Integral Flows on Bidirected Eulerian Graphs

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Abstract

It is proved that every flow-admissible bidirected eulerian graph has a nowhere-zero 4-flow f and a circle C, where either C is unbalanced or $|E(C)| \geqslant 3$, such that $|f(e)| = 3$ only if $e \in E(C)$. It is also proved, with an interesting new method, that in a flow-admissible connected bidirected graph, the support of any $\mathbf{Z}_2$-flow is contained in the support of a 5-flow.

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1 Introduction

A signed graph with orientations on its edges is called a bidirected graph. In 1983, Bouchet [1] conjectured that every flow-admissible bidirected graph has a nowhere-zero 6-flow.

Bouchet's Conjecture([1]). Every flow-admissible bidirected graph has a nowhere-zero 6-flow.

In the same paper, Bouchet [1] obtained the first progress by showing that every flow-admissible bidirected graph has a nowhere-zero 216-flow. The number 216 was improved to 30 by Zýka [12], to 12 by DeVos [3], and then to 11 by DeVos et al. [4]. Bouchet's Conjecture has been confirmed for some families of graphs, for example, Kotzig-graphs by Schubert and Steffen [9], series-parallel graphs by Kaiser and Rollová [5], graphs with two negative edges by Wang et al. [10], graphs without edge-disjoint unbalanced circles by Lu et al. [6]. For certain families of graphs, the flow number 6 may be lowered. Raspaud and Zhu [8] proved that every 4-edge-connected bidirected graph has a nowhere-zero 4-flow. Máčajová and Škoviera [7] proved that every bidirected eulerian graph has a nowhere-zero 4-flow.

Theorem 1. *([7]) Every flow-admissible bidirected eulerian graph has a nowhere-zero 4-flow.*

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The flow number 4 in Theorem 1 above is best possible. There are infinitely many bidirected eulerian graphs that have no nowhere-zero 3-flows. In this paper, we strengthen Theorem 1 by showing the existence of a special 4-flow.

Theorem 2. *Let G be a bidirected eulerian graph. If G is flow-admissible, then G has a nowhere-zero 4-flow f and a circle C, where either C is unbalanced or $|E(C)| \geqslant 3$, such that $|f(e)| = 3$ only if $e \in E(C)$.*

The support of a flow f is $S(f) = \{e \in E(G) : f(e) \neq 0\}$. A key step in the proof of the 11-flow-theorem [4] is to have a 3-flow whose support contains the support of a $\mathbf{Z}_2$-flow f when $S(f)$ contains an even number of negative edges [4, Lemma 3.1], which was originally obtained by Cheng et al. [2, Theorem 1.5]. In the last section, we consider the case that $S(f)$ contains an odd number of negative edges and obtain the following theorem.

Theorem 3. *Let f be a $\mathbf{Z}_2$-flow in a flow-admissible connected bidirected graph G. Suppose that $S(f)$ contains an odd number of negative edges. Then there is a 5-flow ϕ of G such that $S(f) \subseteq S(\phi)$.*

Theorem 3 is best possible in the following sense. Let G be the bidirected graph obtained from three disjoint negative loops L_i, where L_i is incident with a vertex x_i, by adding a new vertex y joined to each x_i with edge yx_i, $1 \leqslant i \leqslant 3$. Let $f(L_i) = 1$ and $f(x_i y) = 0$, $1 \leqslant i \leqslant 3$. Then f is a $\mathbf{Z}_2$-flow of G and there is no 4-flow ϕ of G such that $S(f) \subseteq S(\phi)$. The proof of Theorem 3 involves an interesting new method, which uses results on bidirected eulerian graphs given in Section 3.

2 Notation and terminology

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the *degree* of v is the number of edges incident with v, a loop being counted twice. G is *eulerian* if it is connected and each vertex has even degree. A *circle* is a minimal nonempty eulerian graph (a connected graph with each vertex of degree 2). Let H_1 and H_2 be subgraphs of G. Suppose that f_1, f_2 are two real-valued functions on $E(H_1)$ and $E(H_2)$, respectively. We may regard f_i as a function on $E(G)$ by considering $f_i(e) = 0$ for each $e \in E(G) \setminus E(H_i)$, $i = 1, 2$, and thus, $f_1 + f_2$ is a function on $E(G)$.

A *signed graph* is a graph G associated with a partition (S^+, S^-) of $E(G)$. An edge $e \in E(G)$ is *positive* if $e \in S^+$ and *negative* if $e \in S^-$. An ordinary graph can be regarded as a signed graph with $S^- = \emptyset$. Let H be a subgraph of a signed graph G. The *sign* of H is the parity of $|E(H) \cap S^-|$; H is *positive* if $|E(H) \cap S^-|$ is even, and *negative* otherwise. When H is a circle, instead of positive and negative, we call it *balanced* and *unbalanced*, respectively. A signed graph is *balanced* if it contains no unbalanced circle.

