The Critical Polynomials of Simple Connected Graphs

Tingting Wang Lu Lu

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Abstract

Let G be a connected graph with n vertices and adjacency matrix $A(G)$. The critical polynomial $d_G(x_1, \ldots, x_n)$ is a degree-n multivariate polynomial defined as the determinant of the matrix $M_G(x_1, \ldots, x_n)$, where

$$M_G(x_1, \ldots, x_n) = \mathrm{Diag}(x_1, \ldots, x_n) - A(G).$$

For any positive integer r, define the set

$$V_{d_G}(r) = \{d_G(x_1, \ldots, x_n) \mid x_i \in \mathbb{Z}_{\geqslant r}, 1 \leqslant i \leqslant n\} \cap \mathbb{Z}_{\geqslant 0},$$

where $\mathbb{Z}_{\geqslant r}$ denotes the set of all integers not less than r. The subset $V_G(r) \subseteq V_{d_G}(r)$ consists the elements u such that there exist $a_1, \ldots, a_n \in \mathbb{Z}_{\geqslant r}$ for which $u = d_G(a_1, \ldots, a_n)$, the matrix $M_G(a_1, \ldots, a_n)$ is positive definite if $u \neq 0$ and positive semi-definite with rank $n-1$ if $u = 0$. Furthermore, the associated group $\Phi_{M_G(a_1,\ldots,a_n)}$ must be cyclic. Motivated by Lorenzini's exploration of whether the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geqslant 0}$ might be finite for typical graphs [J. Number Theory 257 (2024) 215–248], we establish that for any simple connected graph G, the subset $V_G(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$. This provides additional evidence in support of Lorenzini's hypothesis that the larger subset $V_{d_G}(2)$ might actually be cofinite in $\mathbb{Z}_{\geqslant 0}$.

Mathematics Subject Classifications: 05C50, 11C20

1 Introduction

Let G be a connected, undirected graph without loops on n vertices $v_1, \ldots, v_n$. Let $A(G)$ denote the adjacency matrix of G. Define

$$M_G(x_1, \ldots, x_n) := \mathrm{Diag}(x_1, \ldots, x_n) - A(G),$$

and let

$$d_G(x_1, \ldots, x_n) := \det\ (M_G) \in \mathbb{Z}[x_1, \ldots, x_n].$$

School of Mathematics and Statistics, Central South University, Changsha, Hunan, 410083, China
Corresponding author: Lu Lu (lulumath@csu.edu.cn).

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The matrix M_G was introduced in [4]. The principal ideal of $\mathbb{Z}[x_1, \ldots, x_n]$ generated by $d_G(x_1, \ldots, x_n)$ is referred to as the n-th *critical ideal* of the graph G. The polynomial $d_G(x_1, \ldots, x_n)$ is called the *critical polynomial* of G in [9]. Furthermore, as shown in [5], two simple graphs G and G' on n vertices are isomorphic if and only if there exists an ordering of the vertices of G and G' such that their critical polynomials $d_G(x_1, \ldots, x_n)$ and $d_{G'}(x_1, \ldots, x_n)$ are identical. Let $\mathbb{Z}_{\geqslant r}$ denote the set of integers greater than or equal to the integer r. For $a_1, \ldots, a_n \in \mathbb{Z}_{\geqslant 1}$, let $M_G(a_1, \ldots, a_n)$ denote the integer matrix obtained from M_G by evaluating M_G at $x_i = a_i$ for $1 \leqslant i \leqslant n$. Such matrices play an important role in geometry. If the matrices of the form $M_G(a_1, \ldots, a_n)$ are positive definite, they are referred to as M-matrices. The *associated group* of $M_G(a_1, \ldots, a_n)$ is defined as follows:

(a) If $\det(M) \neq 0$, the associated group $\Phi_{M_G(a_1,\ldots,a_n)} := \mathbb{Z}^n / \operatorname{Im}(M)$ has order $|\det(M)|$.

