The Frobenius-Mex Theorem and the Crank-Mex Theorem

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Abstract

The mex, as a relatively new partition statistic, has attracted great attention over the past five years and has led to a number of interesting developments. The Frobenius-Mex theorem and the Crank-Mex theorem stand out as two of the most significant results in this area. In this paper, we not only further refine these two theorems in distinct ways, but also provide bijective proofs for all of our refinements. Notably, our work answers Andrews and Newman's request for a bijection underlying their refinement of the Crank-Mex theorem.

Mathematics Subject Classifications: 05A17, 11P84

1 Introduction

A partition λ of n is a finite weakly decreasing sequence of positive integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$$

such that $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. We write $|\lambda| = n$ if λ is a partition of n. The terms λ_i are called the parts of λ, and the number of parts of λ is called the length of λ, denoted $\ell(\lambda)$. The conjugate of λ is defined by $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_t)$, where λ'_i is the number of parts of λ that are greater than or equal to i. We usually write λ in the following compact form

$$\lambda = (\lambda_1^{f_1}, \lambda_2^{f_2}, \ldots, \lambda_k^{f_k}),$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_k$, f_i is the frequency of λ_i in λ, and the superscript f_i may be omitted if it equals one.

The Ferrers graph of a partition λ is a left-justified array of n dots with λ_i dots in the i-th row. The Durfee square of λ is the largest square of dots in the upper left-hand

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corner of the Ferrers graph of λ. The size of the Durfee square of λ is the number of dots on one side, denoted $d(\lambda)$. The Durfee rectangle of λ is the largest rectangle of size $r \times (r+1)$ that fits inside the Ferrers graph of λ. We say that $r+1$ is the long side (i.e., horizontal length) of the Durfee rectangle, and r is the short side (i.e., vertical height). We call the short side the size of the Durfee rectangle of λ, denoted $r(\lambda)$. In fact, $r(\lambda)$ is the size of the Durfee square of the Ferrers graph obtained by removing the first column from that of λ. See Figure 1 for an illustration.

Figure 1: The Durfee square and Durfee rectangle.

Draw a line L along the main diagonal of the Durfee square of λ. Let a_i be the number of dots in the i-th row to the right of L, and let b_j be the number of dots in the j-th column below L. Clearly, $|\lambda| = d + \sum_{i=1}^{d} a_i + \sum_{j=1}^{d} b_j$, where d is short for $d(\lambda)$. It is obvious that $a_1 > a_2 > \cdots > a_d \geqslant 0$ and $b_1 > b_2 > \cdots > b_d \geqslant 0$. The two-rowed array

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_d \\ b_1 & b_2 & \cdots & b_d \end{pmatrix}$$

is called the Frobenius symbol of λ. For example, the Frobenius symbol of $(8, 6, 5, 4, 4, 2, 1)$ is

$$\begin{pmatrix} 7 & 4 & 2 & 0 \\ 6 & 4 & 2 & 1 \end{pmatrix}.$$

Note that only the last entry in each row may be zero; the remaining entries must be positive. Verification of the following result is straightforward.

Proposition 1. *The Frobenius symbol of a partition λ has 0 on its top row if and only if the column to the right of the Durfee square is shorter than the Durfee square side.*

In recent years, a notably simple statistic has emerged in the study of integer partitions, commonly known as the mex, which refers to the smallest positive integer that does not occur in a partition. We use $\mathrm{mex}(\lambda)$ to denote the mex of λ. For instance, if $\lambda = (10, 7, 6, 3^2)$ and $\pi = (8, 3, 2^2, 1)$, then $\mathrm{mex}(\lambda) = 1$ and $\mathrm{mex}(\pi) = 4$. The mex is a useful statistic, which has many fruitful applications. See [6, 10] for example.

The following result, established by Andrews [1], links the mex to Frobenius symbol, a relationship now referred to as the Frobenius-Mex theorem.

Theorem 2. *The number of partitions of n with even (resp., odd) mex equals the number of partitions of n whose Frobenius symbol has (resp., no) 0 on the top row.*

Kang, Li and Wang [10] refined the Frobenius-Mex theorem in the following manner.

Theorem 3. *The partitions of n into k parts and with even (resp., odd) mex are equidistributed with the partitions of n into k parts whose Frobenius symbol has (resp., no) 0 on the top row.*

It is surprising that there exists an elegant bridge connecting the mex and Dyson's crank. Before stating the details, let us assemble the essential notions and notation [3].

For a partition λ, let $\omega(\lambda)$ be the number of parts equal to 1, $\mu(\lambda)$ be the number of parts greater than $\omega(\lambda)$, and $g(\lambda)$ be the largest part. Then the crank of λ, denoted $\mathrm{crank}(\lambda)$, is defined by

$$\mathrm{crank}(\lambda) = \begin{cases} g(\lambda), & \text{if } \omega(\lambda) = 0; \\ \mu(\lambda) - \omega(\lambda), & \text{if } \omega(\lambda) > 0. \end{cases}$$

For instance, $\mathrm{crank}(7, 3^5, 2) = 7$ and $\mathrm{crank}(7, 5, 4^2, 3, 2, 1^4) = 2 - 4 = -2$.

The following result, now known as the Crank-Mex theorem, was established independently by Andrews and Newman [7], and by Hopkins and Sellers [9].

Theorem 4. *The number of partitions of n with even (resp., odd) mex equals the number of partitions of n with negative (resp., nonnegative) crank.*

Andrews and Newman recently [8] established a refinement of the Crank-Mex theorem.

Theorem 5. *The number of partitions of n with even (resp., odd) mex and k parts that are not ones is equal to the number of partitions of n with negative (resp., nonnegative) crank and k parts that are not ones.*

Although not apparent from the definition of the crank, it is true that there are as many partitions of n with negative crank as there are partitions of n with positive crank; see [3] for the reasoning of this fact. Hence, the Crank-Mex theorem can also be stated as follows.

