Monomial Stability of Frobenius Images

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Submitted: Jun 24, 2025; Accepted: Jun 8, 2026; Published: Sep 25, 2026

Abstract

We study representation stability in the sense of Church, Ellenberg, and Farb [Duke Math. J., 164(9), 2015] through the lens of symmetric function theory and the different symmetric function bases. We show that a sequence, $(F_n)_n$, where F_n is a homogeneous symmetric function of degree n, has stabilizing Schur coefficients if and only if it has stabilizing monomial coefficients. More generally, we develop a framework for checking when stabilizing coefficients transfer from one symmetric function basis to another. We also see how one may compute representation stable ranges from the monomial expansions of the F_n.

As applications, we reprove and refine the representation stability of diagonal harmonics, DH_n. We also observe new representation stability phenomenon of the Garsia-Haiman modules. This establishes certain stability properties of the modified Macdonald polynomials, $\tilde{H}_{\mu^{(n)}}[X;q,t]$ and the modified q,t-Kostka numbers, $\tilde{K}_{\mu^{(n)},\nu[n]}(q,t)$, for arbitrary sequences of partitions with $\mu^{(n)} \vdash n$ and $\mu^{(n)} \subseteq \mu^{(n+1)}$.
Mathematics Subject Classifications: 05E05, 05E10, 20C30

1 Introduction

In their seminal work, Church, Ellenberg, and Farb [CEF15] created an effective framework called representation stability for studying various previously observed phenomena. The typical example involves a naturally occurring sequence $(V_n)_{n\in\mathbb{N}}$, where each V_n is an A_n-module for an algebra A_n, typically the group algebra of the symmetric group $\mathbf{k}[S_n]$ over some field $\mathbf{k}$. Despite the fact that different n yield representations of different algebras, these sequences may have decompositions into irreducibles that surprisingly "stabilize" in some appropriate sense. As an example, consider the configuration space of n distinct points in $\mathbb{C}$, $\mathrm{Conf}_n(\mathbb{C})$. Then the natural action of S_n on this space induces an action on the cohomology $H^k(\mathrm{Conf}_n(\mathbb{C}))$, and hence yields an S_n-representation. Letting $\mathbb{S}^\lambda$ denote the irreducible representation of S_n corresponding to partition $\lambda \vdash n$ and taking

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 https://doi.org/10.37236/14309

$k = 1$, the first few representations admit the following decompositions

$$\begin{aligned}
H^1(\mathrm{Conf}_2(\mathbb{C})) &\cong \mathbb{S}^{(2)} \oplus \mathbb{S}^{(1,1)} \\
H^1(\mathrm{Conf}_3(\mathbb{C})) &\cong \mathbb{S}^{(3)} \oplus \mathbb{S}^{(2,1)} \\
H^1(\mathrm{Conf}_4(\mathbb{C})) &\cong \mathbb{S}^{(4)} \oplus \mathbb{S}^{(3,1)} \oplus \mathbb{S}^{(2,2)} \\
H^1(\mathrm{Conf}_5(\mathbb{C})) &\cong \mathbb{S}^{(5)} \oplus \mathbb{S}^{(4,1)} \oplus \mathbb{S}^{(3,2)} \\
H^1(\mathrm{Conf}_6(\mathbb{C})) &\cong \mathbb{S}^{(6)} \oplus \mathbb{S}^{(5,1)} \oplus \mathbb{S}^{(4,2)}
\end{aligned}$$

and this pattern continues with $H^1(\mathrm{Conf}_n(\mathbb{C})) \cong \mathbb{S}^{(n)} \oplus \mathbb{S}^{(n-1,1)} \oplus \mathbb{S}^{(n-2,2)}$ for all $n \geqslant 4$. Note the multiplicity of $\mathbb{S}^{(n-|\lambda|,\lambda_1,\cdots,\lambda_{l(\lambda)})}$ has stabilized to a constant for each possible λ and we will state the precise definition of the phenomenon in the preliminaries. In the example above the stable multiplicities are 1 for $\lambda = \emptyset, (1), (2)$ and zero otherwise. Church and Farb [CF13] also observed this phenomenon with the cohomology of pure braid groups, homology of certain Torelli subgroups, and cohomology of flag varieties among many other examples.

The major contribution of Church, Ellenberg, and Farb [CEF15] was the categorification of representation stability. It turns out that all the examples of $(V_n)_{n\in\mathbb{N}}$ mentioned have natural connective maps $\varphi_n : V_n \to V_{n+1}$, which satisfy certain properties making them compatible with representations in the sequence. The authors of [CEF15] identified the sequences $(V_n, \varphi_n)_{n\in\mathbb{N}}$ with functors from the category of finite sets with injections, **FI**, to the category of $\mathbb{C}$-modules; such functors are called **FI**-modules. They were then able to show that a sequence exhibits representation stability if and only if the corresponding functor is finitely generated in the category of **FI**-modules.

Furthermore, they provided an explicit range $n \geqslant \text{stab-deg}(V_\bullet) + \text{wt}(V_\bullet)$ after which point the multiplicities of the $\mathbb{S}^{(n-|\lambda|,\lambda_1,\cdots,\lambda_{l(\lambda)})}$ would stabilize. However, this lower bound may not be sharp and the quantities $\text{stab-deg}(V_\bullet)$, $\text{wt}(V_\bullet)$ known as stability degree and weight, respectively, may be difficult to compute. One of the major goals of this paper is to provide ways of getting around needing to compute stability degree in particular.

Many authors have generalized the work of Church, Ellenberg, and Farb to other categories ([Gad17] [GL15] [GW18] [Lau21] to name a few) as well as working on finding explicit stable decompositions and bounds for different kinds of stable ranges of given sequences $(V_n)_{n\in\mathbb{N}}$ [Bah24]. One inviting approach for combinatorialists is to use the Frobenius characteristic map, which provides a bridge between representations of S_n and symmetric functions (see [Sag01] Section 4.7). For instance, Hersch and Reiner [HR17] worked heavily with higher Lie characters in the algebra of symmetric functions in order to find stable ranges for the cohomology of configuration spaces of points in $\mathbb{R}^d$.

Another benefit of working with the Frobenius image of representation is that we may have some formula describing the symmetric function corresponding to a representation, when the irreducible decomposition or even character values may not be known. For example, the diagonal harmonics have no known combinatorial descriptions for their irreducible decompositions but their Frobenius images have an explicit monomial formula proved by Carlsson and Mellit [CM18] and first conjectured in [HHL+05b]. Similarly, the modified Macdonald polynomials, $\widetilde{H}_\mu[X; q, t]$, which are the Frobenius images of certain S_n-modules called Garsia-Haiman modules, have no known combinatorial formula for their Schur expansion, but have a monomial formula given in [HHL05a]. Both of these examples turn out to exhibit representation stability.

We describe a method for leveraging these monomial expansions to prove representation stability and provide stable ranges that may not be sharp but are easy to compute from the combinatorics of the monomial expansions. One of the main results of this paper is that a sequence of symmetric functions has stabilizing Schur expansions if and only if it has stabilizing monomial expansion, but the monomial expansions may not stabilize uniformly (Proposition 14). We also study how stability of expansions transfers between pairs of symmetric function bases in general, i.e. for bases that are not the Schur or monomial ones. In Proposition 11, we provide necessary and sufficient conditions on the change of basis coefficients to get stability transfer between a pair of bases. This proposition is then systematically applied to each pair of classical symmetric function bases. We identify the monomial and "class function" basis, p_λ/z_λ, as being particularly compatible with Schur functions yielding Theorem 19, the main result. In the second half of the paper, we apply the theorem to find stable ranges of some diagonal harmonics and Garsia-Haiman modules. We obtain refinements of stability results for the diagonal coinvariant algebras by finding stable ranges for individual Specht modules in Proposition 24. We also generalize the work of Mbirika and Tymoczko [MT20] from Garsia-Procesi modules (related to the Hall-Littlewood polynomials) to the more general Garsia-Haiman modules (related to the modified Macdonald polynomials) in Proposition 34.

Section 2 covers the preliminaries including necessary results for torsion-free **FI**-modules and combinatorial formulas for the change of basis coefficients between the classical symmetric function bases. Section 3 proves the conditions for stability transfer and applies it to the classical bases, concluding with the main result. Sections 4 and 5 cover the applications to coinvariant algebras and Macdonald polynomials. It turns out that the stable range in Theorem 19(a) can be slightly sharpened and proved in a different way, although this will not lead to any sharper ranges in the applications. Hence, we relegate this alternative proof to the Appendix. If the reader is only interested in the applications to harmonics, coinvariant algebras and Garsia-Haiman modules, we recommend the reader read the Appendix in place of Section 3.

Acknowledgments

I would like to thank Jennifer Wang, Nir Gadish, George Seelinger, and my advisor Jim Haglund for helpful discussions and insightful comments.

2 Preliminaries and definitions

We will denote the irreducible representation of S_n, called a *Specht module*, indexed by partition $\lambda \vdash n$ as $\mathbb{S}^\lambda$. If λ is a partition of some m and $n - |\lambda| \geqslant \lambda_1$, then let $\lambda[n] := (n-|\lambda|, \lambda_1, \ldots, \lambda_{l(\lambda)})$. Note that every partition can be put into the form $\lambda[n]$ for a unique λ and n. We will sometimes use λ to refer to the set of cells in its Young diagram and let $H \subseteq_{\text{diag}} \lambda$ be subset of these cells, while $\mu \subseteq_{\text{set}} \lambda$ will denote a subset of the multi-set of parts of λ, and $\mu \subseteq \lambda$ will denote a sub-partition. As usual, $l(\lambda)$ is the *length* of a partition and λ' is the *conjugate partition* to λ. Partitions will always be drawn and referred to in French notation.

We summarize the definition of representation stability introduced by Church, Ellenberg, and Farb in [CEF15]. Let **FI** be the category of finite sets with injections as morphisms. An ***FI**-module* is a functor, F, from **FI** to the category of $\mathbb{C}$-modules, $\mathbf{Mod}_{\mathbb{C}}$. Throughout the paper, we will work with the field $\mathbb{C}$, although it may be replaced by any characteristic 0 field. Letting $[n] = \{1, \ldots, n\}$ and $\iota_n : [n] \to [n+1]$ the injection mapping i to i, we can set $V_n = F([n])$ and $\varphi_n = F(\iota_n)$. Then F induces an S_n-module structure on V_n by considering bijections on $[n]$ and furthermore we get a sequence

$$V_0 \xrightarrow{\varphi_0} V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} \cdots \tag{1}$$

of S_n-modules for varying n. By considering the standard embedding $S_n \hookrightarrow S_{n+1}$ by letting $\sigma \in S_n$ fix $n+1$, the connective maps $\varphi_n : V_n \to V_{n+1}|_{S_n}$ are S_n-equivariant. It is not hard to verify that an equivalent way of thinking about **FI**-modules is taking a sequence $(V_n \in \mathbf{Mod}_{\mathbb{C}[S_n]})_{n\in\mathbb{N}}$ with connective maps φ_n as in (1) that are S_n-equivariant and also satisfy

$$(n+1\ n+2) \cdot \varphi_{n+1} \circ \varphi_n = \varphi_{n+1} \circ \varphi_n, \tag{2}$$

a condition that is forced by functoriality. For this reason, we will think about **FI**-modules and sequences (1) with S_n-equivariant φ_n satisfying (2), interchangeably.

Definition 1. Consider a sequence $(V_n \in \mathbf{Mod}_{\mathbb{C}[S_n]})_{n\in\mathbb{N}}$ with decompositions

$$V_n \cong \bigoplus_\lambda (\mathbb{S}^{\lambda[n]})^{\oplus c_{\lambda,n}},$$

where λ ranges over all partitions and $c_{\lambda,n} = 0$, when $\lambda[n]$ is not defined. We say that $V_\bullet$ is *representation multiplicity stable (RMS)*, if for all partitions λ, there is an N_λ, such that for all $n \geqslant N_\lambda$, the coefficients $c_{\lambda,n} = c_{\lambda,n+1}$ have stabilized.

We say that $V_\bullet$ is *uniformly representation multiplicity stable (URMS)*, if there is an N, such that for all $n \geqslant N$ and all λ, the coefficients $c_{\lambda,n} = c_{\lambda,n+1}$ have stabilized. This N is called a *URMS stable range.*

Following [CEF15], a sequence $(V_n \in \mathbf{Mod}_{\mathbb{C}[S_n]})_{n\in\mathbb{N}}$ with S_n-equivariant connective maps φ_n satisfying (2) is said to be *uniformly representation stable (URS)*, if there is a

URMS stable range, N, such that the following two additional conditions hold for $n \geqslant N$: the map φ_n is injective and $S_{n+1} \cdot \varphi_n(V_n) = V_{n+1}$. We will call such an N a *URS stable range*.

Notice that the URMS condition does not depend on any conditions on connective maps between the V_n, while the URS condition does. See [Bah24] for a summary of other types of stable ranges that have appeared in the literature.

The authors of [CEF15] showed that an **FI**-module is finitely generated if and only if the corresponding sequence (V_n, φ_n) is uniformly representation stable (Proposition 3.3.3 [CEF15]). In addition, they show that the category of **FI**-modules is noetherian and hence, any **FI**-submodule of a finitely generated **FI**-module is finitely generated (Theorem 1.3 [CEF15]). We will use this fact to see the representation stability of sequences of Garsia-Haiman modules in Section 5.

One important quantity related to **FI**-modules is the *weight*, which may be defined for any sequence of S_n-modules $V_\bullet = (V_n \in \mathrm{Mod}_{S_n})_{n\in\mathbb{N}}$ as follows

$$\mathrm{wt}(V_\bullet) := \sup_{\lambda,\ n} \left\{ |\lambda| :\ [\mathbb{S}^{\lambda[n]}, V_n] \neq 0 \right\},$$

where the supremum is taken over partitions λ and $n \geqslant |\lambda| + \lambda_1$ and we recall that $[V, W] = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[S_n]}(V, W)$ for S_n-modules V, W. Note that by definition, this quantity is finite for URMS sequences. The weight quantity will be crucial to the statement of several results in the paper and so we are interested in computing or at least bounding the weight. See the end of Section 2.1 for a discussion of one good approach.

2.1 Torsion-free FI-modules

We say an **FI**-module is *torsion-free*, if the image of any morphism is an injective linear map. This is just equivalent to the the φ_n in the corresponding sequence $(V_n, \varphi_n)_{n\in\mathbb{N}}$ being injective, since the functor necessarily maps isomorphisms in **FI** to bijective linear maps.

Recall the definition of the *central primitive idempotents*: for $\lambda \vdash n$

$$\mathrm{X}^\lambda = \frac{f_\lambda}{n!} \sum_{\sigma\in S_n} \chi^\lambda(\sigma)\sigma \in Z(\mathbb{C}[S_n]),$$

which satisfy $(\mathrm{X}^\lambda)^2 = \mathrm{X}^\lambda$ and $\mathrm{X}^\lambda \mathrm{X}^\mu = 0$ for λ, μ distinct partitions of n. Note that we can naturally view X^λ as an element $\mathbb{C}[S_m]$ for $m \geqslant n$ and hence apply it to elements of S_m-representations. In particular, if V is an S_m-representation and $\lambda \vdash n \leqslant m$, we have that X^λ is an S_n-equivariant projection from V onto an S_n-submodule of $V|_{S_n} = \mathrm{Res}^{S_m}_{S_n}(V)$ that is S_n-isomorphic to $(\mathbb{S}^\lambda)^{\oplus c}$, where $c = [V|_{S_n}, \mathbb{S}^\lambda]$. We will use these operators in the proof of the following lemma.

Lemma 2. *(Monotonicity lemma) Let $(V_n, \varphi_n)_n$ be a sequence arising from a torsion-free* ***FI****-module, i.e. each φ_n is injective. Then for all partitions λ and $n \geqslant |\lambda| + \lambda_1$,*

$$\left[V_n, \mathbb{S}^{\lambda[n]}\right] \leqslant \left[V_{n+1}, \mathbb{S}^{\lambda[n+1]}\right]. \tag{3}$$

Proof. Recall the definition of the natural torsion-free **FI**-module, $(\mathbb{S}^{\lambda[\bullet]}, \psi_\bullet)$, called $V(\lambda)$ in [CEF15]. Here $\mathbb{S}^{\lambda[n]}$ is taken to be the zero module for $n < |\lambda| + \lambda_1$ and for $n \geqslant |\lambda| + \lambda_1$, $\psi_n : \mathbb{S}^{\lambda[n]} \to \mathbb{S}^{\lambda[n+1]}$ is the unique (up to scaling) injective S_n-equivariant map between the two modules.

As seen in [CEF15], given two **FI**-modules one may take (internal) *Kronecker tensor products* of the underling S_n-modules and the connective maps to get a new **FI**-module. Since tensor products of injective maps are injective, we get a torsion-free **FI**-module

$$\cdots \xrightarrow{\varphi_{n-1}\otimes\psi_{n-1}} V_n \otimes \mathbb{S}^{\lambda[n]} \xrightarrow{\varphi_n\otimes\psi_n} V_{n+1} \otimes \mathbb{S}^{\lambda[n+1]} \xrightarrow{\varphi_{n+1}\otimes\psi_{n+1}} \cdots$$

and the self-duality of irreducible S_n-representations implies

$$\begin{aligned} [V_n \otimes \mathbb{S}^{\lambda[n]}, \mathbb{S}^{(n)}] &= \dim \operatorname{Hom}_{S_n}(V_n \otimes \mathbb{S}^{\lambda[n]}, \mathbb{S}^{(n)}) \\ &= \dim \operatorname{Hom}_{S_n}(V_n, \mathbb{S}^{\lambda[n]} \otimes \mathbb{S}^{(n)}) \\ &= [V_n, \mathbb{S}^{\lambda[n]}], \end{aligned} \tag{4}$$

where we are using $\mathbb{S}^{\lambda[n]} \otimes \mathbb{S}^{(n)} \cong \mathbb{S}^{\lambda[n]}$ since a tensor product with a trivial representation does not alter the isomorphism type of the representation.

Hence, it suffices to prove the lemma for the case $\lambda = \emptyset$, since if one has for all torsion-free **FI**-modules, $(W_n, \tau_n)_n$

$$[W_n, \mathbb{S}^{(n)}] \leqslant [W_{n+1}, \mathbb{S}^{(n+1)}],$$

then taking $W_n = V_n \otimes \mathbb{S}^{\lambda[n]}$ and $\tau_n = \varphi_n \otimes \psi_n$ yields

$$[V_n, \mathbb{S}^{\lambda[n]}] \leqslant [V_{n+1}, \mathbb{S}^{\lambda[n+1]}]$$

via (4).

The $\lambda = \emptyset$ case can be restated in terms of invariants: given a torsion-free **FI**-module $(V_n, \varphi_n)_n$,

$$\dim((V_n)^{S_n}) \leqslant \dim((V_{n+1})^{S_{n+1}}).$$

Suppose for contradiction the statement failed for some $(V_n, \varphi_n)_n$ and there was a basis of S_n-invariants of V_n, $\{v_1, v_2, \ldots, v_c\}$, but $\dim((V_{n+1})^{S_{n+1}}) < c$. Then consider the S_{n+1}-submodule of V_{n+1} generated by $\varphi_n(v_1), \cdots, \varphi_n(v_c)$:

$$W := S_{n+1}\langle \varphi_n(v_1), \cdots, \varphi_n(v_c) \rangle.$$

Since $\mathrm{Span}_{\mathbb{C}}(\varphi_n(v_1), \cdots, \varphi_n(v_c))$ forms an S_n-module isomorphic to $(\mathbb{S}^{(n)})^{\oplus c}$, W is S_{n+1}-isomorphic to some quotient of

$$S_{n+1} \otimes_{S_n} (\mathbb{S}^{(n)})^{\oplus c} = \mathrm{Ind}_{S_n}^{S_{n+1}} \left((\mathbb{S}^{(n)})^{\oplus c}\right) \cong_{S_{n+1}} (\mathbb{S}^{(n+1)})^{\oplus c} \oplus (\mathbb{S}^{(n,1)})^{\oplus c},$$

by the branching rule. Hence

$$W \cong_{S_{n+1}} (\mathbb{S}^{(n+1)})^{\oplus a} \oplus (\mathbb{S}^{(n,1)})^{\oplus b},$$

for some $a, b \leqslant c$. Note that $a = \dim(W^{S_{n+1}}) \leqslant \dim((V_{n+1})^{S_{n+1}}) < c$. Then applying the projection operator, $\mathrm{X}^{(n+1)}$, to $\mathrm{Span}_{\mathbb{C}}(\varphi_n(v_1), \cdots, \varphi_n(v_c))$ yields a subspace of an a-dimensional space and so there is a non-trivial linear relation:

$$\alpha_1 \cdot \mathrm{X}^{(n+1)}\varphi_n(v_1) + \cdots + \alpha_c \cdot \mathrm{X}^{(n+1)}\varphi_n(v_c) = 0.$$

Let $v = \alpha_1 \cdot v_1 + \cdots + \alpha_c \cdot v_c \neq 0$, since the v_i form a basis. Since v is an S_n-invariant, $S_{n+1}\langle\varphi_n(v)\rangle$ is isomorphic to a quotient of $\mathbb{S}^{(n+1)} \oplus \mathbb{S}^{(n,1)}$. But since $\mathrm{X}^{(n+1)}\varphi_n(v) = 0$ and φ_n is injective, we conclude that

$$S_{n+1}\langle\varphi_n(v)\rangle \cong_{S_{n+1}} \mathbb{S}^{(n,1)}.$$

While an exceptional case can occur for small n, we will demonstrate the contradiction when $n \geqslant 3$ and return to the case $n < 3$ later.

