Note on a Conjecture of Talagrand: Expectation Thresholds vs. Fractional Expectation Thresholds

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Abstract

We show that a restricted version of a conjecture of M. Talagrand on the relation between "expectation thresholds" and "fractional expectation thresholds" follows easily from a strong version of a second conjecture of Talagrand, on "selector processes". The selector process conjecture was proved by Park and Pham, and the quantitative strengthening used here is due to Bednorz, Martynek, and Meller.

Mathematics Subject Classifications: 05C80

1 Introduction

The purpose of this note is to point out that a restricted version of a conjecture of M. Talagrand (Conjecture 1 below) is an easy consequence of [1], a quantitative strengthening of [5, Theorem 1.5]. We first briefly recall definitions. (For a less hurried introduction to the problem, see e.g. [2, 4].)

For a finite set V, 2^V is the power set of V. An $\mathcal{F} \subseteq 2^V$ is *increasing* if $B \supseteq A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$. For $\mathcal{G} \subseteq 2^V$ we use $\langle \mathcal{G} \rangle$ for the increasing family generated by $\mathcal{G}$, namely $\{B \subseteq V : \exists A \in \mathcal{G}, B \supseteq A\}$. *We assume throughout that* $|V| = n$ *and* $\mathcal{F} \subseteq 2^V$ *is increasing.*

Say $\mathcal{F}$ is *p-small* if there is a $\mathcal{G} \subseteq 2^V$ such that

$$\langle \mathcal{G} \rangle \supseteq \mathcal{F} \text{ and } \sum_{S \in \mathcal{G}} p^{|S|} \leqslant 1/2,$$

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and set $q(\mathcal{F}) = \max\{p : \mathcal{F} \text{ is } p\text{-small}\}$. Say $\mathcal{F}$ is *weakly p-small* if there is a $\lambda : 2^V \to \mathbb{R}^+(:= [0,\infty))$ such that

$$\sum_{S\subseteq I} \lambda_S \geqslant 1 \ \forall I \in \mathcal{F} \quad \text{and} \quad \sum_S \lambda_S p^{|S|} \leqslant 1/2, \tag{1}$$

and set $q_f(\mathcal{F}) = \max\{p : \mathcal{F} \text{ is weakly } p\text{-small}\}$. If there is λ as in (1) supported on sets of size at most r, we say $\mathcal{F}$ is *weakly (p,r)-small.*

We are interested in the following conjecture of Talagrand [6, Conjecture 6.3].

Conjecture 1. There is a universal $L > 0$ such that for every finite V and increasing $\mathcal{F} \subseteq 2^V$,

$$q(\mathcal{F}) \geqslant q_f(\mathcal{F})/L.$$

That is, weakly p-small implies (p/L)-small.

Our goal here is the following statement, which says that for any r, weakly (p,r)-small does imply (p/L)-small, but with L depending on r. The case $r = 2$, which was proved in [4], was one of a pair of test cases for Conjecture 1 suggested in [6]. (The other was resolved in [2].) The simplicity of the present argument versus the rather difficult [4] is a nice illustration of the power of the selector process results we are using.

Theorem 2. *For any r, there is a J such that any weakly (Jp,r)-small $\mathcal{F}$ is p-small.*

This turns out to be an easy consequence of Lemma 3 below, which is essentially [1, Lemma 3.4]. For the statement of the lemma, we need a little more preparation. Let $\mathcal{F} \subseteq 2^V$ be given, and for each $I \in \mathcal{F}$, suppose we are given weights $(\mu_I(v))_{v\in I}$ with $\mu_I(v) \geqslant 0$ and

$$\mu_I(I) := \sum_{v\in I} \mu_I(v) = 1.$$

Say $X \subseteq V$ is *c-bad* ($c > 0$) if

$$\sup_{I\in\mathcal{F}} \mu_I(X \cap I) < c.$$

Lemma 3. *For each positive integer r there is a C such that if $\mathcal{F}$ is not p-small and W is chosen uniformly from the m-element subsets of V, then*

$$\mathbb{P}\left(W \text{ is } \left(1 - \frac{1}{2r}\right)\text{-bad}\right) < 2\sum_{t=1}^{n} \left(C\frac{np}{m}\right)^t.$$

When $r = 1$ this follows from [1, Lemma 3.4], and direct extension of the argument of [1] gives Lemma 3 with $C = (2er)^2$.

