Permanents, Matchings, Symmetric Group Characters, and the *Problème des Ménages*

Paul Renteln

Submitted: Feb 8, 2026; Accepted: Aug 17, 2026; Published: Sep 25, 2026

Abstract

We explain some observed patterns in the Frobenius characteristics of a certain permanent related to the *ménage* numbers, and thereby provide a natural algebraic explanation for a classical number sequence of Moreau, first published by Lucas in 1891 in connection with the *problème des ménages*.

Mathematics Subject Classifications: 05A15, 05E05, 05C70, 20C30

1 Introduction

Let $\mathfrak{S}_n$ be the symmetric group on an n element set, and let d_n denote the number of derangements on $\mathfrak{S}_n$, namely the permutations without fixed points. Let J_n (respectively, I_n) denote the all-ones matrix of size n (respectively, the identity matrix of size n), and write

$$K_n := J_n - I_n.$$

Recall that the permanent ("the determinant without the signs") of an $n \times n$ matrix M is given by

$$\operatorname{per} M = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} M_{i,\sigma(i)}.$$

It follows directly from the definition that

$$d_n = \operatorname{per} K_n.$$

The problem of counting derangements belongs to the vast subject of enumerating permutations with restricted positions. [1] Thus, d_n enumerates all permutations σ of $\mathfrak{S}_n$ such that $\sigma(i) \neq i$. Many of the problems arising in this area can be expressed in terms of permanents of various 0-1 matrices.

Department of Physics, California State University, San Bernardino, CA 92407 (prenteln@csusb.edu).

[1] See, *e.g.*, [20], or the references in [7].

 https://doi.org/10.37236/15190

For instance, the famous *problème des ménages* of Lucas [2] [12] asks for the number m_n of permutations $\sigma \in \mathfrak{S}_n$ such that $\sigma(i) \neq i$ and $\sigma(i) \neq i+1 \bmod n$. [3] We refer to the m_n as *ménage* numbers [4], and by definition of the permanent they are given by

$$m_n = \operatorname{per}(K_n - Z_n), \tag{1.1}$$

where Z_n denotes the cyclic matrix with entries $(Z_n)_{ij} = \delta_{j,i+1 \bmod n}$, where δ_{ij} is the usual Kronecker delta.

Although these permanental representations are concise, they are by no means convenient. Computing with permanents is notoriously difficult, even for 0-1 matrices [25]. Historically, the problem of derangements was solved well before the advent of permanents by Monmort [16], who introduced the powerful tool of inclusion-exclusion. The *problème des ménages* took considerably longer to solve; the first closed formula was only obtained by Touchard in 1934 [24]. The *ménages* problem has a long and storied history. [5]

Inspired by the permanental expressions above, we could consider the following quantity:

$$F(\tau) := \operatorname{per}(K_n - P_\tau), \tag{1.2}$$

where $\tau \in \mathfrak{S}_n$ is now an arbitrary permutation, and P_τ is its associated permutation matrix. More generally, one could ask for

$$\operatorname{per}(K_n - P_{\tau_1} - P_{\tau_2} - \cdots).$$

Problems of this kind have been considered previously, most notably by Shevelev, who calls them 'generalized Touchard problems'. [6]

The present work originated from an attempt to explain a curious observation regarding (1.2). From the definition, it is easy to see that the permanent of a matrix M is invariant under arbitrary (independent) permutations of its rows and columns. In particular, if σ is an arbitrary permutation and P_σ its associated permutation matrix,

$$\operatorname{per} P_\sigma M P_\sigma^T = \operatorname{per} M.$$

As $JP_\sigma = P_\sigma J = J$ and $P_\sigma P_\sigma^T = I$, we see that

$$F(\tau) = F(\sigma\tau\sigma^{-1}),$$

[2] In its original formulation, this was expressed by Lucas as the number of ways to seat n couples around a circular table, men and women seated alternately, so that no man sits next to his wife. In fact, the problem was considered earlier by Tait in his attempt to enumerate various types of knots, and recursive formulae were obtained Cayley and Muir.

[3] We imagine seating the n women clockwise in alternate seats around the table, and label their positions from 1 to n. Then assign the number of woman i to both her spouse and to the seat to the right of her.

[4] Technically, these are the *reduced ménage* numbers, where the actual *ménage* numbers are $2n! \cdot m_n$, as there are $2n!$ ways to first seat the women.

[5] For an extensive list of references, see [18].

[6] For a survey, see [21]. For an approach to permanents using zeons, see [4] and [8].

That is, F is a class function on the symmetric group. As such, it admits an expansion in terms of the irreducible characters χ^λ of $\mathfrak{S}_n$:

$$F = \sum_{\lambda \vdash n} a_\lambda \chi^\lambda \tag{1.3}$$

where the sum is over all partitions λ of n. Our goal is to investigate the numbers a_λ.