By the functoriality of **FI**-modules, one has

$$(n+1\ n+2) \cdot \varphi_{n+1} \circ \varphi_n(v) = \varphi_{n+1} \circ \varphi_n(v)$$

and so $\varphi_{n+1} \circ \varphi_n(v)$ is a non-trivial $(S_n \times S_2)$- invariant in V_{n+2}. By the Pieri rule, the only irreducible representations of S_{n+2} with a non-trivial $(S_n \times S_2)$-invariant are $\mathbb{S}^{(n+2)}, \mathbb{S}^{(n+1,1)}, \mathbb{S}^{(n,2)}$. Furthermore, the $(S_n \times S_2)$-invariant within these irreps is unique up to scaling. Hence, φ_{n+1} induces an injective S_{n+1}-equivariant map from $S_{n+1}\langle\varphi_n(v)\rangle \cong_{S_{n+1}} \mathbb{S}^{(n,1)}$ to $\mathbb{S}^{\mu}$:

$$\rho : S_{n+1}\langle\varphi_n(v)\rangle \to \mathbb{S}^{\mu}|_{S_{n+1}},$$

for some $\mu = (n+2), (n+1,1), (n,2)$ such that it maps $\varphi_n(v)$ to the unique (up to scaling) $(S_n \times S_2)$-invariant of $\mathbb{S}^{\mu}$, call it f. Note that $\mu \neq (n+2)$ since there is no injective S_{n+1}-equivariant map from $\mathbb{S}^{(n,1)} \to \mathbb{S}^{(n+2)}$ by the branching rule.

In the case $\mu = (n,2)$, since we are taking $n \geqslant 3$, the partition, $(n-1,2)$, is well-defined and $\mathrm{X}^{(n-1,2)}\varphi_n(v) = 0$, since $S_{n+1}\langle\varphi_n(v)\rangle \cong \mathbb{S}^{(n,1)}$ doesn't contain $\mathbb{S}^{(n-1,2)}$ as a submodule. We can argue that

$$0 \neq \mathrm{X}^{(n-1,2)}f = \mathrm{X}^{(n-1,2)}\rho(\varphi_n(v)) = \rho(\mathrm{X}^{(n-1,2)}\varphi_n(v)),$$

creating a contradiction. Decompose the restriction

$$\mathbb{S}^{(n,2)}|_{S_{n+1}} = U \oplus W,$$

where U is isomorphic to $\mathbb{S}^{(n,1)}$ and W is isomorphic to $\mathbb{S}^{(n-1,2)}$. Then we can write $f = g + h$ for $g \in U$ and $h \in W$. Suppose that $h = 0$ and $f = g \in U$. Then for any $\sigma \in S_{n+2}$, we may write $\sigma = \pi \circ (n+1\ n+2)$ or $\sigma = \pi$ for some $\pi \in S_{n+1}$. Since $g = f$ is an $(S_n \times S_2)$-invariant (so that $(n+1\ n+2) \cdot g = g$) and $S_{n+1} \cdot U \subset U$, we get $\sigma \cdot g \in U$ in either case. But then $S_{n+2}\langle g\rangle = U \subsetneq \mathbb{S}^{(n,2)}$, contradicting irreducibility of $\mathbb{S}^{(n,2)}$. Hence $0 \neq h = \mathrm{X}^{(n-1,2)} f$.

In the case $\mu = (n+1, 1)$, we can similarly argue that $\mathrm{X}^{(n+1)}\varphi_n(v) = 0$, but $\mathrm{X}^{(n+1)} f \neq 0$, yielding another contradiction. To see this consider the restriction

$$\mathbb{S}^{(n+1,1)}|_{S_{n+1}} \cong U \oplus W,$$

where U is isomorphic to $\mathbb{S}^{(n,1)}$ and W is isomorphic to $\mathbb{S}^{(n+1)}$. Then we can write $f = g+h$ for $g \in U$ and $h \in W$. Again, we can argue that if $h = 0$, then $S_{n+2}\langle g\rangle = U \subsetneq \mathbb{S}^{(n+1,1)}$. Hence $h = \mathrm{X}^{(n+1)} f \neq 0$. Hence all three possible cases for μ yield a contradiction, forcing our initial assumption that $\dim((V_{n+1})^{S_{n+1}}) < \dim((V_n)^{S_n})$ to be false for $n \geqslant 3$.

Finally, we cover the cases $n < 3$. The case $n = 0$ is trivial. In the case $n = 1$, we are forced to have a S_2-equivariant map from $\mathbb{S}^{(1,1)}$ to $\mathbb{S}^{(2,1)}$ that maps the generator of $\mathbb{S}^{(1,1)}$ to the $(S_1 \times S_2)$-invariant of $\mathbb{S}^{(2,1)}$, but this is not an S_2-equivariant map by the $\mu = (n+1, 1)$ case above. For $n = 2$, we again can rule out $\mu = (3, 1)$, but it turns out that there is in fact a S_3-equivariant map from $\mathbb{S}^{(2,1)}$ to $\mathbb{S}^{(2,2)}$ mapping the $(S_2 \times S_1)$-invariant of $\mathbb{S}^{(2,1)}$ to the $(S_2 \times S_2)$-invariant of $\mathbb{S}^{(2,2)}$. In this case, we just consider the argument for the next connective map, φ_4, which induces an S_4-equivariant map from $\mathbb{S}^{(2,2)}$ to $\mathbb{S}^{\mu}|_{S_4}$ for $\mu \vdash 5$, sending the $(S_2 \times S_2)$- invariant, f, to an $(S_2 \times S_3)$-invariant, g, unique up to scalar multiple, by functoriality of the **FI**-module. This combined with the branching rule forces $\mu = (3, 2)$, but this would not yield a S_4-equivariant map, since one can easily check that $\mathrm{X}^{(3)} f = 0$, but $\mathrm{X}^{(3)} g \neq 0$ by the branching rule. □

Remark 3. We interpret the proof above as proving the following representation theoretic fact for $n \neq 2$. Suppose V_{n+1} is an S_{n+1}-module and V_n is an S_n-invariant subspace. Then to show that $[V_n, \mathbb{S}^{\lambda[n]}] \leqslant [V_{n+1}, \mathbb{S}^{\lambda[n+1]}]$, it suffices to find an S_{n+2}-module V_{n+2} containing V_{n+1} (with compatible S_{n+1} action on V_{n+1}), such that $(n+1\ n+2)$ acts as the identity on V_n.

This yields the following easy corollary, which establishes that for torsion-free **FI**-modules it suffices to show that Specht module multiplicities have stabilized in order to establish representation stability in the sense of Church, Ellenberg, and Farb.

Corollary 4. *Let $(V_n, \varphi_n)_n$ be a sequence arising from a finitely generated torsion-free **FI**-module. Then a URMS stable range is also a URS stable range.*

Proof. If N is a URMS stable range, then we have stable decompositions for $n \geqslant N$

$$V_n \cong_{S_n} \bigoplus_{\lambda} (\mathbb{S}^{\lambda[n]})^{\oplus c_\lambda}$$

for $c_\lambda \in \mathbb{N}$ independent of n. Since the maps φ_n are automatically injective, we only have to verify that $W := S_{n+1}\langle \varphi_n(V_n)\rangle$ agrees with V_{n+1} for $n \geqslant N$. We can look at the modified torsion-free **FI**-module with corresponding sequence

$$\cdots \xrightarrow{\varphi_{n-1}} V_n \xrightarrow{\varphi_n} W \xrightarrow{\varphi_{n+1}|_W} V_{n+2} \xrightarrow{\varphi_{n+2}} \cdots$$

Since $W \subseteq V_{n+1}$, we have that $[W, \mathbb{S}^{\lambda[n+1]}] \leqslant c_\lambda$ for all relevant λ. On the other hand, applying the Lemma 2 to the sequence above implies $[W, \mathbb{S}^{\lambda[n+1]}] \geqslant c_\lambda$. This forces us to conclude that $W \cong \bigoplus_\lambda (\mathbb{S}^{\lambda[n+1]})^{\oplus c_\lambda} \cong V_{n+1}$ and so $W = V_{n+1}$ as claimed. □

Another nice property of a torsion-free **FI**-module, with sequence $(V_n, \varphi_n)_n$, is that the function $n \mapsto \dim(V_n)$ is closely related to the weight of the **FI**-module. It is well-known that $\dim\left(\mathbb{S}^{\lambda[n]}\right)$ is agree with a polynomial in n of degree $|\lambda|$ for sufficiently large n (see [CZ24] for explicit combinatorial formulas for this polynomial). In general, knowing the function $n \mapsto \dim(V_n)$ does not necessarily give us information about the weight, nor does it allow us to conclude representation stability. For instance, the non-torsion-free **FI**-module corresponding to the sign representation

$$V_n = \mathbb{S}^{(1^n)}, \ \varphi_n = 0,$$

has $\mathrm{wt}(V_\bullet) = \infty$ but $\dim(V_n) = 1$, a degree 0 polynomial. We bring attention to the fact that we cannot make take consecutive φ_n to be non-zero while respecting (2). As another example, one can have a torsion-free **FI**-module

$$V_n = (\mathbb{S}^{(n)})^{\oplus p(n)},$$

with φ_n taken to be arbitrary injective linear maps and $p(n)$ is any non-decreasing polynomial. The $\mathrm{wt}(V_\bullet) = 0$, but $p(n)$ can have any degree.

However, for finitely generated, torsion-free **FI**-modules, it follows by Lemma 2 that $\dim(V_n)$ is eventually a polynomial in n of degree $\mathrm{wt}(V_\bullet)$. Even without establishing that the sequence $(V_n, \varphi_n)_n$ is finitely generated (URS), we can still bound the weight by using its dimension as seen in the following lemma.

Lemma 5. *Let $(V_n, \varphi_n)_n$ be a sequence arising from a torsion-free **FI**-module, such that $\dim(V_n) = O(n^k)$, for some $k \in \mathbb{N}$. Then the weight of the **FI**-module is $\leqslant k$.*

Proof. The hypothesis implies that $\limsup_{n\to\infty} \frac{\dim(V_n)}{n^k} < \infty$. Now suppose for contradiction that the weight of the **FI**-module is not $\leqslant k$, so there is some $\lambda \vdash K > k$ and $n_0 \geqslant |\lambda| + \lambda_1$, such that a copy of $\mathbb{S}^{\lambda[n_0]}$ appears in the decomposition of V_{n_0}. By Lemma 2, a copy of $\mathbb{S}^{\lambda[n]}$ will appear in V_n for all $n > n_0$ and so

$$\dim(V_n) = \dim(\mathbb{S}^{\lambda[n]}) + f(n),$$

where $f(n)$ is some positive function of n corresponding to the dimension of the quotient space. But since $\dim(\mathbb{S}^{\lambda[n]})$ is positive and eventually polynomial in n in degree $> k$,

$$\limsup_{n\to\infty} \frac{\dim(V_n)}{n^k} \geqslant \limsup_{n\to\infty} \frac{\dim(\mathbb{S}^{\lambda[n]})}{n^k} = \infty,$$

a contradiction. □

One might ask if for a finitely generated, torsion-free **FI**-module the sharpest stable range $n \geqslant N$, for which $\dim(V_n)$ starts to agree with a polynomial in n, agrees with the sharpest URS stable range. We note that the **FI**-module, $(\mathbb{S}^{(1^k)[\bullet]}, \psi_\bullet)$, appearing in the proof of the monotonicity lemma has $\dim(\mathbb{S}^{(1^k)[n]}) = \binom{n-1}{k}$ stably agreeing with a polynomial for $n \geqslant 1$. However, the sharpest URS stable range is $n \geqslant k+1$, since otherwise $(1^k)[n]$ is undefined. Understanding how these two stable ranges relate, may be an interesting further problem.

2.2 Symmetric function theory

We refer the reader to [Sta24] and [Mac98] for an introduction to the theory of symmetric functions. Let Λ denote the algebra of *symmetric functions* over $\mathbb{C}$ in variables $x_1, x_2, \ldots$ and let $\Lambda^{(n)}$ denote the n-th degree homogeneous component.

Recall some of the classical homogeneous bases for Λ, which are indexed by partitions with $u_\lambda \in \Lambda^{(n)}$ whenever $\lambda \vdash n$. Let $(m_\lambda)_\lambda$ denote the *monomial symmetric function* basis, $(h_\lambda)_\lambda$ denote the *complete homogeneous* basis, $(e_\lambda)_\lambda$ denote the *elementary* basis, $(p_\lambda)_\lambda$ denote the *power sum* basis, and $(s_\lambda)_\lambda$ denote the *Schur function* basis.

Define the *Grothendieck ring* of $(S_n)_{n\in\mathbb{N}^+}$, $R = \oplus_n R^{(n)}$, where $R^{(n)}$ is the free $\mathbb{C}$-module generated by isomorphism classes of irreducible representations of S_n. Letting $[V]$ denote the isomorphism class of an S_n-representation, we can identify $[V \oplus W]$ with $[V] + [W]$ in $R^{(n)}$, thereby allowing us to interpret the isomorphism class of any S_n-representation as an element of $R^{(n)}$ by Maschke's theorem. The product in the ring is given by the *induced tensor product*: if V is an S_n-representation and W is an S_m-representation, then $[V] \cdot [W] := \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(V \otimes W)$, interpreting $S_n \times S_m$ as a Young subgroup of S_{n+m}.

Then one can define the *Frobenius characteristic map*, $\mathcal{F} : R^{(n)} \to \Lambda^{(n)}$, to be the map that sends $\mathcal{F}(\mathbb{S}^\lambda)$ to s_λ and is extended by linearity. Then by taking the direct sum over graded components yields a map $\mathcal{F} : R \to \Lambda$ which is a graded $\mathbb{C}$-algebra isomorphism of the Grothendieck ring and the algebra of symmetric functions.

Taking the natural inner product on characters of S_n-representations, $\mathcal{F}$ induces an inner product on $\Lambda^{(n)}$ called the *Hall scalar inner product*, which we will just denote by $\langle \cdot, \cdot \rangle$. A pair of homogeneous basis $(u_\lambda \in \Lambda^{(n)})_{\lambda \vdash n}$, $(v_\lambda \in \Lambda^{(n)})_{\lambda \vdash n}$, are *dual* if $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda,\mu}$. It is well-known that $(h_\lambda)_\lambda$, $(m_\mu)_\mu$ form a dual basis pair, as do $(p_\lambda)_\lambda$, $(p_\mu / z_\mu)_\mu$, and the Schur functions are self-dual, being the images of the irreducible representations of S_n. Here,

$$z_\mu := \mu_1 \mu_2 \cdots \mu_{l(\mu)} \cdot \prod_{j \geqslant 1} c_j(\mu)!$$

for $\mu = (\mu_1, \cdots, \mu_{l(\mu)})$, where $c_j(\mu)$ is the number of parts of μ of size j. This is the order of the centralizer of a $\sigma \in S_n$ of cycle type μ.

We have the following important identity for dual basis pairs. If $(u_\lambda), (v_\mu)$ are a dual pair and $(f_\lambda), (g_\mu)$ are a dual pair and we have a change of basis

$$u_\lambda = \sum_{\mu \vdash n} A_{\lambda,\mu} f_\mu$$

for $A_{\lambda,\mu} \in \mathbb{C}$, then the dual (or adjoint) identity is

$$g_\mu = \sum_{\lambda \vdash n} A_{\lambda,\mu} v_\lambda.$$

We now go through the various change of basis transformations between these bases. See [ER91] for a more comprehensive overview of the change of basis coefficients. We start with some of the following change of bases

$$\begin{aligned}
h_\lambda &= \sum_{\mu \vdash n} IM_{\lambda,\mu} m_\mu \\
e_\lambda &= \sum_{\mu \vdash n} BM_{\lambda,\mu} m_\mu \\
p_\lambda &= \sum_{\mu \vdash n} |OB_{\lambda,\mu}|\, m_\mu \\
h_\lambda &= \sum_{\mu \vdash n} (-1)^{n-l(\mu)} |B_{\lambda,\mu}|\, e_\mu \\
p_\lambda &= \sum_{\mu \vdash n} (-1)^{l(\lambda)-l(\mu)} w(B_{\mu,\lambda}) h_\mu \\
s_\lambda &= \sum_{\mu \vdash n} K_{\lambda,\mu} m_\mu.
\end{aligned}$$

Here $IM_{\lambda,\mu}$ counts the number of $l(\lambda) \times l(\mu)$ matrices with natural number entries, $A \in M_{l(\lambda),l(\mu)}(\mathbb{N})$, such that the sum of the entries in the i-th row is λ_i and the sum of the entries in the j-th column is μ_j. Similarly, $BM_{\lambda,\mu}$ is the number of such matrices only using the entries $0, 1$. We will refer to the coefficients of the inverse change of basis transformation $IM^{-1}_{\lambda,\mu}$ (resp. $BM^{-1}_{\lambda,\mu}$), which have combinatorial formulas in terms of *primitive bi-brick tableaux* which can be found in [KR06].

The so-called *ordered brick tableaux*, $OB_{\lambda,\mu}$ are maps f from the distinguished parts of λ to the distinguished parts of μ such that the parts in the pre-image of each μ_i sum to μ_i. The set of *brick tableaux*, $B_{\lambda,\mu}$ consists of laying "bricks" specified by the parts of μ, where equal sized parts of μ are not distinguished, into the rows of λ. The *weight* of a brick tableau, T, given by $w(T)$ is the product of the rightmost parts of μ in each row of λ. The total weight, $w(B_{\mu,\lambda}) = \sum_{T \in B_{\mu,\lambda}} w(T)$. See Fig. 1.

The *Kostka numbers* $K_{\lambda,\mu} := \#\mathrm{SSYT}(\lambda, \mu)$ count the number of semi-standard Young tableau of shape λ and content μ. Since these are natural numbers, it follows that the

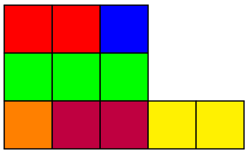

Figure 1: A brick tableau of shape $\lambda = (5, 3^2)$ and content $\mu = (3, 2^3, 1^2)$. The weight is $1 \cdot 3 \cdot 2 = 6$.

Frobenius image of any S_n-module, will be a non-negative sum of Schur functions and hence also a non-negative sum of monomial symmetric functions. Furthermore, the monotonicity observed in Lemma 2 is passed down to the coefficients of the monomial expansion as well.

Corollary 6. *Let $(V_n, \varphi_n)_n$ be a sequence arising from a torsion-free **FI**-module. Then the Schur coefficients of the Frobenius images satisfy*

$$\langle \mathcal{F}(V_n), s_{\lambda[n]} \rangle \leqslant \langle \mathcal{F}(V_{n+1}), s_{\lambda[n+1]} \rangle, \tag{5}$$

for partition λ and $n \geqslant |\lambda| + \lambda_1$.

If $\mu \vdash n$ and $\hat{\mu} \vdash n+1$ with $\mu \subset \hat{\mu}$, then the monomial coefficients of the Frobenius images satisfy

$$\langle \mathcal{F}(V_n), h_\mu \rangle \leqslant \langle \mathcal{F}(V_{n+1}), h_{\hat{\mu}} \rangle \tag{6}$$

and so in particular

$$\langle \mathcal{F}(V_n), h_{\eta[n]} \rangle \leqslant \langle \mathcal{F}(V_{n+1}), h_{\eta[n+1]} \rangle, \tag{7}$$

for a partition η with $|\eta| + \eta_1 \leqslant n$.

Proof. The first inequality is an immediate consequence of (5) and Lemma 2. Since $\langle s_{\lambda[n]}, h_\mu \rangle = K_{\lambda[n],\mu}$ by the duality of (h_μ) and (m_μ), we have that

$$\begin{aligned} \langle \mathcal{F}(V_n), h_\mu \rangle &= \sum_\lambda K_{\lambda[n],\mu} \cdot \langle \mathcal{F}(V_n), s_{\lambda[n]} \rangle \\ &\leqslant \sum_\lambda K_{\lambda[n+1],\hat{\mu}} \cdot \langle \mathcal{F}(V_{n+1}), s_{\lambda[n+1]} \rangle \\ &= \langle \mathcal{F}(V_{n+1}), h_{\hat{\mu}} \rangle \end{aligned}$$

using the first inequality in the statement and the inequality $K_{\lambda[n],\mu} \leqslant K_{\lambda[n+1],\hat{\mu}}$. This inequality on Kostka numbers follows by a direct combinatorial injection from $\mathrm{SSYT}(\lambda[n], \mu)$ to $\mathrm{SSYT}(\lambda[n+1], \hat{\mu})$. The additional cell $\hat{\mu}/\mu$ specifies an extra label, a, missing from the content of $T \in \mathrm{SSYT}(\lambda[n], \mu)$. Hence, we can define the injective map to locate the leftmost entry in the bottom row of T that is less than a and insert a cell labeled a immediately to the right, shifting all relevant cells to the right as seen in the example below:

$$\begin{array}{|c|c|c|c} \hline 2 & 3 & 5 & \\ \hline 1 & 1 & 4 & \multicolumn{1}{c|}{6} \\ \hline \end{array} \longmapsto \begin{array}{|c|c|c|cc} \hline 2 & 3 & 5 & & \\ \hline 1 & 1 & 2 & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{6} \\ \hline \end{array}$$

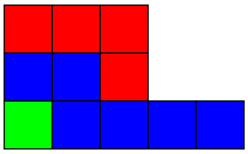

Figure 2: An example of special rim hook tableau of shape $\lambda = (5,3,3)$ and content $\mu = (6,4,1)$, with the special rim hooks highlighted in red,blue, and green. The sign is $(-1)^2 = 1$.