Let G be a signed graph. Following [1], we consider that each edge $e = xy \in E(G)$ is constituted of two distinct half-edges e^x and e^y, having a single end x and y respectively. The set of half-edges of G is denoted by $H(G) = \{e^v : \ e \in E(G) \text{ and } v \in V(G)\}$.

Thus, $|H(G)| = 2|E(G)|$. A *bidirected graph* is a signed graph G with orientations on its half-edges, associated with a mapping, $\tau : H(G) \to \{+1, -1\}$, such that for a half-edge $e^v \in H(G)$, $\tau(e^v) = 1$ if e^v is oriented *away from* v and $\tau(e^v) = -1$ if e^v is oriented *towards* v, and moreover, for each edge $e = xy \in E(G)$, if e is negative, then $\tau(e^x) = \tau(e^y) = 1$ or $\tau(e^x) = \tau(e^y) = -1$; if e is positive, then $\tau(e^x) = 1$ and $\tau(e^y) = -1$, or $\tau(e^x) = -1$ and $\tau(e^y) = 1$.

Let G be a bidirected graph and let A be an abelian group. Suppose that $f : E(G) \to A$ is a function from $E(G)$ to A (f can be considered as a function from $H(G)$ to A in which $f(e^x) = f(e^y) = f(e)$ for every edge $e = xy \in E(G)$). The *boundary* of f at a vertex $v \in V(G)$ is defined by

$$\partial f(v) = \sum_{e^v \in H(v)} \tau(e^v) f(e^v)$$

where $H(v)$ is the set of half-edges which have v as their end. Equivalently,

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where $E^+(v)$ ($E^-(v)$) is the set of edges away from (towards) v. The *support* of f is $S(f) = \{e \in E(G) : f(e) \neq 0\}$. The function f is *nowhere-zero* if $S(f) = E(G)$.

Definition 4. Let G be a bidirected graph and f a function from $E(G)$ to an abelian group A. The function f is an *A-flow* of G if $\partial f(v) = 0$ for all $v \in V(G)$. For a positive integer k, a *k-flow* of G is a $\mathbf{Z}$-flow f of G with $|f(e)| < k$ for each edge $e \in E(G)$; a $\mathbf{Z}_k$-flow is also called a *modulo k-flow*. We say that G is *flow-admissible* if it has a nowhere-zero $\mathbf{Z}$-flow.

For bidirected graphs, there are two operations of special importance: (i) *switching operation* at a vertex v: change the orientation of every half-edge incident with v, that is, switch the sets of positive and negative edges incident with v; (ii) *reversing operation* on an edge e: reverse the orientation of e, that is, change the orientations of both half-edges of e. The definitions imply that an A-flow f yields a new A-flow by changing $f(e)$ into $-f(e)$ after a reversing operation on e; it remains an A-flow after a switching operation at v (see [1]).

3 Flows on bidirected eulerian graphs

If a bidirected graph is obtained from another by a series of switching and reversing operations, we say that the two bidirected graphs are *flow-equivalent*. Note that these operations are reversible. Thus, for simplicity, from now on we do not distinguish between flow-equivalent bidirected graphs.

It was first proved by Xu and Zhang [11]that any positive bidirected eulerian graph has a nowhere-zero 2-flow, a simpler proof of which was given by Máčajová and Škoviera [7].

Lemma 5. *([11]) Every positive bidirected eulerian graph has a nowhere-zero 2-flow.*

In this paper, we need the following slightly stronger version.

Lemma 6. *Let G be a bidirected eulerian graph. If G is positive, then, for any given $e \in E(G)$ and $\alpha \in \{\pm 1\}$, there is a nowhere-zero 2-flow f of G such that $f(e) = \alpha$.*

Proof. Let $C = v_0 e_1 v_1 e_2 \ldots v_{m-1} e_m v_m$ be an eulerian tour of G, where $e_1 = e$, $m = |E(G)|$, and $v_m = v_0$. We treat all v_i as distinct vertices, $1 \leqslant i \leqslant m$, that is, consider C as a circle obtained from G by splitting vertices along the eulerian tour. Define a function $f : E(G) \to \{1, -1\}$ as follows: let $f(e_1) = \alpha$, choose $f(e_2) \in \{\pm 1\}$ such that $\partial f(v_1) = 0$, suppose that $f(e_i)$ has been defined, choose $f(e_{i+1}) \in \{\pm 1\}$ such that $\partial f(v_i) = 0$, $1 \leqslant i \leqslant m - 1$. Then, since G is positive, $\partial f(v_m) = 0$, and so f is a nowhere-zero 2-flow of G. □

For negative bidirected eulerian graphs, Máčajová and Škoviera [7] obtained the following useful result.

