A Variation of the Morris Constant Term

Guoce Xin[a] Chen Zhang[b]

Submitted: Oct 21, 2024; Accepted: Aug 8, 2026; Published: Sep 25, 2026

Abstract

The Morris constant term identity is important due to its equivalence with the well-known Selberg integral. We find a variation of the Morris constant term, denoted $h_n(t)$, in the study of the Ehrhart polynomial $H_n(t)$ of the n-th Birkhoff polytope, which consists of all doubly stochastic matrices of order n. The constant term $h_n(t)$ corresponds to a particular constant term in the study of $H_n(t)$. We give a characterization of $h_n(t)$ as a polynomial of degree $(n-1)^2$ with additional nice properties involving the Morris constant term. We also construct a recursion for $h_n(t)$ using a technique that is similar to the one used by Baldoni-Silva and Vergne, and by Xin in their proofs of the Morris constant term identity. Using this method, we obtain explicit formulas of $h_n(t)$ for $3 \leqslant n \leqslant 29$ without difficulty.

Mathematics Subject Classifications: 05A19, 03D80, 05-08, 05E05

1 Introduction

We are interested in the following form of the Morris constant term identity, which was first proved by Baldoni-Silva and Vergne using total residue in [4], and then given a simplified proof in [26].

Theorem 1 ([4, 26]). *If $k_1, k_2, k_3 \in \mathbb{N}$ and $k_1 + k_2 \geqslant 2$, then*

$$\operatorname*{CT}_{\boldsymbol{x}} M(n; k_1, k_2, k_3) = \prod_{j=0}^{n-1} \frac{\Gamma(1 + \frac{k_3}{2})\Gamma(k_1 + k_2 - 1 + (n + j - 1)\frac{k_3}{2})}{\Gamma(1 + (j + 1)\frac{k_3}{2})\Gamma(k_1 + j\frac{k_3}{2})\Gamma(k_2 + j\frac{k_3}{2})}.$$

where $\operatorname*{CT}_{\boldsymbol{x}} f$ denotes the constant term of the Laurent series f with respect to the variables $\boldsymbol{x} := (x_1, x_2, \dots, x_n)$, and

$$M(n; k_1, k_2, k_3) := \frac{1}{\prod_{i=1}^n x_i^{k_1-1} \prod_{i=1}^n (1 - x_i)^{k_2} \prod_{1\leqslant i<j\leqslant n}(x_i - x_j)^{k_3}}. \tag{1}$$

In particular, we agree that $M(0; k_1, k_2, k_3) = 1$.

[a] School of Mathematical Sciences, Capital Normal University, Beijing 100048, PR China (guoce_xin@163.com).

[b] Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, PR China (ch_enz@163.com).

Here and what follows, rational functions in $\boldsymbol{x}$ are explained as their Laurent series expansions according to $1 > x_1 > x_2 > \cdots > x_n > 0$, unless specified otherwise.

Morris's original constant term identity (stated in his PhD thesis [17]) refers to the negative even k_3 case. It is equivalent to the well-known Selberg integral [19]. See [10] for a nice overview of the importance of the Selberg integral. Indeed, Morris stated a q-constant term identity, not just the $q = 1$ case. Zeilberger [29] first considered the case $(k_1, k_2, k_3) = (1, 2, 1)$ of Theorem 1, and proved that the volume of the Chan–Robbins–Yuen polytope may be expressed as a product of Catalan numbers. Theorem 1 was further studied in [16].

In this paper, we mainly study a variation of the Morris constant term defined in (2). This constant term arises as a particular case of the Ehrhart polynomial $H_n(t)$ of the n-th Birkhoff polytope $\mathcal{B}_n$.

For a positive integer n, $\mathcal{B}_n$ is usually described as the set of all $n \times n$ *doubly stochastic* matrices, where a matrix $B \in \mathbb{R}^{n\times n}$ is said to be doubly stochastic if all elements of B are nonnegative and the sum of each row and each column is equal to 1. The $H_n(t)$ counts the number of the nonnegative integral matrices such that every row sum and every column sum equal to the nonnegative integer t. This has been studied by many researchers; see, e.g., the papers [1, 4, 5, 6, 7, 8, 18, 28] for details. It was conjectured by Anand, Dumir and Gupta [2] and proved by Ehrhart [9] and Stanley [20] that $H_n(t)$ is a polynomial in t of degree $(n-1)^2$ for any fixed positive integer n. The following theorem holds.

Theorem 2 ([20, 21]). *For any positive integer n, we have*

$$\sum_{t\geqslant 0} H_n(t)y^t = \frac{1}{(1-y)^{(n-1)^2+1}} \sum_{i=0}^{(n-1)(n-2)} a_i y^i,$$

where the a_i's are nonnegative integers satisfying $a_0 = 1$ and $a_i = a_{(n-1)(n-2)-i}$ for all i.

It is the first conjecture stated in [22] that the sequence $(a_0, a_1, \dots, a_{(n-1)(n-2)})$ is unimodal, i.e., $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{\lfloor (n-1)(n-2)/2\rfloor}$. Athanasiadis [3] proved this conjecture using McMullen's g-theorem [15].

The formulas $H_1(t) = 1$ and $H_2(t) = t + 1$ are trivial. The first nontrivial case was computed by MacMahon [14] as $H_3(t) = 3\binom{t+3}{4} + \binom{t+2}{2}$. The record is held by Beck and Pixton [5], who obtained an explicit expression for $H_9(t)$ and the leading coefficient of $H_{10}(t)$ (corresponding to the volume of $\mathcal{B}_{10}$) using residue computation. For a positive integer n, $H_n(t)$ can be written as [5, 28]

$$H_n(t) = \sum_{\boldsymbol{m}} \binom{n}{\boldsymbol{m}} \operatorname*{CT}_{\boldsymbol{x}} \mathcal{H}^{\boldsymbol{m}},$$

where $\boldsymbol{m} := (m_1, m_2, \dots, m_n)$ ranges over all weak compositions of n of length n (that is, the m_i's are nonnegative integers such that $\sum_{i=1}^n m_i = n$), and

$$\mathcal{H}^{\boldsymbol{m}} := \prod_{i=1}^{n} \frac{x_i^{(m_i-1)t}}{\prod_{j=1, j\neq i}^{n} (1 - x_j/x_i)^{m_i}}.$$

Among all these $\boldsymbol{m}$, the case $\boldsymbol{m} = (2, 1^{n-2}, 0)$ is the hardest to evaluate. We observe that $\operatorname{CT}_{\boldsymbol{x}} \mathcal{H}^{(2,1^{n-2},0)}$ is similar to the Morris constant term. To be precise,

$$\mathcal{H}^{(2,1^{n-2},0)} = (-1)^{\binom{n-1}{2}} \frac{(x_n/x_1)^{-t} \prod_{i=2}^{n-1}(x_i/x_1)^{2i-n} \prod_{i=1}^{n-1}(1 - x_n/x_i)}{\prod_{i=2}^{n}(1 - x_i/x_1)^3 \prod_{2\leqslant i<j\leqslant n}(1 - x_j/x_i)^2}.$$

Since $\mathcal{H}^{(2,1^{n-2},0)}$ is homogeneous, we may set $x_1 = 1$ when computing $\operatorname{CT}_{\boldsymbol{x}} \mathcal{H}^{(2,1^{n-2},0)}$. That is, we can write

$$h_n(t) := (-1)^{\binom{n-1}{2}} \operatorname{CT}_{\boldsymbol{x}} \mathcal{H}^{(2,1^{n-2},0)} = \operatorname{CT}_{\boldsymbol{x}} \bar{\mathcal{H}}_n, \tag{2}$$

where

$$\bar{\mathcal{H}}_n := \frac{(1 - x_n) \prod_{i=2}^{n-1} x_i^{n-1} \prod_{i=2}^{n-1}(x_i - x_n)}{x_n^t \prod_{i=2}^{n}(1 - x_i)^3 \prod_{2\leqslant i<j\leqslant n}(x_i - x_j)^2}. \tag{3}$$

Now we present our main result, which will be proved in Section 3.

Theorem 3. *Let $h_n(t)$ and $M(n; k_1, k_2, k_3)$ be as in (2) and (1), respectively. For any $n \geqslant 2$, we have*

(1) $h_n(t)$ is a polynomial in t of degree $(n-1)^2$ with leading coefficient $\frac{1}{(n-1)^2!} \operatorname{CT}_{\boldsymbol{x}} M(n-1; 1, 1, 2)$.

(2) $h_n(t) = (-1)^{n-1} h_n(-t - n)$.

(3) Let $N = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$. Then $h_n(t) = P_n(t) \prod_{i=-N-n+1}^{N-1}(t-i)$, where $P_n(t)$ is a polynomial in t of degree $(n - 1)(n - 2) - 2N$.

(4) Let N be as in part (3). Then

$$h_n(N) = \begin{cases} h_{2r}(r(r-1)) = \left(\operatorname{CT}_{\boldsymbol{x}} M(r-1; 3, 1, 2)\right)^2, & \text{if } n = 2r; \\ h_{2r-1}((r-1)^2) = \left(\operatorname{CT}_{\boldsymbol{x}} M(r-1; 2, 1, 2)\right)^2, & \text{if } n = 2r-1. \end{cases}$$

It is trivial that $h_2(t) = t + 1$. In what follows, we always assume $n \geqslant 3$.

According to Theorem 3, for a fixed n, we can construct the polynomial $h_n(t)$ by the method of undetermined coefficients. By part (1) we need the values of $h_n(t)$ at $(n-1)^2$ distinct t's; Part (2) cuts this number by half; Part (3) gives some roots of $h_n(t)$; Part (4) gives $h_n(N)$. This leaves us to compute $h_n(t)$ at $\binom{n-1}{2} - N - 1$ different $t > N$. Direct computation of these $h_n(t)$ is already hard when $n \geqslant 8$. Instead, we will derive a recursion for $h_n(t)$ with respect to t in Section 4. The recursion is derived using a technique similar to that in the proof of the Morris constant term identity.