In this example $\lambda = (3)$, $n = 7$, and $\mu = (2, 1^5)$, $\hat{\mu} = (2^2, 1^4)$. Then (7) follows immediately for $\mu = \eta[n]$ and $\hat{\mu} = \eta[n+1]$. $\square$

For the inverse change of basis we have the *inverse Kostka numbers*

$$m_\mu = \sum_{\lambda \vdash n} K^{-1}_{\mu,\lambda} s_\lambda,$$

a formula for which is given as follows. A *rim hook* of λ is a set $H \subseteq_{\text{diag}} \lambda$ of cells encountered by taking a walk in λ consisting of east and south steps, such that the removal of these cells still leaves a partition shape. A *special rim hook* is a rim hook that contains a cell from the first row of λ. A *special rim hook tableau* of shape λ with content μ, is obtained by successively removing special rim hooks $H_1, H_2, \ldots, H_{l(\mu)}$ from λ until one gets the empty partition such that the $|H_1|, |H_2|, \ldots, |H_{l(\mu)}|$ are a rearrangement of the parts of μ. The *sign* of a special rim hook tableau is $\text{sgn}(T) = \prod_i (-1)^{\#\text{ south steps in } H_i}$. Let $\text{SRHT}(\lambda, \mu)$ denote the set of special rim hook tableaux of shape λ and content μ. See Fig. 2. Then

$$K^{-1}_{\mu,\lambda} = \sum_{T \in \text{SRHT}(\lambda,\mu)} \text{sgn}(T).$$

Recall the character change of basis

$$s_\lambda = \sum_{\mu \vdash n} \chi^\lambda_\mu \frac{p_\mu}{z_\mu},$$

where χ^λ_μ is the evaluation of the character of $\mathbb{S}^\lambda$ at a permutation of cycle type μ (this follows from the fact that the Frobenius map sends the indicator class function of the conjugacy class of cycle type μ to p_μ/z_μ). These coefficients are given by the Murnaghan-Nakayama rule

$$\chi^\lambda_\mu = \sum_{T \in \text{RHT}(\lambda,\alpha)} \text{sgn}(T),$$

where α is any *fixed* rearrangement of μ and $\text{RHT}(\lambda, \alpha)$ is the set of rim hook tableau of shape λ and content α. This is the set of rim hook tableaux where we do not require the hooks to be special (they can start in arbitrary columns of λ), but the hooks must be removed in the order specified by α: $(|H_1|, \ldots, |H_{l(\mu)}|) = (\alpha_1, \ldots, \alpha_{l(\mu)})$. See Table 1, 2 for the remaining change of basis coefficients for basis pairs that we did not mention here.

3 Transferring stability between symmetric function bases

The aim of this section is to provide a careful analysis of how one can use the change of basis coefficients between two symmetric function basis to deduce how stability properties of expansions in one symmetric function basis imply stability properties of expansions in another basis. Since we consider stabilizing expansions of homogeneous $F_n \in \Lambda^{(n)}$ as n increases, it makes sense to restrict our attention to homogeneous bases of symmetric functions indexed by partitions. First, we generalize the notion of stability with respect to a homogenous basis.

Definition 7. Let $(f_\nu)_\nu$ be a basis for Λ with ν running through the set of partitions and $f_\nu \in \Lambda^{(|\nu|)}$. Consider a sequence $(F_n)_n$ with $F_n \in \Lambda^{(n)}$ and expansion in the basis, f, given by

$$F_n = \sum_\lambda d_{\lambda,n} f_{\lambda[n]},$$

where the sum ranges over all partitions λ and we take $d_{\lambda,n} = 0$ whenever $\lambda[n]$ is not defined, i.e. $n < |\lambda| + \lambda_1$. Define the $f_{\lambda[\bullet]}$*-range* of $F_\bullet$, $\mathrm{rg}^f_\lambda(F_\bullet)$, to be the smallest n such that $d_{\lambda,n} = d_{\lambda,n+k}$ for all $k \geqslant 0$ and let $\mathrm{rg}^f_\lambda(F_\bullet) = \infty$ when no such n exists. We say $F_\bullet$ is f*-stable* if $\mathrm{rg}^f_\lambda(F_\bullet) < \infty$ for all λ.

Similarly, define the *uniform* f*-range* of $F_\bullet$, $\mathrm{rg}^f(F_\bullet) = \max_\lambda \Big(\mathrm{rg}^f_\lambda(F_\bullet)\Big)$. We say $F_\bullet$ is *uniformly* f*-stable* if $\mathrm{rg}^f(F_\bullet) < \infty$.

Define the f*-support* of $F_\bullet$ as

$$\mathrm{supp}_f(F_\bullet) := \{\lambda : \ \exists\, n,\ d_{\lambda,n} \neq 0\}$$

and the f*-weight* of $F_\bullet$ as

$$\mathrm{wt}_f(F_\bullet) := \max\{|\lambda| : \ \exists\, n,\ d_{\lambda,n} \neq 0\}.$$

The concept of weight is useful to us since it allows us to upgrade f-stability to uniform f-stability as can be seen in the following proposition.

Proposition 8. *A sequence $F_\bullet$ with $F_n \in \Lambda^{(n)}$ is uniformly f-stable if and only if it is f-stable and has finite weight, $wt^f(F_\bullet) < \infty$.*

Proof. Let us first show that if $F_\bullet$ is f-stable and has finite weight, then it is uniformly f-stable, i.e. the values, $\mathrm{rg}^f_\lambda(F_\bullet)$, are bounded across all λ. By the definition of range, $\mathrm{rg}^f_\lambda(F_\bullet) = 0$ for $\lambda \notin \mathrm{supp}_f(F_\bullet)$, since $d_{\lambda,n} = 0$ for all n. Note also that $\mathrm{wt}_f(F_\bullet) < \infty$ is equivalent to $\mathrm{supp}_f(F_\bullet)$ being a finite set. Then an f-stable sequence, $F_\bullet$, has $\mathrm{rg}^f_\lambda(F_\bullet) < \infty$ for all $\lambda \in \mathrm{supp}_f(F_\bullet)$, a finite set, and $\mathrm{rg}^f_\lambda(F_\bullet) = 0$ for all $\lambda \notin \mathrm{supp}_f(F_\bullet)$. By taking the maximum of these values, only finitely many of which are non-zero, yields a single range

N, such that $d_{\lambda,n} = d_{\lambda,N}$ for all $n \geqslant N$ and all partitions λ. Thus, $F_\bullet$ is uniformly f-stable with

$$\mathrm{rg}^f(F_\bullet) = \max_{\lambda \in \mathrm{supp}_f(F_\bullet)} \mathrm{rg}^f_\lambda(F_\bullet) = \max_{|\lambda| \leqslant \mathrm{wt}_f(F_\bullet)} \mathrm{rg}^f_\lambda(F_\bullet).$$

For the converse, we clearly have that uniform f-stability implies f-stability. To see that uniform f-stability implies finite weight, suppose we have a sequence $F_\bullet$ that is uniformly f-stable with $\mathrm{rg}^f(F_\bullet) = N$. It follows that F has finite support (and hence finite weight) from

$$\mathrm{supp}_f(F_\bullet) \subset \{\lambda : \lambda[N] \text{ is defined}\} = \{\lambda : |\lambda| + \lambda_1 \leqslant N\}.$$

This is because in our convention $d_{\lambda,N} = 0$ when $\lambda[N]$ is not defined and by uniform stability of the coefficients: $d_{\lambda,n} = d_{\lambda,N} = 0$ for all $n \geqslant N$ (and clearly $d_{\lambda,n} = 0$ for $n < N$ since then $\lambda[n]$ is still undefined). Hence, $F_\bullet$ is uniformly f-stable if and only if it is f-stable and $\mathrm{wt}_f(F_\bullet) < \infty$. □

Example 9. We illustrate the above definitions with an example; the Frobenius image of the *standard representation*. The standard representation of S_n is a vector space spanned by n basis vectors with S_n permuting the vectors in the natural way. The Frobenius image is $F_n = h_{(n-1,1)}$ for $n \geqslant 2$ and $F_1 = h_1$ and $F_0 = h_0$. We will consider expanding this function with respect to the Schur basis and monomial basis. In the Schur basis,

$$F_n = h_{(n-1,1)} = s_{(n-1,1)} + s_{(n)},$$

and so may take the relevant

$$d_{\lambda,n} = \begin{cases} 1, & \lambda = \emptyset \\ 1, & \lambda = (1),\ n \geqslant 2 \\ 0, & \text{otherwise.} \end{cases}$$

Hence the stable ranges are $\mathrm{rg}^s_\emptyset(F_\bullet) = 0$, $\mathrm{rg}^s_{(1)}(F_\bullet) = 2$, and 0 for all other λ. The sequence is uniformly s-stable with $\mathrm{rg}^s(F_\bullet) = 2$. The s-support of the sequence is $\mathrm{supp}_s(F_\bullet) = \{\emptyset, (1)\}$ and the s-weight is 1.

If instead, we consider expansions with respect to the monomial basis,

$$h_{(n-1,1)} = \sum_\lambda IM_{\lambda[n],(n-1,1)} m_{\lambda[n]},$$

the relevant coefficients are now $d_{\lambda,n} = IM_{\lambda[n],(n-1,1)}$. We will see later that these stabilize for each λ and so $F_\bullet$ is m-stable. However, $IM_{(1^{n-1})[n],(n-1,1)} = IM_{(1^n),(n-1,1)} = n$ implying $(1^{n-1}) \in \mathrm{supp}_m(F_\bullet)$ for all $n \geqslant 1$. This implies that $F_\bullet$ has infinite weight and so it cannot be uniformly m-stable.

We now put forth the relevant definitions to classify the different ways that stable expansions in one basis relates to stable expansions in another. If the reader is interested only in the stability transfer from the monomial basis to the Schur basis, then they may simply read the following definition, Example 12, and Lemma 15, or go to the Appendix for a self-contained alternative proof.

Definition 10. Given a pair of symmetric function basis $(g_\lambda)_\lambda$, $(f_\mu)_\mu$. We say that *(uniform) stability transfers* from f to g, if whenever a sequence $(F_n)_n$ is (uniformly) f-stable, then it is also (uniformly) g-stable. We say there is *weak stability transfer* from f to g, if whenever $(F_n)_n$ is uniformly f-stable, then it is g-stable. We say *finite weight transfers* from f to g, if whenever $(F_n)_n$ has $\mathrm{wt}_f(F_\bullet) < \infty$, then $\mathrm{wt}_g(F_\bullet) < \infty$.

Let $(g_\lambda)_\lambda$, $(f_\mu)_\mu$ be a pair of homogeneous bases with change of basis coefficients $A_{\lambda,\mu}$, i.e.

$$f_\mu = \sum_{\lambda \vdash |\mu|} A_{\mu,\lambda} g_\lambda,$$

and fix the convention that $A_{\mu[n],\lambda[n]} = 0$, whenever $\mu[n]$ or $\lambda[n]$ is not defined. Let $N^{f\to g}_{\mu,\lambda}$ be the smallest n such that $A_{\mu[n],\lambda[n]} = A_{\mu[n+k],\lambda[n+k]}$ for all $k \geqslant 0$ and let $N^{f\to g}_{\mu,\lambda} = \infty$ if no such n exists. Define the *change of basis support sets*

$$J^{f\to g}_\lambda := \{\mu : \exists\, n,\ A_{\mu[n],\lambda[n]} \neq 0\},$$

$$I^{f\to g}_\mu := \{\mu : \exists\, n,\ A_{\mu[n],\lambda[n]} \neq 0\}.$$

The reason for writing $f \to g$ in the superscript and not $g \to f$ is that when studying stability transfer from f to g, we actually have to look at expansions of f in terms of g as we will see in the following theorem. The proof of this theorem will also include inequalities describing how stable ranges and weight bounds in one basis imply stable ranges and weight bounds in another.

Theorem 11. *Let $(g_\lambda)_\lambda$, $(f_\mu)_\mu$ be a pair of homogeneous bases with change of basis coefficients $A_{\mu,\lambda}$. Consider the following three conditions:*

1. $N^{f\to g}_{\mu,\lambda} < \infty$*, for all* μ, λ
2. $|J^{f\to g}_\lambda| < \infty$*, for all* λ
3. $|I^{f\to g}_\mu| < \infty$*, for all* μ

Then

- *There is <u>finite weight transfer</u> from f to g if and only if condition <u>3 holds</u>*
- *There is <u>weak stability transfer</u> from f to g if and only if condition <u>1 holds</u>*
- *There is <u>stability transfer</u> from f to g if and only if conditions <u>1 and 2 hold</u>*

- *There is* <u>*uniform stability transfer*</u> *from f to g if and only if conditions* <u>*1 and 3 hold*</u>

Proof. (finite weight transfer from f to g iff cond. 3 holds): If condition 3 failed for some μ, then

$$F_n := \begin{cases} f_{\mu[n]}, & n \geqslant |\mu| + \mu_1 \\ 0, & \text{else} \end{cases}$$

would have finite f-weight but $F_n = \sum_\lambda A_{\mu[n],\lambda[n]} g_{\lambda[n]}$ would have infinite g-weight. On the other hand, if condition 3 holds, then for any $(F_n)_n$ with finite f-weight,

$$\begin{aligned} F_n &= \sum_{|\mu| \leqslant \mathrm{wt}_f(F_\bullet)} d_{\mu,n} f_{\mu[n]} \\ &= \sum_\lambda \left(\sum_{|\mu| \leqslant \mathrm{wt}_f(F_\bullet)} d_{\mu,n} A_{\mu[n],\lambda[n]} \right) g_{\lambda[n]}. \end{aligned}$$

The coefficient coresponding to $g_{\lambda[n]}$ is zero unless $A_{\mu[n],\lambda[n]} \neq 0$ for some μ. This forces $\lambda \in \bigcup_{|\mu| \leqslant \mathrm{wt}_f(F_\bullet)} I_\mu^{f \to g}$, which is independent of n and a finite set by condition 3. Hence,

$$\mathrm{wt}_g(F_\bullet) \leqslant \max \left\{ |\lambda| : \ \lambda \in \bigcup_{|\mu| \leqslant \mathrm{wt}_f(F_\bullet)} I_\mu^{f \to g} \right\} < \infty. \tag{8}$$

(weak stability transfer from f to g iff cond. 1): If condition 1 failed for some pair μ, λ, so that $A_{\mu[n],\lambda[n]}$ never stabilized. Then the f-stable sequence form the previous case

$$F_n := \begin{cases} f_{\mu[n]}, & n \geqslant |\mu| + \mu_1 \\ 0, & \text{else} \end{cases}$$

is uniformly f-stable with uniform stable range $n \geqslant |\mu| + \mu_1$, but $F_n = \sum_\nu A_{\mu[n],\nu[n]} g_{\nu[n]}$ is not g-stable since $\mathrm{rg}^g_\lambda(F_\bullet) = \infty$. Hence, condition 1 is necessary for weak stability transfer from f to g.

Conversely, if condition 1 holds and we have a uniformly f-stable sequence

$$\begin{aligned} F_n &= \sum_{|\mu| \leqslant \mathrm{wt}_f(F_\bullet)} d_{\mu,n} f_{\mu[n]} \\ &= \sum_\lambda \left(\sum_{|\mu| \leqslant \mathrm{wt}_f(F_\bullet)} d_{\mu,n} A_{\mu[n],\lambda[n]} \right) g_{\lambda[n]}, \end{aligned}$$

then the coefficient of $g_{\lambda[n]}$ in F_n,

$$\sum_{|\mu| \leqslant \mathrm{wt}_f(F_\bullet)} d_{\mu,n} A_{\mu[n],\lambda[n]},$$

is guaranteed to have stabilized once $n \geqslant \mathrm{rg}^f_\mu(F_\bullet)$ (so that $d_{\mu,n}$ has stabilized) and $n \geqslant N^{f\to g}_{\mu,\lambda}$ (so that $A_{\mu[n],\lambda[n]}$ has stabilized) over all μ with $|\mu| \leqslant \mathrm{wt}_f(F_\bullet)$. So we get an upper bound on the corresponding range for the g expansion

$$\begin{aligned}\mathrm{rg}^g_\lambda(F_\bullet) &\leqslant \max_{|\mu|\leqslant \mathrm{wt}_f(F_\bullet)} \left(\max\left(\mathrm{rg}^f_\mu(F_\bullet), N^{f\to g}_{\mu,\lambda}\right)\right) \\ &= \max\left(\mathrm{rg}^f(F_\bullet), \max_{|\mu|\leqslant \mathrm{wt}_f(F_\bullet)}\left(N^{f\to g}_{\mu,\lambda}\right)\right) < \infty,\end{aligned} \tag{9}$$

which is finite when F_n is uniformly f-stable (so that $\mathrm{rg}^f(F_\bullet) < \infty$) and condition 1 holds (so that each $N^{f\to g}_{\mu,\lambda} < \infty$).

(stability transfer from f to g iff cond. 1 and 2 hold): If for some pair μ, λ, the value $A_{\mu[n],\lambda[n]}$ did not stabilize, then the sequence of F_n defined in the first part would be f-stable, but not g-stable since the coefficient of $g_{\lambda[n]}$ in F_n is $A_{\mu[n],\lambda[n]}$. This shows the necessity of condition 1.

Seeing the necessity of condition 2 for stability transfer is somewhat technical. Suppose for contradiction that stability transfers from f to g, but condition 2 is not true. Then for some λ, there are infinitely many μ such that $A_{\mu[n],\lambda[n]} \neq 0$ for some n. We may pick an infinite sequence of these μ: $\mu^{(1)}, \mu^{(2)}, \ldots$ with $|\mu^{(i)}| < |\mu^{(i+1)}|$ and for each $\mu^{(i)}$ pick an $n^{(i)}$ such that $A_{\mu^{(i)}[n^{(i)}],\lambda[n^{(i)}]} \neq 0$. Since the $\mu^{(i)}[n^{(i)}]$ are defined, $|\mu^{(i)}| < |\mu^{(i+1)}|$ forces the values $n^{(i)} \geqslant |\mu^{(i)}| + \mu^{(i)}_1$ to diverge to infinity. Consequently, we may drop terms from the sequence and reindex so that without loss of generality $n^{(i)} + 1 < n^{(i+1)}$ which ensures that $\{n^{(i)} : i\}$ is missing infinitely many values from $\mathbb{N}$. Now construct $(F_n)_n$ by letting $F_n = 0$ unless $n \in \{n^{(i)} : i\}$ in which case we let $F_{n^{(i)}} = f_{\mu^{(i)}[n^{(i)}]}$. The sequence is f-stable, since any μ shows up in the sequence $(\mu^{(i)})_i$ at most once and so the coefficient of $f_{\mu[n]}$ is eventually 0 for any fixed μ. However, the coefficient of $g_{\lambda[n]}$ is

$$\begin{cases} A_{\mu^{(i)}[n^{(i)}],\lambda[n^{(i)}]}, & n = n^{(i)}, \text{ for some } i \\ 0, & \text{else}\end{cases}$$

oscillates between being zero and non-zero since $\{n^{(i)} : i\}$ misses infinitely many values of $\mathbb{N}$ due to the fact that $n^{(i)} + 1 < n^{(i+1)}$. This contradicts stability transfer from f to g implying the necessity of condition 2.

Now, we argue that conditions 1 and 2 are sufficient for stability to transfer from f to g. Let $(F_n)_n$ be an f-stable sequence with expansion in f

$$F_n = \sum_\mu d_{\mu,n} f_{\mu[n]} = \sum_\lambda \left(\sum_{\mu\in J^{f\to g}_\lambda} d_{\mu,n} A_{\mu[n],\lambda[n]}\right) g_{\lambda[n]}$$

The coefficient of $g_{\lambda[n]}$, $\sum_{\mu\in J^{f\to g}_\lambda} d_{\mu,n} A_{\mu[n],\lambda[n]}$, is guaranteed to have stabilized once for each term of the sum (for each $\mu \in J^{f\to g}_\lambda$), the value $d_{\mu,n}$ has stabilized (which occurs once $n \geqslant \mathrm{rg}^f_\mu(F_\bullet))$ and the value $A_{\mu[n],\lambda[n]}$ has stabilized (which occurs once $n \geqslant N^{f\to g}_{\mu,\lambda}$). Hence,

$$\mathrm{rg}^g_\lambda(F_\bullet) \leqslant \max_{\mu \in J^{f\to g}_\lambda} \left(\max \left(\mathrm{rg}^f_\mu(F_\bullet),\ N^{f\to g}_{\mu,\lambda} \right) \right) < \infty, \tag{10}$$

which is finite since the maximum is taken over a finite set, $J^{f\to g}_\lambda$, by condition 2, the values $N^{f\to g}_{\mu,\lambda} < \infty$ by condition 1, and the values $\mathrm{rg}^f_\mu(F_\bullet) < \infty$ since $F_\bullet$ is f-stable. Thus, $F_\bullet$ being f-stable implies that it is also g-stable.

(uniform stability transfer from f to g): The necessity of condition 1 follows from the weak stability transfer case. To see the necessity of condition 3, suppose that uniform f-stability implies uniform g-stability for all $(F_n)_n$. Take the uniformly f-stable sequence from the first case so that the coefficient of $g_{\lambda[n]}$ in F_n is $A_{\mu[n],\lambda[n]}$. Then this sequence is uniformly g-stability and hence has finite support by Proposition 8. But this finite support is precisely

$$\mathrm{supp}_g(f_{\mu[\bullet]}) = \{\lambda : \exists n,\ A_{\mu[n],\lambda[n]} \neq 0\} = I^{f\to g}_\mu$$

and so condition 3 follows from uniform stability transfer.

Conversely, if condition 1 and 3 hold and $F_\bullet$ is uniformly f-stable, then condition 1 implies that $F_\bullet$ is g-stable by weak stability transfer. Furthermore, condition 3 implies that $F_\bullet$ has finite g-weight by finite weight transfer. Hence, $F_\bullet$ is uniformly g-stable and by putting inequalities (8) and (9) together, we get

$$\mathrm{rg}^g(F_\bullet) = \max_{|\lambda| \leqslant \mathrm{wt}_g(F_\bullet)} (\mathrm{rg}^g_\lambda(F_\bullet)) \leqslant \max \left(\mathrm{rg}^f(F_\bullet),\ \max_{\substack{|\mu| \leqslant \mathrm{wt}_f(F_\bullet) \\ \lambda \in I^{f\to g}_\mu}} \left(N^{f\to g}_{\mu,\lambda} \right) \right) < \infty. \tag{11}$$

$\square$

Due to the importance of our main result, we present the following example of how conditions 1 and 2 imply stability transfer from the monomial basis to the Schur basis, using the analysis of the inverse Kostka numbers provided by Lemma 15. This is effectively just the third part of the proof Theorem 11 for $f = m$ and $g = s$.