Lemma 7. *([7]) Every flow-admissible negative bidirected eulerian graph contains at least two edge-disjoint unbalanced circles.*

Let T be a tree and $S \subseteq V(T)$ with $|S| \geqslant 2$. A subtree T' of T is odd (even) if $|V(T') \cap S|$ is odd (even). The following technical lemma is needed in the proof of Theorem 2.

Lemma 8. *Let T be a tree and $S \subseteq V(T)$ with $|S| \geqslant 2$. If $|S|$ is odd, then at least one of the following properties holds.*

(i) There are two proper subtrees T_1 and T_2 with $V(T_1) \cap V(T_2) = \emptyset$ and $V(T_1) \cup V(T_2) = V(T)$ such that T_1 is even, T_2 is odd, and $|V(T_2) \cap S| > 1$;

(ii) There is $r \in S$ and two proper subtrees T_1 and T_2 with $V(T_1) \cap V(T_2) = \{r\}$ and $V(T_1) \cup V(T_2) = V(T)$ such that both T_1 and T_2 are even;

(iii) There is $r \in V(T) \setminus S$ such that $T - r$ has at least three components, each of which is odd.

Proof. Let P be a shortest path in T connecting two distinct vertices of S, say $P = v_1 v_2 \ldots v_n$, then $V(P) \cap S = \{v_1, v_n\}$. Since $|S|$ is odd, we have $|S| \geqslant 3$, and so $T - V(P) \neq \emptyset$. Let B be a component of $T - V(P)$. If B is even, then (i) holds with $T_1 = B$ and $T_2 = T - V(B)$; if B is odd and there is $v \in \{v_1, v_n\}$ joined to B by an edge e, then (ii) holds with $r = v$, $T_1 = B + e$, and $T_2 = T - V(B)$. Assume thus that each component of $T - V(P)$ is odd and joined to neither v_1 nor v_n.

Let p and q be the smallest and largest integers such that $d_T(v_p) \geqslant 3$ and $d_T(v_q) \geqslant 3$, respectively, where $2 \leqslant p \leqslant q \leqslant n - 1$. If $p \neq q$, then $T - v_p v_{p+1}$ has exactly two components, each of which contains at least two vertices of S, and so (i) holds. Assume thus that $p = q$. Then (iii) holds with $r = v_p$. □

Theorem 9. *Suppose that G is a bidirected eulerian graph with a circle decomposition $\mathcal{C} = \{C_1, C_2, \cdots, C_m\}$, which contains at least two unbalanced circles. Then there is a nowhere-zero 4-flow f of G such that for every $C \in \mathcal{C}$, $|f(e)| = |f(e')| \leqslant 2$ if $e, e' \in E(C)$, with possibly one exception $C^* \in \mathcal{C}$ in which case either C^* is unbalanced or $|E(C^*)| \geqslant 3$.*

Proof. If this is not true, choose G to be a counterexample with $|E(G)|$ minimum. Clearly, G is negative, for otherwise G has a nowhere-zero 2-flow by Lemma 5, and the theorem holds trivially. Construct a graph H with $V(H) = \{c_1, c_2, \cdots, c_m\}$, where c_i corresponds to C_i, $1 \leqslant i \leqslant m$, there is an edge joining c_i and c_j if and only if the corresponding C_i and C_j share a common vertex. Note that a connected subgraph of H corresponds to an eulerian subgraph of G that is the union of circles from $\mathcal{C}$. Let S be the set of vertices of H which correspond to unbalanced circles in $\mathcal{C}$. By the given condition, $|S| \geqslant 2$, and since G is negative, $|S|$ is odd. Let T be a spanning tree of H. Applying Lemma 8 to T and S, we have the following three cases.

(i) There are two proper eulerian subgraphs G_1 and G_2 of G with $E(G_1) \cap E(G_2) = \emptyset$ and $E(G_1) \cup E(G_2) = E(G)$ such that G_1 is positive and G_2 is negative containing at least two unbalanced circles of $\mathcal{C}$. Note that $E(G_i)$ is a union of circles of $\mathcal{C}$, $i = 1, 2$. Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_i$ is a circle decomposition of G_i, $i = 1, 2$. By Lemma 5, G_1 has a nowhere-zero 2-flow f_1. Clearly, for every $C \in \mathcal{C}_1$, $|f_1(e)| = |f_1(e')| = 1$ if $e, e' \in E(C)$. By the minimality of G, G_2 with $\mathcal{C}_2$ as a circle decomposition has a nowhere-zero 4-flow f_2 with the required properties. Then $f_1 + f_2$ is a nowhere-zero 4-flow of G with the required properties.