Theorem 6. *The number of partitions of n with even (resp., odd) mex is the same as the number of partitions of n with positive (resp., nonpositive) crank.*

More recently, Lin and Liu [14] gave a refinement of the above theorem.

Theorem 7. *The number of partitions of n into k parts with even (resp., odd) mex equals the number of partitions of n with positive (resp., nonpositive) crank and largest part k.*

To elucidate the structural properties of the mex, Li, Wang and Xu [12] introduced a new statistic. Given a partition λ, we group the parts of λ, working from the largest to the smallest, into blocks each of which contains at most 2 distinct parts. If $g(\lambda) - 1$ appears in λ, then all the parts of size $g(\lambda)$ and $g(\lambda) - 1$ form a block containing two distinct parts; otherwise, all the parts of size $g(\lambda)$ inhabit a block with only one distinct part. Remove this block from λ and iterate this operation on the remaining parts. Continue this process until eventually every part of λ lies in a unique block. The weight of a block is defined to be 1 if it is a block which contains only 1's, and 2 otherwise. We define the block index of λ, denoted $b(\lambda)$, to be the sum of the weight of its blocks. For example, if

$$\lambda = (12^4, 9^2, 8^3, 7, 5^6, 4^3, 3, 2, 1^5),$$

then it has 6 blocks, namely $(12^4), (9^2, 8^3), (7), (5^6, 4^3), (3, 2)$ and (1^5). Thus,

$$b(\lambda) = 2 + 2 + 2 + 2 + 2 + 1 = 11.$$

Letting $\pi = (13^3, 12^2, 9^3, 7^4, 6, 4^2, 2^5, 1^6)$, then the blocks of π are $(13^3, 12^2)$, (9^3), $(7^4, 6)$, (4^2) and $(2^5, 1^6)$. So $b(\pi) = 2 + 2 + 2 + 2 + 2 = 10$.

We now reveal a relationship between the block index and mex. Given a partition λ, no part larger than $\mathrm{mex}(\lambda)$ will stay in the same block with a part less than $\mathrm{mex}(\lambda)$. Meanwhile, the parity of $b(\lambda)$ depends on whether the 1's form a block by themselves. If $\mathrm{mex}(\lambda) = 2k + 1$, then each of the integers $1, 2, \ldots, 2k$ must appear as a part in λ. Therefore, all the parts of size $2i - 1$ and $2i$ (for $1 \leqslant i \leqslant k$) form a block, which implies that $b(\lambda)$ is even. If $\mathrm{mex}(\lambda) = 2k$, then all the parts of size $2i-2$ and $2i-1$ (for $2 \leqslant i \leqslant k$) form a block, and the 1's form a block by themselves. Thus, $b(\lambda)$ is odd. We now conclude the following result.

Proposition 8. *For any partition λ, $\mathrm{mex}(\lambda)$ and $b(\lambda)$ have opposite parity.*

We now turn our attention to the number of blocks of λ, which is essentially identical to another statistic known as the 2-measure. Let $m(\lambda)$ denote the 2-measure of a partition λ, defined to be the length of the longest subsequence of parts in λ in which any two parts of the subsequence differ by at least 2. For instance, if $\lambda = (11, 8, 7^2, 4, 2^3, 1)$, then $m(\lambda) = 4$ with $(11, 8, 4, 2)$, $(11, 8, 4, 1)$, $(11, 7, 4, 2)$ or $(11, 7, 4, 1)$ as the corresponding maximal subsequence. See [2] for more information. Extracting the maximum element for each block of λ produces a longest subsequence in which any two integers differ by at least 2. This uniquely determined sequence is called the dominating sequence of λ, denoted $\Gamma(\lambda)$. In the running example, the blocks of λ are $(11), (8, 7^2), (4), (2^3, 1)$, so $\Gamma(\lambda) = (11, 8, 4, 2)$. It is easy to see that the length of $\Gamma(\lambda)$, i.e., the number of blocks of λ, is equal to $m(\lambda)$. Let us sum up this information as a proposition.

Proposition 9. *For any partition λ, the number of blocks of λ is $m(\lambda)$. Furthermore, we have $m(\lambda) = \lceil b(\lambda)/2 \rceil$.*

The 2-measure has an unexpected connection with the Durfee square. Andrews, Chern and Li [4], as well as Lin and Lin [13], independently discovered the following result.

Theorem 10. *The number of partitions of n into k parts and with 2-measure m equals the number of partitions of n into k parts and with Durfee square of size m.*

The main objective of this paper is to investigate the Frobenius-Mex theorem and Crank-Mex theorem by incorporating the 2-measure and Durfee square.

First, we unify Theorem 3 and Theorem 10 together as follows.

Theorem 11. *The set of partitions of n into k parts and with even (resp., odd) mex and 2-measure m is equidistributed with the set of partitions of n into k parts whose Frobenius symbol has (resp., no) 0 on the top row, and Durfee square has size m.*

Second, we provide a bijective proof of a recent alternative refinement of the Frobenius-Mex theorem due to Wang, Yang, and Yang [15].

Theorem 12. *The number of partitions of n with even mex, k parts greater than 1, and 2-measure m equals the number of partitions of n with 0 in the top row of the Frobenius symbol, $k+1$ parts greater than 1, and Durfee square of size m. Similarly, the number of partitions of n with odd mex, k parts, and 2-measure m equals the number of partitions with no 0 in the top row, largest part $k+1$, and Durfee square of size m.*

Remark 13. Note that the partition (1^n) is a partition with even mex, and with 0 on the top row of its Frobenius symbol, but with no parts greater than one. We make the convention that as a partition with 0 on the top row of its Frobenius symbol, (1^n) has one part greater than one, thereby ensuring that Theorem 12 holds for the case $k = 0$.