Example 12. Given a sequence of monomial expansions, we have

$$F_n = \sum_\mu d_{\mu,n} m_{\mu[n]} = \sum_\lambda \left(\sum_{\substack{\mu \\ |\mu| \leqslant |\lambda|}} d_{\mu,n} K^{-1}_{\mu[n],\lambda[n]} \right) s_{\lambda[n]},$$

where the sum only needs to be taken over $|\mu| \leqslant |\lambda|$, since otherwise $K^{-1}_{\mu[n],\lambda[n]}$ vanishes, which is established in Lemma 15 and stated there as $J^{m\to s}_\lambda = \{\mu : |\mu| \leqslant |\lambda|\}$. Thus, the Schur coefficient of $s_{\lambda[n]}$, $\sum_{\substack{\mu \\ |\mu| \leqslant |\lambda|}} d_{\mu,n} K^{-1}_{\mu[n],\lambda[n]}$, will stabilize, as soon as all $d_{\mu,n}$ and

$K^{-1}_{\mu[n],\lambda[n]}$ stabilize for all $|\mu| \leqslant |\lambda|$. By Lemma 15 the $K^{-1}_{\mu[n],\lambda[n]}$ stabilize once $n \geqslant 2|\lambda|$ and (10) becomes

$$\mathrm{rg}^s_\lambda(F_\bullet) \leqslant \max\left(2|\lambda|, \quad \max_{\substack{\mu \\ |\mu|\leqslant|\lambda|}} \mathrm{rg}^m_\mu(F_\bullet)\right),$$

which is precisely (22) derived later.

The dual nature of conditions 2 and 3 is captured in this following corollary.

Corollary 13. *Suppose we have a pair of homogeneous bases $(f_\mu)_\mu$, $(g_\lambda)_\lambda$ with corresponding dual bases $(f^*_\mu)_\mu, (g^*_\lambda)_\lambda$ with respect to the Hall scalar inner product. Then the following are equivalent:*

- *Stability transfers from f to g*
- *Uniform stability transfers from g^* to f^**

Proof. Let $A = (A_{\mu,\lambda})_\lambda$ be the change of basis matrix from g to f: $f_{\mu[n]} = \sum_\lambda A_{\mu[n],\lambda[n]} g_{\lambda[n]}$. Then the adjoint A^T gives the change of basis

$$g^*_{\lambda[n]} = \sum_\mu A^T_{\lambda[n],\mu[n]} f^*_{\mu[n]} = \sum_\mu A_{\mu[n],\lambda[n]} f^*_{\mu[n]}.$$

It is clear that condition 1 holds for A if and only if it holds for A^T. The claim follows since condition 2 for the $A_{\mu[n],\lambda[n]}$ is equivalent to condition 3 holding for $A^T_{\mu[n],\lambda[n]} = A_{\lambda[n],\mu[n]}$. Note that another consequence of this is that

$$N^{f\to g}_{\mu,\lambda} = N^{g^*\to f^*}_{\lambda,\mu} \tag{12}$$

since $A_{\mu[n],\lambda[n]}$ and $A_{\lambda[n],\mu[n]}$ stabilize at the same range. □

3.1 Application to the classical bases

In this section, we consider the classical symmetric function basis, $m_\lambda, s_\lambda, h_\lambda, e_\lambda, p_\lambda, p_\lambda/z_\lambda$ and determine which of conditions 1,2,3 from Theorem 11 hold for each ordered pair. Tables 1, 2 summarizes these computations.

Proposition 14. *Let $F = (F_n \in \Lambda^{(n)})_n$. Then the following statements hold:*

- *F is s-stable if and only if it is m-stable*
- *F is uniformly s-stable if and only if it is uniformly h-stable (dual to previous statement by Corollary 13)*
- *F is h-stable if and only if it is p/z-stable*
- *F is uniformly m-stable if and only if it is uniformly p-stable (dual to previous statement by Corollary 13)*

We have bounds on the following relevant change of basis coefficient stable ranges:

$$N^{s\to m}_{\mu,\lambda},\ N^{m\to s}_{\mu,\lambda} \leqslant 2|\lambda|, \tag{13}$$
$$N^{p/z\to h}_{\mu,\lambda},\ N^{h\to p/z}_{\mu,\lambda} \leqslant 2|\lambda|+1 \tag{14}$$

as well as having bounds on the relevant support sets $J^{f\to g}_{\lambda}$ and $I^{f\to g}_{\mu}$:

$$J^{s\to m}_{\lambda},\ J^{m\to s}_{\lambda}, J^{p/z\to h}_{\lambda},\ J^{h\to p/z}_{\lambda} \subseteq \{\mu:\ |\mu|\leqslant|\lambda|\} \tag{15}$$
$$I^{p\to m}_{\mu},\ I^{m\to p}_{\mu}, I^{h\to s}_{\mu},\ I^{s\to h}_{\mu} \subseteq \{\lambda:\ |\lambda|\leqslant|\mu|\} \tag{16}$$

	m_μ	s_μ	h_μ
m_λ		$K_{\mu,\lambda}$: 1,2	$IM_{\mu,\lambda}$: 1
s_λ	$K^{-1}_{\mu,\lambda}$: 1,2		$K_{\lambda,\mu}$: 1,3
h_λ	$IM^{-1}_{\mu,\lambda}$: -	$K^{-1}_{\lambda,\mu}$: 1,3	
e_λ	$BM^{-1}_{\mu,\lambda}$: -	$K^{-1}_{\lambda,\mu'}$: -	$(-1)^{n-l(\lambda)}\|B_{\lambda,\mu}\|$: 2
p_λ	$(-1)^{l(\lambda)-l(\mu)}\dfrac{w(B_{\mu,\lambda})}{z_\lambda}$: 1,3	$\dfrac{\chi^\mu_\lambda}{z_\lambda}$: -	$\dfrac{\|OB_{\lambda,\mu}\|}{z_\lambda}$: 2
p_λ/z_λ	$(-1)^{l(\lambda)-l(\mu)}w(B_{\mu,\lambda})$: 3	χ^μ_λ : 1	$\|OB_{\lambda,\mu}\|$: 1,2

Table 1: Verification of which conditions from Theorem 11 hold for pairs of classical symmetric function bases. The (f_μ, g_λ)-entry indicates the change of basis coefficients from $f_\mu = \sum_\lambda A_{\mu,\lambda} g_\lambda$ as well as listing which of the conditions hold. We record - if all conditions 1,2,3 fail. Diagonal entries are left out of these tables.

As a warm up, we will analyze the Kostka and inverse Kostka numbers to show (13-16) for the change of basis between the Schur basis and the monomial basis.

Lemma 15. *The Kostka numbers, $K_{\mu[n],\lambda[n]}$, stabilize once $n\geqslant 2|\lambda|$ with stable values*

$$K_{\mu[n],\lambda[n]} = \begin{cases} \langle s_\mu h_{|\lambda|-|\mu|}, h_\lambda\rangle, & |\lambda|\geqslant|\mu| \\ 0, & \textit{else} \end{cases},$$

so that $N^{s\to m}_{\mu,\lambda}\leqslant 2|\lambda|$. We have that $K_{\mu[n]\lambda[n]}=0$ unless $|\mu|\leqslant|\lambda|$ so that $J^{s\to m}_{\lambda}\subseteq\{\mu:\ |\mu|\leqslant|\lambda|\}$. The set, $I^{s\to m}_{\mu}$, is infinite since it contains all $\lambda=(1^k)$.

The inverse Kostka numbers, $K^{-1}_{\mu[n],\lambda[n]}$, stabilize once $n\geqslant 2|\lambda|$ and so $N^{m\to s}_{\mu,\lambda}\leqslant 2|\lambda|$. We also have that $K^{-1}_{\mu[n],\lambda[n]}=0$ unless $|\mu|\leqslant|\lambda|$ so $J^{m\to s}_{\lambda}\subseteq\{\mu:\ |\mu|\leqslant|\lambda|\}$. The set, $I^{m\to s}_{\mu}$, is infinite since it contains all $\lambda=(1^k)$.

	e_μ	p_μ	p_μ/z_μ
m_λ	$BM_{\mu,\lambda}$: 1	$\lvert OB_{\mu,\lambda}\rvert$: 1,3	$\lvert OB_{\mu,\lambda}\rvert/z_\mu$: 3
s_λ	$K_{\lambda',\mu}$: 1	χ^λ_μ : 1	χ^λ_μ/z_μ : -
h_λ	$(-1)^{n-l(\lambda)}\lvert B_{\lambda,\mu}\rvert$: 2	$(-1)^{l(\mu)-l(\lambda)}w(B_{\lambda,\mu})$: 2	$(-1)^{l(\mu)-l(\lambda)}w(B_{\lambda,\mu})/z_\mu$: 1,2
e_λ		$(-1)^{n-l(\lambda)}w(B_{\lambda,\mu})$: 2	$(-1)^{n-l(\lambda)}w(B_{\lambda,\mu})/z_\mu$: 2
p_λ	$(-1)^{n-l(\lambda)}\frac{\lvert OB_{\lambda,\mu}\rvert}{z_\lambda}$: 2		$\frac{\delta_{\mu,\lambda}}{z_\mu}$: 2,3
p_λ/z_λ	$(-1)^{n-l(\lambda)}\lvert OB_{\lambda,\mu}\rvert$: 2	$\delta_{\mu,\lambda}z_\lambda$: 2,3	

Table 2: Verification of which conditions from Theorem 11 hold for pairs of classical symmetric function bases. The (f_μ, g_λ)-entry indicates the change of basis coefficients from $f_\mu = \sum_\lambda A_{\mu,\lambda} g_\lambda$ as well as listing which of the conditions hold. We record - if all conditions 1,2,3 fail. Diagonal entries are left out of these tables.

We note the similarity between the first half of the statement in Lemma 15 and Theorem 3.3 in [Col17]. The proof of this statement uses the same map on standard Young tableaux as defined in the proof of that theorem.

Proof. First we argue that if $|\lambda| < |\mu|$, then $K_{\mu[n],\lambda[n]} = 0$ assuming $\mu[n], \lambda[n]$ are defined. We have that $n - |\lambda| > n - |\mu|$ and so the number of cells labeled 1 in any $T \in \mathrm{SSYT}(\mu[n], \lambda[n])$ exceeds the size of the bottom row of T, which would yield a contradiction.

Thus, when showing that the $K_{\mu[n],\lambda[n]}$ stabilize for $n \geqslant 2|\lambda|$, we may restrict to the case $|\lambda| \geqslant |\mu|$ (otherwise the coefficients are always zero). Let $K^{(n)} := \mathrm{SSYT}(\mu[n], \lambda[n])$. Define the following map $\kappa_n : K^{(n)} \to K^{(n+1)}$. Given a $T \in K^{(n)}$, slide the bottom row to the right by one cell and insert a new cell containing a 1 in the bottom left corner. This is a special case of the map defined in the proof of Corllary 6. See the following example:

$$\begin{array}{|c|c|c|c} \hline 2 & 3 & 4 & \\ \hline 1 & 1 & 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \longmapsto \begin{array}{|c|c|c|c|c|} \hline 2 & 3 & 4 & \multicolumn{2}{c}{} \\ \hline 1 & 1 & 1 & 2 & 3 \\ \hline \end{array}$$

Clearly, the map is injective and well-defined, implying $K_{\mu[n],\lambda[n]} \leqslant K_{\mu[n+1],\lambda[n+1]}$. We show that the map is surjective for $n \geqslant 2|\lambda| \geqslant 2|\mu|$. In this case,

$$n \geqslant |\mu| + \mu_1, |\lambda| + \lambda_1,$$

so that both $\mu[n], \lambda[n]$ will are well-defined. For any $T \in K^{(n+1)}$, the number of cells labeled 1 in T is

$$n + 1 - |\lambda| > |\lambda| \geqslant |\mu| \geqslant \mu_1$$

and these cells are all left justified in the bottom row. Removing the bottom left cell and sliding the remaining entries of the bottom row to the left still yields a valid tableau $T' \in K^{(n)}$ that maps to $T \in K^{(n+1)}$ since the first μ_1 cells of the bottom row of T' are forced to be labeled 1 by the assumption on n. Hence, $K_{\mu[n],\lambda[n]}$ stabilizes for $n \geqslant 2|\lambda|$.

For these sufficiently large n, one may take any $T \in K^{(n)}$, remove all cells labeled 1 and decrement the remaining labels by 1 to get a semi-standard Young tableau of skew shape ν and content λ, where ν can be viewed as disjoint union of shapes μ and $(|\lambda|-|\mu|)$. This bijection between elements of $K^{(n)}$ and skew tableaux implies $K_{\mu[n],\lambda[n]} = K_{\nu,\lambda}$, a *skew-Kostka number*. Then

$$K_{\mu[n],\lambda[n]} = K_{\nu,\lambda} = \langle s_\nu, h_\lambda \rangle = \langle s_\mu h_{|\lambda|-|\mu|}, h_\lambda \rangle.$$

Now we focus on the inverse Kostka numbers, let $L^{(n)} := \mathrm{SRHT}(\lambda[n], \mu[n])$ be the set of special rim hook fillings of shape $\lambda[n]$ and content $\mu[n]$, so that

$$K^{-1}_{\mu[n],\lambda[n]} = \sum_{T \in L^{(n)}} \mathrm{sgn}(T).$$

If $|\lambda| < |\mu|$, then no special rim hook fillings exist. This is because there needs to exist a rim hook containing both a cell in the first row and the rightmost cell in the bottom row of $\lambda[n]$. Hence, there has to exist a rim hook of length at least $n - |\lambda|$, but the longest rim hook only has length $n - |\mu| < n - |\lambda|$.

We will show that the inverse Kostka number stabilize once $n \geqslant 2|\lambda|$ and as before we restrict to the case $|\mu| \leqslant |\lambda|$. If $n \geqslant 2|\lambda|$, then we have $n - |\lambda| \geqslant |\lambda| \geqslant \mu_1$ which leads to two cases. For the first case, let $n - |\lambda| > \mu_1$. We construct a map $\varphi_n : L^{(n)} \to L^{(n+1)}$ that extends the rim hook containing the rightmost cell of the bottom row, c, to contain the newly added cell, $c' = \lambda[n+1]/\lambda[n]$. Note that this map is clearly injective and preserves sign since no vertical crossings are introduced. See Fig. 3.

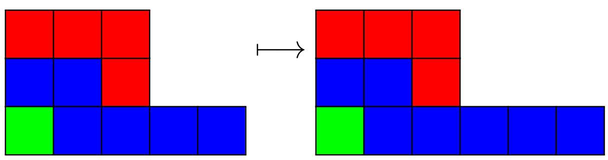

Figure 3: An application of φ_{11} to an element of $L^{(11)}$ with $\mu[11] = (6,4,1)$ and $\lambda[11] = (5,3,3)$. The sign of both special rim hook tableau is -1.

We must check that the map is well-defined, i.e. that the content of the new special rim hook tableau is $\mu[n+1]$. Within $T \in L^{(n)}$, only the special rim hook of length $n - |\mu|$ is long enough to contain c since the second longest hook has length $\mu_1 < n - |\lambda|$. Thus

c' is added to the hook of length $n - |\mu|$ yielding a special rim hook tableau with content $\mu[n+1]$. It easily follows that the map, φ_n, is surjective and so

$$K^{-1}_{\mu[n],\lambda[n]} = \sum_{T \in L^{(n)}} \operatorname{sgn}(T) = \sum_{T' \in L^{(n+1)}} \operatorname{sgn}(T') = K^{-1}_{\mu[n+1],\lambda[n+1]}.$$

In the remaining case $n - |\lambda| = \mu_1$, which combined with $n - |\lambda| \geqslant |\lambda| \geqslant \mu_1$ implies that $n = 2|\lambda|$. Additionally, $|\mu| \leqslant |\lambda|$ implies $|\mu| = \mu_1$ and $\mu[n] = (n - |\mu|, \mu_1) = (|\lambda|, |\lambda|)$. We need to show that the values have stabilized beyond this n: $K^{-1}_{\mu[n+k],\lambda[n+k]} = K^{-1}_{\mu[n],\lambda[n]}$ for all $k \geqslant 0$. The value $K^{-1}_{\mu[n+k],\lambda[n+k]}$ corresponds to the sum over special rim hook fillings of shape $\lambda[n+k]$ of content $\mu[n+k] = (|\lambda| + k, |\lambda|)$. Since $|\lambda| + k$ is the only special rim hook capable of reaching the rightmost cell of $\lambda[n+k]$ and the length of the bottom row of $\lambda[n+k]$ is $|\lambda| + k$, this rim hook precisely occupies the cells of the bottom row. Hence, upon removing this signless rim hook we see that $K^{-1}_{\mu[n+k],\lambda[n+k]} = K^{-1}_{(|\lambda|),\lambda}$, independent of $k \geqslant 0$. □

Proof of Proposition 14 and verification of table entries. We begin by handling the remaining entries of Tables 1, 2 involving Kostka or inverse Kostka numbers. The coefficients for the change of basis $e_{\mu[n]} \to s_{\lambda[n]}$, $K_{\lambda[n]',\mu[n]}$, are zero for sufficiently large n, but $K_{(1^n)',(n)} = 1$ and $K_{(n)',(1^n)} = 1$ implying conditions 3 and 2 fail. The inverse change of basis $s_{\mu[n]} \to e_{\lambda[n]}$, $K^{-1}_{\lambda[n],\mu[n]'}$, has the issue that $K^{-1}_{(n),(n)'} = (-1)^{n-1}$ does not stabilize and so condition 1 fails and one may also check that conditions 2 and 3 fail.

We observe that all entries having a sign of the form $(-1)^{n-l(\lambda)}$ will oscillate in sign as n grows and be non-zero if we let $\lambda = \mu = \emptyset$. Hence condition 1 fails for all such entries.

Now we check the entries involving brick tableaux, $B_{\lambda,\mu}$. Although a bit surprising it turns out that the change of basis coefficients for $p/z \to h$:

$$(-1)^{l(\mu[n]) - l(\lambda[n])} \frac{w(B_{\lambda[n],\mu[n]})}{z_{\mu[n]}} \tag{17}$$

satisfy conditions 1 and 3. For this change of basis coefficient to be non-zero the largest part of size, $n - |\mu|$, must be at least as great as the longest brick, $n - |\lambda|$. Hence, this coefficient will be zero unless $|\mu| \leqslant |\lambda|$ and so the support set $J^{p/z \to h}_{\lambda} \subseteq \{\mu : \ |\mu| \leqslant |\lambda|\}$ implies that condition 3 holds. To see the stabilization of this coefficient, we take $n \geqslant 2|\lambda| + 1$, and note that for non-zero coefficients $n - |\mu| \geqslant n - |\lambda| > |\lambda| \geqslant |\mu| \geqslant \mu_1$ so that

$n-|\mu|$ is distinct from the parts of μ and so $z_{\mu[n]} = z_\mu \cdot (n-|\mu|)$. Now, we claim that

$$
\begin{aligned}
(-1)^{l(\lambda[n])-l(\mu[n])}\frac{w(B_{\lambda[n],\mu[n]})}{z_{\mu[n]}} &= (-1)^{l(\lambda)-l(\mu)}\frac{w(B_{\lambda[n],\mu[n]})}{z_\mu\cdot(n-|\mu|)} && (18)\\
&= \frac{(-1)^{l(\lambda)-l(\mu)}}{z_\mu\cdot(n-|\mu|)}\sum_{\substack{\nu\subseteq_{\text{set}}\lambda\\ |\nu|=|\lambda|-|\mu|}} w(B_{\lambda\setminus\nu,\mu})\cdot w(B_{\nu\cup(n-|\lambda|),(n-|\mu|)}) && (19)\\
&= \frac{(-1)^{l(\lambda)-l(\mu)}}{z_\mu\cdot(n-|\mu|)}\sum_{\substack{\nu\subseteq_{\text{set}}\lambda\\ |\nu|=|\lambda|-|\mu|}} w(B_{\lambda\setminus\nu,\mu})\cdot(n-|\mu|)\cdot|\mathcal{R}(\nu)| && (20)\\
&= \frac{(-1)^{l(\lambda)-l(\mu)}}{z_\mu}\sum_{\substack{\nu\subseteq_{\text{set}}\lambda\\ |\nu|=|\lambda|-|\mu|}} w(B_{\lambda\setminus\nu,\mu})\cdot|\mathcal{R}(\nu)|, && (21)
\end{aligned}
$$

where $\mathcal{R}(\nu)$ is the set of compositions whose parts rearrange to ν. Line (19) follows since the largest brick, $n-|\lambda| > \mu_1$, must go in the bottom row. The remaining $|\lambda|-|\mu|$ spaces in the bottom row must be filled by bricks given by some $\nu \subseteq_{\text{set}} \lambda$. Upon specifying this ν, the brick fillings of the bottom row and the rest of $\mu[n]$ are independent of each other (in the sense of Cartesian products), allowing the w-sum to factor as seen in line (19). To see line (20), we have that

$$
w(B_{\nu\cup(n-|\lambda|),(n-|\mu|)}) = \sum_{T\in\mathcal{R}(\nu\cup(n-|\lambda|))} w(T),
$$

where $w(T)$ is the size of the last part in the rearrangement, T. Since $n-|\lambda|$ is greater than all parts of ν, all cyclic rearrangements are distinct. Letting $T\sim T'$ indicate that T and T' are cyclic rearrangements of each other and $[T]$ denote the corresponding equivalence class, we get

$$
\begin{aligned}
\sum_{[T]\in\mathcal{R}(\nu\cup(n-|\lambda|))/\sim}\left(\sum_{T'\in[T]} w(T')\right) &= \sum_{[T]\in\mathcal{R}(\nu\cup(n-|\lambda|))/\sim}(n-|\mu|)\\
&= (n-|\mu|)\cdot|(\mathcal{R}(\nu\cup(n-|\lambda|))/\sim)|\\
&= (n-|\mu|)\cdot\frac{1}{l(\nu)+1}\binom{l(\nu)+1}{1, c_1(\nu), c_2(\nu),\ldots}\\
&= (n-|\mu|)\binom{l(\nu)}{c_1(\nu), c_2(\nu),\ldots}\\
&= (n-|\mu|)\cdot|\mathcal{R}(\nu)|,
\end{aligned}
$$

since all equivalence classes have size $l(\nu)+1$. Thus, condition 1 holds as well as condition 3. Note that the stable range $n \geqslant 2|\lambda|+1$ cannot be improved to $n\geqslant 2|\lambda|$, since we can consider $\lambda = \mu = (k)$ for any k. Then the coefficient (17) for $n = 2k$ is

$$\frac{w(B_{(k,k),(k,k)})}{z_{(k,k)}} = \frac{k^2}{2! \cdot k^2} = \frac{1}{2},$$

while the coefficient (17) for $n = 2k + 1$ is

$$\frac{w(B_{(k+1,k),(k+1,k)})}{z_{(k+1,k)}} = \frac{(k+1)k}{(k+1)k} = 1.$$

Also note that by letting $\mu = \emptyset$ and $\lambda = (1^{n-1})$, we see that condition 2 fails. By Corollary 13, the adjoint change of basis $m \to p$ satisfies conditions 1 and 2 but not 3. We also see how the factor $z_{\mu[n]}$ is necessary for stabilization of (17) and so the change of basis $p \to h$ does not have stabilizing coefficients.