Using the standard inner product on class functions on $\mathfrak{S}_n$, namely

$$(\varphi, \psi)_{CF} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \varphi(\sigma)\psi(\sigma) \tag{1.4}$$

we can write

$$a_\lambda = (F, \chi^\lambda)_{CF}. \tag{1.5}$$

Rather than compute this directly, however, we will instead compute the Frobenius characteristic of F. [7] Recall that the order of the centralizer of a permutation with cycle type $\mu = 1^{i_1} 2^{i_2} \cdots n^{i_n}$ (written in multiplicity notation) is

$$z_\mu = 1^{i_1} i_1! 2^{i_2} i_2! \cdots n^{i_n} i_n!.$$

We write $\psi(\sigma)$ for the cycle type of σ. If $\psi(\sigma) = \mu$ and f is a class function, we typically denote $f(\sigma)$ by f_μ. For any partition $\mu \vdash n$, $p_\mu := p_1^{i_1} p_2^{i_2} \cdots p_n^{i_n}$ is the power sum symmetric function, where $p_k := \sum_i x_i^k$. The *Frobenius characteristic* of a class function f on $\mathfrak{S}_n$ is

$$\operatorname{ch} f = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f(\sigma) p_{\psi(\sigma)} = \sum_\mu z_\mu^{-1} f_\mu p_\mu. \tag{1.6}$$

The ring Λ of symmetric functions admits a natural (Hall) inner product in which the Schur functions s_λ constitute an orthonormal basis:

$$(s_\mu, s_\nu)_\Lambda = \delta_{\mu\nu}$$

where $\delta_{\mu\nu}$ is the Kronecker delta. The Frobenius map is an isometry between the space of class functions equipped with the standard inner product (1.4) and the space Λ of symmetric functions equipped with the Hall inner product:

$$(f, g)_{CF} = (\operatorname{ch} f, \operatorname{ch} g)_\Lambda. \tag{1.7}$$

According to the Murnaghan–Nakayama rule ([23], Cor. 7.17.5)

$$s_\lambda = \sum_\mu z_\mu^{-1} \chi_\mu^\lambda p_\mu, \tag{1.8}$$

from which we obtain

$$\operatorname{ch} \chi^\lambda = s_\lambda. \tag{1.9}$$

[7] For an overview of symmetric function theory, see [13] or [23].

Therefore, by (1.5) and (1.7),

$$a_\lambda = (F, \chi^\lambda)_{CF} = (\operatorname{ch} F, s_\lambda)_\Lambda. \tag{1.10}$$

With those preliminaries out of the way, we now present in Table 1 the results of a computation using Stembridge's package SF for MAPLE. The data in Table 1 exhibit striking regularities, which are collected together in our main result, Theorem 1.

n	$\operatorname{ch} F$
1	$-s_1$
2	$s_2 + s_{1,1}$
3	$-s_{2,1}$
4	$3\,s_4 + s_{3,1} + s_{2,2}$
5	$13\,s_5 - s_{3,2}$
6	$83\,s_6 + 3\,s_{5,1} + s_{4,2} + s_{3,3}$
7	$592\,s_7 + 13\,s_{6,1} - s_{4,3}$
8	$4821\,s_8 + 83\,s_{7,1} + 3\,s_{6,2} + s_{5,3} + s_{4,4}$
9	$43979\,s_9 + 592\,s_{8,1} + 13\,s_{7,2} - s_{5,4}$
10	$444613\,s_{10} + 4821\,s_{9,1} + 83\,s_{8,2} + 3\,s_{7,3} + s_{6,4} + s_{5,5}$
11	$4934720\,s_{11} + 43979\,s_{10,1} + 592\,s_{9,2} + 13\,s_{8,3} - s_{6,5}$
12	$59661255\,s_{12} + 444613\,s_{11,1} + 4821\,s_{10,2} + 83\,s_{9,3} + 3\,s_{8,4} + s_{7,5} + s_{6,6}$

Table 1: Frobenius characteristics of $F(\tau) = \operatorname{per}(K_n - P_\tau)$ for $\mathfrak{S}_n$

Theorem 1. *Let $K_n := J_n - I_n$. For any permutation $\tau \in \mathfrak{S}_n$, define $F(\tau) := \operatorname{per}(K_n - P_\tau)$. Define $a_\lambda := (\operatorname{ch} F, s_\lambda)$, where $\operatorname{ch} F$ is the Frobenius characteristic of F and s_λ is the Schur function indexed by the partition $\lambda \vdash n$. Set $a_n := a_{(n)}$. Then the following hold:*

(a) *For $n \geqslant 2$,*

$$a_n = n a_{n-1} + a_{n-2} + 2(-1)^n, \tag{1.11}$$

with initial conditions $a_1 = -1$ and $a_2 = 1$. In particular, we have $a_n = \theta_{n+1}$, where $\{\theta_n\}$ is a sequence first discovered by M. Moreau ([12], p. 495).

(b) *The coefficients a_λ vanish if λ has more than two parts.*

(c) *For $i - 1, j \geqslant 0$, $a_{(i+1,j+1)} = a_{(i,j)}$, where $a_{(i,0)} := a_i$.*

(d) *Define $b_k(\tau)$ by*

$$\sum_k b_k(\tau) z^k = (1 - z) \prod_{i=1}^{n} (1 + z^i)^{c_i(\tau)},$$

where $c_i(\tau)$ is the number of cycles of length i in τ. Then

$$F(\tau) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n-2k} b_k(\tau). \tag{1.12}$$

In particular, if τ is a single n-cycle (with $n \geqslant 2$) then $F(\tau) = m_n = a_n - a_{n-2}$.

In Sections 2, 3, and 4 we assemble the various tools we shall require in what follows. Then, in Sections 5, 6, and 7 we use these tools to prove Theorem 1.