(ii) There is an unbalanced circle $C^* \in \mathcal{C}$ and two proper eulerian subgraphs G_1 and G_2 of G with $E(G_1) \cap E(G_2) = C^*$ and $E(G_1) \cup E(G_2) = E(G)$ such that both G_1 and G_2 are positive. Let $\mathcal{C} \setminus \{C^*\} = \mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_i \cup \{C^*\}$ is a circle decomposition of G_i, $i = 1, 2$. By Lemma 5, each G_i has a nowhere-zero 2-flow f_i, $i = 1, 2$. Then $f = f_1 + 2f_2$ is a nowhere-zero 4-flow of G such that for every $C \in \mathcal{C} \setminus \{C^*\}$ and any $e, e' \in E(C)$, $|f(e)| = |f(e')| = 1$ if $C \in \mathcal{C}_1$ and $|f(e)| = |f(e')| = 2$ if $C \in \mathcal{C}_2$. Note that C^* is unbalanced. Thus f is indeed a 4-flow with the required properties.

(iii) There is a balanced circle $C^* \in \mathcal{C}$ such that $G - E(C^*)$ consists of edge-disjoint negative eulerian subgraphs $B_1, B_2, \ldots, B_t$, $t \geqslant 3$, such that $V(B_i) \cap V(C^*) \neq \emptyset$, $1 \leqslant i \leqslant t$. Since G is negative and C^* is balanced, we have that t is odd. Let

$$u_i \in V(B_i) \cap V(C^*), \quad 1 \leqslant i \leqslant t.$$

Set $\mathcal{C} \setminus \{C^*\} = \cup_{i=1}^{t} \mathcal{C}_i$, where $\mathcal{C}_i$ is a circle decomposition of B_i, $1 \leqslant i \leqslant t$.

If $u_i = u_j$ for all $1 \leqslant i \neq j \leqslant t$, let $u = u_1 = u_2 = \cdots = u_t$. Let H_i be the positive eulerian graph obtained from B_i by adding a new negative loop L_i at u_i $(= u)$, the two half-edges of L_i being away from u_i, $1 \leqslant i \leqslant t$. By Lemma 6, for any given $\alpha_i \in \{\pm 1\}$, H_i has a nowhere-zero 2-flow f_i with $f_i(L_i) = \alpha_i$, $1 \leqslant i \leqslant t$. Since $t \geqslant 3$ is odd, we may choose $\alpha_i \in \{\pm 1\}$ such that $2\alpha_1 + \sum_{i=2}^{t} \alpha_i = 0$, so that $2f_1 + \sum_{i=2}^{t} f_i$ gives a 3-flow ϕ of G, after deleting all the loops L_i, such that $S(\phi) = E(G) \setminus E(C^*)$, and for every $C \in \mathcal{C} \setminus \{C^*\}$ and any $e, e' \in E(C)$, $|\phi(e)| = |\phi(e')| = 2$ if $C \in \mathcal{C}_1$ and $|\phi(e)| = |\phi(e')| = 1$ if $C \in \cup_{i=2}^{t} \mathcal{C}_i$. Let f_0 be a 2-flow of G with $S(f_0) = C^*$. Then $\phi + f_0$ is a nowhere-zero 3-flow, and so a 4-flow, of G with the required properties. Assume thus that there are i, j such that $u_i \neq u_j$, and so $|E(C^*)| \geqslant 2$.

Case 1. $|E(C^*)| = 2$, say $E(C^*) = \{e_1, e_2\}$ and $V(C^*) = \{x, y\}$. Since C^* is balanced, e_1 and e_2 have the same sign, either both positive or both negative. By switching and reversing operations, we may assume that e_1 and e_2 are both positive and the orientations of both are from x to y.