To consider the reverse change of basis $h \to p/z$, we recall that $OB_{\lambda,\mu}$ can be interpreted as the set of brick tableaux of shape μ, where the bricks from λ have distinguished labels, but their order within the rows of μ is irrelevant. As expected, $|OB_{\lambda[n],\mu[n]}| = 0$ unless $|\lambda| \geqslant |\mu|$ and trust that the reader can check that the coefficients stabilizes once $n \geqslant 2|\lambda|+1$, since this would force the largest brick into the bottom row. Again, we can't say in general that $n \geqslant 2|\lambda|$ is a stable range since one can take $\lambda = \mu = (k)$, for which $|OB_{(k,k),(k,k)}| = 2$, but $|OB_{(k+1,k),(k+1,k)}| = 1$.

Next, we consider $IM_{\mu,\lambda}$ and $BM_{\mu,\lambda}$ which both satisfy condition 1. For sufficiently large n, $IM_{\mu[n],\lambda[n]} = IM_{\mu[n+1],\lambda[n+1]}$ since there is a bijection between the $(l(\mu)+1) \times (l(\lambda)+1)$ matrices being enumerated that adds a 1 to the $(1,1)$-entry. The map is surjective since this entry is non-zero for large n. We have that $BM_{\mu[n],\lambda[n]}$ satisfies condition 1 since this value is zero for sufficiently large n, since the $(1,1)$-entry of any contributing matrix must eventually exceed 1, which is prohibited for boolean matrices. On the other hand conditions 2 and 3 fail since $IM_{(n),(1^n)} = BM_{(n),(1^n)} = 1$ and the change of basis matrices are symmetric.

As for the inverse change of basis $m \to h$ (resp. $m \to e$), we may simply use the fact that $m_{(n)} = p_{(n)}$ and the previously analyzed $p \to h$ (resp. $p \to e$) to see that condition 1 fails. One can check that conditions 2 and 3 fail since $IM^{-1}_{(1^n),(n)} = IM^{-1}_{(n),(1^n)} \neq 0$ and $BM^{-1}_{(1^n),(n)} = BM^{-1}_{(n),(1^n)} \neq 0$, which follows from the combinatorial formula for IM^{-1}, BM^{-1} involving primitive bi-brick tableau introduced in Theorem 1 of [KR06].

Finally, we look at the beloved character values, χ^{μ}_{λ}. It is easy to see that $\chi^{\mu[n]}_{\lambda[n]}$ stabilizes, since we can insist that a rim hook tableau removes the longest hook, $n - |\lambda|$, first. For n sufficiently large, it will be forced to use the bottom row of $\mu[n]$ and will thus start at the bottom rightmost cell with the rest of the this length $n - |\lambda|$ rim hook's placement being forced. Then we can establish $\chi^{\mu[n]}_{\lambda[n]} = \chi^{\mu[n+1]}_{\lambda[n+1]}$ by using a sign-preserving bijection that extend this longest rim hook by one cell to the right akin to the bijection used in Lemma 15. Conditions 2 and 3 fail, however, since $\chi^{(1^n)}_{(n)} = (-1)^{n-1}$ and $\chi^{(n)}_{(1^n)} = 1$. The remaining cases follow at once. $\square$

Corollary 16. *All sequences* $(F_n \in \Lambda^{(n)})_n$ *satisfy* $wt_s(F_\bullet) = wt_h(F_\bullet)$ *and* $wt_m(F_\bullet) = wt_p(F_\bullet)$.

Proof. By Proposition 14, $I_\mu^{s\to h} \subseteq \{\lambda : |\lambda| \leqslant |\mu|\}$. Substituting this into inequality (8), one gets

$$\mathrm{wt}_h(F_\bullet) \leqslant \max\left\{ |\lambda| : \lambda \in \bigcup_{|\mu|\leqslant \mathrm{wt}_s(F_\bullet)} I_\mu^{s\to h} \right\} \leqslant \mathrm{wt}_s(F_\bullet).$$

Applying the same argument to the other support bounds in (16), it follows that $\mathrm{wt}_s(F_\bullet) \leqslant \mathrm{wt}_h(F_\bullet)$ as well as the pair of inequalities establishing $\mathrm{wt}_m(F_\bullet) = \mathrm{wt}_p(F_\bullet)$. □

Remark 17. It may appear as if whenever conditions 1 and 2 hold for a change of basis matrix, A, then they also hold for the inverse, A^{-1}, as seen in Tables 1, 2. This is just a coincidence and an artifact of the change of basis matrices being particularly nice lower triangular matrices (with respect to an appropriate basis ordering).

To see that this is not true in general, fix any total order on partitions, $\prec$, such that $|\mu| + \mu_1 < |\lambda| + \lambda_1$ implies $\mu \prec \lambda$. Index the partitions with respect to this order $\mu^{(1)} \prec \mu^{(2)} \prec \ldots$ and observe that by the choice of ordering, the partitions of n are precisely the partitions in the tuple

$$L_n = (\mu^{(1)}[n], \mu^{(2)}[n], \cdots \mu^{(p(n))}[n]),$$

where $p(n)$ is the partition number. Define a new ordering on partitions by concatenating the tuples $L = L_0 L_1 L_2 \cdots$ and define a matrix, A, indexed by partitions ordered by L:

$$A_{\mu^{(i)}[n],\mu^{(j)}[n]} := \begin{cases} 1, & i = j, j+1, \text{ and } \mu^{(i)}[n], \mu^{(j)}[n] \text{ defined} \\ 0 & \text{else.} \end{cases}$$

so that each restriction of rows and columns to partitions of n is

$$A^{(n)} := A_{L_n,L_n} = J_{p(n)}(1)^T,$$

a $p(n) \times p(n)$ lower triangular *Jordan block* with eigenvalue 1. One may also view A as the block diagonal concatenation

$$A = \mathrm{diag}(A^{(0)}, A^{(1)}, A^{(2)}, \cdots).$$

This matrix can be interpreted as a change of basis matrix between some homogeneous symmetric function bases with respect to the order on partitions given by L. The matrix is invertible since it is block lower triangular with 1's on the diagonal and the inverse can be explicitly given by inverting each Jordan block

$$A^{-1}_{\mu^{(i)}[n],\mu^{(j)}[n]} := \begin{cases} (-1)^{i-j}, & i \geqslant j, \text{ and } \mu^{(i)}[n], \mu^{(j)}[n] \text{ defined} \\ 0 & \text{else.} \end{cases}.$$

It follows that $A_{\mu[n],\lambda[n]}$ and $A^{-1}_{\mu[n],\lambda[n]}$ are both stable as soon $\mu[n], \lambda[n]$ are defined, which implies that both A and A^{-1} satisfy condition 1. By looking at columns of the Jordan blocks, we see that A satisfies condition 2 with $J_{\mu^{(j)}} = \{\mu^{(j)}, \mu^{(j+1)}\}$ finite. However, $A^{-1}_{\mu^{(i)}[n],\mu^{(j)}[n]} \neq 0$ as long as $i \geqslant j$ and one takes an appropriately large n implying condition 2 fails.

By taking the transpose of A, we get a matrix for which conditions 1 and 3 hold, but $(A^T)^{-1}$ fails condition 3.

3.2 Concluding uniform Schur stability

We can now apply the inequalities (8-11) to get the following corollary, which is relevant for concluding Schur stability and thus can help establish representation stability.

Corollary 18. *(a) Let $(F_n \in \Lambda^{(n)})_n$ be any sequence of homogenous symmetric functions. The sequence is uniformly s-stable if and only if $wt_s(F_\bullet) < \infty$ and it is m-stable. In this case, one has a bound on the stable range*

$$rg^s_\lambda(F_\bullet) \leqslant \max\left(2 \cdot |\lambda|, \max_{|\mu| \leqslant |\lambda|} rg^m_\mu(F_\bullet)\right) \tag{22}$$

and uniform stable range

$$rg^s(F_\bullet) \leqslant \max\left(2 \cdot wt_s(F_\bullet), \max_{|\mu| \leqslant wt_s(F_\bullet)} rg^m_\mu(F_\bullet)\right). \tag{23}$$

(b) A sequence $(F_n \in \Lambda^{(n)})_n$ is uniformly s-stable if and only if $wt_s(F_\bullet) < \infty$ and it is p/z-stable. In this case, one has a bound on the uniform stable range

$$rg^s(F_\bullet) \leqslant \max\left(2 \cdot wt_s(F_\bullet) + 1, \max_{|\mu| \leqslant wt_s(F_\bullet)} rg^{p/z}_\mu(F_\bullet)\right) \tag{24}$$

Proof. By Proposition 8, uniform s-stability is equivalent to having $\mathrm{wt}_s(F_\bullet) < \infty$ and s-stability (and s-stability is equivalent to m-stability by Proposition 14). The inequality (22) follows by applying (10) to $g = s$ and $f = m$:

$$\begin{aligned}
\mathrm{rg}^s_\lambda(F_\bullet) &\leqslant \max_{\mu \in J^{m\to s}_\lambda} \left(\max\left(\mathrm{rg}^m_\mu(F_\bullet), N^{m\to s}_{\mu,\lambda}\right)\right) \\
&\leqslant \max_{|\mu| \leqslant |\lambda|} \left(\max\left(\mathrm{rg}^m_\mu(F_\bullet), 2|\lambda|\right)\right) \\
&= \max\left(2 \cdot |\lambda|, \max_{|\mu| \leqslant |\lambda|} \mathrm{rg}^m_\mu(F_\bullet)\right),
\end{aligned}$$

using $N^{m\to s}_{\mu,\lambda} \leqslant 2|\lambda|$ from (13) and $J^{m\to s}_\lambda = \{\lambda : |\lambda| \leqslant |\mu|\}$ from (15). To get (23), we take the maximum over $|\lambda| \leqslant \mathrm{wt}_s(F_\bullet)$:

$$\mathrm{rg}^s(F_\bullet) = \max_{|\lambda| \leqslant \mathrm{wt}_s(F_\bullet)} \mathrm{rg}^s_\lambda(F_\bullet) \leqslant \max\left(2 \cdot \mathrm{wt}_s(F_\bullet), \max_{|\mu| \leqslant \mathrm{wt}_s(F_\bullet)} \mathrm{rg}^m_\mu(F_\bullet)\right).$$

For part (b), we have that uniform s-stability implies p/z-stability since condition 1 holds as seen in the $s \to p/z$ entry of Tables 1, 2. For the converse of part (b), we have that p/z-stability of $F_\bullet$ implies h-stability by Proposition 14 and we can again apply (10) to get

$$\mathrm{rg}^h_\nu(F_\bullet) \leqslant \max_{\mu \in J^{p/z\to h}_\nu} \left(\max\left(\mathrm{rg}^{p/z}_\mu(F_\bullet), N^{p/z\to h}_{\mu,\nu}\right)\right) \leqslant \max\left(2|\nu|+1,\ \max_{|\mu|\leqslant|\nu|} \mathrm{rg}^{p/z}_\mu(F_\bullet)\right)$$

using $N^{p/z\to h}_{\mu,\nu} = 2|\nu|+1$ by (13) and $J^{p/z\to h}_\nu = \{\mu : |\mu| \leqslant |\nu|\}$ by (15). By Corollary 16, $\mathrm{wt}_h(F_\bullet) = \mathrm{wt}_s(F_\bullet) < \infty$. By Proposition 8, the sequence $F_\bullet$ being h-stable and having finite h-weight implies that it is uniformly h-stable. By Proposition 14, it is thus uniformly s-stable. Finally to get (24) we first observe that since $\mathrm{wt}_s(F_\bullet) = \mathrm{wt}_h(F_\bullet)$,

$$\begin{aligned}
\mathrm{rg}^h(F_\bullet) &= \max_{|\nu|\leqslant \mathrm{wt}_h(F_\bullet)} \mathrm{rg}^h_\nu(F_\bullet) \\
&= \max_{|\nu|\leqslant \mathrm{wt}_s(F_\bullet)} \mathrm{rg}^h_\nu(F_\bullet) \\
&\leqslant \max_{|\nu|\leqslant \mathrm{wt}_s(F_\bullet)} \left(\max\left(2|\nu|+1,\ \max_{|\mu|\leqslant|\nu|} \mathrm{rg}^{p/z}_\mu(F_\bullet)\right)\right) \\
&\leqslant \max\left(2\cdot \mathrm{wt}_s(F_\bullet)+1, \max_{|\mu|\leqslant \mathrm{wt}_s(F_\bullet)} \mathrm{rg}^{p/z}_\mu(F_\bullet)\right)
\end{aligned}$$

and also observe that

$$\max_{\substack{|\nu|\leqslant \mathrm{wt}_h(F_\bullet)\\ \lambda\in I^{h\to s}_\nu}} N^{h\to s}_{\nu,\lambda} \leqslant 2\cdot \mathrm{wt}_h(F_\bullet)$$

since $N^{h\to s}_{\nu,\lambda} = N^{s\to m}_{\lambda,\nu} \leqslant 2|\lambda|$ by (13) and the duality equation (12) and $I^{h\to s}_\nu = \{\lambda : |\lambda| \leqslant |\nu|\}$ by (16). Substituting these into (11) for $g = s$ and $f = h$, we get

$$\begin{aligned}
\mathrm{rg}^s(F_\bullet) &\leqslant \max\left(\mathrm{rg}^h(F_\bullet), \max_{\substack{|\nu|\leqslant \mathrm{wt}_h(F_\bullet)\\ \lambda\in I^{h\to s}_\nu}} N^{h\to s}_{\nu,\lambda}\right) \\
&\leqslant \max\left(2\cdot \mathrm{wt}_s(F_\bullet)+1, \max_{|\mu|\leqslant \mathrm{wt}_s(F_\bullet)} \mathrm{rg}^{p/z}_\mu(F_\bullet)\right).
\end{aligned}$$

□

We now state the main results as easy corollaries being applied to a sequence S_n-modules V_n, where $F_n = \mathcal{F}(V_n)$ and hence $\mathrm{wt}_s(F_\bullet) = \mathrm{wt}(V_\bullet)$. Note that if $\sigma \in S_k$ has cycle type μ, then for $n \geqslant |\mu| + \mu_1$, $\chi^{V_n}(\sigma \cdot (k+1\ k+2\ \cdots\ n))$ is precisely the coefficient of $p_{\mu[n]}/z_{\mu[n]}$ in the p/z expansion of $\mathcal{F}(V_n)$.

Theorem 19. *(a) A sequence of S_n-modules $V = (V_n \in \boldsymbol{Mod}_{S_n})_n$ is URMS if and only if $wt(V_\bullet) < \infty$ and the sequence of Frobenius images $\mathcal{F}(V_\bullet)$ is m-stable. In this case, the multiplicity of $\mathbb{S}^{\lambda[n]}$ in V_n stabilizes once*

$$n \geqslant \max\left(2\cdot|\lambda|, \max_{|\mu|\leqslant|\lambda|} rg^m_\mu(\mathcal{F}(V_\bullet))\right). \tag{25}$$

and a uniform stable range of $V_\bullet$ is given by

$$n \geqslant \max\left(2\cdot wt(V_\bullet), \max_{|\mu|\leqslant wt(V_\bullet)} rg^m_\mu(\mathcal{F}(V_\bullet))\right). \tag{26}$$

(b) A sequence of S_n-modules $V = (V_n \in \boldsymbol{Mod}_{S_n})_n$ is URMS if and only if $wt(V_\bullet) < \infty$ and for every $k \leqslant wt(V)$ and every $\sigma \in S_k$,

$$\chi^{V_n}(\sigma \cdot (k+1\ k+2\ \cdots\ n))$$

stabilizes for sufficiently large n. In this case, a uniform stable range of $V_\bullet$ is given by

$$n \geqslant \max\left(2\cdot wt(V_\bullet)+1, \max_{|\mu|\leqslant wt(V_\bullet)} rg^{p/z}_\mu(\mathcal{F}(V_\bullet))\right), \tag{27}$$

where $rg^{p/z}_\mu(\mathcal{F}(V_\bullet))$ is the minimum n such that the character value

$$\chi^{V_\bullet}(\sigma \cdot (|\mu|+1\ |\mu|+2\ \cdots\ n)),$$

has stabilized, where σ is any element of $S_{|\mu|}$ with cycle type μ. □

Note that by polynomiality of characters of $\mathbb{S}^{\lambda[n]}$ (see Theorem 3.3.4 in [CEF15], for instance), any URMS sequence will eventually have

$$\chi^{V_\bullet}(\sigma \cdot (|\mu|+1\ |\mu|+2\ \cdots\ n)),$$

stabilize for fixed k and fixed $\sigma \in S_k$ since the character is determined by a polynomial in cycles of size at most $\mathrm{wt}(V_\bullet)$. However, this result uses the stable range $n \geqslant$ stab-deg$(V_\bullet)+$ $\mathrm{wt}(V_\bullet)$ and does not extract a stable range from the character values, unlike (27) in our result.

Remark 20. This remark is concerned with the sharpness of (25-27). As explained in the Appendix, one can omit the $2|\lambda|$ (resp. $2\cdot \mathrm{wt}(V_\bullet)$) from (25) (resp. (26)) for series of Schur-positive symmetric functions, although this will not have a significant impact on Sections 4 and 5. Even with the $2|\lambda|$ dropped we may not have a sharp range. As an example consider

$$\mathcal{F}(\mathrm{Ind}^{S_n}_{S_{\nu[n]}}(1)) = h_{\nu[n]} = \sum_\lambda K_{\lambda[n],\nu[n]}\cdot s_{\lambda[n]} = \sum_\mu IM_{\nu[n],\mu[n]}\cdot m_\mu[n],$$

where $IM_{\nu[n],\mu[n]}$ is stable once $n \geqslant |\nu|+|\mu|$ and this bound is sharp. Hence for the case $\nu = (1^2)$, the Schur coefficient of $s_\lambda[n]$ when $\lambda = (1^2)$ is stable once

$$n \geqslant \max_{|\mu|\leqslant|\lambda|}(|\nu|+|\mu|) = 4$$

by (22). However, the stable value $\langle F_n, s_{(1^2)[n]}\rangle = K_{(1^2)[n],(1^2)[n]} = 1$ is attained earlier at $n = 3$. Notice that for $n \geqslant 4$, the sum $\sum_{|\mu|\leqslant|\lambda|} d_{\mu,n}K^{-1}_{\mu[n],\lambda[n]}$ expands as

$$1\cdot 1 + 3\cdot(-1) + 7\cdot 1 + 4\cdot(-1) = 1,$$

while the sum at $n = 3$ has different terms

$$1 \cdot 1 + 3 \cdot (-2) + 6 \cdot 1 = 1.$$

Hence the bound (40) may not be sharp in general.

Unlike the improved range in Lemma 41, one does need the $2 \cdot \mathrm{wt}(V) + 1$ term in the uniform range (27). Consider the URMS sequence defined as

$$V_n = \begin{cases} 0, & n < 4 \\ \mathbb{S}^{(2,2)} \oplus \mathbb{S}^{(3,1)} \oplus \mathbb{S}^{(4)} \oplus \left(\mathbb{S}^{(2,1^2)}\right)^{\oplus 2}, & n = 4 \\ \left(\mathbb{S}^{(n-2,1^2)}\right)^{\oplus 2}, & n \geqslant 5 \end{cases},$$

which clearly has $\mathrm{wt}(V) = 2$ and $\mathrm{rg}^s(\mathcal{F}(V_\bullet)) = 5 = 2 \cdot \mathrm{wt}(V) + 1$. However, one can check that the character values on the conjugacy classes, $\chi^{V_4}(C_{\mu[4]}) = \chi^{V_n}(C_{\mu[n]})$, agree for $|\mu| \leqslant \mathrm{wt}(V) = 2$ and $n \geqslant 5$. Listing the character values in a vector for $\mu = \emptyset, (1), (2), (1^2)$, we get

$$(\chi^{V_4}(\mu[4]))_{|\mu| \leqslant 2} = \begin{array}{c} \mu = \emptyset \\ (1) \\ (2) \\ (1,1) \end{array} \overset{\mathbb{S}^{(2,2)}}{\begin{pmatrix} 0 \\ -1 \\ 2 \\ 0 \end{pmatrix}} + \overset{\mathbb{S}^{(3,1)}}{\begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}} + \overset{\mathbb{S}^{(4)}}{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}} + 2 \overset{\mathbb{S}^{(2,1,1)}}{\begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}} = 2 \overset{\mathbb{S}^{(n-2,1,1)}}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}} = (\chi^{V_n}(\mu[n]))_{|\mu| \leqslant 2},$$

where the n on the right hand side is $\geqslant 5$. Hence, it is not sufficient to just have character values stabilizing in order to conclude that Schur multiplicities have stabilized.

We conclude the section by focusing Theorem 19 on the particular case of torsion-free **FI**-modules.

Theorem 21. *A sequence $(V_n, \varphi_n)_n$ arising from a torsion-free **FI**-module is URS, if and only if, $\dim(V_n) = O(n^k)$ for some k and for each partition μ, the coefficient of $m_{\mu[n]}$ in the monomial expansion is bounded from above:*

$$\sup_n \langle \mathcal{F}(V_n), h_{\mu[n]} \rangle < \infty.$$

Furthermore, it suffices to check that these upper bounds exist for $|\mu| \leqslant k$.