2 The matchings polynomial approach to computing permanents

As mentioned in the introduction, direct computation with permanents is tricky. Although it is not too difficult to obtain the recurrence relation for the derangement numbers d_n using the Laplace expansion of per K_n, carrying out the same process for anything more involved rapidly leads to unpleasant complexities.

Instead, we will employ the beautiful method expounded by Godsil in his excellent monograph [6]. An *r-matching* in a n-vertex graph G is a set of r edges, no two of which have a vertex in common. The number of r-matchings is $p(G, r)$. Set $p(G, 0) = 1$ and define the *matchings polynomial* of G by

$$\mu(G, x) = \sum_{r \geqslant 0} (-1)^r p(G, r) x^{n-2r}. \tag{2.1}$$

We will require a few basic examples of matchings polynomials. It follows immediately from the definition that, if G is the empty graph on n vertices, $\mu(G, x) = x^n$. For other graphs, the following simple proposition is very useful.

Proposition 2. *(e.g., [6], Thm 1.1) The matchings polynomial satisfies the following identities:*

(i) $\mu(G \cup H, x) = \mu(G, x)\mu(H, x)$.

(ii) $\mu(G, x) = \mu(G \backslash e, x) - \mu(G \backslash uv, x)$ *if* $e = \{u, v\}$ *is an edge of* G.

Proof. (Sketch)

(i) Every r-matching in $G \cup H$ is a pair consisting of an s-matching in G and an $r - s$ matching in H, so

$$p(G \cup H, r) = \sum_{s=0}^{r} p(G, s) p(H, r - s).$$

Now plug into (2.1).

(ii) Fix an edge $e = \{u, v\} \in G$. The r-matchings of G either contain e or not. Those that do are counted by $p(G\backslash uv, r-1)$; those that do not are counted by $p(G, r) = p(G\backslash e, r)$. Hence,

$$p(G, r) = p(G\backslash e, r) + p(G\backslash uv, r-1). \qquad \square$$

Again, substitute into (2.1).

Let P_n be the path graph on n vertices and let e be the first edge in the path. From the proof of Proposition 2 we have

$$p(P_n, r) = p(P_{n-1}, r) + p(P_{n-2}, r-1).$$

As

$$p(P_0, r) = \begin{cases} 1, & \text{if } r = 0, \text{ and} \\ 0, & \text{otherwise,} \end{cases}$$

the unique solution to the recurrence is[8]

$$p(P_n, r) = \binom{n-r}{r}.$$

Hence,

$$\mu(P_n, x) = \sum_{r \geqslant 0} (-1)^r \binom{n-r}{r} x^{n-2r}. \tag{2.2}$$

From $p(P_n, r)$ we can obtain $p(C_n, r)$, where C_n is the cycle graph on n vertices. Another application of the proof of Proposition 2 gives[9]

$$p(C_n, r) = p(P_n, r) + p(P_{n-2}, r-1) = \binom{n-r}{r} + \binom{n-r-1}{r-1} = \frac{n}{n-r}\binom{n-r}{r}.$$

Thus,

$$\mu(C_n, x) = \sum_{r \geqslant 0} (-1)^r \frac{n}{n-r} \binom{n-r}{r} x^{n-2r}. \tag{2.3}$$

Note that (2.3) remains valid even at $n = 1$ and $n = 2$, provided we interpret C_1 as a single vertex, and C_2 as a *digon* (two vertices connected by two edges). In the former case, we simply get $\mu(C_1, x) = x$, whereas in the latter case we have $\mu(C_2, x) = x^2 - 2$, directly from the definition (2.1), and both agree with the expansion on the right hand side of (2.3).

[8] Godsil gives the following combinatorial argument instead: lay out the path left to right. Contract each edge in a given r-matching onto its left endpoint. This gives a path on $n - r$ vertices with r of them distinguished. Conversely, given a path on $n - r$ vertices with r distinguished edges, we can reconstruct an r-matching on P_n.

[9] A direct combinatorial proof is also possible (*e.g.*, [22], Lemma 2.3.4).

The theory of matchings polynomials is intimately connected to the theory of rook polynomials. [10]. Let G be a spanning subgraph of the complete bipartite graph $K_{n,n}$. The *rook polynomial* of G is [11]

$$\rho(G,x) := \sum_{r=0}^{n} (-1)^r p(G,r) x^{n-r}. \tag{2.4}$$

Evidently,

$$\rho(G,x^2) = \mu(G,x). \tag{2.5}$$

If G is a spanning subgraph of $K_{n,n}$, then its *bipartite complement* $\widetilde{G}$ is the graph obtained from $K_{n,n}$ by removing the edges of G. Write $\mathrm{pm}(G)$ for the number of perfect matchings of G. This brings us to the following important result. [12]

Proposition 3 (Godsil [6], Thm. 3.1)**.** *Let G be a spanning subgraph of $K_{n,n}$. Then*

$$\mathrm{pm}(\widetilde{G}) = \int_0^\infty \rho(G,x)\, e^{-x}\, dx \tag{2.6}$$

Godsil leaves the proof of this result as an exercise, so we offer one here, mimicking his proof of ([6], Thm. 2.2). In fact, the proof yields something slightly more general, as (2.6) holds when G contains digons. In that case $\widetilde{G}$ is not a simple graph, but rather can be viewed as an edge weighted graph in which some edges have weight -1. (See Remark 4 below.)

