Size Conditions for Pancyclicity of t-Tough Graphs

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Abstract

The toughness of a connected noncomplete graph G is

$$\tau(G) = \min\left\{\frac{|S|}{c(G-S)} : S \subseteq V(G),\ c(G-S) \geqslant 2\right\},$$

where $c(G)$ is the number of components of G; as usual, $\tau(K_n) = \infty$. In 1973, Bondy proposed the metaconjecture that almost every nontrivial condition implying Hamiltonicity should also imply pancyclicity, apart from a simple family of exceptional graphs. Recently, Benediktovich [*Discrete Applied Mathematics* 365 (2025), 130–137] confirmed Bondy's metaconjecture for t-tough graphs when $t \in \{1, 2, 3\}$ by using conditions on the size, the spectral radius, and the signless Laplacian spectral radius. This paper confirms Bondy's metaconjecture for t-tough graphs when $t \geqslant 4$ by means of conditions on the size, the spectral radius, the signless Laplacian spectral radius, the distance spectral radius, and the distance signless Laplacian spectral radius. More precisely, if a t-tough graph G has order $n > 10t - 3$ and size $m \geqslant \binom{n-2t}{2} + 3t^2$, then G is pancyclic.

Mathematics Subject Classifications: 05C35, 05C50

1 Introduction

Throughout this paper, all graphs are finite, simple, and undirected. For a graph G and a vertex set $S \subseteq V(G)$, let $c(G-S)$ denote the number of components of $G-S$. A connected graph G is called t-tough if

$$t\, c(G-S) \leqslant |S|$$

for every vertex cut S of G. The toughness of a connected noncomplete graph G is

$$\tau(G) = \min\left\{\frac{|S|}{c(G-S)} : S \subseteq V(G),\ c(G-S) \geqslant 2\right\},$$

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and G is t-tough precisely when $\tau(G) \geqslant t$; as usual, $\tau(K_n) = \infty$. Chvátal introduced toughness in 1973 as a simple numerical measure of how strongly the pieces of a graph are held together [11]. Since every Hamiltonian graph is 1-tough, toughness has naturally become an important parameter in the study of spanning cycles.

The relation between toughness and Hamiltonicity has been studied from several directions. Toughness by itself does not force Hamiltonicity at the level $t = 2$, as shown by Bauer, Broersma, and Veldman [1]; consequently, additional assumptions involving the number of edges, the degree sequence, forbidden induced subgraphs, or spectral parameters are required. Broersma, Patel, and Pyatkin [5] studied Hamiltonicity of tough $2K_2$-free graphs, while Ota and Sanka [22] and Shi and Shan [24] obtained related results for restricted graph classes. Cai et al. [7] and Fan, Lin, and Lu [14] established sufficient conditions for Hamiltonicity of tough graphs in terms of the size and spectral radius. More recently, Shan and Tanyel [25] obtained a degree-sequence condition for Hamiltonicity in sufficiently tough graphs. These results show that toughness becomes particularly effective when it is combined with a suitable density or degree condition.

A graph is Hamiltonian if it contains a spanning cycle, and it is pancyclic if it contains a cycle of every length from 3 to its order. Every pancyclic graph is Hamiltonian, but the converse is false in general. In 1973, Bondy [3] proposed the following influential metaconjecture.

Bondy's metaconjecture. Almost any nontrivial condition on a graph that implies Hamiltonicity also implies pancyclicity, possibly with a simple family of exceptional graphs.

The classical minimum-degree theorem illustrates this principle. Dirac's condition $\delta(G) \geqslant n/2$ guarantees Hamiltonicity, and Bondy's strengthening shows that a graph satisfying this condition is either pancyclic or the balanced complete bipartite graph $K_{n/2,n/2}$. The bipartite graph is the natural exceptional family because it contains no odd cycle. Degree and degree-sequence versions of this phenomenon were developed, among others, by Schmeichel and Hakimi [23] and Hoàng [16]. In another direction, Erdős [13] conjectured that a sufficiently large Hamiltonian graph with bounded independence number is pancyclic; a strong form of this conjecture was recently proved by Draganić, Correia, and Sudakov [12]. Thus the passage from Hamiltonicity to pancyclicity remains a central theme in the study of cycles.