(b) If $\det(M) = 0$ and $\operatorname{rank}(M) = \rho < n$, then the associated group $\Phi_{M(a_1,\ldots,a_n)} := \mathbb{Z}^n / \operatorname{Im}(M)$ is isomorphic to the torsion subgroup of $\mathbb{Z}^n / \operatorname{Im}(M)$.

For the matrix M, there exisit two matrices P and Q in $GL_n(\mathbb{Z})$ such that PMQ is a diagonal matrix $N = Diag(d_1, \ldots, d_\rho, 0, \ldots, 0)$ where $d_i \mid d_{i+1}$ for $i = 1, \ldots, \rho - 1$. This matrix is called the *Smith normal form* of M. The group Φ_M is isomorphic to the torsion part of $\mathbb{Z}^n / \operatorname{Im} N$. Thus Φ_M is cyclic if and only if $\rho = 1$ or $d_{\rho-1} = 1$. An *arithmetical structure* on G is a pair (M, R), where $M = M_G(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in \mathbb{Z}_{\geqslant 1}$, and $R^\top = (r_1, \ldots, r_n)$ is a positive integer vector with $\gcd(r_1, \ldots, r_n) = 1$ satisfying $MR = 0$. Lorenzini [9] showed that M is a positive semi-definite matrix of rank $n - 1$, and the associated group is called the *critical group* of G. Recently, several notable papers (e.g., [6, 3, 1]) have studied arithmetical structures and critical groups on various families of graphs, including paths, cycles, bidents, paths with a double edge, and stars. In particular, Lorenzini in [9] proved that for every connected graph G there exists an arithmetical structure (M, R) on G such that its critical group is trivial.

For any polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$ and an integer $r \geqslant 1$, consider the following set of non-negative values

$$V_f(r) := \{f(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in \mathbb{Z}_{\geqslant r}\} \cap \mathbb{Z}_{\geqslant 0}.$$

Since $d_G(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$, for a non-negative integer r, the set $V_{d_G}(r)$ is studied. To focus on a more specific case, the subset $V_G(r) \subset V_{d_G}(r)$ is defined. The number $u \in \mathbb{Z}_{\geqslant 0}$ belongs to $V_G(r)$ if and only if there exist $a_1, \ldots, a_n \in \mathbb{Z}_{\geqslant r}$ such that:

(i) $u = d_G(a_1, \ldots, a_n)$.

(ii) If $d_G(a_1, \ldots, a_n) \neq 0$, then the matrix $M_G(a_1, \ldots, a_n)$ is positive definite; if $d_G = 0$, then the matrix M_G is positive semi-definite of rank $n - 1$.

(iii) The associated group $\Phi_{M_G(a_1,\ldots,a_n)}$ is cyclic.

As usual, we always write C_n, P_n, and K_n for the cycle, the path, and the complete graph on n vertices, respectively. The star on $n + 1$ vertices is denoted as S_n, and the non-pendent vertex is called the *center* of S_n. The *extended cycle* C_n^+ is obtained from C_n by attaching a new vertex to one vertex of C_n. The *extended star* S_n^+ is obtained from S_n by attaching a new vertex to a pendent vertex of S_n. The *broom* $B_{l,k}$ is the graph obtained by attaching k new vertices to one endpoint of P_l, such endpoint is called the *center* and the other endpoint is called the *handle* of $B(l,k)$ (see Fig. 2 where u_1 is the handle and u_l is the center). Clearly, $B_{l,0}$ is just the path P_l with the handle being one of the endpoints, and $B_{1,k}$ is just the star S_k with the handle being the center. The graph D_n $(n \geqslant 4)$ is obtained from the path $P_{n-1} = v_1v_2 \ldots v_{n-1}$ by adding an additional vertex attached to v_2. The graph $\tilde{D}_n$ is obtained from D_n by adding a new vertex attached to v_{n-2}. If $n = 4$, then $\tilde{D}_n$ is just the star S_4. For $6 \leqslant n \leqslant 8$, the graph $\tilde{E}_n$ is the specific tree on $n+1$ vertices shown in Fig. 1. The graph E_n is obtained from $\tilde{E}_n$ by deleting the pendant vertex colored red. The graphs P_n $(n \geqslant 2)$, D_n $(n \geqslant 4)$, and E_n $(6 \leqslant n \leqslant 8)$ are called *Dynkin diagrams* on n vertices. The graphs $\tilde{D}_n$ $(n \geqslant 4)$ and $\tilde{E}_n$ $(6 \leqslant n \leqslant 8)$ with $n+1$ vertices are called *extended Dynkin diagrams* or *affine Dynkin diagrams*.