Proof. We proceed by induction. Label the integral by $I(G)$. If G is the empty graph on $2n$ vertices, then $\widetilde{G} = K_{n,n}$, and the number $\mathrm{pm}(\widetilde{G})$ of perfect matchings of $\widetilde{G}$ equals the number of permutations on n elements, namely $n!$. As $\mu(G,x) = x^{2n}$ we have $\rho(G,x) = x^n$, so as $\int_0^\infty x^n\, e^{-x}\, dx = n!$, the theorem holds. (This requires that we set $\mathrm{pm}(\varnothing) = 1$.)

Now assume that G has at least one edge $e = \{u,v\}$, and suppose the theorem holds for any subgraph of G. Then by (ii) of Proposition 2 combined with (2.4) and (2.5),

$$I(G) = I(G\backslash e) - I(G\backslash uv),$$

so, by the inductive hypothesis,

$$I(G) = \mathrm{pm}(\widetilde{G\backslash e}) - \mathrm{pm}(\widetilde{G\backslash uv}).$$

[10] As we use only the terminology of rook theory and nothing more, we merely refer the interested reader to [20, 22]

[11] This is Godsil's definition. Riordan calls these 'associated rook polynomials'.

[12] A special case of this formula appears as Problem 18 in Chapter 8 of [20]. Hints of the formula can be found in [24] and [26]. In this generality, it seems to have appeared first in [7], but it was Godsil who recognized its effectiveness in the theory of permutations with restricted positions. For possible generalizations, see [1].

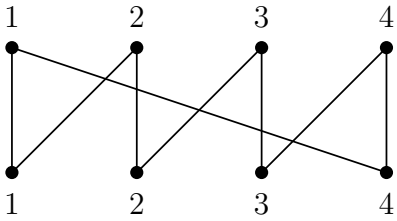


Figure 1: The *ménage* number m_4 is the number of perfect matchings in the complement of C_8 in $K_{4,4}$.

On the other hand, conditioning on the presence or absence of the edge e in $\widetilde{G\backslash e} = \widetilde{G} \cup e$, we obtain

$$\mathrm{pm}(\widetilde{G\backslash e}) = \mathrm{pm}(\widetilde{G}\backslash uv) + \mathrm{pm}(\widetilde{G}).$$

But $\widetilde{G\backslash uv} = \widetilde{G}\backslash uv$, as any edges incident with u and v in G are absent from both graphs (both are subgraphs of $K_{n-1,n-1}$). The theorem follows. □

In the *ménages* problem, we wish to compute the number of permutations σ satisfying $\sigma(i) \neq i, i+1 \bmod n$. This is equivalent to asking for the number of perfect matchings in the bipartite complement of the cycle C_{2n} (see Figure 2). This can also be seen from (1.1), by viewing the matrix $K_n - Z_n$ as a bipartite adjacency matrix, where the rows of $K_n - Z_n$ label the vertices on the bottom of Figure 2 and the columns of $K_n - Z_n$ label the vertices on the top. Thus, applying Proposition 3 and using (2.3) and (2.5), Godsil obtains ([6], p. 9) the familiar result

$$m_n = \int_0^\infty \mu(C_{2n}, x^{1/2})\, e^{-x}\, dx = \sum_{r \geqslant 0} (-1)^r \frac{2n}{2n-r} \binom{2n-r}{r} (n-r)!. \tag{2.7}$$

The second formula was first obtained by Touchard [24], after many earlier attempts by others such as Cayley, Muir, and Lucas. [13]

Remark 4. *As noted in the discussion following* (2.3)*, the second formula in* (2.7) *remains valid even at $n = 1$ (both sides equal -1). We can see this as well from the proof of Proposition 3. If G were a digon, then the right hand side of*

$$\mathrm{pm}(\widetilde{G}) = \mathrm{pm}(\widetilde{G\backslash e}) - \mathrm{pm}(\widetilde{G\backslash uv})$$

would read $0 - 1$, giving $m_1 = \mathrm{pm}(\widetilde{G}) = -1$ as before. Hence, as promised, Proposition 3 holds when G contains digons, provided we interpret $\widetilde{G}$ as a kind of generalized bipartite complement, in which removing two edges from a single edge leaves a single negatively weighted edge.

[13] See also [9], [10], and the derivations in [3], [20] and [22]. Most of these derivations require subtle combinatorial reasoning, whereas Godsil's elegant approach supplies the answer with ease.

At this point we recall the well-known and oft-observed fact that the matchings polynomial for a cycle is essentially the Chebyshev polynomial of the first kind (*cf.*, [20], p. 223)

$$\mu(C_n, 2x) = 2T_n(x). \tag{2.8}$$

Thus, setting

$$t_n(x) := \mu(C_{2n}, x^{1/2}) = 2T_{2n}(\sqrt{x}/2), \tag{2.9}$$

we find, using (2.7), that

$$m_n = \int_0^\infty t_n\, e^{-x}\, dx. \tag{2.10}$$

Remarkably, (2.10) admits a very neat generalization, providing us with a powerful tool to aid us in our proof of Theorem 1. [14]