Proof. The forward direction follows since URS sequences have $\dim(V_n) = O(n^k)$ and the $\mathcal{F}(V_n)$ are s-stable and hence m-stable (so in particular the monomial coefficients are bounded from above). For the converse direction, we have by Lemma 5 for tor-free **FI**-modules that $\mathrm{wt}(V_\bullet) \leqslant k$. Then we can apply Theorem 19 to show that $(V_n, \varphi_n)_n$ is URMS (and hence URS by Corollary 4) if we can show that the sequences of monomial coefficinets $(\mathcal{F}(V_n), h_{\mu[n]} \rangle)_n$ stabilize for all $|\mu| \leqslant k$. By Lemma 6, these sequences are non-decreasing sequences of natural numbers, and so they will stabilize if and only if they are bounded from above. $\square$

4 Applications to r-diagonal harmonics

In this section, we consider sequences of multi-graded S_n-representations called r-diagonal harmonics, which are closely related to r-diagonal coinvariant algebras, the representation stability of which was studied in [CEF15]. We recall the definition of the r-diagonal harmonics (see Bergeron [Ber09] for exposition and proofs). Let A_r denote the set of r-tuples, $(a_1, \cdots, a_r)$, such that $a_i \in \mathbb{N}$ and at least one $a_i \neq 0$. The *r-diagonal harmonics*,

$$H_n^{(r)} := \left\{ f \in \mathbb{C}[X_n^{(1)}, \ldots, X_n^{(r)}] : \ \sum_{k=1}^{n} \left(\frac{\partial}{\partial_{x_k^{(1)}}} \right)^{a_1} \cdots \left(\frac{\partial}{\partial_{x_k^{(r)}}} \right)^{a_r} f = 0, \ (a_1, \cdots, a_r) \in A_r \right\}.$$

By the symmetry of the definition, $H_n^{(r)}$ is invariant under the diagonal action of S_n and hence forms a multi-graded S_n-submodule of $\mathbb{C}[X_n^{(1)}, \ldots, X_n^{(r)}]$.

4.1 Connection to r-diagonal coinvariant algebras

In this subsection, we define r-diagonal coinvariant algebras and describe how they relate to r-diagonal harmonics. See the exposition of Bergeron [Ber09] for more details on the combinatorics of coinvariant spaces.

Let $X_n^{(i)}$ be the variable set $x_1^{(i)}, \ldots, x_n^{(i)}$ for $i = 1, \cdots, r$. The algebra, $\mathbb{C}[X_n^{(1)}, \ldots, X_n^{(r)}]$, has r-multi-grading given by degree, $\deg(x_k^{(i)}) = 1$. Let S_n act on $\mathbb{C}[X_n^{(1)}, \cdots, X_n^{(r)}]$ via the diagonal action by permuting each set of variables simultaneously, i.e. $\sigma \cdot x_k^{(i)} = x_{\sigma(k)}^{(i)}$. Let $\mathbb{C}[X_n^{(1)}, \ldots, X_n^{(r)}]_+^{S_n}$ be the subset of invariants under this action with no constant term. The ideal generated by this subset, $\left(\mathbb{C}[X_n^{(1)}, \ldots, X_n^{(r)}]_+^{S_n}\right)$, is often called the *Hilbert ideal* for the action. Then the *r-diagonal coinvariant algebra* is defined by taking the quotient by the Hilbert ideal,

$$R_n^{(r)} := \mathbb{C}[X_n^{(1)}, \ldots, X_n^{(r)}] / \left(\mathbb{C}[X_n^{(1)}, \ldots, X_n^{(r)}]_+^{S_n}\right).$$

Since the Hilbert ideal is homogeneous, this algebra inherits the multi-grading of $\mathbb{C}[X_n^{(1)}, \ldots, X_n^{(r)}]$ and since the Hilbert ideal is S_n-invariant, we get an induced S_n-action on $R_n^{(r)}$. Note that for $J = (j_1, \ldots, j_r)$, the multi-graded component $[R_n^{(r)}]_J$ is also a S_n-representation, since the action preserves grading.

It is well known that the quotient map

$$f \mapsto f + (\mathbb{C}[X_n^{(1)}, \ldots, X_n^{(r)}]_{S_n}^{+})$$

from $H_n^{(r)}$ to $R_n^{(r)}$ is a multi-grading preserving S_n-equivariant map and in fact

$$H_n^{(r)} \cong R_n^{(r)}$$

as multi-graded S_n-modules.

Note that the representations $R_n^{(r)}$ do not admit an **FI**-module structure but rather a ***co-FI**-module* structure. In any case, we can still talk about the properties like weight and being URMS when considering sequences of S_n-representations for varying n. Note that while $R_n^{(r)}$ is not URMS, for any fixed $J = (j_1, \dots, j_r)$, the graded component $[R_n^{(r)}]_J$ is in fact URMS, a celebrated result of [CEF15].

At the end of section 5.1 in [CEF15], the authors were able to use the **co-FI**-module structure of $R_n^{(r)}$ to prove the following statement which is of crucial importance to us in this section. As far as the author knows, the stable range is folklore but an explicit yet technical proof can be found in Section 4.1 of [Bah24].

Proposition 22 ([CEF15] Proposition 5.1.4, [Bah24] Theorem F)**.** *For any fixed $J = (j_1, \dots, j_r)$, the sequence in n, $[R_n^{(r)}]_J$ is URMS and has weight bounded above*

$$wt([H_n^{(r)}]_J) = wt([R_n^{(r)}]_J) \leqslant |J|,$$

where $|J| := j_1 + \cdots + j_r$. Furthermore, the URMS stable range is

$$n \geqslant 2|J|.$$

This stable range as well as other types of "stable ranges" not mentioned in this paper are studied in greater depth for the r-diagonal coinvariant algebras in [Bah24]. In a different direction, there is "stability" result in which n is kept fixed and r is allowed to vary (see Theorem 2.1 of [Ber11]).

It then follows from Proposition 22 and the graded isomorphism, $H_n^{(r)} \cong R_n^{(r)}$, that for any $J = (j_1, \cdots, j_r)$, the S_n-modules $[H_n^{(r)}]_J$ form a URMS sequence with weight $\leqslant |J|$ and stable range $n \geqslant 2|J|$.

4.2 Harmonics as FI-modules

Unlike the diagonal coinvariant algebras, the diagonal harmonics, $H_n^{(r)}$, do have a natural **FI**-module structure defined as follows. Let

$$i_n : \mathbb{C}[X_n^{(1)}, \dots, X_n^{(r)}] \to \mathbb{C}[X_{n+1}^{(1)}, \dots, X_{n+1}^{(r)}]$$

be the standard injective homomorphism sending $x_k^{(i)} \mapsto x_k^{(i)}$ for $i = 1, \dots, r$ and $k = 1, \dots, n$. These maps are clearly S_n-homomorphisms satisfying $(n+1\ n+2) \cdot i_{n+1} \circ i_n = i_{n+1} \circ i_n$. From this it follows that the sequence of polynomial algebras, $\mathbb{C}[X_{n+1}^{(1)}, \dots, X_{n+1}^{(r)}]$, arises from a torsion-free **FI**-module. Then the $H_n^{(r)}$ will correspond to an **FI**-submodule since one can easily check that

$$i_n(H_n^{(r)}) \subseteq H_{n+1}^{(r)}$$

follows immediately from the definition. Hence, we obtain the following corollary of Corollary 4 and Proposition 22.

Corollary 23. *For any $J = (j_1, \cdots, j_r)$, the sequence $[H_n^{(r)}]_J$ with connective maps i_n arises from a finitely generated torsion-free **FI**-module with weight $\leqslant |J|$ and a URS stable range $n \geqslant 2|J|$.* □

The representation theory of the diagonal harmonics (and hence also diagonal coinvariant algebras) is notoriously difficult. The classical case $r = 1$, has been well understood (see [Ber09] for an exposition) and there are combinatorial descriptions of the distribution of multiplicities of Specht modules throughout the graded components (See Eq (29)). For the case $r = 2$, there is a monomial formula (see [CM18]) for the graded Frobenius image but no combinatorial description of the Schur expansion, making this case perfect for our methods. For the case $r \geqslant 3$, there are no proven formulas for the Frobenius image.

We study the cases $r = 1, 2$ through the monomial expansions of Frobenius images, allowing us to reprove the stable range in Proposition 22 by simply using the weight bound

$$\mathrm{wt}([H_n^{(r)}]_J) \leqslant |J|, \tag{28}$$

which follows from the $[H_n^{(r)}]_J$ being a submodule of $\left[\mathbb{C}[X_n^{(1)}, \ldots, X_n^{(r)}]\right]_J$. This approach avoids needing to compute the stability-degree as advocated by [CEF15] and the homological methods used by [Bah24]. We also get a refinement of Proposition 22 in the case $r = 2$.

4.3 $r = 1$: classical harmonics

We will denote $H_n^{(1)}$ simply by H_n and let q track the degree. It is well-known (see section 8.5 of [Ber09] and (1.89) in [Hag08]) that the graded Frobenius image of H_n is given by

$$\mathcal{F}(H_n; q) = \sum_{\lambda[n] \vdash n} \left(\sum_{T \in \mathrm{SYT}(\lambda[n])} q^{\mathrm{maj}(T)} \right) s_{\lambda[n]} = \sum_{\mu[n] \vdash n} \begin{bmatrix} n \\ \mu[n] \end{bmatrix}_q m_{\mu[n]}, \tag{29}$$

where $\begin{bmatrix} n \\ \mu[n] \end{bmatrix}_q$ is the *q-multinomial coefficient* and maj is the *major index statistic* on standard Young tableaux

$$\mathrm{maj}(T) = \sum_{j \text{ below } j+1 \text{ in } T} j$$

We emphasize that we are working in French notation for the above sum. Note that the sequence $(H_n)_n$ is not URMS since

$$\langle \mathcal{F}(H_n), s_{\lambda[n]} \rangle = |\mathrm{SYT}(\lambda[n])| \to \infty$$

for $\lambda \neq \varnothing$.

We will see how $[H_n]_i$ is URMS upon taking the i-th graded component. The stability of the multiplicities of the Specht modules in $[H_n]_i$ can be seen directly by looking at

$$\langle \mathcal{F}([H_n]_i), s_{\lambda[n]} \rangle = \#\{T \in \mathrm{SYT}(\lambda[n]) : \ \mathrm{maj}(T) = i\},$$

and as seen in [CEF15], the map $\{T \in \mathrm{SYT}(\lambda[n]) : \mathrm{maj}(T) = i\} \to \{T \in \mathrm{SYT}(\lambda[n+1]) : \mathrm{maj}(T) = i\}$ that attaches a cell labeled $n+1$ to the end of a $T \in \mathrm{SYT}(\lambda[n])$ becomes a bijection for $n \geqslant i+|\lambda|$. The weight bound (28) implies that only λ with $|\lambda| \leqslant i$ contribute to $[H_n]_i$. Hence, the degree-i harmonics, $[H_n]_i$, form a uniformly representation stable sequence with URS stable range $n \geqslant 2i$.

Alternatively, we may look at the monomial expansion and apply our result, Theorem 19. The monomial coefficients are given by

$$\langle \mathcal{F}([H_n]_i), h_{\mu[n]} \rangle = [q^i] \begin{bmatrix} n \\ \mu[n] \end{bmatrix}_q = \#\{w \in \mathcal{R}(1^{n-|\mu|}, 2^{\mu_1}, \ldots, (l+1)^{\mu_l}) : \mathrm{inv}(w) = i\},$$

where $[q^i]$ denotes taking the coefficient of q^i and $l = l(\mu)$. The set on the right hand side consists of words with content $1^{n-|\mu|}, 2^{\mu_1}, \ldots, (l+1)^{\mu_l}$ and i inversions. Now consider the map

$$\begin{aligned} &\{w \in \mathcal{R}(1^{n-|\mu|}, 2^{\mu_1}, \ldots, (l+1)^{\mu_l}) : \mathrm{inv}(w) = i\} \to \\ &\{w \in \mathcal{R}(1^{n+1-|\mu|}, 2^{\mu_1}, \ldots, (l+1)^{\mu_l}) : \mathrm{inv}(w) = i\} \end{aligned}$$

that maps $w \mapsto 1w$. We claim that for $n \geqslant |\mu|+i$ and $n \geqslant |\mu|+\mu_1$, this map is a bijection, i.e. that

$$\mathrm{rg}^m_\mu(\mathcal{F}([H_n]_i) \leqslant |\mu| + \max(i, \mu_1).$$

To see that it is surjective, we observe that any $w \in \mathcal{R}(1^{n+1-|\mu|}, 2^{\mu_1}, \ldots,)$ with $\mathrm{inv}(w) = i$, has $w_1 = 1$. If $w_1 \neq 1$, then the initial entry, w_1, would contribute at least $n+1-|\mu| > i$ inversions and so we can't have $\mathrm{inv}(w) = i$.

Since the weight of $[H_n]_i$ is bounded by i, we have a stable range from Eq (25) given by

$$n \geqslant \max\left(2i, \max_{|\mu|\leqslant i} |\mu| + \max(i, \mu_1)\right) = 2i,$$

which agrees with the one given in [CEF15] using their stability-degree formula as well as the explicit Schur expansion.

4.4 $r = 2$: classical diagonal harmonics

We will denote $H_n^{(2)}$ by DH_n throughout this section and refer to the algebra as the diagonal coinvariant algebra. From Proposition 22, the bi-graded components, $[DH_n]_{(i,j)}$, form a URS sequence for fixed i, j with URS stable range $n \geqslant 2(i+j)$. Our goal in this subsection is to reproduce this stable range by looking at the monomial expansion of corresponding Frobenius image, $\mathcal{F}([DH_n]_{(i,j)})$.

Letting q track the $X_n^{(1)}$ degree and t track the $X_n^{(2)}$ degree, the bi-graded Frobenius image is known to be

$$\mathcal{F}(DH_n; q, t) = \nabla e_n,$$

where ∇ is a certain operator defined on the modified Macdonald polynomials, $\tilde{H}_\mu[X;q,t]$, see [Hai02] and Theorem 2.17.1 [Hag08]. While it is still an open problem to find a combinatorial formula for the Schur expansion of ∇e_n, Carlsson and Mellit [CM18] proved a combinatorial formula for the monomial expansion of ∇e_n. For any composition $\alpha \vDash n$, they show

$$\langle \nabla e_n, h_\alpha \rangle = \sum_{\substack{P \in P_n \\ P\ \alpha\text{-shuffle}}} q^{\operatorname{dinv}(P)} t^{\operatorname{area}(P)}, \tag{30}$$

where all the relevant terms will be subsequently defined. In particular, when α is of partition shape, the above yields a formula for the monomial coefficient of m_α.

We now go through what all the terms in (30) mean. The set P_n denotes the set of *labeled Dyck paths* in an n-by-n grid. Recall that a Dyck path is a lattice walk in an n-by-n grid consisting of $(0,1)$ and $(1,0)$ as steps (east and north steps), starting at $(0,0)$, ending at (n,n), and staying weakly above the diagonal $y = x$. We will encode Dyck paths as words in letters N, E to denote the north (resp. east) steps taken in the lattice walk. For example the red path in Fig. 4 as an example is the Dyck path $NNEENNENEENE$.

A *labeled Dyck path* consists of a Dyck path and a bijection f from the set of north steps of the path to $[n]$, such that runs of consecutive north steps are mapped to increasing $i \in [n]$. Visually this can be represented by drawing labels to the right of each north step so that labels are increasing up columns (see Fig. 4).

The *reading word* of a labeled Dyck path is obtained by scanning southwest along diagonals, starting from the diagonal $y = x + n$, then scanning $y = x + n - 1$ and so on. Recording the labels encountered while scanning yields the reading word $\sigma(P) \in S_n$. A permutation, σ, is an *α-shuffle* if the values $1, 2, \cdots, \alpha_1$ occur in the same relative order in the one-line notation of σ, the values $\alpha_1 + 1, \cdots, \alpha_1 + \alpha_2$ occur in the same relative order, and so on. We say P is an α-shuffle if $\sigma(P)$ is an α-shuffle.

Finally, we describe the statistics, dinv and area, on the $P \in P_n$. The *area statistic* simply counts the number of cells that are entirely contained between the Dyck path and diagonal $y = x$. The *dinv statistic* counts the number of diagonal inversions. A *diagonal inversion* of a labeled Dyck path is a pair of cells, b, c, immediately right of a north step, with corresponding labels i, j, such that one of two following conditions are satisfied. Suppose that b has the greater label, i.e. $i > j$. The first condition is that b and c are in the same diagonal $y = x + s$ and b is further north than c. The second alternative condition is that c is on a diagonal of the form $y = x + s$, b is in a diagonal $y = x + s + 1$, and b is further south than c.

We now state the main result of this subsection. The following proposition may be viewed a refinement of Proposition 22 of Church, Farb, and Ellenberg, since it finds individual stable ranges for the multiplicities of Specht modules in $[DR_n]_{(i,j)}$.

Proposition 24. *The coefficient of $\mathbb{S}^{\lambda[n]}$ in $[DH_n]_{(i,j)}$ stabilizes once $n \geqslant \max(|\lambda| + i + j, 2|\lambda|)$.*

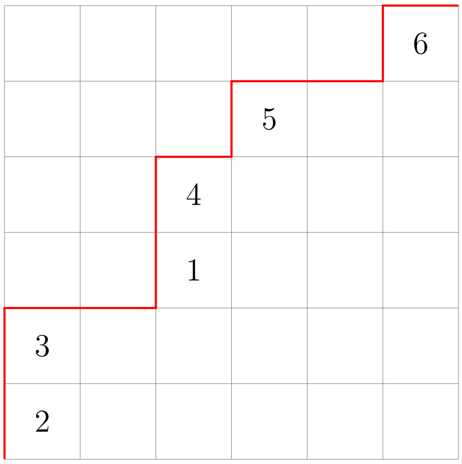

Figure 4: A labeled Dyck path, P, with reading word $\sigma(P) = 543612$, which is a α-shuffle for $\alpha = (2, 1, 1, 2)$ as well as all refinements of this composition. The area of P is 3 and $\operatorname{dinv}(P) = 6$ with the following label pairs contributing diagonal inversions: $(2, 6), (1, 6), (3, 4), (4, 5), (3, 5), (1, 3)$. Hence P contributes to the monomial coefficients of $m_{(2^2,1^2)}$, $m_{(2,1^3)}$, $m_{(1^4)}$ in $[DR_6]_{(5,3)}$.

Proof. For our analysis, it will be convenient to use a rearrangement of the parts of $\mu[n]$. Define the composition $\alpha^{(n)} = (\mu_1, \cdots, \mu_l, n - |\mu|)$ for $n \geqslant |\mu| + \mu_1$. Then (30) tells us that

$$\left\langle \mathcal{F}\left([DH_n]_{(i,j)}\right), h_{\mu[n]} \right\rangle = [q^i t^j] \left\langle \mathcal{F}(DH_n; q, t), h_{\mu[n]} \right\rangle \tag{31}$$

$$= [q^i t^j] \left\langle \mathcal{F}(DH_n; q, t), h_{\alpha^{(n)}} \right\rangle \tag{32}$$

$$= \#\{P \in P_n : \ P \text{ is an } \alpha^{(n)}\text{-shuffle}, \ \operatorname{dinv}(P) = i, \ \operatorname{area}(P) = j\} \tag{33}$$

Our goal is to find a range $n \geqslant M_\mu$ for which the above quantity has stabilized as this will allow us to apply the bound (25) in Theorem 19.

Let $D^{(n)}$ denote the set being enumerated in (33). To see when $|D^{(n)}|$ stabilizes, we can construct connective maps that are eventually bijections. Define the map $\psi_n : D^{(n)} \to D^{(n)}$ as follows. Given a labeled Dyck path P with underlying Dyck path π, extend the Dyck path π to the Dyck path, $NE\pi$, on an $(n + 1)$-by-$(n + 1)$ grid. Keep the same labeling, while adding $n + 1$ next to this newly created north step. See Fig. 5 for an example.

The new reading word can be obtained from $\sigma(P)$ by adding an $n + 1$ to the end of its one-line notation. Hence, $\psi_n(P)$ is an $\alpha^{(n+1)}$-shuffle whenever P is an $\alpha^{(n)}$-shuffle. One can easily verify that $\operatorname{area}(\pi) = \operatorname{area}(NE\pi)$ and $\operatorname{dinv}(\psi_n(P)) = \operatorname{dinv}(P)$ since there are no new area-contributing cells and $n + 1$ does not form any diagonal inversion pairs. Hence, $\psi_n : D^{(n)} \to D^{(n+1)}$ is well-defined.

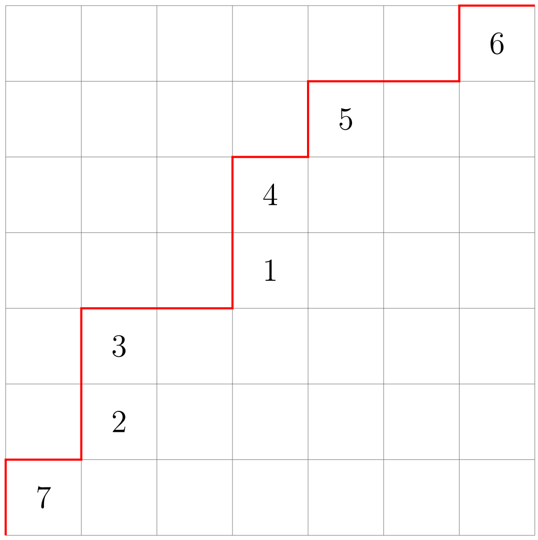

Figure 5: The labeled Dyck path obtained from the one in Fig. 4 after applying ψ_6.

The map is clearly injective but not necessarily surjective for small n. We claim the map is surjective, once $n \geqslant i + j + |\mu|$ and $n \geqslant |\mu| + \mu_1$, i.e. that

$$\mathrm{rg}_{\mu}^{m}(\mathcal{F}([DH_n]_{(i,j)})) \leqslant |\mu| + \max(i + j, \mu_1).$$

Let $P \in D^{(n+1)}$. Since $\mathrm{area}(P) = j$, there are at least $n + 1 - j$ rows with no area contributing cells. Hence there are at least $n + 1 - j$ cells on the diagonal $y = x$.

Now the reading word of P must contain the values $|\mu| + 1, \ldots, n, n + 1$ in the same relative order, since the reading word is an $\alpha^{(n+1)}$-shuffle. Suppose for sake of contradiction that the bottom leftmost label of P (i.e. the last entry of the reading word) is $a \neq n + 1$. Then the relative order of $|\mu|+1, \ldots, n, n+1$ forces $a \leqslant |\mu|$. Looking for potential diagonal inversion pairs among labels in the diagonal $y = x$, the cell containing a is southwest of at least $n - j$ other cells on the diagonal $y = x$. Since $a \leqslant |\mu|$, at most $|\mu| - 1$ of these cells have a label less than a. Hence there are at least

$$n - j - (|\mu| - 1) > i = \mathrm{dinv}(P)$$

diagonal inversion pairs, a contradiction. Hence the final entry of the reading word must be $n + 1$.