Spectral methods provide another useful source of sufficient conditions. Conditions involving the adjacency spectral radius or the signless Laplacian spectral radius have been used to guarantee Hamiltonian and related spanning properties; see, for example, [15, 20, 26, 27, 30, 31]. Distance-based matrices have also been used in this setting. Zhou and Wang [29] studied distance signless Laplacian conditions for Hamiltonian properties, while other relationships between toughness and adjacency, signless Laplacian, and distance spectra were obtained in [6, 8, 9, 10, 18, 21]. These works motivate the use of several spectral parameters as consequences of a common edge-size condition.

Recently, Benediktovich [2] confirmed Bondy's metaconjecture for t-tough graphs when $t \in \{1, 2, 3\}$, using conditions on the size, the adjacency spectral radius, and the signless Laplacian spectral radius. The purpose of the present paper is to continue this line of investigation for every integer $t \geqslant 4$. We first prove that the same natural edge threshold guarantees Hamiltonicity and then apply Hoàng's pancyclicity criterion. Although that criterion yields the alternative that the graph is pancyclic or bipartite, the bipartite case is excluded here by the size hypothesis, which forces $m > n^2/4$. Consequently, our main result has the stronger conclusion that G is pancyclic.

The paper keeps the following simple structure. Section 2 proves the size condition in Theorem 1 and retains the explicit verification for $t = 4$. Section 3 derives the four spectral consequences in Theorem 2; the routine proofs of parts (i) and (ii) are omitted, while the distance-spectral arguments in parts (iii) and (iv) are given in full.

Theorem 1. *Let G be a t-tough connected graph, where $t \geqslant 4$ is an integer. Suppose that G has order $n > 10t - 3$ and size m. If*

$$m \geqslant \binom{n-2t}{2} + 3t^2, \tag{1}$$

then G is pancyclic.

The size theorem immediately yields several spectral consequences. In the statement below, $\lambda_1(G)$ denotes the spectral radius of G, $q(G)$ denotes the signless Laplacian spectral radius, $\lambda_1(D(G))$ denotes the distance spectral radius, and $\eta_1(G)$ denotes the distance signless Laplacian spectral radius. The corresponding matrix definitions are postponed to Section 3, where they are first needed. Parts (i) and (ii) extend the corresponding results of Benediktovich [2], while parts (iii) and (iv) give distance-spectral analogues.

Theorem 2. *Let G be a t-tough connected graph, where $t \geqslant 4$ is an integer, and suppose that G has order $n > 10t - 3$. If one of the following conditions holds:*

(i) $\lambda_1(G) \geqslant \sqrt{n^2 - (4t+2)n + 10t^2 + 2t + 1}$;

(ii) $q(G) \geqslant \dfrac{n^2 - (4t+1)n + 10t^2 + 2t}{n-1} + n - 2$;

(iii) $\lambda_1(D(G)) \leqslant n + 4t - 1 - \dfrac{10t^2 + 2t}{n}$;

(iv) $\eta_1(G) \leqslant 2n + 8t - 2 - \dfrac{20t^2 + 4t}{n}$,

then G is pancyclic.

2 Proof of Theorem 1

We begin with the notation and auxiliary results needed in the proof. For a graph G, write $n = |V(G)|$ and $m = |E(G)|$. The degree of a vertex v is denoted by $d(v)$, and the minimum and maximum degrees are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The cycle and complete graph of order r are denoted by C_r and K_r. For two vertex-disjoint graphs G_1 and G_2, their disjoint union is written $G_1 + G_2$, and their join is denoted by $G_1 \vee G_2$. If $G_1 = \cdots = G_s = H$, we write sH for their disjoint union. A bipartite graph with bipartition (X, Y) is denoted by $G[X, Y]$, and $K_{r,s}$ denotes the complete bipartite graph with part sizes r and s.