Theorem 5. *Suppose that $\tau \in \mathfrak{S}_n$ has cycle type $\mu = 1^{i_1}2^{i_2}\cdots n^{i_n}$, and let $F_\mu := F(\tau) = \mathrm{per}(K_n - P_\tau)$. Define*

$$t_\mu := t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}. \tag{2.11}$$

Then

$$F_\mu = \int_0^\infty t_\mu\, e^{-x}\, dx. \tag{2.12}$$

Proof. Computing $\mathrm{per}(K_n - P_\tau)$ is equivalent to computing the number of perfect matchings in the (generalized) bipartite complement of

$$C_2^{i_1} \cup C_4^{i_2} \cup \cdots \cup C_{2n}^{i_n},$$

where all the unions are disjoint, and $C_{2k}^{i_k}$ denotes i_k disjoint copies of C_{2k}. By (i) of Proposition 2, the matchings polynomial of the disjoint union is simply the product of the matchings polynomials of all its constituents. The theorem now follows by combining this observation with (2.5), (2.6), (2.8), and (2.9). □

3 The coefficients a_λ

Using (1.5) and (2.12) we obtain

$$a_\lambda = (F, \chi^\lambda) = \sum_\mu z_\mu^{-1} \chi_\mu^\lambda\, F_\mu = \int_0^\infty \left(\sum_\mu z_\mu^{-1} \chi_\mu^\lambda\, t_\mu \right) e^{-x}\, dx.$$

Comparing this to the Murnaghan-Nakayama rule (1.8)

$$s_\lambda = \sum_\mu z_\mu^{-1} \chi_\mu^\lambda\, p_\mu$$

[14] A closely related formula appears at the end of Touchard's paper [24]. For another application, see [15].

we see that

$$a_\lambda = \int_0^\infty S_\lambda e^{-x}\,dx, \tag{3.1}$$

where we have defined

$$S_\lambda := s_\lambda\big|_{p_i=t_i}. \tag{3.2}$$

By the Jacobi–Trudi identity ([23], Thm 7.16.1)

$$s_\lambda = \det h_{\lambda_i-i+j}\big|_{i,j=1}^n,$$

where h_m is the complete homogenous symmetric function. As

$$s_m := s_{(m)} = h_m$$

we may also write

$$s_\lambda = \det s_{\lambda_i-i+j}\big|_{i,j=1}^n,$$

It follows immediately that

$$S_\lambda = \det S_{\lambda_i-i+j}\big|_{i,j=1}^n. \tag{3.3}$$

In particular, computation of S_λ, and thus of a_λ, reduces to knowledge of the S_m.

4 The polynomials S_m

In this section we obtain various formulae for the S_m. To begin, we need some facts about Chebyshev polynomials. Define the *shifted Chebyshev polynomials*

$$T_m^*(x) := T_m(2x-1).$$

These obey the recurrence ([14], Eq. 1.23)

$$T_m^*(x) = 2(2x-1)T_{m-1}^*(x) - T_{m-2}^*(x) \tag{4.1}$$

with initial conditions

$$T_0^*(x) = 1 \quad \text{and} \quad T_1^*(x) = 2x-1. \tag{4.2}$$

Moreover, we have ([14], Eq. 1.24)

$$T_m^*(x) = T_{2m}(\sqrt{x})$$

so that by (2.9),

$$t_m(x) = 2T_m^*(x/4). \tag{4.3}$$

Combining (4.1) with (4.3) yields

$$t_m(x) = (x-2)t_{m-1}(x) - t_{m-2}(x) \tag{4.4}$$

with initial conditions

$$t_0(x) = 2 \quad \text{and} \quad t_1(x) = x-2. \tag{4.5}$$

Next, recall Newton's formulae written in terms of Schur functions:

$$ms_m = \sum_{r=1}^{m} p_r s_{m-r}.$$

Evaluating both sides at $p_r = t_r$ yields

$$mS_m = \sum_{r=1}^{m} t_r S_{m-r}. \tag{4.6}$$

Now substitute (4.4) into (4.6) to get

$$\begin{aligned} mS_m &= (x-2)S_{m-1} + \sum_{r=2}^{m} t_r S_{m-r} \\ &= (x-2)S_{m-1} + \sum_{r=2}^{m} ((x-2)t_{r-1} - t_{r-2})S_{m-r} \\ &= (x-2)S_{m-1} + (x-2)\sum_{r=1}^{m-1} t_r S_{m-1-r} - \sum_{r=0}^{m-2} t_r S_{m-2-r} \\ &= m(x-2)S_{m-1} - mS_{m-2}, \end{aligned}$$

whence we conclude that

$$S_m(x) = (x-2)S_{m-1} - S_{m-2}(x) \tag{4.7}$$

with initial conditions

$$S_0(x) = 1 \quad \text{and} \quad S_1(x) = t_1 = x - 2. \tag{4.8}$$

Note that we may also write (4.7) as

$$S_m(x) = t_1 S_{m-1} - S_{m-2}(x) \tag{4.9}$$

From the recurrence we obtain the ordinary generating function for the S_m's in the usual manner:

$$\begin{aligned} S(z) :&= \sum_{m=0}^{\infty} S_m z^m \\ &= 1 + t_1 z + \sum_{m=2}^{\infty} (t_1 S_{m-1} - S_{m-2}) z^m \\ &= 1 + t_1 z + t_1 \sum_{m=1}^{\infty} S_m z^{m+1} - \sum_{m=0}^{\infty} S_m z^{m+2} \\ &= 1 + t_1 z + t_1 z(S(z) - 1) - z^2 S(z), \end{aligned}$$