Third, we establish a refinement of Theorem 5, and construct a combinatorial proof in response to Andrews and Newman's request [8].

Theorem 14. *The number of partitions of n with even (resp., odd) mex, k parts that are not ones and 2-measure m is equal to the number of partitions of n with negative (resp., nonnegative) crank, k parts that are not ones and Durfee rectangle of size $m-1$ (resp., m).*

Remark 15. Konan [11] also gave a combinatorial proof of the Crank-Mex theorem, but his approach differs substantially from ours. Specifically, Konan's method applies only to the even-mex case of Theorem 5; see [5] for further details.

Finally, we further refine Theorem 7 as follows.

Theorem 16. *The number of partitions of n with even (resp., odd) mex, k parts and 2-measure m is the same as the number of partitions of n with positive (resp., nonpositive) crank, the largest part k and Durfee square of size m.*

The rest of the paper is organized as follows. In Section 2, we construct two bijections, the composition of which yields a combinatorial proof of Theorem 11. The goal of Section 3 is to prove Theorem 12 combinatorially. In Section 4, we give a bijective proof of Theorem 14, answering Andrews and Newman's request for a bijection to explain their refinement of the Crank-Mex theorem. Finally, Section 5 presents a combinatorial proof of Theorem 16.

2 Proof of Theorem 11

A gap-free partition is a partition which has 1 as its smallest part and any two adjacent parts differ by at most one. We always use $\mathbf{s} = (s_1, s_2, \ldots,)$ to denote a gap-free partition, and use $n_o(\lambda)$ to denote the number of odd parts in λ. Let $\mathcal{P}$ and $\mathcal{S}$ denote the set of partitions and gap-free partitions, respectively.

Theorem 17. *There exists a bijection φ from $\mathcal{P}$ to $\mathcal{S}$ such that $\ell(\lambda) = n_o(\mathbf{s})$, $b(\lambda) = g(\mathbf{s})$, $m(\lambda) = \lceil g(\mathbf{s})/2 \rceil$ and*

$$|\lambda| = \sum_{i \geqslant 1} \lceil s_i/2 \rceil,$$

where $\lambda \in \mathcal{P}$ and $\mathbf{s} = \varphi(\lambda) \in \mathcal{S}$.

Proof. Given a partition $\lambda \in \mathcal{P}$, we define an operation τ on each block of λ as follows.

- For a block of weight 2, say $(t^a, (t-1)^b)$ where $a \geqslant 1$ and $b \geqslant 0$, we change it to be $(t^{a-1}, (t-1)^{b+1})$. Namely, we change a part of size t to be a part of size $t-1$.
- For a block of weight 1, then it must be (1^a) where $a \geqslant 1$. We change it to be (1^{a-1}). Namely, we remove a part of size 1.

Let $\tau(\lambda)$ denote the produced new partition. Clearly, we have $|\tau(\lambda)| = |\lambda| - m(\lambda)$. Furthermore, $b(\tau(\lambda)) = b(\lambda) - 1$ if the last block of λ is $(2, 1^a)$ or (1); otherwise $b(\tau(\lambda)) = b(\lambda)$.

For example, if $\lambda = (12^5, 10^8, 9^3, 8^3, 7^6, 5, 4^4, 3^2, 2^5, 1)$, we group its parts into blocks

$$(12^5)(10^8, 9^3)(8^3, 7^6)(5, 4^4)(3^2, 2^5)(1),$$

and then change its blocks as follows

$$\begin{aligned}
(12^5) &\to (12^4, 11),\\
(10^8, 9^3) &\to (10^7, 9^4),\\
(8^3, 7^6) &\to (8^2, 7^7),\\
(5, 4^4) &\to (4^5),\\
(3^2, 2^5) &\to (3, 2^6),\\
(1) &\to \emptyset.
\end{aligned}$$

Thus, $\tau(\lambda) = (12^4, 11, 10^7, 9^4, 8^2, 7^7, 4^5, 3, 2^6)$ and $b(\tau(\lambda)) = 10$.

We now repeat this transformation until the partition is completely emptied. As a result, we produce a nonincreasing sequence $\mathbf{s} = (s_1, s_2, s_3 \ldots)$ where

$$s_1 = b(\lambda), s_2 = b(\tau(\lambda)), s_3 = b(\tau^2(\lambda)), \ldots.$$

Note that $\mathbf{s}$ is a gap-free partition, which is the desired $\varphi(\lambda)$.

To see the algorithm more clearly, we treat the case where $\lambda = (7, 6, 3, 2, 1^3)$, and illustrate the transformations as follows

$$\begin{aligned}
(7, 6, 3, 2, 1^3) &\xrightarrow{-5} (6^2, 2^2, 1^2)\\
&\xrightarrow{-4} (6, 5, 2, 1^3) \xrightarrow{-4} (5^2, 1^4)\\
&\xrightarrow{-3} (5, 4, 1^3) \xrightarrow{-3} (4^2, 1^2) \xrightarrow{-3} (4, 3, 1) \xrightarrow{-3} (3^2)\\
&\xrightarrow{-2} (3, 2) \xrightarrow{-2} (2^2) \xrightarrow{-2} (2, 1) \xrightarrow{-2} (1^2)\\
&\xrightarrow{-1} (1) \xrightarrow{-1} \emptyset,
\end{aligned}$$

where the symbol $\lambda \xrightarrow{-b(\lambda)}$ means that λ is reduced by $b(\lambda)$. Therefore,

$$\varphi(\lambda) = (5, 4, 4, 3, 3, 3, 3, 2, 2, 2, 2, 1, 1).$$

We next show that the procedure φ is invertible.