For a nonnegative integer r, the r-closure $C_r(G)$ is obtained by repeatedly joining nonadjacent vertices whose current degree sum is at least r, until no such pair remains. Bondy and Chvátal proved that $C_r(G)$ is uniquely determined [4].

Lemma 3 (Bondy–Chvátal [4]). *A graph G of order n is Hamiltonian if and only if $C_n(G)$ is Hamiltonian.*

Let $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$ be the nondecreasing degree sequence of G. We use $(0^{x_0}, 1^{x_1}, \ldots, \Delta^{x_\Delta})$ to denote a degree sequence in which x_i vertices have degree i. For a positive integer t, define the predicate $P(t)$ as follows:

$$P(t): \quad \text{for every integer } k < n/2, \quad d_k \leqslant k \implies d_{n-k+t} \geqslant n - k.$$

Lemma 4 (Shan–Tanyel [25]). *Let $t \geqslant 4$ be an integer. If a t-tough graph G satisfies $P(t)$, then G is Hamiltonian.*

Lemma 5 (Hoàng [16]). *If a t-tough graph G satisfies $P(t)$ and is Hamiltonian, then G is pancyclic or bipartite.*

Lemma 6 (Hoàng [16]). *If G is a noncomplete t-tough graph, then $\delta(G) \geqslant 2t$.*

Lemma 7 (Jung [19]). *Let G be a non-Hamiltonian graph of order at least 11. Then at least one of the following holds:*

(1) *there are nonadjacent vertices x, y such that $d(x) + d(y) \leqslant |V(G)| - 5$;*

(2) *there is a set $S \subseteq V(G)$ such that $c(G - S) \geqslant |S| + 1$.*

Corollary 8. *Let G be a t-tough non-Hamiltonian graph of order at least 11, where $t \geqslant 1$. Then there are nonadjacent vertices x, y such that*

$$d(x) + d(y) \leqslant |V(G)| - 5.$$

Proof. If the second alternative of Lemma 7 held, then

$$t \leqslant \tau(G) \leqslant \frac{|S|}{c(G - S)} \leqslant \frac{|S|}{|S| + 1} < 1,$$

a contradiction. Hence the first alternative must hold. □

Lemma 9 (Hoàng [16]). *Let G be a Hamiltonian graph of order n. If more than $n/3$ vertices of G have degree greater than $n/2$, then G is pancyclic.*

Lemma 10 (Benediktovich [2]). *A 2-tough graph G is Hamiltonian if and only if its $(n-1)$-closure $C_{n-1}(G)$ is Hamiltonian.*

Proof of Theorem 1. First observe that the size condition excludes the bipartite case. Indeed,

$$\binom{n-2t}{2} + 3t^2 - \frac{n^2}{4} = \frac{(n-4t-1)^2 + 4t^2 - 4t - 1}{4} > 0,$$

where the last inequality follows from $t \geqslant 4$. Hence (1) gives $m > n^2/4$. On the other hand, every bipartite graph of order n has at most $\lfloor n^2/4 \rfloor$ edges. Therefore G is not bipartite.

Suppose, to the contrary, that G is not pancyclic. If G satisfied $P(t)$, then Lemma 4 would imply that G is Hamiltonian, and Lemma 5 would then imply that G is pancyclic or bipartite. Since G is not bipartite, this would force G to be pancyclic, a contradiction. Hence G does not satisfy $P(t)$.