For two graphs G and H, their *disjoint union* $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *join* $G \vee H$ of them is the graph obtained from $G \cup H$ by adding all possible edges between G and H. Furthermore, the graph $K_1 \vee G$ is also called the *cone* of G, and also denoted by $C(G)$.

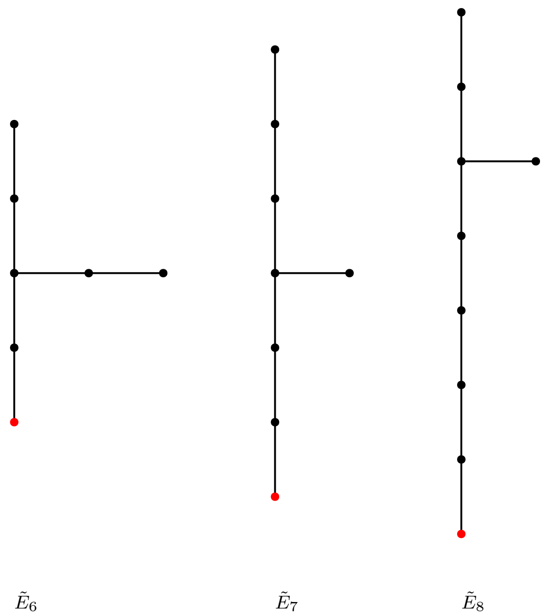

Figure 1: $\tilde{E}_n$, $6 \leqslant n \leqslant 8$

Motivated by geometric problems, Lorenzini [9] investigated the properties of the sets $V_{d_G}(r)$ and $V_G(r)$ for $r \in \{1, 2\}$.

Theorem 1 ([9])**.** *Let G be a connected graph containing an induced subgraph H such that $1 \in V_H(2)$. Then $V_G(2)$ contains $\mathbb{Z}_{>0}$. In particularly, if G is a connected simple graph, then $\mathbb{Z}_{>0} \subset V_G(2)$, except possibly if G is a tree, a cycle C_n, a complete bipartite* *graph $K_{p,q}$, a complete graph K_n with $n \leqslant 13$, an extended cycle C_n^+ with $n \leqslant 7$, or the cone $C(P_3)$.*

Theorem 1 characterize a large family of graphs with $V_G(2) \supset \mathbb{Z}_{>0}$. For the other classes of graphs, this property is far from being proved. So, $V_{d_G}(2)$ is considered for these graphs. For a subset $S \subseteq \mathbb{Z}_{\geqslant 0}$ and a positive integer l, let $S(l) := \{0, 1, 2, \ldots, l\} \cap S$ and let $s(l) \colon = |S(l)|$. The *density* $d(S)$ of S is defined as

$$d(S) := \lim_{l \to \infty} \frac{s(l)}{l}.$$

If $d(S) = 1$, then S is *dense* in $\mathbb{Z}_{\geqslant 0}$.

Theorem 2 ([9])**.** *Let G be one of the following graphs:*

(i) G is a Dynkin diagram, an extended Dynkin diagram, a star S_n, or an extended star S_n^+.

(ii) G is a cycle C_n or an extended cycle C_n^+.

(iii) G is the cone $C(P_3)$, the complete graph K_n, the complete bipartite graph $K_{2,n}$ or $K_{3,n}$.