We consider another operation $\xrightarrow{+w}$ on λ, adding a weight w to λ where w is either $b(\lambda)$ or $b(\lambda)+1$. We define this operation by saying that $\lambda \xrightarrow{+w} \pi$ if π is obtained from λ in the following manner. We first group the parts of λ into blocks in a very similar manner to that in the definition of block index. But this time we start from the smallest part, and take into account the parity of w. More precisely, if w is even, we group the parts in the ordinary manner, and change each block $((t+1)^a, t^b)$ to $((t+1)^{a+1}, t^{b-1})$ where $a \geqslant 0$ and $b \geqslant 1$. If w is odd, we let all 1's inhabit a block regardless of whether 2 exists or not, and group the remaining parts in the ordinary manner. We change the block (1^a) to (1^{a+1}) where $a \geqslant 0$, and change the other blocks as above.

For instance, let $\lambda = (12^4, 11, 10^7, 9^4, 8^2, 7^7, 4^5, 3, 2^6)$. Clearly, $b(\lambda) = 10$. If $w = 10$, then we group the parts into blocks

$$(12^4, 11)(10^7, 9^4)(8^2, 7^7)(4^5)(3, 2^6),$$

and change the blocks as follows

$$\begin{aligned} (12^4, 11) &\to (12^5), \\ (10^7, 9^4) &\to (10^8, 9^3), \\ (8^2, 7^7) &\to (8^3, 7^6), \\ (4^5) &\to (5, 4^4), \\ (3, 2^6) &\to (3^2, 2^5). \end{aligned}$$

Thus, $\lambda \xrightarrow{+w} (12^5, 10^8, 9^3, 8^3, 7^6, 5, 4^4, 3^2, 2^5)$. If $w = 11$, then we group the parts in the same way but add an empty block (1^0) this time, and change the blocks as follows

$$\begin{aligned} (12^4, 11) &\to (12^5), \\ (10^7, 9^4) &\to (10^8, 9^3), \\ (8^2, 7^7) &\to (8^3, 7^6), \\ (4^5) &\to (5, 4^4), \\ (3, 2^6) &\to (3^2, 2^5), \\ (1^0) &\to (1). \end{aligned}$$

Thus, $\lambda \xrightarrow{+w} (12^5, 10^8, 9^3, 8^3, 7^6, 5, 4^4, 3^2, 2^5, 1)$.

Given a gap-free partition $\mathbf{s} = (s_1, s_2, \ldots,)$, we view each part of $\mathbf{s}$ as a weight, for which we perform the operation $\xrightarrow{+w}$ successively, starting from the empty partition and adding the weights from 1 to s_1. Finally, we obtain a partition λ. For example, taking

$$\mathbf{s} = (5, 4, 4, 3, 3, 3, 3, 2, 2, 2, 2, 1, 1),$$

then the process of adding weights is

$$\begin{aligned}
\emptyset &\xrightarrow{+1} (1) \xrightarrow{+1} (1^2)\\
&\xrightarrow{+2} (2,1) \xrightarrow{+2} (2^2) \xrightarrow{+2} (3,2) \xrightarrow{+2} (3^2)\\
&\xrightarrow{+3} (4,3)(1) \xrightarrow{+3} (4^2)(1^2) \xrightarrow{+3} (5,4)(1^3) \xrightarrow{+3} (5^2)(1^4)\\
&\xrightarrow{+4} (6,5)(2,1^3) \xrightarrow{+4} (6^2)(2^2,1^2)\\
&\xrightarrow{+5} (7,6)(3,2)(1^3).
\end{aligned}$$

Therefore, $\lambda = (7,6,3,2,1^3)$.

Finally, we analyze the behavior of the statistics under consideration.

We know from the definition of τ that $\ell(\tau(\lambda)) = \ell(\lambda) - 1$ if and only if $b(\lambda)$ is odd. Hence, the length of λ is equal to the number of times that $b(\tau^i(\lambda))$ is odd during the process of performing the operation τ, which equals the number of odd parts in $\mathbf{s}$. Therefore, $\ell(\lambda) = n_o(\mathbf{s})$. In addition, since $m(\lambda) = \lceil b(\lambda)/2 \rceil$ and $b(\lambda) = s_1 = g(\mathbf{s})$, we have $m(\lambda) = \lceil g(\mathbf{s})/2 \rceil$. Because $|\lambda| = |\tau(\lambda)| + m(\lambda) = |\tau(\lambda)| + \lceil s_1/2 \rceil$, iterating this relation yields that

$$|\lambda| = \lceil s_1/2 \rceil + \lceil s_2/2 \rceil + \cdots ,$$

which completes the proof. □

Remark 18. The bijection φ is adapted from the approach in [16], where Wang and Zhang established a rather intricate result.

Remark 19. The bijection φ can be used to prove Theorem 1.9 in [12], but requiring deeper and more sophisticated analysis. The reader may complete this task independently or consult the arguments presented in [16].

Theorem 20. *There exists a bijection ϕ from $\mathcal{S}$ to $\mathcal{P}$ such that $n_o(\mathbf{s}) = \ell(\pi)$, $\lceil g(\mathbf{s})/2 \rceil = d(\pi)$ and*

$$\sum_{i \geqslant 1} \lceil s_i/2 \rceil = |\pi|,$$

where $\mathbf{s} \in \mathcal{S}$ and $\pi = \phi(\mathbf{s})$. Moreover, $g(\mathbf{s})$ is odd if and only if the column to the right of the Durfee square of π is shorter than the Durfee square side.

Proof. For any partition $\mathbf{s} \in \mathcal{S}$, the image $\phi(\mathbf{s})$ is constructed stepwise through the successive addition of rows or columns to the Ferrers graph.