In particular, we obtain $h_{10}(t) = P_{10}(t) \prod_{i=-29}^{19}(t - i)$, where $P_{10}(t)$ is a polynomial of degree 32 with leading coefficient 32137511231440656961920000/81! that is too complicated to fully state here. The explicit formulas for $h_n(t)$ for $3 \leqslant n \leqslant 29$ are available at [27]. Data shows that $P_n(t)$ is divisible by $t(t + n)$ when $n \geqslant 4$.

The rest of this paper is structured as follows. In Section 2, we introduce two versions of Jacobi's change of variables formula, and a constant term concept in [28]. These will be the main tools in our proof of Theorem 3. In Section 3, we complete the proof of Theorem 3 by dividing it into some lemmas. In Section 4, we introduce a new set of constant terms $D_n(\ell, t, k_1, k_2, k_3)$ involving elementary symmetric functions. Then we obtain a relation among them and use these relations to derive a recursion for $h_n(t)$. By combining with Theorem 3, we can compute the explicit formula for $h_n(t)$ for any n. We provide Examples 25 and 27 to explain our method. We also give a conjectured nice formula for a tri-diagonal determinant (See Conjecture 28). Section 5 gives a conjecture about the generating function of $h_n(t)$.

2 Jacobi's change of variables formula and a constant term concept

In this section, we introduce Jacobi's change of variables formulas and some basic constant term operations. These will be used in our proof.

We use both constant term and residue, though they are equivalent in some sense. In series level, they are defined by

$$\operatorname*{CT}_{x} \sum_{m\in\mathbb{Z}} a_m x^m = a_0, \qquad \operatorname*{Res}_{x} \sum_{m\in\mathbb{Z}} a_m x^m = a_{-1},$$

so that $\operatorname*{CT}_{x} x f(x) = \operatorname*{Res}_{x} f(x)$. However formal Laurent series does not form a ring. For instance $\sum_{m\geqslant 0} x^m \cdot \sum_{m\geqslant 0} x^{-m}$ is not well-defined.

We work instead in the field of iterated Laurent series $K((x_1))\cdots((x_n))$ (see [25]), where a total ordering on the variables determines the unique series expansion of a rational function. In this framework, "convergence" is understood in the formal sense that every coefficient in the expansion is a finite sum, guaranteed by the well-orderedness of the support (the fundamental structure theorem). This is precisely the sense in which we interpret constant term identities involving multiple variables; in particular, constant term operators with respect to different variables commute, provided the order is specified.

Thus, when writing $\operatorname*{CT}_{\boldsymbol{x}} f(\boldsymbol{x})$, the total order must be clear from the context. In this paper, we specify the order by $1 > x_1 > x_2 > \cdots > x_n > 0$ for the primary variables, which gives the expansion

$$\frac{1}{1 - x_i/x_j} = \begin{cases} \displaystyle\sum_{m\geqslant 0} (x_i/x_j)^m, & \text{if } i > j; \\ \displaystyle\sum_{m\geqslant 0} -(x_i/x_j)^{-m-1}, & \text{if } i < j. \end{cases}$$

Other orderings (e.g., involving auxiliary variables) will be specified when needed. All multi-variable constant terms in the sequel are understood in the iterated Laurent series field with the ordering indicated in each instance.

We also need the residue notation at a particular point. A rational function $f(x)$ may be expanded as a Laurent series at $x = x_0$. Denote the coefficient of $(x - x_0)^j$ by $[(x - x_0)^j]f(x)$, and the residue of $f(x)$ at $x = x_0$ by

$$\operatorname*{Res}_{x=x_0} f(x) := [(x - x_0)^{-1}]f(x).$$

We distinguish $\operatorname*{Res}_{x_i=0} f(x)$ from $\operatorname*{Res}_{x_i} f(x)$: they agree only when x_i is treated as the smallest variable.

2.1 Jacobi's change of variables formula

We need the following Jacobi's change of variables formula in the one variable case.

Lemma 4 ([12])**.** *Let c be a complex number. Suppose the function $g(y)$ is holomorphic in a neighborhood of $y = c$ and suppose $f(x)$ is meromorphic in a neighborhood of $x = g(c)$. If $\big(\partial_y g(y)\big)\big|_{y=c} \neq 0$, then*

$$\operatorname*{Res}_{x=g(c)} f(x) = \operatorname*{Res}_{y=c} f(g(y))\partial_y g(y),$$

where $\partial_y := \frac{\partial}{\partial y}$.

Lemma 5. *Let $F(x_1, x_2, \dots, x_n)$ be a rational function. Then we have*

$$\operatorname*{CT}_{\boldsymbol{x}} F(x_1, x_2, \dots, x_n) = \operatorname*{CT}_{\boldsymbol{y}} \frac{1}{\prod_{i=1}^n (1 - y_i)} F\Big(-\frac{y_1}{1 - y_1}, -\frac{y_2}{1 - y_2}, \dots, -\frac{y_n}{1 - y_n}\Big),$$

where the right hand side adopt the total ordering specified by $1 > y_1 > y_2 > \cdots > y_n > 0$.

Proof. The lemma follows by applying Lemma 4 with $x_i = -\frac{y_i}{1-y_i}$ successively for all i:

$$\begin{aligned}
\operatorname*{CT}_{\boldsymbol{x}} F(x_1, x_2, \dots, x_n) &= \operatorname*{Res}_{\boldsymbol{x}} \frac{1}{\prod_{i=1}^n x_i} F(x_1, x_2, \dots, x_n) \\
&= \operatorname*{Res}_{\boldsymbol{y}} \frac{\prod_{i=1}^n \partial_{y_i}(-\frac{y_i}{1-y_i})}{\prod_{i=1}^n (-\frac{y_i}{1-y_i})} F\Big(-\frac{y_1}{1 - y_1}, -\frac{y_2}{1 - y_2}, \dots, -\frac{y_n}{1 - y_n}\Big) \\
&= \operatorname*{Res}_{\boldsymbol{y}} \frac{1}{\prod_{i=1}^n y_i(1 - y_i)} F\Big(-\frac{y_1}{1 - y_1}, -\frac{y_2}{1 - y_2}, \dots, -\frac{y_n}{1 - y_n}\Big) \\
&= \operatorname*{CT}_{\boldsymbol{y}} \frac{1}{\prod_{i=1}^n (1 - y_i)} F\Big(-\frac{y_1}{1 - y_1}, -\frac{y_2}{1 - y_2}, \dots, -\frac{y_n}{1 - y_n}\Big). \qquad \square
\end{aligned}$$

Let $F(\boldsymbol{x})$ be a formal Laurent series. We define the action of a nonsingular integer matrix $W = (w_{ij})_{n\times n}$ on $F(\boldsymbol{x})$ by

$$WF(\boldsymbol{x}) := W \circ F(\boldsymbol{x}) = F(y_1, y_2, \dots, y_n),$$

where $y_i = \boldsymbol{x}^{We_i} = x_1^{w_{1i}} \cdots x_n^{w_{ni}}$, and e_i is the standard i-th unit vector in $\mathbb{C}^n$ for each i. Notably, $W\boldsymbol{x}^\alpha = \boldsymbol{x}^{W\alpha}$.

For the multi-variable case, we need the next version of Jacobi's change of variables formula.

Lemma 6. *Suppose W is a nonsingular integer matrix and $F(\boldsymbol{x})$ is an iterated Laurent series in $\boldsymbol{x}$. Then we have*

$$\operatorname*{CT}_{\boldsymbol{x}} F(\boldsymbol{x}) = \operatorname*{CT}_{\boldsymbol{x}}(WF(\boldsymbol{x})),$$

whenever $WF(\boldsymbol{x})$ converges in the same iterated Laurent series field.

Note that the convergence is not automatic: under the order $1 > x_1$, the change of variables $x_1 \mapsto x_1^{-1}$ sends $\sum_{m\geqslant 0} x_1^m$ to $\sum_{m\geqslant 0} x_1^{-m}$, which is not a Laurent series.

Proof of Lemma 6. By linearity, we can assume $F(\boldsymbol{x}) = \boldsymbol{x}^\alpha$. Then $WF(\boldsymbol{x}) = \boldsymbol{x}^{W\alpha}$. Since $W\alpha = \mathbf{0}$ if and only if $\alpha = \mathbf{0}$, the lemma follows.

See [26] for a general version of this lemma, where the convergence issue is addressed in full. In our applications, however, convergence is easy to verify. $\square$

2.2 A constant term concept

For positive integers $q_1, q_2, \ldots, q_n$, let

$$f(x) = \frac{L(x)}{\prod_{i=1}^n (1 - x/u_i)^{q_i}}, \tag{4}$$

where $L(x)$ is a Laurent polynomial in x, and the u_i's do not depend on x. Assume the PFD (short for partial fraction decomposition) of $f(x)$ with respect to x is given by

$$f(x) = L_1(x) + \sum_{i=1}^n \frac{A_i(x)}{(1 - x/u_i)^{q_i}}, \tag{5}$$

where $L_1(x)$ is a Laurent polynomial in x, and $A_i(x)$ is a polynomial in x of degree less than q_i for each i. Define

$$\operatorname*{CT}_{x=u_i} f(x) := A_i(0). \tag{6}$$

Then we have the following lemmas.

Lemma 7. *Let $f(x)$ be as in (4) with PFD as in (5). Then*

$$\operatorname*{CT}_{x} f(x) = \operatorname*{CT}_{x} L_1(x) + \sum_{i=1}^n \chi(x/u_i < 1) \operatorname*{CT}_{x=u_i} f(x). \tag{7}$$

In particular, if $f(x)$ is proper in x, i.e., the degree of the numerator in x is less than that of the denominator, then $L_1(x)$ only contains negative powers in x. Consequently,

$$\operatorname*{CT}_{x} f(x) = \sum_{i=1}^n \chi(x/u_i < 1) \operatorname*{CT}_{x=u_i} f(x). \tag{8}$$

Proof. Equation (7) follows by the following observation:

$$\operatorname*{CT}_{x} \frac{A_i(x)}{(1-x/u_i)^{q_i}} = \begin{cases} A_i(0), & \text{if } x/u_i < 1; \\ 0, & \text{if } x/u_i > 1. \end{cases}$$

This follows since for $x/u_i > 1$, the series expansion

$$\frac{A_i(x)}{(1-x/u_i)^{q_i}} = \frac{A_i(x)}{(-x/u_i)^{q_i}(1-u_i/x)^{q_i}} = (-u_i/x)^{q_i} A_i(x) \sum_{l\geqslant 0} \binom{l+q_i-1}{l} (u_i/x)^l$$

only contains negative powers of x.