Since $t \geqslant 3$ is odd, without loss of generality and by relabeling, we may assume that $x = u_1 = u_2 = \cdots = u_\ell$ and $y = u_{\ell+1} = u_{\ell+2} = \cdots = u_t$, where $\ell \geqslant 1$ is odd and $t - \ell \geqslant 2$ is even. As before, let H_i be the positive eulerian graph obtained from B_i by adding a new negative loop L_i at u_i, the two half-edges of L_i being away from u_i, $1 \leqslant i \leqslant t$. By Lemma 5, for any given $\alpha_i \in \{\pm 1\}$, H_i has a nowhere-zero 2-flow f_i with $f_i(L_i) = \alpha_i$, $1 \leqslant i \leqslant t$. Choose $\alpha_1 = 1$, and since ℓ is odd, we may choose $\alpha_i \in \{\pm 1\}$, $2 \leqslant i \leqslant \ell$, such that $\sum_{i=2}^{\ell} \alpha_i = 0$. Since $t - \ell \geqslant 2$ is even, choose $\alpha_{\ell+1} = \alpha_{\ell+2} = -1$, and $\alpha_i \in \{\pm 1\}$, $\ell + 3 \leqslant i \leqslant t$, such that $\sum_{i=\ell+3}^{t} \alpha_i = 0$. Then $2f_1 + \sum_{i=2}^{t} f_i$ gives a function ϕ, after deleting all the loops L_i, such that $S(\phi) = E(G) \setminus E(C^*)$, $\partial\phi(x) = -4$ and $\partial\phi(y) = 4$. Extend ϕ to a nowhere-zero 4-flow f of G with $f(e) = \phi(e)$ if $e \in E(G) \setminus \{e_1, e_2\}$ and $f(e_1) = f(e_2) = 2$. Then, for every $C \in \mathcal{C}$ and any $e, e' \in E(C)$, $|f(e)| = |f(e')| \leqslant 2$, without exception.

Case 2. $|E(C^*)| \geqslant 3$. Let $G_1 = B_1$, $G_2 = B_2 \cup C^*$, and $G_3 = G \setminus (B_1 \cup B_2)$. Note that $E(C^*) \subseteq E(G_3)$. Since C^* is balanced and G is negative, each G_i is a negative eulerian subgraph of G with $u_1 \in V(G_1) \cap V(G_2) \cap V(G_3)$. Let H_i be the positive eulerian graph obtained from G_i by adding a new negative loop L_i at u_1, the two half-edges of L_i being directed away from u_1, $1 \leqslant i \leqslant 3$. By Lemma 6, H_1 has a nowhere-zero 2-flow f_1 with $f_1(L_1) = 1$, and H_i has a nowhere-zero 2-flow f_i with $f_i(L_i) = -1$, $i = 2, 3$. Then $2f_1 + f_2 + f_3$ gives a 3-flow ϕ, after deleting the loops L_i, of G such that $E(G) \setminus E(C^*) \subseteq S(\phi)$ and $\phi(e) \in \{0, \pm 2\}$ for each $e \in E(C^*)$. Let f_4 be a 2-flow of G with $S(f_4) = C^*$. Then $f = \phi + f_4$ is a nowhere-zero 4-flow of G. For every $C \in \mathcal{C} \setminus \{C^*\}$ and any $e, e' \in E(C)$, $|f(e)| = |f(e')| = 2$ if $C \in \mathcal{C}_1$ and $|f(e)| = |f(e')| = 1$ if $C \in \cup_{i=2}^{t} \mathcal{C}_i$, where $\mathcal{C}_i$ is a circle decomposition of B_i defined before, $1 \leqslant i \leqslant t$. Thus f is indeed a nowhere-zero 4-flow of G with the required properties. This completes the proof of the theorem. $\square$

Now, Theorem 2 in the introduction follows from Theorem 9 as the following corollary.

Corollary 10. *Let G be a bidirected eulerian graph. If G is flow-admissible, then G has a nowhere-zero 4-flow f and a circle C, where either C is unbalanced or $|E(C)| \geqslant 3$, such that $|f(e)| = 3$ only if $e \in E(C)$.*

Proof. If G is positive, by Lemma 5, G has a nowhere-zero 2-flow, and the corollary holds trivially. Assume thus that G is negative. By Lemma 7, G has two edge-disjoint unbalanced circles, and so it has a circle decomposition that contains these two unbalanced circles as members. The corollary follows from Theorem 9. $\square$

The technical lemma below will be needed in the next section.

Lemma 11. *Let G be a bidirected eulerian graph. Suppose that G is negative and flow-admissible. If there is a circle C containing all the negative edges of G, then G has two 2-flows f_1 and f_2 such that $E(C) \subseteq S(f_1) \cup S(f_2)$, $S(f_1) \cap S(f_2) \subseteq E(C)$, and $S(f_i) - E(C)$ consists of vertex-disjoint paths with ends in $V(C)$, $i = 1, 2$.*

Proof. By Lemma 7, G has two edge-disjoint unbalanced circles, say C_1 and C_2. Since C contains all the negative edges of G, $E(C_i) \cap E(C) \neq \emptyset$, and so $C_i \neq C$, $i = 1, 2$.