Assume that $\mathbf{s} = (s_1, s_2, \ldots, s_k)$. Start from the Ferrers graph of the empty partition. For i from k to 1, we append a row of length $(s_i + 1)/2$ to the Ferrers graph if s_i is odd, and a column of length $s_i/2$ otherwise. Let π be the partition represented by the resulting Ferrers graph.

For example, letting $\mathbf{s} = (5, 4^2, 3^4, 2^4, 1^2)$, then the process of producing $\phi(\mathbf{s})$ is depicted in Figure 2.

It is obvious that $\sum_{i \geqslant 1} \lceil s_i/2 \rceil = |\pi|$. Since we add a row only when we encounter an odd s_i, we have $n_o(\mathbf{s}) = \ell(\pi)$.

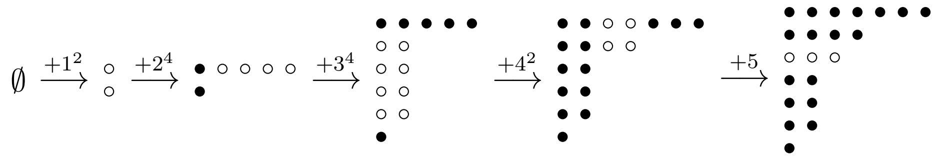


Figure 2: An illustration of ϕ.

We claim that when adding a row of length t, the Ferrers graph has a Durfee square of size t, and the column to the right of the Durfee square has length strictly less than t; when adding a column of length t, the Ferrers graph has a Durfee square of size t, and the column to the right of the Durfee square has length exactly t; This can be proved by induction on t. For the inductive step, observe that: if we add a row of length t, the next step will involve adding either another row of length t or a new column of length t; and if we add a column of length t, the next step will involve adding either another column of length t or a new row of length $t+1$.

We now conclude that if $g(\mathbf{s}) = 2m-1$, then $d(\pi) = m$ and the length of the column to the right of the Durfee square is less than m; if $g(\mathbf{s}) = 2m$, then $d(\pi) = m$ and the length of the column to the right of the Durfee square is equal to m.

Finally, we show that ϕ is invertible. Given a partition $\pi \in \mathcal{P}$, assume that $d(\pi) = m$. If the column to the right of the Durfee square has length less than m, then we remove a row of length m (since the last row of the Durfee square must have length m for this case) and output an odd integer $2m-1$; otherwise, we remove a column of length m and output an even integer $2m$. Repeat this operation until the Ferrers graph becomes empty. $\square$

Employing Propositions 8 and 1, the composition of the maps φ and ϕ provides a combinatorial proof of Theorem 11.

Remark 21. Combining the bijections φ and ϕ also leads to a bijective proof of Theorem 1.8 in [12]. We leave this as an exercise for the interested reader.

3 Proof of Theorem 12

For convenience, let $\ell_{\geqslant j}(\lambda)$ denote the number of parts of λ greater than or equal to j.

Given a partition λ with even mex, assume that $\ell(\lambda) = l$, $\ell_{\geqslant 2}(\lambda) = k$, and $m(\lambda) = m$. Clearly, λ must have 1 as its parts. Thus, $l > k$, and all 1's inhabit a block by themselves. Removing the first column from the Ferrers graph of λ, we get a new partition, denoted $\overline{\lambda}$. It is easy to see that $\mathrm{mex}(\overline{\lambda})$ is odd, $\ell(\overline{\lambda}) = \ell_{\geqslant 2}(\lambda) = k$, and $m(\overline{\lambda}) = m-1$. Applying the composition of φ and ϕ to $\overline{\lambda}$ produces a partition, say $\widetilde{\lambda}$. According to Theorem 11, we know that $\ell(\widetilde{\lambda}) = \ell(\overline{\lambda}) = k$, $d(\widetilde{\lambda}) = m(\overline{\lambda}) = m-1$, and the column to the right of the Durfee square of $\widetilde{\lambda}$ has length $m-1$. We remove this column and place it beneath the Durfee square of $\widetilde{\lambda}$, yielding a partition $\overline{\pi}$. Obviously, $\ell(\overline{\pi}) = \ell(\widetilde{\lambda}) + 1 = k+1$ and $d(\overline{\pi}) = d(\widetilde{\lambda}) = m-1$. Because $l > k$, we can append a column of length l to the left of the Ferrers graph of $\overline{\pi}$. Let π be the produced partition. It is clear that $|\pi| = |\lambda|$,

$\ell_{\geqslant 2}(\pi) = \ell(\overline{\pi}) = k+1$, $d(\pi) = d(\overline{\pi}) + 1 = m$, and the column to the right of the Durfee square of π has length less than m. Clearly, each step above is invertible. See Figure 3 for an example.

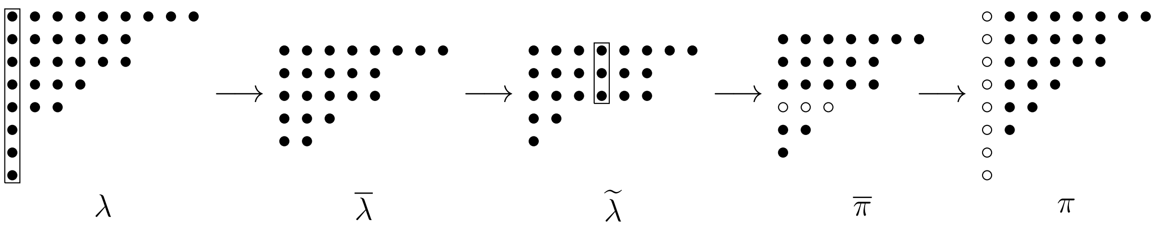


Figure 3: The case for even mex.