Equation (8) holds by $\operatorname*{CT}_{x} L_1(x) = 0$ provided $f(x)$ is proper. ☐

Lemma 8. *Let $f(x)$ be as in (4) with PFD as in (5). Then*

$$\operatorname*{CT}_{x=u_i} f(x) = -\operatorname*{Res}_{x=u_i} x^{-1} f(x).$$

Proof. Suppose $A_i(x) = \sum_{l=0}^{q_i-1} a_{i,l} x^l$, where the $a_{i,l}$'s are not all zero. By the definition of $\operatorname*{CT}_{x=u_i} f(x)$ (See (6)), it is sufficient to show that

$$\operatorname*{Res}_{x=u_i} x^{-1} f(x) = [(x-u_i)^{-1}] x^{-1} f(x) = -a_{i,0}.$$

For any integer r, we can expand x^r at $x = u_i$ as follows:

$$x^r = u_i^r (1-(1-x/u_i))^r = \begin{cases} \displaystyle\sum_{k=0}^{r} \binom{r}{k} u_i^{r-k} (x-u_i)^k, & \text{if } r \geqslant 0; \\ \sum_{k\geqslant 0} \binom{k-r-1}{k} (-1)^k u_i^{r-k} (x-u_i)^k, & \text{if } r < 0. \end{cases}$$

It is clear that

$$\operatorname*{Res}_{x=u_i} x^r = 0,$$

and

$$\operatorname*{Res}_{x=u_i} \frac{x^r}{(1-x/u_i)^{q_i}} = (-u_i)^{q_i} [(x-u_i)^{q_i-1}]\, x^r = \begin{cases} -1, & \text{if } r = -1; \\ 0, & \text{if } 0 \leqslant r \leqslant q_i - 2. \end{cases}$$

By linearity, we have

$$\begin{aligned} \operatorname*{Res}_{x=u_i} x^{-1} f(x) &= \operatorname*{Res}_{x=u_i} \left(x^{-1} L_1(x) + \sum_{j=1, j\neq i}^{n} \frac{x^{-1} A_j(x)}{(1-x/u_j)^{q_j}} \right) + \operatorname*{Res}_{x=u_i} \frac{x^{-1} A_i(x)}{(1-x/u_i)^{q_i}} \\ &= 0 + \sum_{l=0}^{q_i-1} a_{i,l} \operatorname*{Res}_{x=u_i} \frac{x^{l-1}}{(1-x/u_i)^{q_i}} = -a_{i,0}, \end{aligned}$$

as desired. ☐

Via Lemma 8, the definition of $\underset{x=u_i}{\mathrm{CT}}$ is extended to general rational functions. In particular, $\underset{x_i=1}{\mathrm{CT}}$ is well-defined in what follows.

Lemma 9. *Let $F(x_1, x_2, \ldots, x_n)$ be a rational function. We have*

$$\underset{x_n=1}{\mathrm{CT}} \cdots \underset{x_2=1}{\mathrm{CT}}\, \underset{x_1=1}{\mathrm{CT}}\, F(x_1, x_2, \ldots, x_n) = \underset{\boldsymbol{y}}{\mathrm{CT}} \frac{\prod_{i=1}^n y_i}{\prod_{i=1}^n (1-y_i)} F(1-y_1, 1-y_2, \ldots, 1-y_n)$$

with order $1 > y_n > y_{n-1} > \cdots > y_1 > 0$ when expanding F as a Laurent series in $\boldsymbol{y}$.

Proof. We successively make the change of variables $x_i = 1 - y_i$ for $i = 1, 2, \ldots, n$ to obtain

$$\begin{aligned}
\underset{x_n=1}{\mathrm{CT}} \cdots \underset{x_2=1}{\mathrm{CT}}\, \underset{x_1=1}{\mathrm{CT}}\, F &= \underset{x_n=1}{\mathrm{Res}} \cdots \underset{x_2=1}{\mathrm{Res}}\, \underset{x_1=1}{\mathrm{Res}}\, \frac{F(x_1, x_2, \ldots, x_n)}{\prod_{i=1}^n (-x_i)} && \text{(by Lemma 8)} \\
&= \underset{y_n}{\mathrm{Res}} \cdots \underset{y_2}{\mathrm{Res}}\, \underset{y_1}{\mathrm{Res}}\, \frac{F(1-y_1, 1-y_2, \ldots, 1-y_n)}{\prod_{i=1}^n (1-y_i)} && \text{(by Lemma 4)} \\
&= \underset{\boldsymbol{y}}{\mathrm{CT}} \frac{\prod_{i=1}^n y_i}{\prod_{i=1}^n (1-y_i)} F(1-y_1, 1-y_2, \ldots, 1-y_n).
\end{aligned}$$

The order is naturally $1 > y_n > y_{n-1} > \cdots > y_1 > 0$. □

In [28, Lemma 2.1] the present authors together with Zhou and Zhong provided an explicit formula for $A_i(0)$. For convenience, we denote

$$g_i(x) = f(x)(1 - x/u_i)^{q_i} = \frac{L(x)}{\prod_{l=1, l\neq i}^n (1 - x/u_l)^{q_l}} \tag{9}$$

for each $i \in \{1, 2, \ldots, n\}$.

Lemma 10 ([28])**.** *Let $f(x)$, $A_i(x)$ and $g_i(x)$ be as in* (4), (5) *and* (9)*, respectively. Then*

$$\underset{x=u_i}{\mathrm{CT}} f(x) = A_i(0) = \frac{(-1)^{q_i-1}}{(q_i-1)!} \Big(\partial_w^{q_i-1} \frac{g_i(wu_i)}{w} \Big)\Big|_{w=1}.$$

In particular, if $q_i = 1$, then

$$A_i(0) = g_i(u_i). \tag{10}$$

Lemma 11. *For integers $p_1, p_2, \ldots, p_n$, positive integers $q_{0,1}, q_{0,2}, \ldots, q_{0,n}$, and nonnegative integers $q_{i,j}$ $(1 \leqslant i < j \leqslant n)$, let*

$$F = \frac{\prod_{i=1}^n x_i^{p_i}}{\prod_{i=1}^n (1-x_i)^{q_{0,i}} \prod_{1\leqslant i<j\leqslant n} (x_i - x_j)^{q_{i,j}}}$$

be proper in x_i for each $1 \leqslant i \leqslant n$, i.e., $p_i < q_{0,i} + \sum_{j=1}^{i-1} q_{j,i} + \sum_{j=i+1}^n q_{i,j}$. Then

$$\underset{\boldsymbol{x}}{\mathrm{CT}}\, F = \underset{x_n=1}{\mathrm{CT}} \cdots \underset{x_2=1}{\mathrm{CT}}\, \underset{x_1=1}{\mathrm{CT}}\, F.$$

Proof. By Lemmas 7 and 10, we have

$$\begin{aligned}
\operatorname*{CT}_{x_1} F &= \operatorname*{CT}_{x_1=1} F \\
&= \frac{(-1)^{q_{0,1}-1}}{(q_{0,1}-1)!}\Big(\partial_w^{q_{0,1}-1}\frac{w^{p_1-1}}{\prod_{i=2}^n (w-x_i)^{q_{1,i}}}\Big)\Big|_{w=1} \cdot \frac{\prod_{i=2}^n x_i^{p_i}}{\prod_{i=2}^n (1-x_i)^{q_{0,i}} \prod_{2\leqslant i<j\leqslant n}(x_i-x_j)^{q_{i,j}}} \\
&= (-1)^{q_{0,1}-1}\Bigg(\sum_{\substack{\sum_{i=1}^n l_i = q_{0,1}-1 \\ l_i\geqslant 0}} \frac{\partial_w^{l_1}}{l_1!} w^{p_1-1} \prod_{i=2}^n \frac{\partial_w^{l_i}}{l_i!}(w-x_i)^{-q_{1,i}}\Bigg)\Bigg|_{w=1} \\
&\qquad \cdot \frac{\prod_{i=2}^n x_i^{p_i}}{\prod_{i=2}^n (1-x_i)^{q_{0,i}} \prod_{2\leqslant i<j\leqslant n}(x_i-x_j)^{q_{i,j}}} \\
&= (-1)^{q_{0,1}-1} \sum_{\substack{\sum_{i=1}^n l_i = q_{0,1}-1 \\ l_i\geqslant 0}} \binom{p_1-1}{l_1} \prod_{i=2}^n \binom{-q_{1,i}}{l_i} \cdot G_{l_1,l_2,\dots,l_n},
\end{aligned}$$

where each

$$G_{l_1,l_2,\dots,l_n} = \frac{\prod_{i=2}^n x_i^{p_i}}{\prod_{i=2}^n (1-x_i)^{q_{0,i}+q_{1,i}+l_i} \prod_{2\leqslant i<j\leqslant n}(x_i-x_j)^{q_{i,j}}}$$

is proper in x_i for all $2 \leqslant i \leqslant n$. The lemma follows by repeating the above process. □

Applying Lemma 11 to $\bar{\mathcal{H}}_n$ (as defined in (3)) yields the following corollary.

Corollary 12. *Let $h_n(t)$ and $\bar{\mathcal{H}}_n$ be as in* (2) *and* (3)*, respectively. Then*

$$h_n(t) = \operatorname*{CT}_{x_n=1} \cdots \operatorname*{CT}_{x_3=1} \operatorname*{CT}_{x_2=1} \bar{\mathcal{H}}_n.$$

We also need the following lemma, which is a consequence of Theorem 1.