from which we conclude (using (4.5)) that

$$S(z) = \frac{1}{1 - t_1 z + z^2} = \frac{1}{(1+z)^2 - xz}. \tag{4.10}$$

From (4.10) we may obtain an explicit formula for $S_m(x)$. We have

$$\begin{aligned} S(z) &= \frac{1}{(1+z)^2} \cdot \frac{1}{1 - (xz/(1+z)^2)} \\ &= \sum_{k=0}^{\infty} \frac{x^k z^k}{(1+z)^{2k+2}} \\ &= \sum_{k=0}^{\infty} x^k z^k \sum_{j=0}^{\infty} \binom{j+2k+1}{j} (-z)^j \end{aligned}$$

and extracting the coefficient of z^m gives

$$S_m(x) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m+k+1}{m-k} x^k. \tag{4.11}$$

5 Proofs of 1(b) and 1(c)

For the reader's convenience, we restate the relevant parts of Theorem 1 here.

Theorem 1(b). *The coefficients a_λ vanish if λ has more than two parts.*

Proof. It suffices to show this for λ having three parts, as the general case is similar. So assume $\lambda = (p, q, r)$. Then by the analogue (3.3) of the Jacobi-Trudi identity,

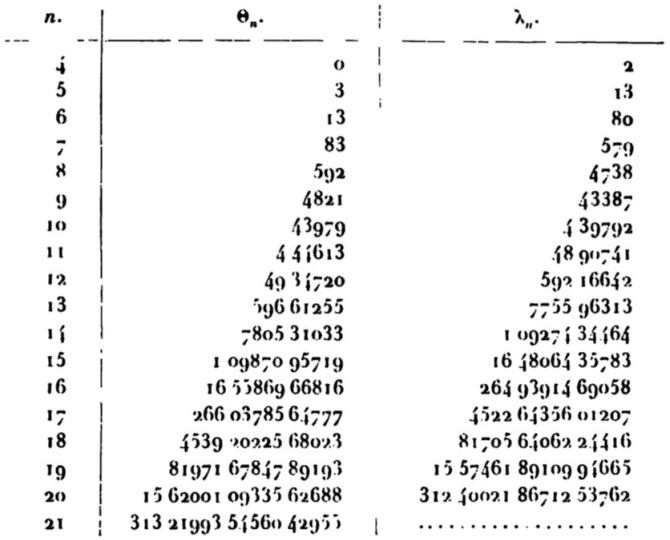
$$S_{(p,q,r)} = \begin{vmatrix} S_p & S_{p+1} & S_{p+2} \\ S_{q-1} & S_q & S_{q+1} \\ S_{r-2} & S_{r-1} & S_r \end{vmatrix}$$

Using the fundamental recurrence (4.9) in the last column gives

$$S_{(p,q,r)} = \begin{vmatrix} S_p & S_{p+1} & t_1 S_{p+1} - S_p \\ S_{q-1} & S_q & t_1 S_q - S_{q-1} \\ S_{r-2} & S_{r-1} & t_1 S_{r-1} - S_{r-2}, \end{vmatrix},$$

which vanishes, as the three columns are linearly dependent. The theorem now follows from (3.1). □

Theorem 1(c). *For $i - 1, j \geqslant 0$, $a_{(i+1,j+1)} = a_{(i,j)}$, where $a_{(i,0)} := a_{(i)}$.*



n.	θ_n.	λ_n.
4	0	2
5	3	13
6	13	80
7	83	579
8	592	4738
9	4821	43387
10	43979	439792
11	444613	4890741
12	4931720	59216642
13	59661255	775596313
14	780531033	10927434764
15	10987095719	164806435783
16	165586966816	2649391469058
17	2660378564777	45226435601207
18	45392022568023	817056406224416
19	819716784789193	15574618910996665
20	15620010933562688	312400218671253762
21	313219935456042955	.....

Figure 2: Table from Lucas (1891).

Proof. By (3.3),

$$S_{(i,j)} = \begin{vmatrix} S_i & S_{i+1} \\ S_{j-1} & S_j \end{vmatrix} = S_i S_j - S_{i+1} S_{j-1}.$$

Note that $S_{(i,0)} = S_i$ (because $S_0 = 1$ by (4.8)). Also, using the recurrence (4.9),

$$\begin{aligned} S_{(i+1,j+1)} - S_{(i,j)} &= S_{i+1}S_{j+1} - S_{i+2}S_j - S_iS_j + S_{i+1}S_{j-1} \\ &= S_{i+1}(t_1 S_j - S_{j-1}) - (t_1 S_{i+1} - S_i)S_j - S_iS_j + S_{i+1}S_{j-1} \\ &= 0. \end{aligned}$$

Hence, the theorem follows from (3.1) again. □

Theorem 1(b) implies that the general form of $\operatorname{ch} F$ is

$$\operatorname{ch} F = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{(n-k,k)} s_{(n-k,k)}, \tag{5.1}$$

in agreement with the data in Table 1. Moreover, by Theorem 1(c), the entirety of Table 1 follows recursively once we know the values of a_n. This is the subject of the next section.