Remark 22. Note that the map from λ to π also preserves the length, i.e., $\ell(\lambda) = \ell(\pi)$. Thus, the identity for even case in Theorem 12 can be further refined as follows: the number of partitions of n into l parts with even mex, k parts greater than 1, and 2-measure m equals the number of partitions of n into l parts with 0 in the top row of the Frobenius symbol, $k+1$ parts greater than 1, and Durfee square of size m.

Given a partition λ with odd mex, assume that $\ell(\lambda) = k$ and $m(\lambda) = m$. We apply the composition of φ and ϕ to λ, producing a partition $\widetilde{\lambda}$ with $\ell(\widetilde{\lambda}) = \ell(\lambda) = k$, $d(\widetilde{\lambda}) = m(\lambda) = m$ and the column to the right of the Durfee square of $\widetilde{\lambda}$ has length m. Take the conjugate $\widetilde{\lambda}'$ of $\widetilde{\lambda}$. Clearly, $g(\widetilde{\lambda}') = \ell(\widetilde{\lambda}) = k$, $d(\widetilde{\lambda}') = d(\widetilde{\lambda}) = m$, and there exists a row of length m below the Durfee square. We remove this row and put it to the right of the Durfee square, producing a partition π. It is clear that $|\pi| = |\lambda|$, $d(\pi) = d(\widetilde{\lambda}') = m$, $g(\pi) = g(\widetilde{\lambda}') + 1 = k+1$, and the column to the right of the Durfee square of π has length m. Each step above is invertible. See Figure 4 for an illustration.

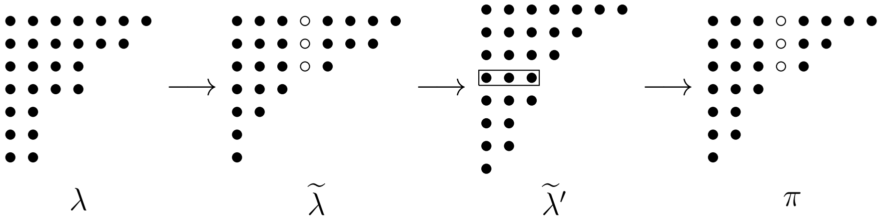


Figure 4: The case for odd mex.

4 Proof of Theorem 14

We begin by presenting a straightforward result, which appears as Lemma 5.1 in [15]. Interested readers are invited to supply their own proofs.

Lemma 23. *A partition λ has negative crank if and only if $\omega(\lambda) > r(\lambda)$.*

Theorem 24. *There is a bijection ψ from $\mathcal{P}$ to $\mathcal{P}$ such that $|\lambda| = |\pi|$ and $\ell_{\geqslant 2}(\lambda) = \ell_{\geqslant 2}(\pi)$ where $\lambda \in \mathcal{P}$ and $\pi = \psi(\lambda)$. Moreover,*

(1) *if $\mathrm{mex}(\lambda)$ is even and $m(\lambda) = m$, then $\mathrm{crank}(\pi) < 0$ and $r(\pi) = m - 1$;*

(2) *if $\mathrm{mex}(\lambda)$ is odd and $m(\lambda) = m$, then $\mathrm{crank}(\pi) \geqslant 0$ and $r(\pi) = m$.*

Proof. Given a partition $\lambda \in \mathcal{P}$, assume that $\ell(\lambda) = l$, $\ell_{\geqslant 2}(\lambda) = k$ and $m(\lambda) = m$.

We next perform operations on the Ferrers graph.

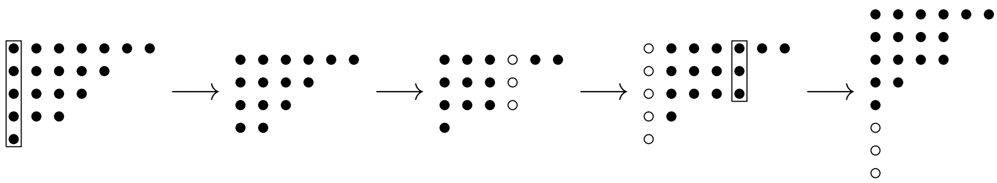
- Removing the first column from the Ferrers graph of λ yields a new partition, denoted $\overline{\lambda}$. Note that the length of the column removed is l, and $\ell(\overline{\lambda}) = \ell_{\geqslant 2}(\lambda) = k$.
- Applying the composition of φ and ϕ to $\overline{\lambda}$, we get a partition $\widetilde{\lambda}$ with $\ell(\widetilde{\lambda}) = \ell(\overline{\lambda}) = k$ and $d(\widetilde{\lambda}) = m(\overline{\lambda})$.
- Appending a column of length l to the left of the Ferrers graph of $\widetilde{\lambda}$, we obtain a new partition $\overline{\pi}$ with $\ell_{\geqslant 2}(\overline{\pi}) = \ell(\widetilde{\lambda}) = k$.

Note that each step above is invertible.

If $\mathrm{mex}(\lambda)$ is even, then λ must have 1 as its parts, and all 1's inhabit a block by themselves. Thus, $l > k$, $\mathrm{mex}(\overline{\lambda})$ is odd, and $m(\overline{\lambda}) = m - 1$. According to Theorem 11, $d(\widetilde{\lambda}) = m(\overline{\lambda}) = m - 1$ and the column to the right of the Durfee square of $\widetilde{\lambda}$ has length $m - 1$. Because $l > k$, a part of size 1 must appear in $\overline{\pi}$, and $\ell_{\geqslant 2}(\overline{\pi}) = \ell(\widetilde{\lambda}) = k$. It is clear that the Durfee rectangle of $\overline{\pi}$ has size $(m-1) \times m$, i.e., $r(\overline{\pi}) = m - 1$, and has a column of length $m - 1$ to its right. We change this column to $m - 1$ parts of size 1. Let π be the produced partition, which is the desired $\psi(\lambda)$.