Lemma 13. *Let $M(n;k_1,k_2,k_3)$ be as in* (1)*. If $k_2 = 1$, then*

$$\operatorname*{CT}_{\boldsymbol{x}} M(n+1;k_1,1,k_3) = \operatorname*{CT}_{\boldsymbol{x}} M(n;k_1,k_3+1,k_3).$$

Proof. We give a constant term proof. We first compute

$$\begin{aligned}
\operatorname*{CT}_{x_1} M(n+1;k_1,1,k_3) &= \operatorname*{CT}_{x_1=1} M(n+1;k_1,1,k_3) && \text{(by (8))} \\
&= M(n+1;k_1,1,k_3)\cdot(1-x_1)\big|_{x_1=1} && \text{(by (10))} \\
&= \frac{1}{\prod_{i=2}^{n+1} x_i^{k_1-1} \prod_{i=2}^{n+1}(1-x_i)^{k_3+1} \prod_{2\leqslant i<j\leqslant n+1}(x_i-x_j)^{k_3}}.
\end{aligned}$$

Relabelling the variables by $x_i = x_{i-1}$ for all $2 \leqslant i \leqslant n+1$ yields

$$\operatorname*{CT}_{x_1} M(n+1;k_1,1,k_3) = M(n;k_1,k_3+1,k_3).$$

The lemma follows. □

Theorem 1 presents the following nontrivial symmetry.

Corollary 14. *Let $M(n;k_1,k_2,k_3)$ be as in* (1)*. Then*

$$\operatorname*{CT}_{\boldsymbol{x}} M(n;k_1,k_2,k_3) = \operatorname*{CT}_{\boldsymbol{x}} M(n;k_2,k_1,k_3).$$

Proof. Clearly $M(n;k_1,k_2,k_3)$ is proper in x_i for each i. Then

$$\begin{aligned}\operatorname*{CT}_{\boldsymbol{x}} M(n;k_1,k_2,k_3) &= \operatorname*{CT}_{x_n=1}\cdots\operatorname*{CT}_{x_2=1}\operatorname*{CT}_{x_1=1} M(n;k_1,k_2,k_3) && \text{(by Lemma 11)}\\ &= \operatorname*{CT}_{\boldsymbol{y}} \frac{1}{\prod_{i=1}^{n}(1-y_i)^{k_1}\prod_{i=1}^{n} y_i^{k_2-1}\prod_{1\leqslant i<j\leqslant n}(y_j-y_i)^{k_3}} && \text{(by Lemma 9)}\\ &= \operatorname*{CT}_{\boldsymbol{x}} M(n;k_2,k_1,k_3),\end{aligned}$$

where the last equation holds by making the change of variables $y_i = x_{n+1-i}$ for all i. $\square$

The symmetry in Corollary 14 was considered by Morales and Shi [16]. They gave two additional proofs using interpretations of $\operatorname*{CT}_{\boldsymbol{x}} M(n;k_1,k_2,k_3)$ by the volume of a flow polytope, and by the number of integer flows of a certain flow polytope.

3 The proof of Theorem 3

Our proof of Theorem 3 relies on two alternative constant term expressions for $h_n(t)$.

The first one is obtained by making the change of variables $x_i = y_1y_2\cdots y_i$ for all i in Lemma 6. We can write (2) as follows:

$$\begin{aligned} h_n(t) &= \operatorname*{CT}_{\boldsymbol{y}} \bar{\mathcal{H}}_n|_{x_i=y_1y_2\cdots y_i}\\ &= \operatorname*{CT}_{\boldsymbol{y}} \frac{y_1^{-t+n-2}y_n^{-t}\prod_{i=2}^{n-1} y_i^{-t+(i-1)(n-i)}\prod_{i=2}^{n-1}(1-y_{i+1}\cdots y_n)}{(1-y_1y_2\cdots y_n)^2\prod_{i=2}^{n-1}(1-y_1y_2\cdots y_i)^3\prod_{2\leqslant i<j\leqslant n}(1-y_{i+1}\cdots y_j)^2}. \end{aligned} \tag{11}$$

Here we shall explain the series expansion as follows. Since the change of variables transforms

$$\frac{1}{1-x_j/x_i} = \sum_{k\geqslant 0}(x_j/x_i)^k \mapsto \sum_{k\geqslant 0}(y_{i+1}\cdots y_j)^k = \frac{1}{1-y_{i+1}\cdots y_j}$$

we can simply treat $0<y_i<1$ in the new constant term expression. Later we will also use the change of variables $y_1 = x_1$ and $y_i = x_i/x_{i-1}$ for all $i \geqslant 2$ to transform back. Then the series expansion will be clear.

Equation (11) has three consequences, as stated in the next three lemmas.

Lemma 15. *Let $n \geqslant 3$ be a positive integer, and $N = \lfloor \frac{(n-1)^2}{4} \rfloor$ be as in part (3) of Theorem 3. Then $h_n(t) = 0$ for any $0 \leqslant t \leqslant N-1$.*

Proof. Let

$$p_i = \begin{cases} -t+n-2, & \text{if } i=1; \\ -t+(i-1)(n-i), & \text{if } 2 \leqslant i \leqslant n-1; \\ -t, & \text{if } i=n. \end{cases}$$

Clearly $p_i \leqslant p_r = -t+(r-1)(n-r) = -t+N$ for any $1 \leqslant i \leqslant n$, where $r = \lfloor \frac{n+1}{2} \rfloor$. If $0 \leqslant t \leqslant N-1$, then $p_r > 0$, i.e., $\bar{\mathcal{H}}_n|_{x_i=y_1y_2\cdots y_i}$ only contains positive powers in y_r, so that $\operatorname*{CT}_{y_r} \bar{\mathcal{H}}_n|_{x_i=y_1y_2\cdots y_i} = 0$. Hence $h_n(t)=0$. □

Lemma 16. *Let $n=2r$ be a positive even integer, and $N = \lfloor \frac{(n-1)^2}{4} \rfloor$ be as in part (3) of Theorem 3. Then*

$$h_n(N) = h_{2r}(r(r-1)) = \big(\operatorname*{CT}_{\boldsymbol{x}} M(r-1;3,1,2)\big)^2.$$

Proof. By (11), we have

$$h_{2r}(r(r-1)) = \operatorname*{CT}_{\boldsymbol{y}} \frac{y_1^{-(r-1)(r-2)} y_n^{-r(r-1)} \prod_{i=2}^{n-1} y_i^{-(r-i)(r-i+1)} \prod_{i=2}^{n-1}(1-y_{i+1}\cdots y_n)}{(1-y_1y_2\cdots y_n)^2 \prod_{i=2}^{n-1}(1-y_1y_2\cdots y_i)^3 \prod_{2\leqslant i<j\leqslant n}(1-y_{i+1}\cdots y_j)^2}.$$

Since $-(r-i)(r-i+1)=0$ when $i=r$ or $i=r+1$, $h_{2r}(r(r-1))$ is a power series in both y_r and y_{r+1}. We can eliminate y_r and y_{r+1} by substituting $y_r = y_{r+1} = 0$ directly. Then using the change of variables $y_1 = x_1$ and $y_i = x_i/x_{i-1}$ for all $i \geqslant 2$, $h_{2r}(r(r-1))|_{y_r=y_{r+1}=0} = F_1 \cdot F_2$, where

$$F_1 = \operatorname*{CT}_{\boldsymbol{x}} \frac{1}{\prod_{i=2}^{r-1} x_i^2 \prod_{i=2}^{r-1}(1-x_i)^3 \prod_{2\leqslant i<j\leqslant r-1}(x_i-x_j)^2},$$
$$F_2 = \operatorname*{CT}_{\boldsymbol{x}} \frac{x_n^{-r(r-1)} \prod_{i=r+1}^{n-1} x_i^{n-1} \prod_{i=r+1}^{n-1}(x_i-x_n)}{\prod_{r+1\leqslant i<j\leqslant n}(x_i-x_j)^2}.$$

Obviously

$$F_1 = \operatorname*{CT}_{\boldsymbol{x}} M(r-2;3,3,2) = \operatorname*{CT}_{\boldsymbol{x}} M(r-1,3,1,2),$$

where the last equation holds by Lemma 13. For F_2, by making the change of variables $x_i = y_i^{-1}$ for all i in Lemma 6 and then making $y_i = x_{n-i}$, we obtain

$$F_2 = \operatorname*{CT}_{\boldsymbol{y}} \frac{1}{\prod_{i=r+1}^{n-1} y_i^2 \prod_{i=r+1}^{n-1}(1-y_i) \prod_{r+1\leqslant i<j\leqslant n-1}(y_i-y_j)^2} = \operatorname*{CT}_{\boldsymbol{x}} M(r-1;3,1,2).$$

Here we clarify the series expansion under the change of variables $x_i \to y_i^{-1} \to x_{n-i}$:

$$\frac{1}{1-x_j/x_i} = \sum_{k\geqslant 0}(x_j/x_i)^k \mapsto \sum_{k\geqslant 0}(y_i/y_j)^k \mapsto \sum_{k\geqslant 0}(x_{n-i}/x_{n-j})^k = \frac{1}{1-x_{n-i}/x_{n-j}}.$$

Thus in the new constant term, we can still use the order $1 > x_1 > x_2 > \cdots > x_n > 0$. This completes the proof. □

Lemma 17. *Let $n = 2r - 1$ be a positive odd integer, and $N = \lfloor \frac{(n-1)^2}{4} \rfloor$ be as in part (3) of Theorem 3. Then*

$$h_n(N) = h_{2r-1}((r-1)^2) = \big(\operatorname*{CT}_{\boldsymbol{x}} M(r-1; 2, 1, 2)\big)^2.$$

Proof. By (11), we have

$$h_{2r-1}((r-1)^2) = \operatorname*{CT}_{\boldsymbol{y}} \frac{y_1^{-(r-2)^2} y_n^{-(r-1)^2} \prod_{i=2}^{n-1} y_i^{-(r-i)^2} \prod_{i=2}^{n-1}(1 - y_{i+1} \cdots y_n)}{(1 - y_1 y_2 \cdots y_n)^2 \prod_{i=2}^{n-1}(1 - y_1 y_2 \cdots y_i)^3 \prod_{2 \leqslant i < j \leqslant n}(1 - y_{i+1} \cdots y_j)^2}.$$

Since $-(r-i)^2 = 0$ only when $i = r$, $h_{2r-1}((r-1)^2)$ is a power series in y_r. Then the remaining work proceeds similar to the proof of Lemma 16. $\square$

The second expression for $h_n(t)$ is obtained by Corollary 12 and Lemma 9. We can also write (2) as follows:

$$h_n(t) = \operatorname*{CT}_{\boldsymbol{x}} \frac{(1 - x_1)^{-t-1} \prod_{i=2}^{n-1}(1 - x_i)^{n-2} \prod_{i=2}^{n-1}(x_1 - x_i)}{x_1 \prod_{i=2}^{n-1} x_i^2 \prod_{1 \leqslant i < j \leqslant n-1}(x_i - x_j)^2}, \tag{12}$$

where we make the change of variables $y_i = x_{n+1-i}$ after applying Lemma 9. It should be clear that we shall still treat $1 > x_1 > x_2 > \cdots > x_{n-1} > 0$ when expanded as Laurent series.