Suppose that C_1 and C_2 have been chosen such that $|E(C_1)\cup E(C_2)\cup E(C)|$ is as small as possible. Let $F_1 = C_1 \oplus C$ and $F_2 = C_2 \oplus C$, where $\oplus$ denotes symmetrical difference. Then $E(C) \subseteq E(F_1)\cup E(F_2)$ since $E(C_1)\cap E(C_2) = \emptyset$. For each $i = 1, 2$, each component of F_i is eulerian; if F_i has an unbalanced circle, then, since F_i is positive, it has two edge-disjoint unbalanced circles C_1' and C_2'; but $|E(C_1') \cup E(C_2') \cup E(C)| < |E(C_1) \cup E(C_2) \cup E(C)|$, contradicting the choice of C_1 and C_2. Assume thus that F_i is balanced, $i = 1, 2$. Since C_i is a circle and $E(C_i)\cap E(C) \neq \emptyset$, we have that $E(F_i) - E(C)$ is a set of vertex-disjoint paths with ends in $V(C)$, $i = 1, 2$. The lemma follows by defining f_i to be a 2-flow of G with $S(f_i) = F_i$, $i = 1, 2$. □

4 Covering the support of a $\mathbf{Z}_2$-flow

As mentioned in the introduction, Theorem 12 below plays a key role in the proof of the 11-flow-theorem [4]. Introducing a new method, we present here a simpler and shorter proof.

Theorem 12. *([2]) Let f be a $\mathbf{Z}_2$-flow of a connected bidirected graph G. If $S(f)$ is positive, then there is a 3-flow ϕ of G such that $S(f) \subseteq S(\phi)$ and $\phi(e) \in \{\pm 2\}$ if and only if $e \in S(\phi) \setminus S(f)$.*

Proof. For each edge $e \in E(G) \setminus S(f)$, add a copy of e parallel to e with the same orientation, that is, duplicate e to form a balanced circle of two edges. Denote by $\widehat{G}$ the resulting bidirected graph. Since each component of $S(f)$ is eulerian, $\widehat{G}$ is eulerian. Since $S(f)$ is positive, so is $\widehat{G}$. By Lemma 5, $\widehat{G}$ has a nowhere-zero 2-flow, which yields a 3-flow ϕ of G with $\phi(e) \in \{\pm 1\}$ if $e \in S(f)$ and $\phi(e) \in \{0, \pm 2\}$ if $e \in E(G) \setminus S(f)$. □

We use the same method to handle the case in which the support of the $\mathbf{Z}_2$-flow contains an odd number of negative edges.

Theorem 13. *Let f be a $\mathbf{Z}_2$-flow in a flow-admissible connected bidirected graph G. Suppose that $S(f)$ is negative and contains two edge-disjoint unbalanced circles. Then there is a 5-flow ϕ of G such that $S(f) \subseteq S(\phi)$, $\phi(e)$ is even for each $e \in E(G) \setminus S(f)$, and $|\phi(e)| \leqslant 3$ for each $e \in S(f)$.*

Proof. As in the proof of Theorem 12, for each edge $e \in E(G) \setminus S(f)$, duplicate e with the same orientation to form a balanced 2-circle (circle of two edges). Denote by $\widehat{G}$ the resulting bidirected graph. Then $\widehat{G}$ is eulerian. Because G is flow-admissible, so is $\widehat{G}$.

Let C' and C'' be two edge-disjoint unbalanced circles in $S(f)$. Let $\mathcal{C}_1$ be a circle decomposition of $S(f)$ such that $C', C'' \in \mathcal{C}_1$ and let $\mathcal{C}_2$ be the set of all the 2-circles resulting from duplicating edges of $E(G)\setminus S(f)$. Then $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ is a circle decomposition of $\widehat{G}$, which contains at least two edge-disjoint unbalanced circles. By Theorem 9, $\widehat{G}$ has a nowhere-zero 4-flow φ such that for every $C \in \mathcal{C}$, $|\varphi(e)| = |\varphi(e')| \leqslant 2$ if $e, e' \in E(C)$, with possibly one exception $C^* \in \mathcal{C}$ in which case either C^* is unbalanced or $|E(C^*)| \geqslant 3$, which means that $C^* \notin \mathcal{C}_2$, and so φ yields a 5-flow ϕ of G with the required properties. □

Theorem 14. *Let f be a $\mathbf{Z}_2$-flow in a flow-admissible connected bidirected graph G. Suppose that $S(f)$ is negative and does not contain two edge-disjoint unbalanced circles. Then there is a 5-flow ϕ of G such that $S(f) \subseteq S(\phi)$ and $|\phi(e)| \leqslant 3$ for each $e \in E(G) \setminus S(f)$.*

Proof. As in the proof of Theorem 13, we obtain from G the flow-admissible connected eulerian graph $\widehat{G}$ by duplicating each edge $e \in E(G) \setminus S(f)$ to form a balanced 2-circle. Since $S(f)$ is negative, it contains at least one unbalanced circle. Let C be an unbalanced circle of $S(f)$ and $H = S(f) - E(C)$.