Then the set $V_{d_G}(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$.

In fact, Theorem 1 shows that for any connected simple graph G, the set $V_G(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$, with the exception of certain graphs. These exceptional cases include trees, cycles C_n, complete bipartite graphs $K_{p,q}$, complete graphs K_n with $n \leqslant 13$, extended cycles C_n^+ with $n \leqslant 7$, and the cone $C(P_3)$. Motivated by these results, we turn our attention to these exceptional classes of graphs, leading to the following conclusions.

Theorem 3. *If T is a tree on n vertices, then $V_T(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$.*

Theorem 4. *If $G \in \{C_n, K_{k,l}, K_n, C_n^+, C(P_3)\}$, then $V_G(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$.*

Combining Theorems 1, 3 and 4, we get the following result immediately.

Theorem 5. *If G is connected simple graph, then $V_G(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$.*

Remark 6. In [9], Lorenzini posed the problem of whether the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geqslant 0}$ is finite for a broad class of finite graphs. Our main result (Theorem 5) addresses this problem by establishing that for any simple connected graph G, the set $V_G(2)$ – and hence $V_{d_G}(2)$ – is dense in $\mathbb{Z}_{\geqslant 0}$. This density result provides a partial resolution to Lorenzini's question and simultaneously generalizes Theorem 2 within the framework of arithmetic progressions.

The remainder of this paper is organized as follows. Section 2 introduces several essential lemmas that form the basis for proving the main results. In Section 3, we present the proof of Theorem 3, while Section 4 is devoted to the proof of Theorem 4.

2 Preliminaries

For a graph G and a vertex $v \in V(G)$, denote by G_v the subgraph obtained from G by deleting the vertex v and all edges incident to v.

Lemma 7 ([9]). *Let G be a connected graph, and let $v \in G$ be a vertex such that the subgraph G_v satisfies one of the following properties:*

(i) The complement of $V_{d_{G_v}}(2)$ in $\mathbb{Z}_{\geqslant 0}$ is finite.

(ii) Up to reordering the vertices of G_v, there exist $a_2, \ldots, a_{n-1} \in \mathbb{Z}_{\geqslant 2}$ such that $d_{G_v}(t, a_2, \ldots, a_{n-1}) = \alpha t - \beta$, where $\alpha \in \mathbb{Z}_{>0}$, $\beta \in \mathbb{Z}$, and $\gcd(\alpha, \beta) = 1$.

(iii) $V_{d_{G_v}}(2)$ contains an infinite subset of pairwise coprime values $\{u_1, u_2, \ldots\}$ such that $\lim_{j\to\infty} \prod_{i=1}^{j} \left(1 - \frac{1}{u_i}\right) = 0$.

Then, $V_{d_G}(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$.

Lemma 8 ([9]). *Let G be a graph on n vertices, and for some choice of $a_2, \ldots, a_n \geqslant 2$, we have $d_G(t, a_2, \ldots, a_n) = \alpha t - \beta$ with $\gcd(\alpha, \beta) = 1$. Let G^+ denote the graph obtained by attaching a new vertex v_0 with $e \geqslant 1$ edges to the vertex v_1. Let $q \geqslant 1$ denote any integer coprime to $e\alpha\beta$. Then, $d_{G^+}(q, t, a_2, \ldots, a_n)$ is of the form $\alpha' t - \beta'$, where $\gcd(\alpha', \beta') = 1$.*

Lemma 9 ([7, Dirichlet's Theorem]). *Let $a, b \in \mathbb{Z}_{\geqslant 1}$ with $\gcd(a, b) = 1$. Then, there are infinitely many prime numbers in the sequence of integers $b, b + a, b + 2a, \ldots, b + ka$ for $k \in \mathbb{Z}_{\geqslant 0}$.*

