussions. I also thank the anonymous reviewer for their careful reading of this manuscript. This project was supported by FAPEMIG and CNPq.

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