Consequently, there exists an integer $k < n/2$ for which

$$d_k \leqslant k \qquad \text{and} \qquad d_{n-k+t} \leqslant n-k-1.$$

We now estimate the degree sum by dividing the degree sequence into the three intervals determined by k and $n-k+t$. The first k degrees are at most k, the next $n-2k+t$ degrees are at most $n-k-1$, and the final $k-t$ degrees are at most $n-1$. Therefore

$$\begin{aligned} 2m &= \sum_{i=1}^{k} d_i + \sum_{i=k+1}^{n-k+t} d_i + \sum_{i=n-k+t+1}^{n} d_i \\ &\leqslant k^2 + (n-2k+t)(n-k-1) + (k-t)(n-1) \\ &= n^2 - n + 3k^2 + (1-2n-t)k \\ &= 2\binom{n-2t}{2} + 6t^2 - (k-2t)(2n-3k-5t-1). \end{aligned} \tag{2}$$

Thus

$$m \leqslant \binom{n-2t}{2} + 3t^2 - \frac{(k-2t)(2n-3k-5t-1)}{2}. \tag{3}$$

Combining this upper bound with the hypothesis $m \geqslant \binom{n-2t}{2} + 3t^2$ gives

$$(k-2t)(2n-3k-5t-1) \leqslant 0. \tag{4}$$

Since G is not pancyclic, it is not complete. Lemma 6 therefore yields $\delta(G) \geqslant 2t$. Together with $\delta(G) \leqslant d_k \leqslant k$, this gives

$$k \geqslant 2t.$$

We distinguish whether equality or strict inequality holds in (4).

Case 1. $(k-2t)(2n-3k-5t-1) = 0$.

There are two possibilities: either $k = 2t$, or $k \neq 2t$ and $2n-3k-5t-1 = 0$.

Subcase 1.1. $k = 2t$.

Here $d_{2t} \leqslant 2t$, $d_{n-t} \leqslant n-2t-1$, and $d_n \leqslant n-1$. Since the lower bound in (1) and the upper bound in (3) coincide, equality holds in every estimate used in the degree-sum calculation. Hence the first $2t$ vertices have degree $2t$, the next $n-3t$ vertices have degree $n-2t-1$, and the final t vertices have degree $n-1$. Thus G has degree sequence

$$\Big((2t)^{2t}, (n-2t-1)^{n-3t}, (n-1)^{t}\Big). \tag{5}$$

We show that the $(n-1)$-closure of G is complete. First,

$$2(n-2t-1) > n-1$$

because $n > 10t-3$. Therefore all vertices of degree $n-2t-1$ become pairwise adjacent in $C_{n-1}(G)$. Each of these vertices also becomes adjacent to every vertex of degree $2t$, since

$$(n-2t-1) + 2t = n-1.$$

The t vertices of degree $n-1$ are already universal. It follows that, after these closure operations, each of the $2t$ low-degree vertices is adjacent to all $n-3t$ middle-degree vertices and all t universal vertices, and hence has degree at least

$$(n-3t) + t = n-2t.$$

Any two remaining low-degree vertices now have degree sum at least $2(n-2t) > n-1$, so they also become adjacent. Consequently, $C_{n-1}(G) = K_n$.

Since $t \geqslant 4$, the graph G is in particular 2-tough. Lemma 10 now implies that G is Hamiltonian. Moreover, the $n-3t$ vertices of degree $n-2t-1$ together with the t universal vertices give $n-2t$ vertices whose degrees exceed $n/2$. Indeed, $n > 10t-3$ implies both $n-2t > n/3$ and $n-2t-1 > n/2$. Lemma 9 therefore implies that G is pancyclic. This contradicts our assumption.

Subcase 1.2. $k \neq 2t$ and $2n-3k-5t-1 = 0$.