Remark 10. If $a \in \mathbb{Z}_{\geqslant 1}$ and $b \in \mathbb{Z}$ with $\gcd(a, b) = 1$, there are also infinitely many primes in the sequence $-b, -b + a, -b + 2a, \ldots, -b + ka$ for $k \in \mathbb{Z}_{\geqslant 0}$. In fact, if $b < 0$, the result follows directly from Lemma 9. If $b > 0$, there exists a positive integer m such that $ma - b > 0$. Moreover, since $\gcd(a, b) = 1$, it follows that $\gcd(a, ma - b) = 1$. Thus, we can express the sequence $-b + ka = (k - m)a + (ma - b)$, where $k \in \mathbb{Z}_{\geqslant 0}$. Applying Lemma 9, we conclude that the sequence contains infinitely many primes.

A set of positive primes $\mathcal{P}$ is said to have Dirichlet density if

$$\lim_{s\to 1} \frac{\sum_{p\in\mathcal{P}} p^{-s}}{\ln(s-1)^{-1}}$$

exists. If the limit exists we set it equal to $d(\mathcal{P})$ and call $d(\mathcal{P})$ the *Dirichlet density* of $\mathcal{P}$. By using this definition, we get the following result.

Lemma 11. *Let G be a graph and assume that there exists a vertex v such that, for some labelling of G_v, there exist $a_1, \ldots, a_{n-2} \in \mathbb{Z}_{\geqslant 2}$ satisfying*

$$d_{G_v}(a_1, \ldots, a_{n-2}, t) = \alpha t - \beta,$$

where $\alpha \in \mathbb{Z}_{\geqslant 1}$, $\beta \in \mathbb{Z}$, and $\gcd(\alpha, \beta) = 1$. Suppose that there exists a subset $\mathcal{P} \subseteq \mathcal{P}_0 = \{p \text{ is a prime } \mid p = \alpha t - \beta, t \geqslant 2\}$ such that the complement of $\mathcal{P}$ in $\mathcal{P}_0$ is finite and such that $\mathcal{P}$ satisfies the following properties:

(i) For any $t_0 \geqslant 2$ with $\alpha t_0 - \beta \in \mathcal{P}$, the matrix $M_{G_v}(a_1, \dots, a_{n-2}, t_0)$ is positive definite.

(ii) For any $t_0 \geqslant 2$ with $\alpha t_0 - \beta \in \mathcal{P}$, the $(n-1)$-th determinant divisor of the matrix $M_G(a_1, \dots, a_{n-2}, t_0, u)$, for every $u \in \mathbb{Z}_{\geqslant 2}$, is equal to 1.

Then $V_G(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$.

Proof. Since $\mathcal{P}_0$ has Dirichlet density $1/\phi(\alpha_1)$ where $\phi(\cdot)$ is the Euler's totient function, we get $\sum_{p\in\mathcal{P}_0} 1/p$ is infinite (see [8, Page 251]). As the complement of P in $\mathcal{P}_0$ is finite, we have $\sum_{p\in\mathcal{P}} 1/p$ is infinite too. According to [11, Lemma 5.1.1], we have

$$\lim_{j\to\infty} \prod_{i=1}^{j} \left(1 - \frac{1}{p_i}\right) = 0,$$

where $p_i \in \mathcal{P}$ for $1 \leqslant i \leqslant j$. For $p_i = \alpha t_i - \beta \in \mathcal{P}$ and $u \in \mathbb{Z}_{\geqslant 2}$, consider the matrix

$$M_G(a_1, \dots, a_{n-2}, t_0, u).$$

Since its $(n-1)$-th leading principal minor is p_i, we get

$$\det M_G(a_1, \dots, a_{n-2}, t_0, u) = p_i u - w_i,$$

for some $w_i \in \mathbb{Z}$.