Since labeled Dyck paths must be increasing up columns and $n + 1$ is at the bottom of its column, $n + 1$ is the only label in its column. Hence, the Dyck path of P is of the form $NE\pi$ and P lies in the image of ψ_n.

It follows that

$$\left\langle \mathcal{F}\left([DH_n]_{(i,j)}\right), h_{\mu[n]} \right\rangle$$

is stable once $n \geqslant i + j + |\mu|$ and so (25) implies that

$$\left\langle \mathcal{F}\left([DH_n]_{(i,j)}\right), s_{\lambda[n]} \right\rangle$$

is stable once

$$n \geqslant \max\left(2|\lambda|, \max_{|\mu|\leqslant|\lambda|}(\max(i+j,\mu_1)+|\mu|)\right) = \max(2|\lambda|,\, i+j+|\lambda|),$$

and this finishes the proof. □

Remark 25. The maps, ψ_n, defined in the proof above are injective for all n and provide a combinatorial proof of inequality (7) for $V_n = [DH_n]_{i,j}$.

Using the weight bound(28), we have weight$([DH_n]_{(i,j)}) \leqslant i+j$. Note that it is in fact possible to show that weight$([DH_n]_{(i,j)}) = i+j$ using combinatorial methods. Since $([DH_n]_{(i,j)})_n$ arises from a torsion-free **FI**-module and has monomial stable Frobenius images, it suffices to show that $\dim([DH_n]_{(i,j)})$ eventually agrees with a polynomial in n of degree $i+j$, in order to conclude that the weight is $i+j$. This precise computation is carried out in Wang's dissertation [Wan25]. Hence, (26) of Theorem 19 applies yielding the following corollary which agrees with the range given in Proposition 22.

Corollary 26. *The (i,j)-bi-graded component of the diagonal harmonics DH_n is URS with URS stable range $n \geqslant 2(i+j)$.*

5 Applications to Garsia-Haiman modules

In this section, we describe S_n-modules known as the Garsia-Haiman modules [GH93], whose bi-graded Frobenius images are $\tilde{H}_\mu$, the modified Macdonald polynomials. We will then show that they have an **FI**-module structure and hence are representation stable by the noetherian property of **FI**-modules. We will then provide an alternative direct approach to showing representation stability from the monomial expansions of their Frobenius images, while obtaining new insights into the combinatorics of Macdonald polynomials (see Corollary 30).

5.1 Garsia-Haiman modules

As before, we take DH_n to denote $H_n^{(2)}$, the diagonal harmonics, and denote the variable sets $X_n^{(1)}$ and $X_n^{(2)}$ instead by $x_1, \cdots, x_n$ and $y_1, \ldots, y_n$.

Given a partition shape $\nu \vdash n$ and a cell c in ν, we define the *co-leg* and *co-arm* of c, $l'(c)$ (resp. $a'(c)$), to be the number of cells below and in the same column as c (resp. left and in the same row as c). Define the ν-bialternant

$$\Delta_\nu = \det\left(x_i^{l'(c)} y_i^{a'(c)}\right)_{i=1,\ldots,|\nu|,\ c\in\nu},$$

where c ranges over some enumeration of the cells of ν and Δ_ν is only defined up to sign since we don't provide an ordering on the cells.

For example, if $\nu = (3,2)$, then the (coleg,coarm) value pairs as we range over the cells are (0,0), (0,1), (0,2),(1,0), (1,1) and the $(3,2)$-bialternant is

$$\Delta_{(3,2)} = \det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_5y_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 \end{pmatrix}.$$

It is known (see solution to Exercise 2.12 [Hag08]) that $\Delta_\nu \in DH_n$ for all $n \geqslant |\nu|$. Let

$$\mathcal{L}(f) := \mathrm{Span}_{\mathbb{C}}(\text{partial derivatives of } f).$$

Since solutions to the differential equations associated with DH_n are closed under partial differentiation (see Exercise 2.12 [Hag08]), we have

$$GH_\nu := \mathcal{L}(\Delta_\nu) \subseteq DH_n$$

is a multi-graded S_n-submodule of DH_n, which is referred to as the *Garsia-Haiman module* corresponding to ν. We verify the following simple lemma.

Lemma 27. *If $\mu \subset \nu$ are pair of partitions, then $GH_\mu \subset GH_\nu$.*

Proof. It suffices to show this for $\mu \subset \nu$ with $\mu \vdash n$ and $\nu \vdash n+1$, differing by a single cell. It is further enough to show that the bialternant $\Delta_\mu \in GH_\nu$, since GH_ν is closed under taking partial derivatives and the action of S_n. Let

$$A = \left(x_i^{l'(c)} y_i^{a'(c)}\right)_{i=1,\ldots,|\mu|,\ c\in\mu^{(n)}},$$

so that $\det(A) \underset{\text{sgn}}{=} \Delta_\mu$, where $\underset{\text{sgn}}{=}$ means equality up to sign. Let $\hat{c}$ be the unique cell ν/μ so that

$$\Delta_\nu \underset{\text{sgn}}{=} \det \begin{bmatrix} & x_1^{l'(\hat{c})} y_1^{a'(\hat{c})} \\ A & \vdots \\ & x_n^{l'(\hat{c})} y_n^{a'(\hat{c})} \\ (x_{n+1}^{l'(c)} y_{n+1}^{a'(c)})_{c\in\mu} & x_{n+1}^{l'(\hat{c})} y_{n+1}^{a'(\hat{c})} \end{bmatrix} \underset{\text{sgn}}{=} x_{n+1}^{l'(\hat{c})} y_{n+1}^{a'(\hat{c})} \det(A) + \sum_{c\in\mu} x_{n+1}^{l'(c)} y_{n+1}^{a'(c)} f_c,$$

for some polynomials $f_c \in \mathbb{C}[x_1, \cdots, x_n, y_1, \cdots, y_n]$, independent of x_{n+1}, y_{n+1}. Then

$$\left(\frac{\partial}{\partial x_{n+1}}\right)^{l'(\hat{c})} \left(\frac{\partial}{\partial y_{n+1}}\right)^{a'(\hat{c})} \Delta_\nu \underset{\text{sgn}}{=} (l'(\hat{c})! \cdot a'(\hat{c})!)\, \Delta_\mu$$

since every cell, c, of μ is either strictly left of $\hat{c}$ (so that $a'(c) < a'(\hat{c})$) or strictly below $\hat{c}$ (so that $l'(c) < l'(\hat{c})$). This verifies that $\Delta_\mu \in GH_\nu$. □

For the remainder of the section we will always let $\mu^{(\bullet)}$ denote a sequence of partitions $\mu^{(0)} \subset \mu^{(1)} \subset \cdots \subset \mu^{(n)} \subset \dots$ such that $\mu^{(n)} \vdash n$. It follows that the sequence

$$GH_{\mu^{(0)}} \subset \cdots \subset GH_{\mu^{(n)}} \subset GH_{\mu^{(n+1)}} \subset \dots$$

with inclusions as connective maps is a sub-**FI**-module of $(DH_n, i_n)_n$, which we will simply denote as $GH_{\mu^{(\bullet)}}$. By Theorem 1.3 of [CEF15], each (i,j)-bi-graded component of $GH_{\mu^{(\bullet)}}$ is a finitely generated **FI**-module with weight $\leqslant i+j$.

5.2 Application of monomial stability to modified Macdonald polynomials

In this subsection, we will show that the bi-graded components of the $GH_{\mu^{(\bullet)}}$ are uniformly representation stable directly by looking at their Frobenius images. Since these modules are torsion-free, it suffices to show that they form a URMS sequence by Corollary 4. We have the following Proposition of Haiman, which motivates the original interest in Garsia-Haiman modules as models of the more well-known modified Macdonald polynomials.

Definition 28 ([Hai01])**.** The *modified Macdonald polynomials*, $\tilde{H}_\nu[X;q,t] \in \Lambda_{\mathbb{C}(q,t)}$, are characterized by

1. $\tilde{H}_\nu[X(1-q);q,t] \in \mathrm{Span}_{\mathbb{Q}(q,t)}\{s_\lambda : \lambda \geqslant \nu\}$
2. $\tilde{H}_\nu[X(1-t);q,t] \in \mathrm{Span}_{\mathbb{Q}(q,t)}\{s_\lambda : \lambda \geqslant \nu'\}$
3. $\langle \tilde{H}_\nu, s_{(n)} \rangle = 1$.

Here, $\geqslant$ refers to the *dominance order* on partitions.

Theorem 29 ([Hai01])**.** *As an ungraded module, GH_ν, is isomorphic to the regular representation of $S_{|\nu|}$. Letting q track the y-weight and t track the x-weight, we have that the bi-graded Frobenius image of the Garsia-Haiman module is*

$$\mathcal{F}(GH_\nu;q,t) = \tilde{H}_\nu[X;q,t],$$

the modified Macdonald polynomials with their monomial expansions described in Theorem 31. So in particular, $\tilde{H}_\nu[X;1,1] = h_{(1^{|\nu|})}$.

As with the diagonal harmonics and ∇e_n, there are no known combinatorial formulas for the Schur decompositions of the modified Macdonald polynomials. The coefficients for the Schur expansion of $\tilde{H}_\nu[X;q,t]$ are referred to as the *modified q,t-Kostka numbers*, $\tilde{K}_{\lambda,\nu}(q,t) \in \mathbb{N}[q,t]$, satisfying

$$\tilde{H}_\nu[X;q,t] = \sum_{\lambda \vdash |\nu|} \tilde{K}_{\lambda,\nu}(q,t)\, s_\lambda[X].$$

So in particular, $\tilde{K}_{\lambda,\nu}(1,1) = K_{\lambda,\nu}$ reduces to an ordinary Kostka number by Proposition 29. We get the following combinatorial inequality as an immediate corollary of applying Corollary 6 to the torsion-free **FI**-module $GH_{\mu^{(\bullet)}}$.

Corollary 30. *Let $\mu^{(1)} \subset \mu^{(2)} \subset \cdots$ be a sequence of partitions where $\mu^{(n)} \vdash n$ and λ is some partition. Then for $n \geqslant |\lambda| + \lambda_1$*

$$\tilde{K}_{\lambda[n+1],\mu^{(n+1)}}(q,t) - \tilde{K}_{\lambda[n],\mu^{(n)}}(q,t) \in \mathbb{N}[q,t]$$

and for $\eta \vdash n$ and $\hat{\eta} \vdash n+1$ with $\eta \subset \hat{\eta}$, we have

$$\langle \tilde{H}_{\mu^{(n+1)}}, h_\eta \rangle - \langle \tilde{H}_{\mu^{(n)}}, h_{\hat{\eta}} \rangle \in \mathbb{N}[q,t].$$

□

With the knowledge that the the monomial coefficients $\langle \mathcal{F}([GH_{\mu^{(n)}}]_{i,j}), h_{\eta[n]} \rangle$ form a non-decreasing sequence of natural numbers, monomial stability will follow as soon as we can show that they are bounded from above. For this we can leverage the following miraculous combinatorial monomial formula provided by Haglund, Haiman, and Loehr [HHL05a].

Theorem 31 ([HHL05a])**.** *The modified Macdonald polynomial for a partition $\nu \vdash n$, is a symmetric function in variable set X over $\mathbb{C}(q,t)$:*

$$\tilde{H}_\nu[X;q,t] = \sum_{\sigma:\nu\to\mathbb{N}} q^{inv(\sigma)} t^{maj(\sigma)} x_\sigma,$$

where the sum is taken over all $\mathbb{N}$-labelings of the cells of ν, called fillings, the statistics inv and maj are described below, and $x_\sigma = \prod_{c\in\nu} x_{\sigma(c)}$.

We thus get the following formula for the monomial coefficients of the bi-graded components of the Garsia-Haiman modules

$$\langle \mathcal{F}([GH_{\mu^{(n)}}]_{i,j}), h_{\eta[n]} \rangle \tag{34}$$
$$= \langle [q^i t^j] \tilde{H}_{\mu^{(n)}}, h_{\eta[n]} \rangle \tag{35}$$
$$= \#\{\sigma : \mu^{(n)} \to \mathbb{N} : \text{content } 0^{n-|\eta|}, 1^{\eta_1}, \ldots, l^{\eta_l},\ \text{inv}(\sigma) = i,\ \text{maj}(\sigma) = j\} \tag{36}$$

where $l = l(\eta)$.

The *reading order* of a diagram ν is the order on cells that goes left-to-right along rows starting from the top row and working down. The *standardization* of a labeling $\sigma : \nu \to \mathbb{N}$ with content $\alpha = (\alpha_0, \alpha_1, \cdots)$, denoted by $\text{st}(\sigma)$, is an injective labeling $\sigma : \nu \to \{1, \cdots, |\nu|\}$ that labels the 0's appearing in ν as $1, 2, \cdots, \alpha_0$ in the order that they appear in the reading order. The 1's appearing in ν are labeled $\alpha_0 + 1, \cdots, \alpha_0 + \alpha_1$ in reading order and so on. A *triple* consists of cells with (row,column) coordinates of the form $u = (r,c)$, $v = (r-1,c)$ and $w = (r, c+k)$ for $r, c, k \geqslant 1$, where u, w must be cells of ν, but v is allowed to be a cell immediately below ν. Extend $\text{st}(\sigma)$ to cells immediately below ν by labeling them $-\infty$. A triple is an *inversion triple* of σ if the labels $\text{st}(\sigma)(u), \text{st}(\sigma)(w), \text{st}(\sigma)(v)$ are decreasing clockwise. The statistic $\text{inv}(\sigma)$ counts the number of inversion triples of σ.

A *descent* of σ is a cell $u = (r+1, c)$ in ν, such that $v = (r, c)$ is also in ν and $\sigma(u) > \sigma(v)$. Letting $\mathrm{Des}(\sigma)$ denote that set of descents we define the *major index statistic* for fillings as

$$\mathrm{maj}(\sigma) := \sum_{u \in \mathrm{Des}(\sigma)} (\mathrm{leg}(u) + 1),$$

where $\mathrm{leg}(u)$ is the number of cells of ν in the same column as u and above u. See Fig. 6 for an example.

1	4	3	
0	2	1	
0	3	0	1

Figure 6: An $\mathbb{N}$-labeling of $\nu = (4, 3, 3)$. The descents occur at cells $(2, 3), (3, 1), (3, 2), (3, 3)$ yielding $\mathrm{maj}(\sigma) = 2 + 1 + 1 + 1 = 5$. The inversion triples are formed by $((3, 2), (3, 1), (3, 3))$, $((1, 2), (1, 3))$, and $((1, 2), (1, 4))$ yielding $\mathrm{inv}(\sigma) = 3$.

Question 32. The formula for the monomial coefficient, (36) begs the question whether or not Corollary 30 (which followed from the non-trivial representation theoretic Lemma 2) can be proven directly. In other words, if $\eta \vdash n$ with length l and $\hat{\eta} \vdash n + 1$ with $\hat{\eta}/\eta$ is a single cell in the a-th row, then

$$\langle \tilde{H}_{\mu^{(n)}}, h_\eta \rangle = \#\{\sigma : \mu^{(n)} \to \mathbb{N} : \text{content } 1^{\eta_1}, \ldots, l^{\eta_l},\ \mathrm{inv}(\sigma) = i,\ \mathrm{maj}(\sigma) = j\}$$

and

$$\langle \tilde{H}_{\mu^{(n+1)}}, h_{\hat{\eta}} \rangle = \#\{\sigma : \mu^{(n)} \to \mathbb{N} : \text{content } 1^{\eta_1}, \ldots, a^{\eta_a + 1}, \ldots l^{\eta_l},\ \mathrm{inv}(\sigma) = i,\ \mathrm{maj}(\sigma) = j\}.$$

Can one describe an injective map from the former set into the latter one. Note that this may be an intricate problem, since fillings can end up looking quite different in order to meet the same inv and maj target values in the new shape $\mu^{(n+1)}$.

Lemma 33. *Fix $i, j \geqslant 0$ and a partition η. We have the following upper bound*

$$[q^i t^j] \langle \tilde{H}_\mu, h_{\eta[n]} \rangle \leqslant \binom{k}{\eta[k]}, \tag{37}$$

where n and $\mu \vdash n$ are arbitrary and $k = (|\eta| + i)(|\eta| + j)$. We interpret the left hand side to be zero when $\eta[n]$ is not defined and interpret $\binom{k}{\eta[k]} = 1$ in the single case when $\eta[k]$ is not defined ($i, j = 0$ and $\eta = (1)$).

Proof. We will show the upper bound holds for any given n and $\mu \vdash n$. Let $A \subseteq_{\mathrm{diag}} \mu$ be the subset of the cells of μ consisting of the rightmost $|\eta| + i$ cells of the bottom row and the leftmost $|\eta| + i$ cells of any higher row. Let $B \subseteq_{\mathrm{diag}} \mu$ be the subset of cells which are

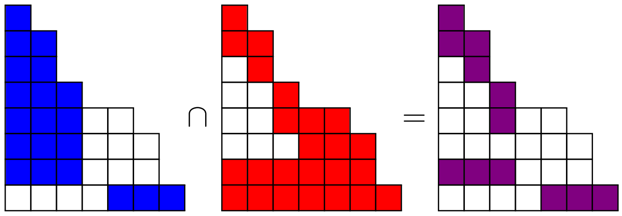

Figure 7: An example of the zone Z for $\mu = (7, 6, 6, 5, 3, 2, 2, 1)$, $|\eta| = 2$, $i = 1$ and $j = 2$, with the set A in blue, the set B in red, and the set Z in purple.

in either the topmost j or bottommost $|\eta|$ cells of a column. Let $Z = A \cap B$. Note that by construction $|Z| \leqslant (|\eta| + i)(|\eta| + j)$. See Fig. 7 for an example.

We claim that any filling $\sigma : \mu \to \mathbb{N}$ contributing to the count $[q^i t^j]\langle \tilde{H}_\mu, h_{\eta[n]}\rangle$, i.e., a filling with content $0^{n-|\eta|}, 1^{\eta_1}, \ldots, l^{\eta_l}$ and $\operatorname{inv}(\sigma) = i$ and $\operatorname{maj}(\sigma) = j$, must have all its non-zero entries in Z.

Suppose for contradiction that σ attached a non-zero label to a cell, c, outside of A. If c is in the first row, then there are at least $|\eta| + i$ cells right of c, with at most $|\eta| - 1$ of them being non-zero. But this means that c creates at least $i + 1$ inversions with the cells labeled zero to its right, a contradiction. Similarly, if c is not in the bottom row then it has at least $|\eta| + i$ columns to its left, of which at least $i + 1$ contain only zeros. But then c forms an inversion triple of the form $\begin{array}{|c|}\hline 0\\\hline 0\\\hline\end{array}\ \boxed{c}$ with each of those columns, yielding the same contradiction. Hence $c \in A$.

Now we will show that $c \in B$. If c is in the top j entries of its column then we are done. Suppose otherwise and observe that c cannot form a descent, or else it would contribute at least $j + 1$ to $\operatorname{maj}(\sigma)$. Hence c is in the bottom row or the cell immediately below c, c', has a non-zero label. But c' cannot form a descent by the same reasoning and we can repeat the argument to conclude that there is a contiguous vertical interval of cells starting from c to the bottom row containing non-zero labels. Hence c must be in the bottom $|\eta|$ rows of μ, implying $c \in B$. Thus, we have shown that if $\sigma(c) > 0$, then $c \in A \cap B = Z$.

Since $[q^i t^j]\langle \tilde{H}_\mu, h_{\eta[n]}\rangle$ is bounded by the total number of ways to fill $\sigma|_Z : Z \to \mathbb{N}$ using content $0^{|Z|-|\eta|}, 1^{\eta_1}, \ldots, l^{\eta_l}$, we have

$$[q^i t^j]\langle \tilde{H}_\mu, h_{\eta[n]}\rangle \leqslant \binom{|Z|}{|Z| - |\eta|, \eta_1, \ldots, \eta_l} \leqslant \binom{k}{\eta[k]},$$

where $k = (|\eta| + i)(|\eta| + j)$. $\square$

Hence, we arrive at the following proposition.

Proposition 34 (Direct proof of $(\boldsymbol{GH_{\mu(\bullet)}})$ stability)**.** *Each (i, j)-bi-graded component of $(GH_{\mu(\bullet)})$ is uniformly representation stable.*

Proof. This follows by Theorem 21, the fact that $\mathrm{wt}(GH_{\mu^{(\bullet)}}) \leqslant i + j$, and the upper bounds on monomial coefficients established in Lemma 33. $\square$

This proposition is an extension of the work of Mbirika and Tymoczko [MT20] on the representation stability of Springer fibers, which is the $j = 0$ case of the proposition since the $[GH_{\mu^{(n)}}]_{(i,0)}$ agree with the components of the Garsia-Procesi modules in [MT20].

5.3 Special case: $\mu^{(n)} = \mu[n]$

In this subsection, we focus on the special case where we fix some μ and explicitly compute a stable range for the sequence of $(GH_{\mu[n]})_n$ for $n \geqslant |\mu| + \mu_1$.

Lemma 35. *The monomial coefficient $\langle [q^i t^j]\tilde{H}_{\mu[n]}, h_{\eta[n]}\rangle$ stabilizes once $n \geqslant \max(|\mu| + \mu_1 + |\eta| + i,\ |\eta| + \eta_1)$.*

Proof. Fix partitions μ, ν and i, j and let $M^{(n)}$ denote the set of being enumerated in (36). Then

$$\langle [q^i t^j]\tilde{H}_{\mu[n]}, h_{\eta[n]}\rangle = |M^{(n)}|$$

and we wish to show $|M^{(n)}|$ stabilizes.