In this subcase,

$$2n = 3k + 5t + 1.$$

Since $k < n/2$, we have $2k < n$, and hence $4k+2 \leqslant 2n$. Since $n > 10t-3$, we also have $20t-6 < 2n$. Combining these inequalities with the displayed identity gives

$$4k+2 \leqslant 3k+5t+1 \quad \text{and} \quad 20t-6 < 3k+5t+1.$$

Thus

$$5t-2 \leqslant k \leqslant 5t-1.$$

Because $n = (3k+5t+1)/2$ is an integer, $k = 5t-2$ is impossible: in that case $3k+5t+1 = 20t-5$ would be odd. Therefore

$$k = 5t-1 \qquad \text{and} \qquad n = 10t-1.$$

Again equality holds throughout the degree-sum estimate. Hence

$$d_1 = \cdots = d_{6t} = 5t-1, \qquad d_{6t+1} = \cdots = d_{10t-1} = 10t-2,$$

so the degree sequence of G is

$$\Big((5t-1)^{6t}, (10t-2)^{4t-1}\Big).$$

If x and y are nonadjacent, neither can have degree $10t-2 = n-1$; hence both have degree $5t-1$, and

$$d(x) + d(y) = 10t - 2 = n - 1 > n - 5.$$

If G were non-Hamiltonian, Corollary 8 would provide a nonadjacent pair with degree sum at most $n-5$, contrary to the displayed equality. Hence G is Hamiltonian. In addition, the $4t-1$ universal vertices satisfy $4t-1 > n/3$ and have degree greater than $n/2$. Lemma 9 shows that G is pancyclic, again contradicting the assumed failure of the conclusion.

Case 2. $(k-2t)(2n-3k-5t-1) < 0$.

Since $k \geqslant 2t$, the first factor is nonnegative; strict negativity therefore forces

$$2n - 3k - 5t - 1 < 0.$$

As $k < n/2$, we have $n \geqslant 2k+1$, and hence

$$4k + 2 \leqslant 2n \leqslant 3k + 5t.$$

It follows that $k \leqslant 5t-2$. Substituting this bound into $n \leqslant (3k+5t)/2$ gives

$$n \leqslant \frac{3(5t-2)+5t}{2} = 10t - 3,$$

contrary to the hypothesis $n > 10t-3$.

Both cases lead to a contradiction. Therefore G is pancyclic. This proves Theorem 1. □

Remark 11. For completeness, we give the separate closure verification for the first new value $t = 4$. This argument is supplementary: the general proof above already covers $t = 4$, but the explicit configurations help illustrate the equality cases in the size estimate.

Proof. Suppose, to the contrary, that G is not Hamiltonian, and let $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$ be its degree sequence. As in the proof of Theorem 1, there is an integer $k < n/2$ such that

$$(k-8)(2n-3k-21) \leqslant 0.$$

By Lemma 6, $\delta(G) \geqslant 8$, and hence $k \geqslant 8$.

Case 1. $(k-8)(2n-3k-21) = 0$.

Subcase 1.1. $k = 8$.

If $k = 8$, then $d_8 \leqslant 8$, $d_{n-4} \leqslant n-9$, and $d_n \leqslant n-1$. The lower bound in (1) and the corresponding upper bound in (3) are equalities. Thus G has degree sequence

$$(8^8, (n-9)^{n-12}, (n-1)^4).$$

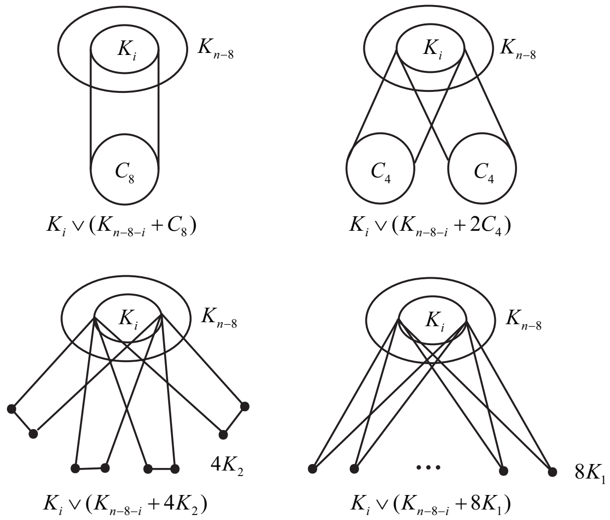

Figure 1: Corresponding graph configurations in the case $t = 4$.