Let $P_i = \{p_i u - w_i \mid u \in \mathbb{Z}_{\geqslant 2} \text{ and } u \geqslant w_i/p_i\}$. Clearly, $P_i \subseteq V_{d_G}(2)$. Since the set P_i forms an arithmetic progression, the density of P_i is $1/p_i$. Moreover, since

$$\det M_G(a_1, \dots, a_{n-2}, t_0, u_0) = p_i u_0 - w_i \in P_i,$$

the matrix M_G is positive definite when $\det M_G > 0$, and positive semi-definite of rank $n-1$ when $\det M_G = 0$. Moreover, the $(n-1)$-th determinant divisor of M_G is equal to 1, which implies that the associated group Φ_{M_G} is cyclic. Thus, $P_i \subseteq V_G(2)$.

Since the values of $\mathcal{P}$ are distinct primes, we find that the union $\bigcup_{i=1}^{j} P_i$ has density greater than

$$1 - \prod_{i=1}^{j} \left(1 - \frac{1}{p_i}\right).$$

Since $\bigcup_{i=1}^{j} P_i \subseteq V_G(2)$ for any j, we get

$$d(V_G(2)) \geqslant \lim_{j\to\infty} \left(1 - \prod_{i=1}^{j} \left(1 - \frac{1}{p_i}\right)\right) = 1.$$

This yields $d(V_G(2)) = 1$, and thus $V_G(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$. □

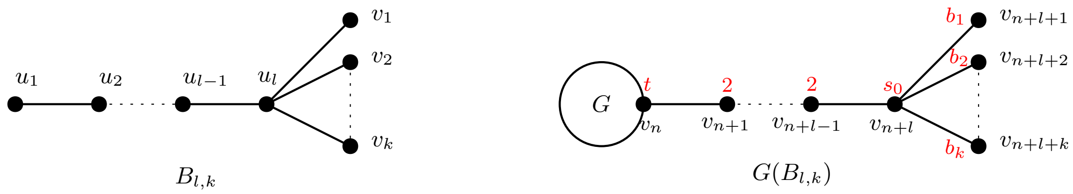

Figure 2: The graphs $B_{l,k}$ and $G(B_{l,k})$ with gluable vertex v_n.

3 Proof of Theorem 3

A graph G with n vertices is called an *initial graph* if, for some labelling of G, there exist integers $a_1, \dots, a_{n-1}$ with each $a_i \geqslant 2$ satisfying the following conditions:

C1. $d_G(a_1, \dots, a_{n-1}, t) = \alpha t - \beta$, where $\alpha \in \mathbb{Z}_{\geqslant 1}$, $\beta \in \mathbb{Z}$, and $\gcd(\alpha, \beta) = 1$; the vertex corresponding to the variable t is called a *gluable vertex*;

C2. The r-th leading principal minor of $M_G(a_1, \dots, a_{n-1}, t)$ is positive for $1 \leqslant r \leqslant n-1$.

In this part, we always assume that G is an initial graph. For integers $l \geqslant 1$ and $k \geqslant 0$, let $G(B_{l,k})$ be the graph obtained from G by attaching the handle of $B_{l,k}$ to the vertex v_n (see Fig.2), where the numbers colored red are the values of x_i corresponding to the vertices v_i which will be used in Lemma 12.

Lemma 12. *There exist values $s_0, b_1, \dots, b_k \in \mathbb{Z}_{\geqslant 2}$ with $s_0 \geqslant 1 + \sum_{i=1}^{k} \frac{1}{b_i}$ such that*

$$d_{G(B_{l,k})}(a_1, \dots, a_{n-1}, t, 2, \dots, 2, s_0, b_1, \dots, b_k) = \alpha_1 t - \beta_1,$$

for some $\alpha_1 \in \mathbb{Z}_{\geqslant 1}$ and $\beta_1 \in \mathbb{Z}$ with $\gcd(\alpha_1, \beta_1) = 1$.