The formula in (12) also yields several results. Lemma 18 states that $h_n(t)$ is a polynomial in t, which allows us to extend the definition of $h_n(t)$ for negative t. Both Lemmas 19 and 20 are based on this representation.

Lemma 18. *For a positive integer n, $h_n(t)$ is a polynomial in t of degree $(n-1)^2$ with leading coefficient $\frac{1}{(n-1)^2!} \operatorname*{CT}_{\boldsymbol{x}} M(n-1; 1, 1, 2)$.*

Proof. We make the change of variables $x_i = y_1 y_2 \cdots y_i$ for all i again. Thus (12) becomes $h_n(t) = \operatorname*{CT}_{\boldsymbol{y}} F_1 \cdot F_2$, where

$$\begin{aligned} F_1 &= \frac{(1 - y_1)^{-t-1} \prod_{i=2}^{n-1}(1 - y_1 y_2 \cdots y_i)^{n-2}}{y_1^{(n-1)^2}}, \\ F_2 &= \frac{\prod_{i=2}^{n-1}(1 - y_2 \cdots y_i)}{\prod_{i=2}^{n-1} y_i^{(n-i)(n-i+1)} \prod_{1 \leqslant i < j \leqslant n-1}(1 - y_{i+1} \cdots y_j)^2}. \end{aligned}$$

Observe that

$$Q(t) = \operatorname*{CT}_{y_1} F_1 = \big[y_1^{(n-1)^2}\big] \left(\sum_{k=0}^{(n-1)^2} \binom{t+k}{k} y_1^k \right) \prod_{i=2}^{n-1} \left(\sum_{k=0}^{n-2} \binom{n-2}{k} (-y_2 \cdots y_i)^k y_1^k \right)$$

is a polynomial in t of degree $(n-1)^2$. The coefficients $[t^\ell]\ Q(t)$ are polynomials in $y_2,\ldots,y_{n-1}$. In particular, the leading coefficient $[t^{(n-1)^2}]\ Q(t)$ is $\frac{1}{(n-1)^2!}$. Therefore,

$$h_n(t)=\sum_{\ell=0}^{(n-1)^2} t^\ell\cdot \operatorname*{CT}_{\boldsymbol{y}}\big([t^\ell]\ Q(t)\big)F_2$$

is a polynomial in t of degree $(n-1)^2$.

For the leading coefficient, we make the change of variables $y_2=x_2$ and $y_i=x_i/x_{i-1}$ for $i>2$ to obtain

$$\begin{aligned}(n-1)^2!\cdot[t^{(n-1)^2}]\ h_n(t)&=\operatorname*{CT}_{\boldsymbol{y}} F_2=\operatorname*{CT}_{\boldsymbol{x}}\frac{1}{\prod_{i=2}^{n-1}x_i^2\prod_{i=2}^{n-1}(1-x_i)\prod_{2\leqslant i<j\leqslant n-1}(x_i-x_j)^2}\\&=\operatorname*{CT}_{\boldsymbol{x}} M(n-2;3,1,2)\\&=\operatorname*{CT}_{\boldsymbol{x}} M(n-2;1,3,2) \qquad\text{(by Corollary 14)}\\&=\operatorname*{CT}_{\boldsymbol{x}} M(n-1;1,1,2) \qquad\text{(by Lemma 13).}\end{aligned}$$

This completes the proof. □

Lemma 19. *For any $-n+1\leqslant t\leqslant -1$, we have $h_n(t)=0$.*

Proof. By (12), we can write

$$\begin{aligned}h_n(t)&=\operatorname*{CT}_{\boldsymbol{x}}\frac{\prod_{i=2}^{n-1}(1-x_i)^{n-2}}{\prod_{i=2}^{n-1}x_i^2\prod_{2\leqslant i<j\leqslant n-1}(x_i-x_j)^2}\operatorname*{CT}_{x_1}\frac{(1-x_1)^{-t-1}}{x_1\prod_{i=2}^{n-1}(x_1-x_i)}\\&=\operatorname*{CT}_{\boldsymbol{x}}\frac{\prod_{i=2}^{n-1}(1-x_i)^{n-2}}{\prod_{i=2}^{n-1}x_i^2\prod_{2\leqslant i<j\leqslant n-1}(x_i-x_j)^2}\operatorname*{CT}_{x_1}\frac{(1-x_1)^{-t-1}x_1^{-n+1}}{\prod_{i=2}^{n-1}(1-x_i/x_1)}.\end{aligned}$$

When $-n+1\leqslant t\leqslant -1$, the expansion of $\frac{(1-x_1)^{-t-1}x_1^{-n+1}}{\prod_{i=2}^{n-1}(1-x_i/x_1)}$ only contains negative powers in x_1. Hence the constant term in x_1 equals 0, which implies that $h_n(t)=0$. □

Lemma 20. *For a positive integer n, $h_n(t)=(-1)^{n-1}h_n(-t-n)$.*

Proof. By applying Lemma 5 to (12) and then making the change of variables $y_i=x_i$ for all i, we obtain

$$h_n(t)=(-1)^{n-1}\operatorname*{CT}_{\boldsymbol{x}}\frac{(1-x_1)^{t+n-1}\prod_{i=2}^{n-1}(1-x_i)^{n-2}\prod_{i=2}^{n-1}(x_1-x_i)}{x_1\prod_{i=2}^{n-1}x_i^2\prod_{1\leqslant i<j\leqslant n-1}(x_i-x_j)^2}.$$

Taking $t=-t-n$ in the above formula and comparing with (12) yield

$$h_n(t)=(-1)^{n-1}h_n(-t-n).$$ □

Now we conclude this section by giving a proof of our main result.

Proof of Theorem 3. Parts (1) and (2) are Lemmas 18 and 20, respectively. Part (3) holds by Lemmas 15 and 19 with the application of part (2). Part (4) follows from Lemmas 16 and 17. □

4 The computation of $h_n(t)$

In this section, we first briefly introduce several formulas about elementary symmetric functions. See [13] or [23] for a comprehensive overview of symmetric functions. Next we transform $h_n(t)$ into $D_n(0, t+n-1, n-2, 1, 2)$, where $D_n(\ell, t, k_1, k_2, k_3)$ is defined in (18) involving elementary symmetric functions. Using the well-known fact $\operatorname*{Res}_{x_i} \partial_{x_i} F = 0$, we derive a relation among $D_n(\ell, t, k_1, k_2, k_3)$ (See Proposition 24). Furthermore, for a fixed n, we can obtain a recursion for $D_n(0, t, n-2, 1, 2)$ (and hence for $h_n(t)$) with respect to t by the method of undetermined coefficients. Finally, we can construct $h_n(t)$ using the recursion and Theorem 3.

4.1 Basic formulas about elementary symmetric functions

Let $X := x_1 + x_2 + \cdots + x_n$, and let

$$e_\ell[X] := \sum_{1 \leqslant i_1 < i_2 < \cdots < i_\ell \leqslant n} x_{i_1} x_{i_2} \cdots x_{i_\ell}$$

be the ℓ-th elementary symmetric function in the variables $x_1, x_2, \ldots, x_n$. By convention, $e_0[X] = 1$, and $e_\ell[X] = 0$ if $\ell < 0$ or $\ell > n$. We need the well-known formula

$$e_\ell[X] = e_\ell[X - x_i] + x_i e_{\ell-1}[X - x_i]. \tag{13}$$

We also need the following known formulas, which will be frequently used later.

Lemma 21. *For any $0 \leqslant \ell \leqslant n$, we have*

$$\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} e_\ell[X - x_{i_1} - \cdots - x_{i_k}] = \binom{n-\ell}{k} e_\ell[X], \tag{14}$$

$$\sum_{i=1}^{n} x_i e_\ell[X - x_i] = (\ell+1) e_{\ell+1}[X]. \tag{15}$$

Proof. Observe that for any $1 \leqslant i_1 < \cdots < i_\ell \leqslant n$, the term $x_{i_1} \cdots x_{i_\ell}$ belongs to $e_\ell[X - x_i]$ exactly when $i \notin \{i_1, \ldots, i_\ell\}$. Since there are $n - \ell$ such indices i, each term of $e_\ell[X]$ appears $n - \ell$ times in $\sum_{i=1}^n e_\ell[X - x_i]$. Thus (14) holds.

Equation (15) holds by an analogous reasoning: for any $1 \leqslant i_1 < \cdots < i_{\ell+1} \leqslant n$, the term $x_{i_1} \cdots x_{i_{\ell+1}}$ belongs to $x_i e_\ell[X - x_i]$ exactly when $i \in \{i_1, \ldots, i_{\ell+1}\}$. Since there are $\ell + 1$ such indices i, each term of $e_{\ell+1}[X]$ appears $\ell + 1$ times in $\sum_{i=1}^n x_i e_\ell[X - x_i]$. This completes the proof. $\square$

Let

$$F(x_i, x_j) := \frac{x_i(1 - x_i)}{x_i - x_j} e_\ell[X - x_i]. \tag{16}$$

We need the following lemma; its proof is a straightforward algebraic check and is omitted.