In addition, we can easily see that $|\pi| = |\lambda|$, $r(\pi) = r(\overline{\pi}) = m - 1$, $\ell_{\geqslant 2}(\pi) = \ell_{\geqslant 2}(\overline{\pi}) = k$, and $\omega(\pi) \geqslant m$, implying $\mathrm{crank}(\pi) < 0$ due to Lemma 23.

See Figure 5 for an illustration.



λ $\overline{\lambda}$ $\widetilde{\lambda}$ $\overline{\pi}$ π

Figure 5: The case for $\omega(\pi) > r(\pi)$.

If $\mathrm{mex}(\lambda)$ is odd, then 2 must appear as a part of λ when 1 is already a part; otherwise we have $\mathrm{mex}(\lambda) = 2$, a contradiction. Thus, λ does not have the block of type (1^a), which implies that $\overline{\lambda}$ and λ have the same number of blocks, i.e., $m(\overline{\lambda}) = m(\lambda) = m$.

We next perform different operations according to the parity of $\operatorname{mex}(\overline{\lambda})$.

If $\operatorname{mex}(\overline{\lambda})$ is even, then the column to the right of the Durfee square of $\widetilde{\lambda}$ has length less than $d(\widetilde{\lambda}) = m(\overline{\lambda}) = m$ due to Theorem 11. We see that the Durfee rectangle of $\overline{\pi}$ has size $m \times (m+1)$, i.e., $r(\overline{\pi}) = m$, and has a column of length less than m to its right. Assume that $\omega(\overline{\pi}) = am + b$ where $0 \leqslant b < m$. We change am parts of size 1 to a columns of length m, and then add them to the right of the Durfee rectangle of $\overline{\pi}$. The yielded partition is the desired $\pi := \psi(\lambda)$.

It is routine to check that $|\pi| = |\lambda|$, $r(\pi) = r(\overline{\pi}) = m$, $\ell_{\geqslant 2}(\pi) = \ell_{\geqslant 2}(\overline{\pi}) = k$, and $\omega(\pi) < m$, implying $\operatorname{crank}(\pi) \geqslant 0$ due to Lemma 23.

See Figure 6 for an example.

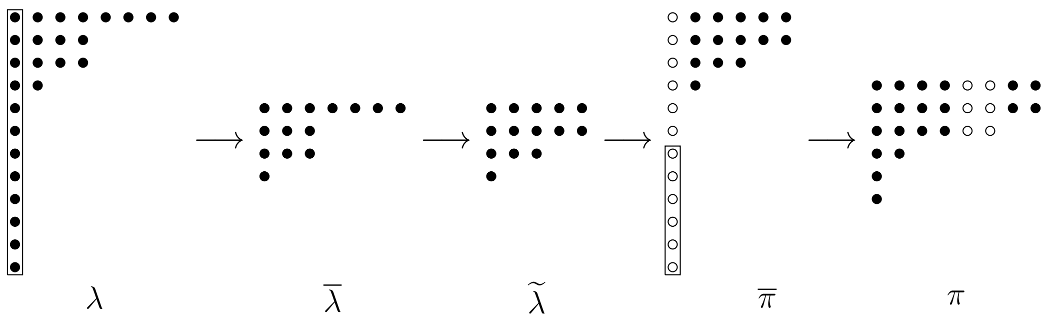


Figure 6: The case for $\omega(\pi) < r(\pi)$.

If $\operatorname{mex}(\overline{\lambda})$ is odd, the column to the right of the Durfee square of $\widetilde{\lambda}$ has length $d(\widetilde{\lambda}) = m(\overline{\lambda}) = m$ due to Theorem 11. We see that the Durfee rectangle of $\overline{\pi}$ has size $m \times (m+1)$, i.e., $r(\overline{\pi}) = m$, and has a column of length m to its right. We change this column to m parts of size 1. The produced partition is denoted π, which is the desired $\psi(\lambda)$.

It is easy to verify that $|\pi| = |\lambda|$, $r(\pi) = r(\overline{\pi}) = m$, $\ell_{\geqslant 2}(\pi) = \ell_{\geqslant 2}(\overline{\pi}) = k$. The condition that both $\operatorname{mex}(\lambda)$ and $\operatorname{mex}(\overline{\lambda})$ are odd implies that $\operatorname{mex}(\lambda) = 1$. Therefore, $\ell(\lambda) = \ell_{\geqslant 2}(\lambda)$, i.e., $l = k$, which tells us that there are no ones in $\overline{\pi}$. Thus, $\omega(\pi) = m$, implying $\operatorname{crank}(\pi) \geqslant 0$ due to Lemma 23.

See Figure 7 for an illustration.

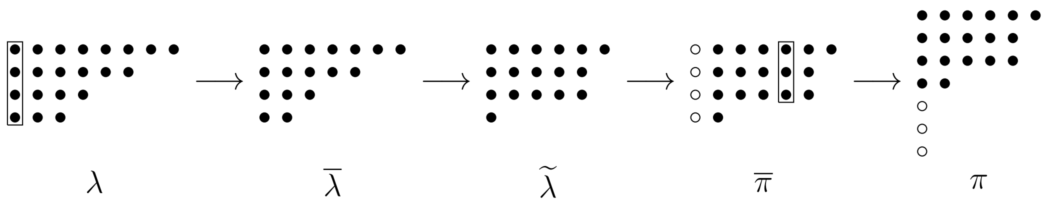


Figure 7: The case for $\omega(\pi) = r(\pi)$.