Let S be the set of the eight vertices of degree 8. By Corollary 8, S contains two nonadjacent vertices. Since G is not Hamiltonian, Lemma 3 implies that $C_n(G)$ is not Hamiltonian. By the definition of the closure, all vertices outside S form a complete graph K_{n-8}. If a vertex of this K_{n-8} is adjacent in $C_n(G)$ to one vertex of S, then it is adjacent to every vertex of S. Moreover, at least four vertices of K_{n-8} are adjacent to all vertices of S.

If exactly four vertices of K_{n-8} are adjacent to all vertices of S, then $C_n(G)$ contains $K_4 \vee (K_{n-12} + C_8)$ or $K_4 \vee (K_{n-12} + 2C_4)$. Each of these graphs is Hamiltonian, and hence so is $C_n(G)$, a contradiction.

If exactly five vertices of K_{n-8} are adjacent to all vertices of S, then $C_n(G)$ contains one of

$$K_5 \vee (K_{n-13} + C_8), \qquad K_5 \vee (K_{n-13} + 2C_4), \qquad K_5 \vee (K_{n-13} + 4K_2).$$

Each displayed graph is Hamiltonian, again a contradiction.

If exactly six vertices of K_{n-8} are adjacent to all vertices of S, then $C_n(G)$ contains one of

$$K_6 \vee (K_{n-14} + C_8), \qquad K_6 \vee (K_{n-14} + 2C_4), \qquad K_6 \vee (K_{n-14} + 4K_2).$$

Each of these three spanning configurations is Hamiltonian. Thus $C_n(G)$ would be Hamiltonian, a contradiction.

If exactly seven vertices of K_{n-8} are adjacent to all vertices of S, then the closure contains

$$K_7 \vee (K_{n-15} + 4K_2).$$

This graph has a Hamiltonian cycle, so this possibility is also excluded.

If exactly eight vertices of K_{n-8} are adjacent to all vertices of S, then

$$C_n(G) \supseteq K_8 \vee (K_{n-16} + 8K_1).$$

The graph on the right is Hamiltonian, again contradicting the choice of G.

Finally, if $i > 8$ vertices of K_{n-8} are adjacent to all vertices of S, then

$$C_n(G) \supseteq K_i \vee (K_{n-8-i} + 8K_1).$$

Since the complete part has more than eight vertices, one can alternate the eight isolated vertices with eight vertices of K_i and insert all remaining vertices of the two complete parts into the resulting cycle. Hence this spanning subgraph is Hamiltonian. This is the final contradiction in Subcase 1.1.

Subcase 1.2. $k \neq 8$ and $2n - 3k - 21 = 0$.

Here $k = (2n - 21)/3 < n/2$ and $n > 37$. Hence $38 \leqslant n \leqslant 41$, and the divisibility condition forces $n = 39$ and $k = 19$. We have $d_{19} \leqslant 19$, $d_{24} \leqslant 19$, and $d_{39} \leqslant 38$. Equality in the size estimates gives

$$\sum_{i=1}^{39} d_i = 1026,$$

so the degree sequence is $(19^{24}, 38^{15})$. No nonadjacent pair x, y satisfies $d(x) + d(y) \leqslant 34 = |V(G)| - 5$. Corollary 8 therefore implies that G is Hamiltonian, a contradiction.

Case 2. $(k - 8)(2n - 3k - 21) < 0$.

In this case, $k \geqslant 9$ and $2n - 3k - 21 < 0$. Since $k < n/2$, we have $n \geqslant 2k + 1$. Consequently,

$$4k + 2 \leqslant 2n \leqslant 3k + 20,$$

which yields $k \leqslant 18$ and $n \leqslant 37$, contrary to $n > 37$.