Since $S(f)$ does not contain two edge-disjoint unbalanced circles, H contains no unbalanced circle, and so H is balanced. By switching operations, we may assume that all the edges of H are positive. For each $v \in V(C) \cap V(H)$, split v into two new vertices $v' \in V(C)$ and $v'' \in V(H)$, where v' is incident with, and only with, the two edges incident with v in C, and add two positive parallel edges joining v' and v'', both oriented from v' to v'', so that v' has degree 4 and v'' has the same degree as v in $\widehat{G}$. Denote by $\widehat{G}'$ the resulting bidirected graph. Then $\widehat{G}'$ is eulerian and flow-admissible. In $\widehat{G}'$, let C' and H' be the subgraphs corresponding to C and H, respectively. Then $V(C') \cap V(H') = \emptyset$, which is the purpose we split $v \in V(C) \cap V(H)$. Let $X_1, X_2, \cdots, X_t$ be the components of H'. Then each X_i is an eulerian subgraph with all edges positive, $1 \leqslant i \leqslant t$. By reversing operations, we may assume that X_i has a directed eulerian tour, say Y_i, so that $\widehat{G}'$ has a 2-flow g_i with $S(g_i) = X_i$ and $g_i(e) = 1$ for each $e \in E(X_i)$, $1 \leqslant i \leqslant t$. Set $g = \sum_{i=1}^{t} g_i$. Let $\widehat{G}^*$ be the graph obtained from $\widehat{G}'$ by contracting each X_i into a single vertex x_i, $1 \leqslant i \leqslant t$. By the construction of $\widehat{G}'$, $x_i \notin V(C')$, $1 \leqslant i \leqslant t$. We distinguish two cases.

Case 1. $\widehat{G}^* \setminus E(C')$ contains unbalanced circles. Let C^* be such an unbalanced circle. We consider the following two subcases.

(i) $V(C^*) \cap V(C') \neq \emptyset$. Let $R = C^* + C'$. Then R is a positive eulerian subgraph with degree 2 or 4. A vertex x has degree 4 in R if and only if $x \in V(C^*) \cap V(C')$. Let f' be a 2-flow of $\widehat{G}^*$ with $S(f') = R$ and $f'(e) = 1$ for each $e \in E(R)$.

If $x_i \in V(R)$, then, since $x_i \notin V(C')$, x_i has degree 2 in R. Let e_1 and e_2 be the two edges incident with x_i in R. Let a_1, a_2 be the ends of e_1 and e_2 in X_i, respectively. Then a_1 and a_2 divide the directed eulerian tour Y_i into two directed segments S_1 and S_2. Note that $g_i(e) = 1$ for each $e \in E(Y_i)$ and all the edges of Y_i are positive. We may extend f' to a 2-flow f_i of $\widehat{G}^* + X_i$ by choosing a segment $S \in \{S_1, S_2\}$ such that $f'(e) = g_i(e) = 1$ for each $e \in E(S)$ and $f'(e) = 0$ for each $e \in E(X_i) \setminus E(S)$. (Sending flow values of f' along the forward direction of Y_i.) Doing this for each $x_i \in V(R)$, we extend f' to a 2-flow f'' of $\widehat{G}'$ such that $f''(e) = f'(e)$ if $e \in E(\widehat{G}^*)$ and $f''(e) = g(e) = 1$ or $f''(e) = 0$ if $e \in \cup_{i=1}^{t} E(X_i)$. Then $\varphi = f'' + g$ is a 3-flow of $\widehat{G}'$ with $S(f) \subseteq S(\varphi)$ and $|\varphi(e)| = 2$ only if $e \in \cup_{i=1}^{t} E(X_i)$.

(ii) $V(C^*) \cap V(C') = \emptyset$. Let $R = C^* + C' + P$, where P is a path intersecting with C^* and C' only at its ends (R is called a long barbell). Let f' be a 3-flow of $\widehat{G}^*$ with $S(f') = R$ such that $f'(e) = 1$ if $e \in E(C^*) \cup E(C')$ and $f'(e) = 2$ if $e \in E(P)$. Note that each vertex has degree 2 or 3 in R, with degree 3 only if it is an end of the path P.

If $x_i \in V(R)$ and x_i has degree 2 in R, then similarly to (i) above, we may extend f'

to a 3-flow f_i of $\widehat{G}^* + X_i$ by sending flow values of f' along the forward direction of Y_i. (If x_i is a vertex of the path P, send flow value 2 along the forward direction of Y_i.)