Lemma 22. *For any distinct $i, j \in \{1, 2, \ldots, n\}$, let $F(x_i, x_j)$ be as in* (16)*. Then we have*

$$F(x_i, x_j) + F(x_j, x_i) = e_\ell[X - x_i - x_j] + e_{\ell+1}[X - x_i - x_j] - e_{\ell+1}[X].$$

4.2 A transformation for $h_n(t)$

For $k_1, k_2, k_3 \in \mathbb{N}$, let

$$\Phi_n(\ell, t, k_1, k_2, k_3) := \frac{e_\ell[Y - y_n] \prod_{i=1}^n x_i^{k_1} \prod_{i=1}^{n-1} (x_i - x_n)}{x_n^t \prod_{i=1}^n (1 - x_i)^{k_2} \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)^{k_3}}, \tag{17}$$

where $y_i = 1 - x_i$ are treated as variables, and $Y := y_1 + y_2 + \cdots + y_n$ (so $e_\ell[Y - y_i]$ is the ℓ-th elementary symmetric function in the variables $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n$). Define

$$D_n(\ell, t, k_1, k_2, k_3) := \operatorname*{Res}_{\boldsymbol{x}} \Phi_n(\ell, t, k_1, k_2, k_3). \tag{18}$$

By the definition of $\Phi_n(\ell, t, k_1, k_2, k_3)$, we have

(1) $\Phi_n(\ell, t, k_1, k_2, k_3) = 0$ if $\ell < 0$ or $\ell > n - 1$;

(2) $x_n^i \Phi_n(\ell, t, k_1, k_2, k_3) = \Phi_n(\ell, t - i, k_1, k_2, k_3)$ for any integer i.

The following lemma transforms the computation of $h_n(t)$ into the computation of $D_n(0, t + n - 1, n - 2, 1, 2)$.

Lemma 23. *For a positive integer n, let $h_n(t)$ and $D_n(\ell, t, k_1, k_2, k_3)$ be as in* (2) *and* (18)*, respectively. Then we have*

$$h_n(t) = D_n(0, t + n - 1, n - 2, 1, 2).$$

Proof. By (18), we have

$$D_n(0, t + n - 1, n - 2, 1, 2) = \operatorname*{Res}_{\boldsymbol{x}} \Phi_n(0, t + n - 1, n - 2, 1, 2) = \operatorname*{CT}_{\boldsymbol{x}} f,$$

where

$$f = \Phi_n(0, t + n - 1, n - 2, 1, 2) \prod_{i=1}^n x_i = \frac{\prod_{i=1}^{n-1} x_i^{n-1} \prod_{i=1}^{n-1} (x_i - x_n)}{x_n^t \prod_{i=1}^n (1 - x_i) \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)^2}.$$

Since f is proper in x_1, by Lemmas 7 and 10, we have

$$\operatorname*{CT}_{x_1} f = \operatorname*{CT}_{x_1 = 1} f = ((1 - x_1) f)\big|_{x_1 = 1}.$$

Then the lemma follows by checking that $((1 - x_1) f)\big|_{x_1 = 1}$ is the same as (3). □

4.3 A recursion for $h_n(t)$

We first derive a relation among $D_n(\ell, t, k_1, k_2, k_3)$'s for fixed k_1, k_2, k_3 and different ℓ and t. Then taking $k_1 = n - 2$, $k_2 = 1$ and $k_3 = 2$ to give a specific relation among $D_n(\ell, t, n-2, 1, 2)$. Furthermore, we derive a recursion for $D_n(0, t, n-2, 1, 2)$ with respect to t by the method of undetermined coefficients.

Let $\Phi_n(\ell, t, k_1, k_2, k_3)$ be as in (17). Let $V := \Phi_n(0, t, k_1, k_2, k_3)$. Then

$$D_n(\ell, t, k_1, k_2, k_3) = \operatorname*{Res}_{\boldsymbol{x}} e_\ell[Y - y_n] V. \tag{19}$$

Proposition 24. *Let $k_1, k_2, k_3 \in \mathbb{N}$, and $D_n(\ell, t, k_1, k_2, k_3)$ be as in* (19)*. Then we have*

$$\begin{aligned} A_{\ell+1} D_n(\ell+1, t, k_1, k_2, k_3) + A_\ell D_n(\ell, t, k_1, k_2, k_3) + \bar{A}_\ell D_n(\ell, t-1, k_1, k_2, k_3) \\ + A_{\ell-1} D_n(\ell-1, t, k_1, k_2, k_3) + \bar{A}_{\ell-1} D_n(\ell-1, t-1, k_1, k_2, k_3) = 0, \end{aligned}$$

where

$$\begin{aligned} A_{\ell+1} &= (\ell+1)\Big(k_2 - k_1 - 3 + \frac{k_3}{2}(2n-2-\ell)\Big), \\ A_\ell &= t - n + (\ell+1)\Big(k_2 - k_1 - 2 + \frac{k_3}{2}(2n-2-\ell)\Big) - (n-\ell)\Big(k_2 - 2 + \frac{k_3}{2}(n-1-\ell)\Big), \\ \bar{A}_\ell &= -\Big(t - n + 1 + (\ell+1)\Big(k_2 - k_1 - 2 + \frac{k_3}{2}(2n-2-\ell)\Big)\Big), \\ A_{\ell-1} &= -\bar{A}_{\ell-1} = -(n-\ell)\Big(k_2 - 1 + \frac{k_3}{2}(n-1-\ell)\Big). \end{aligned}$$

Proof. Recall that $y_i = 1 - x_i$. Since $e_\ell[Y - y_i] x_i (1 - x_i) V$ is a formal Laurent series in x_i for any $1 \leqslant i \leqslant n$, we have $\operatorname{Res}_{x_i} \partial_{x_i} e_\ell[Y - y_i] x_i (1 - x_i) V = 0$. Hence

$$\operatorname{Res}_{\boldsymbol{x}} \sum_{i=1}^{n} \partial_{x_i} e_\ell[Y - y_i] x_i (1 - x_i) V = 0.$$

It can be shown that

$$\begin{aligned} &\partial_{x_i} e_\ell[Y - y_i] x_i (1 - x_i) V \\ &= \begin{cases} e_\ell[Y - y_i] V \Big((k_1+1)(1-x_i) + (k_2-1)x_i - \frac{x_i(1-x_i)}{x_n - x_i} + k_3 \sum\limits_{j=1, j\neq i}^{n} \frac{x_i(1-x_i)}{x_j - x_i}\Big), & \text{if } i < n; \\ e_\ell[Y - y_n] V \Big((k_1 + 1 - t)(1-x_n) + (k_2-1)x_n + (k_3-1) \sum\limits_{j=1}^{n-1} \frac{x_n(1-x_n)}{x_j - x_n}\Big), & \text{if } i = n. \end{cases} \end{aligned}$$

Now we take the sum over all i. The computation is naturally divided into three parts, denoted S_1, S_2, S_3 respectively. In detail,

$$\begin{aligned} S_1 &= -t(1-x_n) e_\ell[Y - y_n] + \sum_{i=1}^{n} (k_1+1)(1-x_i) e_\ell[Y - y_i] \\ &= -t y_n e_\ell[Y - y_n] + \sum_{i=1}^{n} (k_1+1) y_i e_\ell[Y - y_i] && \text{(by } y_i = 1 - x_i) \\ &= -t y_n e_\ell[Y - y_n] + (k_1+1)(\ell+1) e_{\ell+1}[Y], && \text{(by (15))} \\ S_2 &= \sum_{i=1}^{n} (k_2-1) x_i e_\ell[Y - y_i] = \sum_{i=1}^{n} (k_2-1)(1-y_i) e_\ell[Y - y_i] && \text{(by } y_i = 1 - x_i) \\ &= (k_2-1)\big((n-\ell) e_\ell[Y] - (\ell+1) e_{\ell+1}[Y]\big), && \text{(by (14) and (15))} \end{aligned}$$

and

$$\begin{aligned}
S_3 &= k_3 \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \frac{x_i(1-x_i)}{x_j - x_i} e_\ell[Y-y_i] - \sum_{i=1}^{n-1} \frac{x_i(1-x_i)}{x_n - x_i} e_\ell[Y-y_i] - \sum_{j=1}^{n-1} \frac{x_n(1-x_n)}{x_j - x_n} e_\ell[Y-y_n] \\
&= k_3 \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} F(y_i, y_j) - \sum_{i=1}^{n-1} F(y_i, y_n) - \sum_{j=1}^{n-1} F(y_n, y_j) \qquad \text{(by } y_i = 1 - x_i\text{)} \\
&= \frac{k_3}{2} \sum_{1\leqslant i\neq j\leqslant n} (F(y_i, y_j) + F(y_j, y_i)) - \sum_{i=1}^{n-1} (F(y_i, y_n) + F(y_n, y_i)) \\
&= \frac{k_3}{2} \sum_{1\leqslant i\neq j\leqslant n} (e_\ell[Y - y_i - y_j] - e_{\ell+1}[Y] + e_{\ell+1}[Y - y_i - y_j]) \\
&\quad - \sum_{i=1}^{n-1} (e_\ell[Y - y_i - y_n] - e_{\ell+1}[Y] + e_{\ell+1}[Y - y_i - y_n]) \qquad \text{(by Lemma 22)} \\
&= \frac{k_3}{2} \big((n-\ell)(n-1-\ell)e_\ell[Y] - (\ell+1)(2n-2-\ell)e_{\ell+1}[Y]\big) - (n-1-\ell)e_\ell[Y-y_n] \\
&\quad + (n-1)e_{\ell+1}[Y] - (n-2-\ell)e_{\ell+1}[Y-y_n]. \qquad \text{(by (14) and (14))}
\end{aligned}$$

By (13), we have $e_{\ell+1}[Y] = e_{\ell+1}[Y-y_n] + y_n e_\ell[Y-y_n]$, which yields

$$-(S_1+S_2+S_3) = A_{\ell+1}e_{\ell+1}[Y-y_n] + (A_\ell + \bar{A}_\ell x_n)e_\ell[Y-y_n] + (A_{\ell-1} + \bar{A}_{\ell-1}x_n)e_{\ell-1}[Y-y_n].$$

Then the lemma follows by $\operatorname{Res}_{\boldsymbol{x}}(S_1 + S_2 + S_3)V = 0$ and $x_n\Phi_n(\ell, t, k_1, k_2, k_3) = \Phi_n(\ell, t - 1, k_1, k_2, k_3)$. □