Suppose $n \geqslant |\mu| + \mu_1 + |\eta| + i$ and $n \geqslant |\eta| + \eta_1$ so that $\mu[n]$ and $\eta[n]$ are well-defined. Define the map $\gamma_n : M^{(n)} \to M^{(n+1)}$ as follows. For $\sigma \in M^{(n)}$, shift the bottom row of the corresponding diagram to the right by one cell and insert a new cell labeled 0 in the bottom left corner. So for instance:

$$\gamma_n : \begin{ytableau} 1 & 0 & 2 \\ 0 & 0 & 4 & 3 & 1 & 0 \end{ytableau} \longmapsto \begin{ytableau} 1 & 0 & 2 \\ 0 & 0 & 0 & 4 & 3 & 1 & 0 \end{ytableau}$$

We have to check that the map is well-defined and surjective. We claim that the first $\mu_1 + 1$ entries of the bottom row of any $\sigma' \in M^{(n+1)}$ are 0. Suppose for contradiction that there was a non-zero value a among these first $\mu_1 + 1$ cells. Then there are at least $n+1-|\mu|-(\mu_1+1) = n-|\mu|-\mu_1$ entries right of a. Of these, at most $|\eta|-1$ are allowed to be non-zero (since a is one of the $|\eta|$ non-zero values in the labeling). Hence a would contribute at least $n - |\mu| - \mu_1 - (|\eta| - 1) > i$ inversion triples formed by taking a, the phantom cell labeled $-\infty$ below it, and a cell labeled 0 to the right of it. This contradicts that $\mathrm{inv}(\sigma') = i$ for $\sigma' \in M^{(n+1)}$.

This implies that if we let σ be obtained from a $\sigma' \in M^{(n+1)}$ by deleting the bottom left cell and shifting the bottom row to the left by one, then $\sigma \in M^{(n)}$ and $\gamma_n(\sigma) = \sigma'$, which implies that γ_n is well-defined and bijective. Indeed, the first μ_1 cells of the bottom row of σ will be labeled 0 and so the status of descents and inversion triples does not change when going from σ' to σ. Thus, σ' and σ have the same inv and maj values. $\square$

Corollary 36. *The coefficient of $\mathbb{S}^{\lambda[n]}$ in $[GH_{\mu[n]}]_{(i,j)}$ stabilizes once $n \geqslant \max(2|\lambda|, |\lambda| + |\mu| + \mu_1 + i)$. Thus, $[GH_{\mu[n]}]_{(i,j)}$ is URS with URS stable range $n \geqslant \max(2(i + j), |\mu| + \mu_1 + 2i + j)$.*

Proof. By Theorem 19, the coefficient of $\mathbb{S}^{\lambda[n]}$ in $[GH_{\mu[n]}]_{(i,j)}$ stabilizes once $n \geqslant 2|\lambda|$ and $\max(|\mu| + \mu_1 + |\eta| + i,\ |\eta| + \eta_1)$ for all $|\eta| \leqslant |\lambda|$. The maximum of all these quantities is $\max(2|\lambda|, |\lambda| + |\mu| + \mu_1 + i))$. Since the weight of $[GH_{\mu[n]}]_{(i,j)}$ is $\leqslant i + j$, we get the stable range in the statement. $\square$

We may restate Corollary 36 as follows. The coefficient $[q^i t^j]\tilde{K}_{\lambda[n],\mu[n]}(q, t)$ is non-zero only if $|\lambda| \leqslant i + j$ and stabilizes once $n \geqslant \max(2|\lambda|, |\lambda| + |\mu| + \mu_1 + i)$.

Example 37. One case in which the q, t-Kostka number is well-understood occurs when $\lambda[n]$ is a hook shape. Let $B_{\mu[n]} = \sum_{c\in\mu[n]} q^{a'(c)}t^{l'(c)}$ so that in particular $B_{\mu[n]} - 1 = \sum_{c\in\mu[n]\setminus(1,1)} q^{a'(c)}t^{l'(c)}$ is the sum over all but the bottom left cell. From [Mac98], we have the plethystic identity

$$\tilde{K}_{1^k[n],\mu[n]}(q, t) = \tilde{K}_{(n-k,1^k),\mu[n]}(q, t) = e_k[B_{\mu[n]} - 1],$$

the elementary symmetric function e_k with variables set to $q^{a'(c)}t^{l'(c)}$ for $c \in \mu[n]\setminus(1, 1)$. We expect the coefficient of $q^i t^j$ to stabilize once $n \geqslant \max(2k, k + |\mu| + \mu_1 + i)$. It in fact stabilizes earlier; once $n \geqslant |\mu| + i + 1$. This is because the additional term contributed by the cell added to the bottom row $B_{\mu[n+1]} - B_{\mu[n]} = q^{n+1-|\mu|-1}t^0$, which cannot contribute to the coefficient of $q^i t^j$, once $n - |\mu| > i$.

As noted by the authors in [GHT99], the formula in Example 37 admits the following plethystic formula generalization to all shapes $\lambda[n]$:

$$\tilde{K}_{\lambda[n],\mu[n]} = \nabla^{-1} s_\lambda \left[\frac{1 - \epsilon(MtB_\mu - 1)}{M} - 1 - \epsilon[n - |\mu|]_q\right]. \tag{38}$$

Since ∇^{-1} introduces negative powers of q and t, one cannot simply repeat the argument in Example 37. However, with more control on the resulting positive and negative powers it may be possible to get a similar stable range to the one obtained above.

Alternatively, in [GT96] a formula of the form

$$\tilde{K}_{\lambda[n],\mu[n]} = k_\lambda[B_{\mu[n]}; q, t], \tag{39}$$

is found where $k_\lambda[X; q, t]$ is an inhomogeneous symmetric function with highest degree $|\lambda|$ and coefficients being Laurent polynomials in q, t. In Example 37, $k_{1^k}[X] = e_k[X - 1]$. Again, the coefficients have negative powers so the argument cannot run through as in the example.

Question 38. Can one alternatively prove Corollary 36 using a careful analysis of equation (38) or (39)?

6 Further work

Similar to Wang's work [Wan25] on computing the polynomial that eventually agrees with $\dim([DH_n]_{(i,j)})$, we can ask about the polynomial corresponding to $\dim([GH_{\mu[n]}]_{(i,j)})$ or more generally, $\dim([GH_{\mu^{(n)}}]_{(i,j)})$. In particular, we know that the degree of the polynomial will agree with $\mathrm{wt}([GH_{\mu^{(\bullet)}}]_{(i,j)})$ and it is interesting to see whether or not this degree attains its maximum possible value, $i+j$, like in the case of DH_n. We are currently in the process of carrying out this computation using the expansion of $\tilde{H}_\mu$ in the symmetric functions of Lascoux, Leclerc, and Thibon [LLT97].

A related object of interest, is the intersection of Garsia-Haiman modules, GH_μ for varying μ, although proofs of the conjectured Frobenius image and even the dimension of the intersection, remain elusive (see [BG99]). Recently, a conjectural monomial formula was presented for the intersection $GH_\mu \cap GH_\lambda$, where μ, λ differ by a cell [KLO23]. It may be interesting to explicitly observe the monotonicity described by Corollary 6, as well as the stability of the monomial coefficients as we grow μ, λ as this would provide further evidence for the validity of this formula.

We saw how for torsion-free **FI**-modules the eventual polynomial dimension of V_n can provide a bound on weight. We ask if one can make similar conclusions for non-torsion-free **FI**-modules, provided we know something about the character values.

Question 39. Suppose that for large n, $\dim(V_n) = p(n)$ for a degree K polynomial p and the character values

$$\chi^{V_n}(\sigma \cdot (k+1 \quad k+2 \quad \cdots \quad n))$$

stabilize for all $\sigma \in S_k$ and all $k \leqslant K$. Is $(V_n)_n$ URMS?

Note how this relates to polynomiality of characters: this provides a converse to Theorem 3.3.4 in [CEF15], while checking less conditions about the character values i.e. not demanding that $\chi^{V_n}(\sigma)$ is a polynomial in cycle counts of σ.

In a different direction, we can also consider combinatorial Hopf algebras other than the symmetric functions. There is an analogue of the Frobenius characteristic map between the Grothendieck ring of the *0-Hecke algebras*, $(H_n(0))_n$, and the *quasi-symmetric functions*, QSym (see [Lau21] or [Hua16] for instance). Laudone [Lau21] introduces the notion of representation stability for sequences, $(V_n)_n$, where V_n is an $H_n(0)$-module, calling them $\mathcal{H}$*-modules*. Instead of having the coefficient of $s_{\lambda[n]}$ stabilizing, the quasi-symmetric Frobenius images have the multiplicity of certain (P, ω)-partition generating functions stabilize (see [Sta24]). The author is currently working on generalizing theory of torsion-free **FI**-modules and stability transfer to $\mathcal{H}$-modules and their quasisymmetric Frobenius images. Additionally, the Grothendieck ring of the *projective 0-Hecke modules* is given by the *non-commutative symmetric functions*, NSym. The author is also currently working on the stability theory for NSym and projective and 0-Hecke modules.

Question 40. Can one generalize the methods developed for transferring stability between symmetric function bases to transferring stability between quasi-symmetric (resp. non-commutative symmetric) bases? Can these methods be applied to conclude 0-Hecke representation stability of sequences in the sense of [Lau21]?

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Appendix

As promised this section provides an alternate proof of Theorem 19(a) and in fact improves the ranges (22) and (23) by dropping the $2|\lambda|$ term.

Proposition 41. *A sequence of S_n-modules $V = (V_n \in \boldsymbol{Mod}_{S_n})_n$ is URMS if $wt(V_\bullet) < \infty$ and the sequence of Frobenius images $\mathcal{F}(V_\bullet)$ is m-stable. In this case, the multiplicity of $\mathbb{S}^{\lambda[n]}$ in V_n stabilizes once*

$$n \geqslant \max_{|\mu| \leqslant |\lambda|} rg_\mu^m(\mathcal{F}(V_\bullet)). \tag{40}$$

and a uniform stable range of $V_\bullet$ is given by

$$n \geqslant \max_{|\mu| \leqslant wt(V_\bullet)} rg_\mu^m(\mathcal{F}(V_\bullet)). \tag{41}$$

Note that because we need to consider all $|\mu| \leqslant |\lambda|$ in (40), we can take $\mu = (|\lambda|)$ which leaves $\mu[n]$ undefined for $n < 2|\lambda|$. In all the applications in Sections 4 and 5, it is possible to have a non-zero monomial coefficient corresponding to this $\mu[n]$ for $n \geqslant 2|\lambda|$ and hence we anyways need $n \geqslant 2|\lambda|$ for (40) to hold.

The proof of Proposition 41 does not rely on the general ideas of stability transfer put forth in Section 3 and instead uses very specific properties of the Kostka numbers as seen in the following key lemma. Recall, that a symmetric function is *Schur-positive* if the coefficients of its Schur expansion are non-negative integers. A degree-n homogeneous symmetric function is the Frobenius image of some S_n-representation if and only if it is Schur-positive.

The following key lemma is stated in slightly greater generality than necessary since it includes a parameter a, which is simply taken to be $a = |\lambda|$ for the sake of proving Proposition 41.

Lemma 42. *Fix n and let $F_{n+i} = \sum_\lambda c_{\lambda,n+i} s_{\lambda[n+i]}$, for $i = 0, 1$ be two Schur-positive functions. Let $F_{n+i} = \sum_\mu d_{\mu,n+i} m_{\mu[n+i]}$ be the corresponding monomial expansions, with the usual convention that $c_{\lambda,n+i} = 0$ (resp. $d_{\mu,n+i} = 0$) when $\lambda[n+i]$ (resp. $\mu[n+i]$) is not defined. If $d_{\mu,n} = d_{\mu,n+1}$ for all $|\mu| \leqslant a$ for some fixed a, then $c_{\lambda,n} = c_{\lambda,n+1}$ for all $|\lambda| \leqslant a$.*

Proof. We note that throughout the proof n and a are fixed and we use the standard convention that $K_{\lambda[n+i],\mu[n+i]} = 0$, whenever $\lambda[n+i]$ or $\mu[n+i]$ is not defined. By expanding $s_{\lambda[n+i]} = \sum_\mu K_{\lambda[n+i],\mu[n+i]} m_{\mu[n+i]}$ and equating $m_{\mu[n+i]}$ coefficients of F_{n+i}, we get the relation

$$d_{\mu,n+i} = \sum_{\substack{\lambda \\ |\lambda| \leqslant |\mu|}} c_{\lambda,n+i} K_{\lambda[n+i],\mu[n+i]},$$

where the sum only needs to be taken over $|\lambda| \leqslant |\mu|$ since otherwise $K_{\lambda[n+i],\mu[n+i]} = 0$ by Lemma 15 (note the roles of λ and μ are reversed in the statement of that lemma). For fixed $|\lambda| \leqslant a$, define the corresponding *Kostka column vector*, $\vec{K}_{\lambda,n+i}$ to be the vector

$(K_{\lambda[n+i],\mu[n+i]})_{|\mu|\leqslant a}$. Then the condition, $d_{\mu,n} = d_{\mu,n+1}$, for all $|\mu| \leqslant a$ establishes the equality of vectors

$$\sum_{|\lambda|\leqslant a} c_{\lambda,n}\vec{K}_{\lambda,n} = \sum_{|\lambda|\leqslant a} c_{\lambda,n+1}\vec{K}_{\lambda,n+1} \tag{42}$$

For $i = 0, 1$, define

$$B_i := \{\lambda : |\lambda| \leqslant a,\ c_{\lambda,n+i} \neq 0\}.$$

We claim that for all $\lambda \in B_1$, $\vec{K}_{\lambda,n} = \vec{K}_{\lambda,n+1}$. Suppose for sake of contradiction there was a $\lambda \in B_1$ such that the vectors disagreed for some entry μ, $|\mu| \leqslant a$. By the injectivity of the map, κ_n, defined in Lemma 15 (with roles of λ and μ swapped), we have that $K_{\lambda[n],\mu[n]} \leqslant K_{\lambda[n+1],\mu[n+1]}$. Hence, for the vector entries to disagree it must be the case that $K_{\lambda[n],\mu[n]} < K_{\lambda[n+1],\mu[n+1]}$.

Since the inequality of these Kostka numbers is strict for this $\lambda \in B_1$,

$$\kappa_n : \mathrm{SSYT}(\lambda[n], \mu[n]) \to \mathrm{SSYT}(\lambda[n+1], \mu[n+1])$$

fails to be surjective and so there is a certain $T \in \mathrm{SSYT}(\lambda[n+1], \mu[n+1])$ not in the image of κ_n. Then it must be the case that the number of cells labeled 1 in T is $n+1-|\mu| \leqslant \lambda_1$, i.e. T has a cell above every cell containing a 1 or else T would be in the image of the map as seen in the proof of Lemma 15. We must also have that $|\mu| \geqslant |\lambda|$, since otherwise $K_{\lambda[n+1],\mu[n]+1} = 0$ and $\mathrm{SSYT}(\lambda[n+1], \mu[n+1])$ would be empty (but it contains T). Hence,

$$|\mu| \geqslant |\lambda| \geqslant \lambda_1 \geqslant n + 1 - |\mu|$$

which may be rewritten as $2|\mu| - (n+1) \geqslant 0$. Using this, we may define $\tilde{\mu} := (n+1-|\mu|, 1^{2|\mu|-(n+1)}) \vdash |\mu|$ and so $\tilde{\mu}[n+1] = (n+1-|\mu|, n+1-|\mu|, 1^{2|\mu|-(n+1)})$. We claim that the set of semi-standard Young tableaux, $\mathrm{SSYT}(\lambda[n+1], \tilde{\mu}[n+1])$, is non-empty, since we can construct a $\tilde{T} \in \mathrm{SSYT}(\lambda[n+1], \tilde{\mu}[n+1])$ by placing $n+1-|\mu|$ cells labeled 1 at the start of the bottom row, then placing $n+1-|\mu|$ cells labeled 2 at the start of the second row ($\lambda_1 \geqslant n+1-|\mu|$ guarantees that there is enough space in $\lambda[n]$ to do this), and filling in the remaining cells of $\lambda[n+1]$ with entries $3, 4, \ldots$ in some order that creates a semi-standard Young tableau in $\mathrm{SSYT}(\lambda[n+1], \tilde{\mu}[n+1])$. See Fig.8.

6	7	8				
2	2	2	2	2	5	
1	1	1	1	1	3	4

Figure 8: Example of a $\tilde{T} \in \mathrm{SSYT}(\lambda[n+1], \tilde{\mu}[n+1])$ for $\lambda = (6, 3)$ and $\tilde{\mu} = (5, 1^6)$, $|\mu| = 11$ and $n = 15$.

Hence, $K_{\lambda[n+1],\tilde{\mu}[n+1]} \neq 0$, but $\tilde{\mu}[n]$ is not well-defined since $n < n+1 = |\tilde{\mu}| + \tilde{\mu}_1$ and so $K_{\nu[n],\tilde{\mu}[n]} = 0$ for all partitions ν. Since $|\tilde{\mu}| = |\mu| \leqslant a$, we must have equality of

$$0 = d_{\tilde{\mu},n} = \sum_{\nu} c_{\nu,n} K_{\nu[n],\tilde{\mu}[n]}$$

and

$$d_{\tilde{\mu},n+1} = \sum_{\nu} c_{\nu,n+1} K_{\nu[n+1],\tilde{\mu}[n+1]}.$$

By Schur-positivity, all the $c_{\nu,n+1} \geqslant 0$ with at least one strictly positive $c_{\lambda,n+1} > 0$ since we are assuming $\lambda \in B_1$ and all the $K_{\nu[n+1],\tilde{\mu}[n+1]} \geqslant 0$ with $K_{\lambda[n+1],\tilde{\mu}[n+1]} > 0$ as was just previously shown. But then $c_{\lambda,n+1} K_{\lambda[n+1],\tilde{\mu}[n+1]} > 0$ and hence $d_{\tilde{\mu},n+1} > 0$, a contradiction since $d_{\tilde{\mu},n+1} = d_{\tilde{\mu},n} = 0$. It follows that if $\lambda \in B_1$, it must be the case that $\vec{K}_{\lambda,n} = \vec{K}_{\lambda,n+1}$.

Next, we claim that the set of vectors, $\{\vec{K}_{\lambda,n} : \lambda \in B_0 \cup B_1\}$, is linearly independent. First, let us verify that $\lambda[n]$ is defined for $\lambda \in B_0 \cup B_1$. If $\lambda \in B_0$, we have that $\lambda[n]$ is defined since $c_{\lambda,n} \neq 0$ and our convention for coefficients would force $c_{\lambda,n} = 0$ if $\lambda[n]$ weren't defined. Similarly, for $\lambda \in B_1$, the fact that $c_{\lambda,n+1} \neq 0$ implies that $\lambda[n+1]$ is defined. We have just seen that $\lambda \in B_1$ implies $\vec{K}_{\lambda,n} = \vec{K}_{\lambda,n+1}$, with $\vec{K}_{\lambda,n+1} \neq \vec{0}$ since $K_{\lambda[n+1],\lambda[n+1]} = 1$ with $\lambda[n+1]$ defined. This forces $K_{\lambda[n],\lambda[n]} = 1$ and so $\lambda[n]$ must be defined as well (or else this entry of the vector would be zero in our convention).

Now, that we have verified that the $\vec{K}_{\lambda,n}$ are non-zero since $\lambda[n]$ is always defined, we can use the classic triangularity of the Kostka number to see their linear independence. Recall that $K_{\lambda[n],\mu[n]} = 0$ whenever $\lambda[n]$ precedes $\mu[n]$ in lexicographic order. Thus, with respect to the lexicographic ordering of $\{\mu[n] : |\mu| \leqslant a\}$, $\vec{K}_{\lambda,n}$ contains a 1 in the $\mu = \lambda$ entry and a 0 in all entries above. Thus, $\{\vec{K}_{\lambda,n} : \lambda \in B_0 \cup B_1\}$ forms a linearly independent set.

Upon looking at only non-zero terms of (42) we get

$$\sum_{\lambda \in B_0} c_{\lambda,n} \vec{K}_{\lambda,n} = \sum_{\lambda \in B_1} c_{\lambda,n+1} \vec{K}_{\lambda,n+1} = \sum_{\lambda \in B_1} c_{\lambda,n+1} \vec{K}_{\lambda,n},$$

where the last equality follows since $\lambda \in B_1$ implies $\vec{K}_{\lambda,n} = \vec{K}_{\lambda,n+1}$. Finally, by linear independence of $\{\vec{K}_{\lambda,n} : \lambda \in B_0 \cup B_1\}$ and the fact that all $c_{\lambda,n+i}$ appearing above are non-zero, the above linear relation implies that $B_0 = B_1$ and $c_{\lambda,n} = c_{\lambda,n+1}$ for all $|\lambda| \leqslant a$ with $\lambda \in B_0 = B_1$. This establishes $c_{\lambda,n} = c_{\lambda,n+1}$, whenever either coefficient is non-zero ($\lambda \in B_0 = B_1$). Similarly, for $|\lambda| \leqslant a$, $c_{\lambda,n} = 0$ is equivalent to $c_{\lambda,n+1} = 0$ since the sets of λ where they are non-zero, B_0, B_1 agree. Thus, $c_{\lambda,n} = c_{\lambda,n+1}$ for all $|\lambda| \leqslant a$ concluding the proof of the lemma. □

Proof of Proposition 41. Take Schur-positive $F_n = \mathcal{F}(V_n)$ and $a = |\lambda|$. Once $n \geqslant \max_{|\mu| \leqslant |\lambda|} \mathrm{rg}^m_\mu(F_\bullet)$ (condition (40)), the monomial coefficients are stable

$$d_{\mu,n} = d_{\mu,n+1} = d_{\mu,n+2} = \cdots$$

for all $|\mu| \leqslant |\lambda| = a$. Repeatedly, applying Lemma 42, we get

$$c_{\lambda,n} = c_{\lambda,n+1} = c_{\lambda,n+2} = \cdots$$

implying that the Schur coefficients have stabilized for this n. Finally, by taking the maximum over $|\lambda| \leqslant \mathrm{wt}_s(F_\bullet)$ we can get a uniform stable range

$$\mathrm{rg}^s(F_\bullet) = \max_{|\lambda| \leqslant \mathrm{wt}_s(F_\bullet)} \mathrm{rg}^s_\lambda(F_\bullet) \leqslant \max_{|\mu| \leqslant \mathrm{wt}_s(F_\bullet)} \mathrm{rg}^m_\mu(F_\bullet),$$

finishing the proof. $\square$