If $x_i \in V(R)$ and x_i has degree 3 in R, then x_i is an end of P in C^*. Let e_j, $1 \leqslant j \leqslant 3$, be the three edges incident with x_i in R, where $e_1 \in E(P)$, $e_2, e_3 \in E(C^*)$, and let y_j be the end of e_j in X_i, $1 \leqslant j \leqslant 3$, say that y_1, y_2, y_3 are around the directed eulerian tour Y_i in that order. Let S_1 be the directed segment of Y_i from y_1 to y_2, S_2 the directed segment of Y_i from y_2 to y_3, and S_3 the directed segment of Y_i from y_3 to y_1. Without loss of generality, we may assume that the half-edge $e_1^{y_1}$ has an orientation towards y_1, $e_2^{y_2}$ away from y_2, and $e_3^{y_3}$ away from y_3, with $f'(e_1) = 2$ and $f'(e_2) = f'(e_3) = 1$. Note that all the edges of Y_i are positive. Extend f' to a 3-flow f_i of $\widehat{G}^* + X_i$ by setting $f_i(e) = f'(e)$ if $e \in E(\widehat{G}^*)$, $f_i(e) = 2$ if $e \in S_1$, $f_i(e) = 1$ if $e \in S_2$, and $f_i(e) = 0$ if $e \in S_3$.

In either case above, we have a 3-flow f_i of $\widehat{G}^* + X_i$ with $f_i(e) = f'(e)$ if $e \in E(\widehat{G}^*)$ and $f_i(e) \in \{0, 1, 2\}$ if $e \in E(X_i)$. Doing this for each $x_i \in V(R)$, we have a 3-flow f'' of $\widehat{G}'$ such that $f''(e) = f'(e)$ if $e \in E(\widehat{G}^*)$ and $f''(e) \in \{0, 1, 2\}$ if $e \in \cup_{i=1}^t E(X_i)$. Then $\varphi = f'' + g$ is a 4-flow of $\widehat{G}'$ with $S(f) \subseteq S(\varphi)$ and $|\varphi(e)| = 3$ only if $e \in \cup_{i=1}^t E(X_i)$.

By the definition of R, for each 2-circle B, obtained by duplicating an edge of G, $|E(B) \cap E(R)| \leqslant 1$. Therefore, φ yields a 4-flow (hence a 5-flow) ϕ of G such that $S(f) \subseteq S(\phi)$ and $|\phi(e)| \leqslant 2$ if $e \in E(G) \setminus S(f)$. This completes the proof of Case 1.

Case 2. $\widehat{G}^* \setminus E(C')$ contains no unbalanced circle. By switching operations, we may assume that all the edges of $\widehat{G}^* \setminus E(C')$ are positive, which means that all the negative edges of $\widehat{G}^*$ are contained in C'. It follows from Lemma 11 that $\widehat{G}^*$ has two 2-flows f_1 and f_2 such that $E(C') \subseteq S(f_1) \cup S(f_2)$, $S(f_1) \cap S(f_2) \subseteq E(C')$, and $S(f_i) - E(C')$ consists of vertex-disjoint paths with ends in $V(C')$, $i = 1, 2$.

As before, we shall extend f_1 and f_2 to flows of $\widehat{G}'$. First, consider f_1. For each $x_i \in V(S(f_1))$, send flow values of f_1 along the directed eulerian tour Y_i, always along the forward direction of Y_i, to obtain a 2-flow φ_1 of $\widehat{G}'$ so that $\varphi_1(e) = f_1(e)$ if $e \in E(\widehat{G}^*)$ and $\varphi_1(e) \in \{0, 1\}$ if $e \in \cup_{i=1}^t E(X_i)$. By the same arguments, f_2 is extended to a 2-flow φ_2 of $\widehat{G}'$ such that $\varphi_2(e) = f_2(e)$ if $e \in E(\widehat{G}^*)$ and $\varphi_2(e) \in \{0, 1\}$ if $e \in \cup_{i=1}^t E(X_i)$. Then $\varphi = g + \varphi_1 + 2\varphi_2$ is a 5-flow of $\widehat{G}'$. Since $S(f_i) - E(C')$ is a set of vertex-disjoint paths with ends in $V(C')$, we see that for each 2-circle B, obtained by duplicating an edge of G, $|E(B) \cap S(f_i)| \leqslant 1$, $i = 1, 2$. Hence, φ yields a 5-flow ϕ of G such that $|\phi(e)| \leqslant 3$ for each $e \in E(G) \setminus S(f)$, and thus ϕ is indeed a 5-flow of G with the required properties. This completes the proof of Theorem 14. □

Combining Theorems 13 and 14 yields Theorem 3.

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