For convenience, let $d_n(\ell, t) := D_n(\ell, t, n-2, 1, 2)$. Substituting $k_1 = n-2$, $k_2 = 1$, $k_3 = 2$ into Proposition 24 yields

$$\begin{aligned}
&(\ell+1)(n-2-\ell)d_n(\ell+1, t) + (t - (n-2\ell)(n-2-\ell) - 1)d_n(\ell, t) \\
&- (t + \ell(n-2-\ell))d_n(\ell, t-1) - (n-\ell)(n-1-\ell)d_n(\ell-1, t) \\
&+ (n-\ell)(n-1-\ell)d_n(\ell-1, t-1) = 0.
\end{aligned} \tag{20}$$

The remaining work is to obtain a recursion for $d_n(0, t)$ with respect to t, which can be completed by the following steps:

(S1) We obtain $n-1$ equations, say $E_0, E_1, \ldots, E_{n-2}$, by substituting $\ell = 0, 1, \ldots, n-2$ into (20) respectively. For any $\ell \leqslant -1$ and $\ell \geqslant n-1$, (20) becomes $0 = 0$. The only nontrivial case is when $\ell = n-1$, for which we can verify $d_n(n-1, t) = 0$ as follows:

$$\begin{aligned}
d_n(n-1, t) &= \operatorname{Res}_{\boldsymbol{x}} \Phi_n(n-1, t, n-2, 1, 2) \\
&= \operatorname*{CT}_{\boldsymbol{x}} \frac{\prod_{i=1}^{n} x_i^{n-1} \cdot \prod_{i=1}^{n-1}(x_i - x_n)}{x_n^t(1-x_n)\prod_{1\leqslant i<j\leqslant n}(x_i - x_j)^2} \\
&= \operatorname*{CT}_{\boldsymbol{x}} \frac{\prod_{i=2}^{n} x_i^{n-1} \cdot \prod_{i=2}^{n-1}(x_i - x_n)}{x_n^t(1-x_n)\prod_{2\leqslant i<j\leqslant n}(x_i - x_j)^2} \operatorname*{CT}_{x_1} \frac{x_1^{-n+2}}{(1 - x_n/x_1)\prod_{i=2}^{n-1}(1 - x_i/x_1)^2}.
\end{aligned}$$

Since $n \geqslant 3$, the rightmost part only contains negative powers in x_1, which yields $d_n(n-1,t) = 0$.

(S2) For each $\ell \in \{0, 1, \dots, n-2\}$, we substitute $t = t - i$ for $i = 0, 1, \dots, n-2-\ell$ into E_ℓ respectively to obtain $\binom{n}{2}$ equations:

$$E_{0,0}, E_{0,1}, \dots, E_{0,n-2}, E_{1,0}, E_{1,1}, \dots, E_{1,n-3}, \dots, E_{n-2,0},$$

where $E_{\ell,i} = E_\ell|_{t=t-i}$.

(S3) We obtain a recursion for $d_n(0,t)$ with respect to t using the method of undetermined coefficients. That is, we consider the equation

$$\sum_{\ell=0}^{n-2} \sum_{i=0}^{n-2-\ell} a_{\ell,i} E_{\ell,i} \tag{21}$$

with undetermined coefficients the $a_{\ell,i}$'s such that $d_n(\ell, t-i)$ vanish for all $\ell > 0$. This gives a linear system described as follows:

$$B(a_{0,0}, a_{0,1}, \dots, a_{0,n-2}, a_{1,0}, a_{1,1}, \dots, a_{1,n-3}, \dots, a_{n-3,0}, a_{n-3,1})^T = a_{n-2,0}\boldsymbol{b}, \tag{22}$$

where B is a $\frac{(n+1)(n-2)}{2} \times \frac{(n+1)(n-2)}{2}$ upper triangular matrix with $B_{i,i} = (\ell+1)(n-2-\ell)$ if $\sum_{j=0}^{\ell-1}(n-1-j) < i \leqslant \sum_{j=0}^{\ell}(n-1-j)$. That is, all the diagonal entries of B are not zero. Hence the linear system in (22) must have a solution. Substituting any nonzero particular solution (say, with respect to $a_{n-2,0} = 1$) into (21) yields an equation in $d_n(0,t), d_n(0,t-1), \dots, d_n(0,t-n+1)$.

We compute the recursion for $d_4(0,t)$ in the following example to explain the above steps.

Example 25. Fix $n = 4$. By (20), we have

$$\begin{aligned}(\ell+1)(2-\ell)d_4(\ell+1,t) + (t - 2(2-\ell)^2 - 1)d_4(\ell,t) - (t + \ell(2-\ell))d_4(\ell,t-1)\\ - (4-\ell)(3-\ell)d_4(\ell-1,t) + (4-\ell)(3-\ell)d_4(\ell-1,t-1) = 0.\end{aligned}$$

Carrying out the above three steps yields

$$\begin{aligned}(t-3)(t-5)^2 d_4(0,t) + (-3t^3 + 28t^2 - 74t + 42)d_4(0,t-1)\\ + (3t^3 - 17t^2 + 19t - 3)d_4(0,t-2) - t^2(t-2)d_4(0,t-3) = 0.\end{aligned}$$

Furthermore, by Lemma 23, we obtain

$$\begin{aligned}t(t-2)^2 h_4(t) + (-3t^3 + t^2 + 13t - 9)h_4(t-1)\\ + (3t^3 + 10t^2 - 2t - 18)h_4(t-2) - (t+1)(t+3)^2 h_4(t-3) = 0.\end{aligned} \tag{23}$$

It is easy to check that

$$h_4(t) = \frac{(t-1)t^2(t+1)(t+2)(t+3)(t+4)^2(t+5)}{60480},$$

which fits (23).

Remark 26. A formula for $D_n(\ell, t, k_1, k_2, k_3)$ for $\ell > 0$ can be obtained, but is not needed here.

For general n, the recursion for $d_n(0, t)$ can be described as $\sum_{i=0}^{n-1} c_i(t) d_n(0, t-i) = 0$, where $c_i(t)$ is a polynomial in t for each i. By Lemma 23, we have

$$h_n(t) = -\frac{1}{c_0(t+n-1)} \sum_{i=1}^{n-1} c_i(t+n-1) h_n(t-i), \tag{24}$$

which gives a recursion for $t > t_0$, where t_0 is the maximum integer root (if it exists) of $c_0(t+n-1)$.

Example 27. Analogous to Example 25, for $n = 5$, we can obtain

$$\begin{aligned} &t(t-4)(t-3)^2 h_5(t) - (4t^4 - 16t^3 - 22t^2 + 126t - 80) h_5(t-1) \\ &+ (6t^4 + 12t^3 - 74t^2 - 80t + 208) h_5(t-2) \\ &- (4t^4 + 32t^3 + 50t^2 - 106t - 208) h_5(t-3) + (t+4)^2 (t+5)(t+1) h_5(t-4) = 0. \end{aligned} \tag{25}$$

Then we can obtain an explicit expression for $h_5(t)$ by the method of undetermined coefficients. By parts (1) and (3) of Theorem 3, we set

$$h_5(t) = \Big(\frac{300}{16!} t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0\Big) \prod_{i=-8}^{3} (t-i).$$

Part (4) of Theorem 3 gives $h_5(4) = 9$. Substituting $t = 5$ into (25) yields $20h_5(5) - 500h_5(4) = 0$ since $h_5(1) = h_5(2) = h_5(3) = 0$. Thus $h_5(5) = 225$. By part (2) of Theorem 3, we have $h_5(-9) = h_5(4) = 9$ and $h_5(-10) = h_5(5) = 225$. Solving

$$\begin{cases} h_5(4) = 479001600(a_0 + 4a_1 + 16a_2 + 64a_3) + \dfrac{160}{91} = 9, \\ h_5(-9) = 479001600(a_0 - 9a_1 + 81a_2 - 729a_3) + \dfrac{32805}{728} = 9, \\ h_5(5) = 6227020800(a_0 + 5a_1 + 25a_2 + 125a_3) + \dfrac{3125}{56} = 225, \\ h_5(-10) = 6227020800(a_0 - 10a_1 + 100a_2 - 1000a_3) + \dfrac{6250}{7} = 225 \end{cases}$$

gives

$$a_0 = 0, \quad a_1 = \frac{1}{34871316480}, \quad a_2 = \frac{127}{348713164800}, \quad a_3 = \frac{1}{6974263296}.$$

Therefore,

$$h_5(t) = \frac{t(t+5)(5t^2 + 25t + 2)}{348713164800} \prod_{i=-8}^{3} (t-i).$$

Computer experiments suggest us to conjecture that $t_0 = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$. For instance, $t_0 = 4$ when $n = 5$. To verify this conjecture, we need a formula for $c_0(t)$. This can be obtained by considering a subsystem of (21). Observe that $d_n(\ell, t)$ only appears in $E_{\ell-1,0}$, $E_{\ell,0}$ and $E_{\ell+1,0}$. Thus we only need to consider $\sum_{\ell=0}^{n-2} a_{\ell,0} E_{\ell,0}$. The coefficient of $d_n(\ell, t)$ in $\sum_{\ell=0}^{n-2} a_{\ell,0} E_{\ell,0}$ is

$$\ell(n-1-\ell)a_{\ell-1,0} + (t - (n-2\ell)(n-2-\ell) - 1)a_{\ell,0} - (n-1-\ell)(n-2-\ell)a_{\ell+1,0}$$

for each $0 \leqslant \ell \leqslant n-2$. In particular, the coefficient of $d_n(0, t)$ is

$$c_0(t) = (t - (n-1)^2)a_{0,0} - (n-1)(n-2)a_{1,0}.$$

We obtain

$$B \begin{pmatrix} a_{0,0} \\ a_{1,0} \\ \vdots \\ a_{n-3,0} \end{pmatrix} = -a_{n-2,0}\boldsymbol{b}$$

by setting the coefficients of $d_n(\ell, t)$ equal to 0 for any $\ell > 0$, where $\boldsymbol{b} = (0, \ldots, 0, -2, t-1)^T$, and B is a $(n-2) \times (n-2)$ upper triangular matrix with

$$B_{i,j} = \begin{cases} i(n-1-i), & \text{if } i = j; \\ t - (n-2i)(n-2-i) - 1, & \text{if } i = j-1; \\ -(n-1-i)(n-2-i), & \text{if } i = j-2; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\det B$ is a constant. Applying Cramer's Rule and the rules of column transformation, we have

$$a_{0,0} = \frac{(-1)^n a_{n-2,0} \det(\beta_2, B', \boldsymbol{b})}{\det B}, \quad a_{1,0} = \frac{(-1)^{n-1} a_{n-2,0} \det(\beta_1, B', \boldsymbol{b})}{\det B},$$

where β_i denotes the i-th column of B for $i = 1, 2$, and B' is obtained from B by removing the first two columns. Consequently,

$$\begin{aligned} c_0(t) &= \frac{(-1)^n a_{n-2,0}}{\det B} \big((t-(n-1)^2) \det(\beta_2, B', \boldsymbol{b}) + (n-1)(n-2) \det(\beta_1, B', \boldsymbol{b}) \big) \\ &= \frac{(-1)^n a_{n-2,0}}{\det B} \det C, \end{aligned}$$

where

$$C = \begin{pmatrix} t-(n-1)^2 & -(n-1)(n-2) & \mathbf{0} & 0 \\ \beta_1 & \beta_2 & B' & \boldsymbol{b} \end{pmatrix}. \tag{26}$$