Finally, we show that the process from $\overline{\pi}$ to π is invertible. If $\omega(\pi) > r(\pi)$, then we perform the inverse operation for the case where $\operatorname{mex}(\lambda)$ is even. If $\omega(\pi) < r(\pi)$, then we perform the inverse operation for the case where $\operatorname{mex}(\lambda)$ is odd and $\operatorname{mex}(\overline{\lambda})$ is even. If $\omega(\pi) = r(\pi)$, then we perform the inverse operation for the case where both $\operatorname{mex}(\lambda)$ and $\operatorname{mex}(\overline{\lambda})$ are odd. $\square$

5 Proof of Theorem 16

We first present a result, which is Lemma 5.1 in [12].

Lemma 25. *A partition λ has nonpositive crank if and only if $\omega(\lambda) \geqslant d(\lambda)$.*

Theorem 26. *There exists a bijection ρ from $\mathcal{P}$ to $\mathcal{P}$ such that $|\lambda| = |\pi|$, $\ell(\lambda) = g(\pi)$, $d(\lambda) = d(\pi)$, and the Frobenius symbol of λ has 0 on its top row if and only if $\operatorname{crank}(\pi) > 0$, where $\pi = \rho(\lambda)$ and $\lambda \in \mathcal{P}$.*

Proof. For any partition λ, we operate on its conjugate λ'. Note that $d(\lambda) = d(\lambda')$, writing d for short. Assume that $\omega(\lambda') = pd + r$ where $p \geqslant 0$ and $0 \leqslant r < d$, and there are q parts of size d below the Durfee square of λ' where $q \geqslant 0$. We first change q parts of size d to be qd parts of size 1, and then change pd parts of size 1 to be p parts of size d. Let π be the yielded partition. Note that $\omega(\pi) = qd + r$ and $d(\pi) = d$.

If the Frobenius symbol of λ has 0 on its top row, then the column to the right of the Durfee square has length less than d. Thus, there are no rows of length d below the Durfee square of λ', i.e., $q = 0$, which implies that $\omega(\pi) = r < d = d(\pi)$. Therefore, $\operatorname{crank}(\pi) > 0$ according to Lemma 25.

If the Frobenius symbol of λ has no 0 on its top row, then the column to the right of the Durfee square has length d. Thus, there exists a row of length d below the Durfee square of λ', i.e., $q > 0$, which implies that $\omega(\pi) = qd + r \geqslant d = d(\pi)$. Thus, $\operatorname{crank}(\pi) \leqslant 0$ due to Lemma 25.

In fact, the map from λ' to π is an involution. Therefore, the process from λ to π is a bijection, which is the desired ρ.

Clearly, we have $|\lambda| = |\lambda'| = |\pi|$, $\ell(\lambda) = g(\lambda') = g(\pi)$, and $d(\lambda) = d(\lambda') = d(\pi)$. $\square$

Example 27. Letting $\lambda = (10, 6, 5, 2^2, 1)$ in which $\ell(\lambda) = 6$, $d(\lambda) = 3$, and the Frobenius symbol has no 0 on its top row, then $\pi = (6, 5, 3^2, 2, 1^7)$ with $g(\pi) = 6$, $d(\pi) = 3$, and $\operatorname{crank}(\pi) = -7$. See Figure 8 for the transformation.

The composition of the maps φ, ϕ and ρ gives a bijective proof of Theorem 16.

Remark 28. Theorem 16 is a refinement of Theorem 1.6 in [12], which is stated in terms of block index.

Remark 29. Since the number of parts of λ greater than or equal to i equals the i-th part of λ', and the map from λ' to π preserves the size of the first d parts, we have

$$\sum_{\lambda \in \mathcal{F}_{=0}} x^{d(\lambda)} y_1^{\ell_{\geqslant 1}(\lambda)} y_2^{\ell_{\geqslant 2}(\lambda)} \cdots y_{d(\lambda)}^{\ell_{\geqslant d(\lambda)}} q^{|\lambda|} = \sum_{\pi \in \mathcal{C}_{>0}} x^{d(\pi)} y_1^{\pi_1} y_2^{\pi_2} \cdots y_{d(\pi)}^{\pi_{d(\pi)}} q^{|\pi|},$$

$$\sum_{\lambda \in \mathcal{F}_{\neq 0}} x^{d(\lambda)} y_1^{\ell_{\geqslant 1}(\lambda)} y_2^{\ell_{\geqslant 2}(\lambda)} \cdots y_{d(\lambda)}^{\ell_{\geqslant d(\lambda)}} q^{|\lambda|} = \sum_{\pi \in \mathcal{C}_{\leqslant 0}} x^{d(\pi)} y_1^{\pi_1} y_2^{\pi_2} \cdots y_{d(\pi)}^{\pi_{d(\pi)}} q^{|\pi|},$$

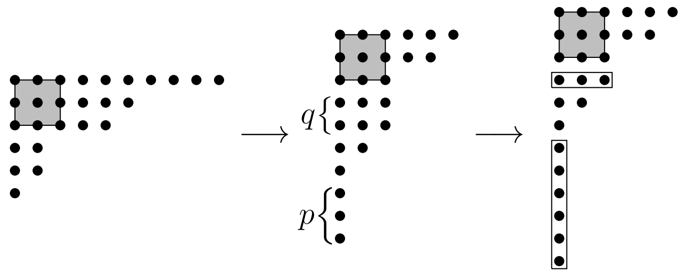


Figure 8: An illustration of ρ.

where $\mathcal{F}_{=0}$ (resp., $\mathcal{F}_{\neq 0}$) denotes the set of partitions whose Frobenius symbol has (resp., no) 0 on its top row, and $\mathcal{C}_{>0}$ (resp., $\mathcal{C}_{\leqslant 0}$) denotes the set of partitions with positive (resp., nonpositive) crank.

Acknowledgements

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