Thus every 4-tough graph of order $n > 37$ satisfying (1) is Hamiltonian. In each equality case above, more than $n/3$ vertices have degree greater than $n/2$; therefore Lemma 9 shows that G is pancyclic. This completes the supplementary verification for $t = 4$. □

3 Proof of Theorem 2

We introduce the spectral notation only at this stage. Let $A(G)$ be the adjacency matrix of G, and let $\lambda_1(G)$ be its largest eigenvalue. The signless Laplacian matrix is $Q(G) = \mathrm{Diag}(d(v_1), \ldots, d(v_n)) + A(G)$, and its largest eigenvalue is denoted by $q(G)$.

For a connected graph G, let $d_{ij}(G)$ be the distance between v_i and v_j, and let $D(G) = (d_{ij}(G))_{n\times n}$ be the distance matrix. Its largest eigenvalue is denoted by $\lambda_1(D(G))$. The transmission of a vertex v is $\mathrm{Tr}(v) = \sum_{u\in V(G)} d(u, v)$. The distance signless Laplacian matrix is

$$Q_D(G) = \mathrm{Diag}(\mathrm{Tr}(v_1), \ldots, \mathrm{Tr}(v_n)) + D(G),$$

and its largest eigenvalue is denoted by $\eta_1(G)$. The Wiener index of G is

$$W(G) = \sum_{1\leqslant i<j\leqslant n} d_{ij}(G) = \frac{1}{2} \sum_{v\in V(G)} \mathrm{Tr}(v).$$

Lemma 12 (Indulal [17])**.** *Let G be a connected graph of order n. Then*

$$\lambda_1(D(G)) \geqslant \frac{2W(G)}{n}.$$

Lemma 13 (Xing–Zhou–Li [28])**.** *Let G be a connected graph of order n. Then*

$$\eta_1(G) \geqslant \frac{4W(G)}{n},$$

with equality if and only if G is transmission-regular.

Proof of Theorem 2. The proofs of parts (i) and (ii) are straightforward and are the same as those of the corresponding results in Benediktovich [2, Theorems 2.2 and 2.3], so we omit them. In each case, the spectral hypothesis reduces to (1), and Theorem 1 implies that G is pancyclic. Thus parts (i) and (ii) extend the corresponding results of Benediktovich for $t \in \{1, 2, 3\}$.

For each vertex $v \in V(G)$,

$$\mathrm{Tr}(v) \geqslant d(v) + 2\Big(n - 1 - d(v)\Big) = 2(n-1) - d(v).$$

Therefore,

$$W(G) = \frac{1}{2} \sum_{v \in V(G)} \mathrm{Tr}(v) \geqslant n(n-1) - m. \tag{6}$$

For part (iii), Lemma 12 and (6) give

$$\lambda_1(D(G)) \geqslant \frac{2W(G)}{n} \geqslant 2(n-1) - \frac{2m}{n}.$$

Combining this inequality with the hypothesis of part (iii), we obtain

$$2(n-1) - \frac{2m}{n} \leqslant n + 4t - 1 - \frac{10t^2 + 2t}{n},$$

which is equivalent to

$$m \geqslant \binom{n-2t}{2} + 3t^2.$$

Theorem 1 now implies that G is pancyclic.

For part (iv), Lemma 13 and (6) give

$$\eta_1(G) \geqslant \frac{4W(G)}{n} \geqslant 4(n-1) - \frac{4m}{n}.$$

Combining this inequality with the hypothesis of part (iv), we obtain

$$4(n-1) - \frac{4m}{n} \leqslant 2n + 8t - 2 - \frac{20t^2 + 4t}{n},$$

and hence

$$m \geqslant \binom{n-2t}{2} + 3t^2.$$

Again, Theorem 1 implies that G is pancyclic, completing the proof. □

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