The matrix C is of size $(n-1) \times (n-1)$. To be precise, $C = (C_{i,j})_{(n-1)\times(n-1)}$, where

$$C_{i,j} = \begin{cases} t - (n-2i+2)(n-1-i) - 1, & \text{if } i = j; \\ (i-1)(n-i), & \text{if } i = j+1; \\ -(n-1-i)(n-i), & \text{if } i = j-1; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\det C$ is just the characteristic polynomial of the $(n-1)\times(n-1)$ matrix $-C|_{t=0}$, it is a polynomial in t of degree $n-1$. It is not hard to compute the determinant for given n. This leads to the following conjecture.

Conjecture 28. Let C be the matrix in (26). Then

$$\det C = \begin{cases} (t-n+1)\prod_{i=0}^{r-2}(t-n+1-r(r-1)+i(i+1))^2, & \text{if } n=2r; \\ (t-n+1)(t-n+1-(r-1)^2)\prod_{i=1}^{r-2}(t-n+1-(r-1)^2+i^2)^2, & \text{if } n=2r-1. \end{cases}$$

For example,

$$\det C = \begin{cases} (t-13)(t-55)^2(t-53)^2(t-49)^2(t-43)^2(t-35)^2(t-25)^2, & \text{if } n=14, \\ (t-14)(t-63)(t-62)^2(t-59)^2(t-54)^2(t-47)^2(t-38)^2(t-27)^2, & \text{if } n=15. \end{cases}$$

Conjecture 28 has been proved in [11]. Thus,

$$t_0 = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor = \begin{cases} r(r-1), & \text{if } n=2r; \\ (r-1)^2, & \text{if } n=2r-1. \end{cases}$$

It follows that parts (3) and (4) of Theorem 3 give initial values of $h_n(t)$ for the recursion (24).

5 A conjecture about the generating function of $h_n(t)$

Theorem 2 suggests one should consider properties of the generating function of $h_n(t)$. It is necessary to clarify some definitions. A real number sequence $(a_0, a_1, \ldots, a_m)$ is said to be *palindromic* if $a_i = a_{m-i}$ for all i. It is said to be *log-concave* (short for *logarithmically concave*) if $a_i^2 \geqslant a_{i-1}a_{i+1}$ for all $1 \leqslant i \leqslant m-1$. Let $b_i = \frac{1}{\binom{m}{i}}a_i$ for all i. Then the sequence $(a_0, a_1, \ldots, a_m)$ is said to be *strongly log-concave* if $b_i^2 \geqslant b_{i-1}b_{i+1}$ for all $1 \leqslant i \leqslant m-1$. See [24] for further details.

We pose the following conjecture.

Conjecture 29. Let $N = \lfloor \frac{(n-1)^2}{4} \rfloor$ and $\hat{N} = \lfloor \frac{(n-2)^2}{2} \rfloor$. Then

$$h_n^*(y) := (1-y)^{(n-1)^2+1}\sum_{t\geqslant 0} h_n(t)y^t = y^N \sum_{i=0}^{\hat{N}} a_i y^i,$$

where the a_i's are positive integers satisfying $a_i = a_{\hat{N}-i}$. Moreover, the polynomial $h_n^*(y)$ is real-rooted. Consequently, $(a_0, a_1, \ldots, a_{\hat{N}})$ is unimodal and strongly log-concave.

We have verified Conjecture 29 for $3 \leqslant n \leqslant 29$. For example,

$$h_5^*(y) = 3y^4(3y^4 + 24y^3 + 46y^2 + 24y + 3).$$

The roots of $h_5^*(y)$ are 0 (with multiplicity 4), $-3\pm\sqrt{6}$, $-1\pm\frac{\sqrt{6}}{3}$. Also the other properties are easy to verify.

Data of $h_n^*(y)$ for $3 \leqslant n \leqslant 29$ is also available at [27].

Acknowledgements

The authors would like to express sincere gratitude for all the suggestions that have improved the presentation of this paper. Guoce Xin was partially supported by the National Natural Science Foundation of China (No. 12571355). Chen Zhang was partially supported by the National Natural Science Foundation of China (No. 12601634), and the Postdoctoral Fellowship Program and China Postdoctoral Science Foundation (No. BX20250066).

References

[1] P. Alexandersson, S. Hopkins and G. Zaimi, *Restricted Birkhoff polytopes and Ehrhart period collapse*, Discrete Comput. Geom. 73 (2025), 62–78.

[2] H. Anand, V. C. Dumir and H. Gupta, *A combinatorial distribution problem*, Duke Math. J. 33 (1966), 757–769.

[3] C. A. Athanasiadis, *Ehrhart polynomials, simplicial polytopes, magic squares and a conjecture of Stanley*, J. Reine Angew. Math. 583 (2005), 163–174.

[4] W. Baldoni-Silva and M. Vergne, *Residues formulae for volumes and Ehrhart polynomials of convex polytopes*, arXiv:math/0103097v1, 2001.

[5] M. Beck and D. Pixton, *The Ehrhart polynomial of the Birkhoff polytope*, Discrete Comput. Geom. 30 (2003), 623–637.

[6] C. S. Chan and D. P. Robbins, *On the volume of the polytope of doubly stochastic matrices*, Exp. Math. 8 (1999), 291–300.

[7] J. A. De Loera, F. Liu and R. Yoshida, *A generating function for all semi-magic squares and the volume of the Birkhoff polytope*, J. Algebr. Comb. 30 (2009), 113–139.

[8] P. Diaconis and A. Gangolli, *Rectangular arrays with fixed margins*, IMA Series on Volumes in Mathematics and its Applications, # 72 Springer-Verlag (1995), 15–41.

[9] E. Ehrhart, *Sur les carrés magiques*, C. R. Acad. Sci. Paris 227 A (1973), 575–577.

[10] P. J. Forrester and S. O. Warnaar, *The importance of the Selberg integral*, B. Am. Math. Soc. 45 (2008), 489–534.

[11] J. Hu and C. Zhang, *A proof of Xin-Zhang's tridiagonal determinant conjecture*, arXiv:2601.03082, 2026.

[12] C. G. J. Jacobi, *De resolutione aequationum per series infinitas*, J. Reine Angew. Math. 6 (1830), 257–286.

[13] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd edition, Clarendon Press, Oxford, 1995.

[14] P. A. MacMahon, *Combinatory Analysis*, Chelsea Publishing Co., New York, 1960.

[15] P. McMullen, *The number of faces of simplicial polytopes*, Israel J. Math. 9 (1971), 559–570.

[16] A. H. Morales and W. Shi, *Refinements and symmetries of the Morris identity for volumes of flow polytopes*, Comptes Rendus. Mathématique 359 (2021), 823–851.

[17] W. G. Morris, *Constant term identities for finite and affine root system*, Ph.D. Thesis, University of Wisconsin, Madison, 1982.

[18] J. Mount, *Fast unimodular counting*, Combin. Probab. Comput. 9 (2000), 277–285.

[19] A. Selberg, *Bemerkninger om et multipelt integral*, Nordisk Mat. Tidskr. 26 (1944), 71–78.

[20] R. P. Stanley, *Linear homogeneous Diophantine equations and magic labelings of graphs*, Duke Math. J. 40 (1973), 607–632.

[21] R. P. Stanley, *Magic labelings of graphs, symmetric magic squares, systems of parameters, and Cohen-Macaulay rings*, Duke Math. J. 43 (1976), 511–531.

[22] R. P. Stanley, Combinatorics and Commutative Algebra, Progress in Mathematics 41, Birkhäuser, Boston, first edition, 1983; second edition, 1996.

[23] R. P. Stanley, *Enumerative Combinatorics (Volume 2)*, Cambridge studies in advanced mathematics, Cambridge, 2012.

[24] R. P. Stanley, *Algebraic Combinatorics: Walks, Trees, Tableaux, and More*, Springer, New York, 2013.

[25] G. Xin, *A fast algorithm for MacMahon's partition analysis*, Electron. J. Combin. 11 (2004), #R58.

[26] G. Xin, *The Ring of Malcev-Neumann Series and The Residue Theorem*, Ph.D. thesis, Brandeis University, `arXiv:math.CO/0405133`, 2004.

[27] G. Xin and C. Zhang, *Morris-Data*, `https://pan.baidu.com/s/16zjRkvXMQfhDK7E7fBhZtQ` with passcode `morr`.

[28] G. Xin, C. Zhang, Y. Zhou and Y. Zhong, *The constant term algebra of type A: the structure*, Adv. Math. 465 (2025), 110154.

[29] D. Zeilberger, *Proof of a conjecture of Chan, Robbins, and Yuen*, Electron. Trans. Numer. Anal. 9 (1999), 147–148.