2 Proof of Theorem 2

We will prove Theorem 2 with $J = 10rC$, where $C = C(r)$ is as in Lemma 3.[1] For the rest of our discussion, X is a uniformly random m-element subset of V, with $m = Jpn/(2r) = 5Cpn$.

Suppose $\lambda : \binom{V}{\leqslant r} \to \mathbb{R}^+$ satisfies (1) with Jp in place of p. We may assume $\lambda_\emptyset = 0$, since otherwise λ' given by $\lambda'_\emptyset = 0$ and $\lambda'_S = \lambda_S/(1-\lambda_\emptyset)$ if $S \neq \emptyset$ also satisfies (1) (with $1/2$ improved to $(1-2\lambda_\emptyset)/(2(1-\lambda_\emptyset))$). We use Lemma 3 in combination with the following easy point.

Observation 4. *With probability at least* $1/2$, $\sum_{S\subseteq X} \lambda_S < 1/(2r)$.

Proof. This follows from Markov's Inequality, once we notice that our assumptions on X and λ imply

$$\mathbb{E}\left[\sum_{S\subseteq X} \lambda_S\right] \leqslant \sum_S (m/n)^{|S|}\lambda_S = \sum_S (Jp/(2r))^{|S|}\lambda_S \leqslant (2r)^{-1}\sum_S (Jp)^{|S|}\lambda_S \leqslant 1/(4r). \quad \square$$

For $I \in \mathcal{F}$, let $\nu_I = \sum_{S\subseteq I} |S|\lambda_S \geqslant 1$ (see (1), recalling that $\lambda_\emptyset = 0$). Define $\mu_I : I \to \mathbb{R}^+$ by

$$\mu_I(v) = \nu_I^{-1} \sum_{v\in S\subseteq I} \lambda_S \quad \forall v \in I$$

and notice that $\mu_I(I) = 1$. Suppose for a contradiction that $\mathcal{F}$ is not p-small. Then by Lemma 3 and our choice of m, the probability that X is $(1-1/(2r))$-bad (with respect to the μ_I's) is less than $1/2$. So the following claim gives the desired contradiction.

Claim 5. *With probability at least* $1/2$, *X is $(1-1/(2r))$-bad with respect to the μ_I's.*

Proof. For any $Y \subseteq V$ and $I \in \mathcal{F}$,

$$\begin{aligned}
\mu_I(Y\cap I) = \nu_I^{-1}\sum_{v\in Y\cap I}\sum_{v\in S\subseteq I}\lambda_S &\leqslant \nu_I^{-1}\sum_{\substack{S\subseteq I\\ S\not\subseteq Y}}(|S|-1)\lambda_S + \nu_I^{-1}\sum_{S\subseteq Y\cap I}|S|\lambda_S \\
&= \nu_I^{-1}\sum_{S\subseteq I}(|S|-1)\lambda_S + \nu_I^{-1}\sum_{S\subseteq Y\cap I}\lambda_S \\
&\leqslant \nu_I^{-1}(1-1/r)\sum_{S\subseteq I}|S|\lambda_S + \nu_I^{-1}\sum_{S\subseteq Y\cap I}\lambda_S \\
&= 1-1/r + \nu_I^{-1}\sum_{S\subseteq Y\cap I}\lambda_S \leqslant 1-1/r + \sum_{S\subseteq Y\cap I}\lambda_S.
\end{aligned}$$

When $Y = X$ we may bound the last sum (for any I) by $\sum_{S\subseteq X}\lambda_S$, which by Observation 4 is less than $(2r)^{-1}$ with probability at least $1/2$; and it follows that with probability at least $1/2$,

$$\sup_{I\in\mathcal{F}} \mu_I(X\cap I) < 1-1/r+(2r)^{-1} = 1-1/(2r). \quad \square$$

[1] A more involved argument, using ideas from the proof of [3, Theorem 8], would allow us to take $J = O(C)$, but for simplicity we settle for the present version.

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