6 Proof of Theorem 1(a)

The earliest appearance of the numbers a_n seems to be in Lucas' 1891 monograph [12], where he attributes their discovery to M. C. Moreau. [15] On p. 495 of [12] we find a

[15] Moreau is perhaps best known for his discovery of the formula enumerating cyclically distinct necklaces. See [17] and ([12], pp. 501-503).

table, reproduced in Figure 2, listing the *ménage* numbers, which Lucas calls λ_n, and some auxiliary numbers θ_n, whose relationship is

$$\lambda_n = \theta_{n+1} - \theta_{n-1}. \tag{6.1}$$

(See also [19].) Moreover, Lucas notes the following recurrence relation [16]

$$\theta_{n+1} = n\theta_n + \theta_{n-1} + 2(-1)^n. \tag{6.2}$$

Comparing Figure 2 with Table 1, we are led to the identification

$$a_n = \theta_{n+1}. \tag{6.3}$$

We prove the general validity of (6.3) by showing that the a_n's satisfy the same recurrence as the θ_{n+1}'s. (It is clear from Table 1 and Figure 2 that $a_3 = \theta_4 = 0$ and $a_4 = \theta_5 = 3$.)

Theorem 1(a). *For $n \geqslant 2$,*

$$a_n = na_{n-1} + a_{n-2} + 2(-1)^n, \tag{1.11}$$

with initial conditions $a_1 = -1$ and $a_2 = 1$. In particular, we have $a_n = \theta_{n+1}$, where $\{\theta_n\}$ is a sequence first discovered by M. Moreau ([12], p. 495).

Proof. Combining (3.1) and (4.11) yields

$$a_n = \int_0^\infty S_n\, e^{-x}\, dx = \sum_{k=0}^{n} (-1)^{n-k} k! \binom{n+k+1}{n-k} = \sum_{k=0}^{n} (-1)^k (n-k)! \binom{2n-k+1}{k} \tag{6.4}$$

By direct computation we find $a_1 = -1$ and $a_2 = 1$. Attempts to use (6.4) directly to demonstrate (1.11) rapidly devolve into a morass of binomial coefficients. Instead, we simply appeal to a result of Riordan ([20], p. 200), who shows that the generating function

$$W_n(z) = \sum_k \binom{2n-k+1}{k} (n-k)!(z-1)^k$$

satisfies the recurrence

$$W_n(z) = nW_{n-1}(z) + (z-1)^2 W_{n-2}(z) + 2(z-1)^n. \tag{6.5}$$

Comparison with (6.4) shows that

$$a_n = W_n(0). \tag{6.6}$$

Now (1.11) follows immediately from (6.5). □

[16] Other, similar recurrences were obtained earlier by Cayley and Muir.

In Lucas' book he introduces the sequence $\{\theta_n\}$ as a mere *aide au calcul*. But our results reveal the reason why this particular sequence shows up naturally in the evaluation of the *ménage* numbers. By the Murnaghan-Nakayama rule, if σ is a cycle of length n in $\mathfrak{S}_n$, then

$$\chi^\lambda(\sigma) = \begin{cases} (-1)^k, & \text{if } \lambda = (n-k, 1^k), \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

So by Theorems 1(b) and 1(c),

$$m_n = F(\sigma) = \sum_\lambda a_\lambda \chi^\lambda(\sigma) = \sum_k (-1)^k a_{(n-k,1^k)} = a_{(n)} - a_{(n-1,1)} = a_n - a_{n-2}. \tag{6.7}$$

Taken together, Theorem 1(a) and (6.7) provide an explanation for (6.1) and (6.2). Moreover, using (1.4) and (1.5) and the fact that $\chi^{(n)}$ is the trivial representation, we see that

$$\theta_{n+1} = a_n = \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} \mathrm{per}(K_n - P_\tau), \tag{6.8}$$

thereby supplying an algebraic interpretation for Moreau's sequence $\{\theta_n\}$.

7 Proof of 1(d)

Given $\mathrm{ch}\, F$ we can obtain the value of $F(\tau) = \mathrm{per}(K_n - P_\tau)$ without having to compute any permanents, simply by knowing the values of the irreducible characters of $\mathfrak{S}_n$. For instance, in the previous section we observed that if τ is a full cycle, $\mathrm{per}(K_n - P_\tau) = a_{(n)} - a_{(n-1,1)}$. For another example, suppose that $\tau = (1)(2)(34)(567)$. Then using Table 1 and MAPLE we find

$$F(\tau) = 592\chi^{(7)}(\tau) + 13\chi^{(6,1)}(\tau) - \chi^{(4,3)}(\tau) = 592 \cdot 1 + 13 \cdot 1 - 1 = 604.$$

To carry out these computations in general one desires a reasonable formula for the character values χ^λ where λ is a two-rowed partition. Of course, one always has the Murnaghan-Nakayama formula. But another approach employs character polynomials. [17] Write $c_i(\tau)$ for the number of cycles of length i in the disjoint cycle decomposition of τ. In an upublished paper, Ashraf derives a formula ([2], Eq. 9) for the character polynomials associated to two-rowed partitions.[18] From his analysis we immediately obtain the following result. [19]

$$\chi^{(n-k,k)}(\tau) = [z^k](1-z) \prod_i (1+z^i)^{c_i(\tau)} =: b_k(\tau). \tag{7.1}$$

We can now prove Theorem 1(d).

[17] For a survey, see [5]; see also [11].
[18] By a result of Frobenius, every irreducible character is a polynomial function of the c_i's.
[19] There is a typographical (sign) error in Ashraf's formula, which is corrected here.

Proof of Theorem 1(d). From (1.10), (5.1), and Theorem 1(c) we have

$$F(\tau) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n-2k} \chi^{(n-k,k)}(\tau),$$

so (7.1) gives

$$F(\tau) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n-2k} b_k(\tau). \tag{1.12}$$

In particular, if τ is a single n-cycle (with $n \geqslant 2$) then $c_n(\tau) = 1$ and $c_i = 0$ $(1 \leqslant i \leqslant n-1)$, so that

$$\begin{aligned} F(\tau) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n-2k} [z^k](1-z)(1+z^n) \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n-2k} [z^k](1-z+z^n-z^{n+1}) \\ &= a_n - a_{n-2}. \end{aligned}$$

□

For $\tau = (1)(2)(34)(567)$ we find $c_1(\tau) = 2$, $c_2(\tau) = 1$, and $c_3(\tau) = 1$, so

$$(1-z)\prod_i (1+z^i)^{c_i(\tau)} = (1-z)(1+z)^2(1+z^2)(1+z^3) = 1+z+z^3-z^5-z^7-z^8.$$

Reading off the coefficients and using Table 1 again we find

$$F(\tau) = \sum_{k=0}^{3} a_{7-2k} b_k(\tau) = a_7 + a_5 + a_1 = 592 + 13 - 1 = 604,$$

as before.

Acknowledgements

I would like to thank the referees for their careful reading of the manuscript and for their very helpful corrections and suggestions, which led to a much improved exposition.

References

[1] M.A. Alekseyev, "Weighted de Bruijn Graphs for the Menage Problem and its Generalizations", in V. Mäkinen, S. Puglisi, and L. Salmela (eds), *Combinatorial Algorithms*. IWOCA 2016. Lecture Notes in Computer Science, vol. 9843. (Springer, NY, 2016).

[2] A.U. Ashraf, "Character polynomials for two rows and hook partitions", arXiv:1812.09377.

[3] L. Comtet, *Advanced Combinatorics* (Dordrecht Reidel, Boston, 1974).

[4] P. Feinsilver and J. McSorley, "Zeons, permanents, the Johnson scheme, and generalized derangements", *Intl. J. Combin.* (2011) Article ID 539030.

[5] A.M Garsia and A. Goupil, "Character polynomials, their q-analogues, and the Kronecker product", *Elect. J. Combin.* **16(2)** (2009) #R19.

[6] C.D. Godsil, *Algebraic Combinatorics* (Chapman and Hall, New York, 1993).

[7] S.A. Joni and G.-C. Rota, "A vector space analogue of permutations with restricted position", *J. Comb. Th.* **A29** (1980) 59-73.

[8] W.B. Jurkat and H.J. Ryser, "Matrix Factorizations of Determinants and Permanents", *J. Alg.* **3** (1966) 1-27.

[9] I. Kaplansky, "Solution of the '*Problème des ménages*'", *Bull. Amer. Math. Soc.* **49** (1943) 784-785.

[10] I. Kaplansky and J. Riordan, "The *problème des ménages*", *Scripta Math.* **12(2)** (1946) 113-124.

[11] A. Kerber, *Applied Finite Group Actions*, 2nd ed. (Springer-Verlag, Berlin, 1999).

[12] E. Lucas, *Théorie des nombres* (Gauthier-Villars et Fils, Paris, 1891).

[13] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed. (Oxford University Press, Oxford, 1995).

[14] J.C. Mason and D.C Handscomb, *Chebyshev Polynomials* (Chapman and Hall/CRC, Boca Raton, 2003).

[15] B.D. McKay and I.M. Wanless, "Maximising the permanent of (0,1)-matrices and the number of extensions of Latin rectangles", *Elect. J. Combin.* **5** (1998) #R11.

[16] P.R. de Monmort, *Essay d'analyse sur les jeux de hazard*, 2nd. ed. (Jacque Quillau, Paris, 1713).

[17] C. Moreau, "Sur les permutations circulaires distinctes", *Nouvelles annales de mathématiques* 2^e *série* **11** (1872) 309-314.

[18] OEIS Foundation Inc. (2026), Entry A000179 in The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A000179.

[19] OEIS Foundation Inc. (2026), Entry A000904 in The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A000904.

[20] J. Riordan, *An Introduction to Combinatorial Analysis* (Wiley, New York, 1958).

[21] V.S. Shevelev, "Some problems in the theory of enumeration of permutations with restricted positions", *J. Soviet Math.* **61(4)** (1992) 2272-2317.

[22] R.P. Stanley, *Enumerative Combinatorics*, Vol. 1, 2nd ed. (Cambridge University Press, Cambridge, 2012).

[23] R.P. Stanley, *Enumerative Combinatorics*, Vol. 2 (Cambridge University Press, Cambridge, 1999).

[24] J. Touchard, "Sur un problème de permutations", *C.R. Acad. Sci. Paris* **198** (1934) 631-633.

[25] L.G. Valiant, "The complexity of computing the permanent", *Th. Comp. Sci.* **8(2)** (1979) 189-201.

[26] M. Wyman and L. Moser, "On the *Problème des ménages*", *Canad. J. Math* **10(3)** (1958) 468-480.