Proof. Consider the corresponding matrix $M_{G(B_{l,k})}(a_1, \dots, a_{n-1}, t, 2, \dots, 2, s_0, b_1, \dots, b_k)$, which has the form

$$\begin{pmatrix}
G_{v_n} & * & & & & & & & & & \\
* & t & -1 & & & & & & & & \\
 & -1 & 2 & -1 & & & & & & & \\
 & & -1 & 2 & -1 & & & & & & \\
 & & & \ddots & \ddots & \ddots & & & & & \\
 & & & & -1 & 2 & -1 & & & & \\
 & & & & & -1 & s_0 & -1 & -1 & \cdots & -1 \\
 & & & & & & -1 & b_1 & & & \\
 & & & & & & -1 & & b_2 & & \\
 & & & & & & \vdots & & & \ddots & \\
 & & & & & & -1 & & & & b_k
\end{pmatrix}.$$

By directed calculations, we get

$$\begin{aligned}
\det\left(M_{G(B_{l,k})}\right) &= f_k(b)\left(s_0-g_k(b)\right)\left(\left(l-1\right)\left(\alpha t-\beta-\alpha\right)+\left(\alpha t-\beta\right)\right)\\
&\quad -f_k(b)\left(\left(l-2\right)\left(\alpha t-\beta-\alpha\right)+\alpha t-\beta\right)\\
&= \left(f_k(b)\left(s_0-g_k(b)\right)l-f_k(b)\left(l-1\right)\right)\alpha t-\left(f_k(b)\left(s_0-g_k(b)\right)l\right.\\
&\quad \left.-f_k(b)\left(l-1\right)\right)\beta-\left(f_k(b)\left(s_0-g_k(b)\right)\left(l-1\right)-f_k(b)\left(l-2\right)\right)\alpha\\
&= \alpha_1 t-\beta_1,
\end{aligned} \tag{1}$$

where

$$f_k(b)=\begin{cases}\prod_{i=1}^{k} b_i, & \text{if } k\geqslant 1,\\ 1, & \text{if } k=0,\end{cases},\quad g_k(b)=\begin{cases}\sum_{i=1}^{k}\frac{1}{b_i}, & \text{if } k\geqslant 1,\\ 0, & \text{if } k=0,\end{cases} \tag{2}$$

and

$$\begin{cases}\alpha_1=\left(f_k(b)\left(s_0-g_k(b)\right)l-f_k(b)\left(l-1\right)\right)\alpha,\\ \beta_1=\left(f_k(b)\left(s_0-g_k(b)\right)l-f_k(b)\left(l-1\right)\right)\beta+\left(f_k(b)\left(s_0-g_k(b)\right)\left(l-1\right)-f_k(b)\left(l-2\right)\right)\alpha.\end{cases}$$

First, we choose $b_1,\ldots,b_k$ to be different primes which are coprime with each of $a_1,\ldots,a_{n-1},\alpha,\beta$ and l. For any prime p satisfying $p\mid f_k(b)l$, we get either $p\mid l$ or $p\mid f_k(b)$. If $p\mid l$, then $p\nmid(l-1)$, and thus $p\nmid\left(f_k(b)g_k(b)l+f_k(b)\left(l-1\right)\right)$ because $\gcd(l,f_k(b))=1$. If $p\mid f_k(b)$, then $p=b_i$ for some i. Since $f_k(b)g_k(b)=\sum_{i=1}^{k}\prod_{j\neq i}b_j$, we get $p\nmid f_k(b)g_k(b)$. This yields $p\nmid\left(f_k(b)g_k(b)l+f_k(b)\left(l-1\right)\right)$ because $\gcd(b_i,l)=1$. Therefore, we conclude that

$$\gcd\left(f_k(b)l,f_k(b)g_k(b)l+f_k(b)(l-1)\right)=1.$$

According to Lemma 9, there exists a number $s_0>g_k(b)+1$ such that

$$f_k(b)ls_0-\left(f_k(b)g_k(b)l+f_k(b)(l-1)\right)=p',$$

where p' is a prime coprime with α. For convenience, denote by

$$q'=f_k(b)\left(s_0-g_k(b)\right)\left(l-1\right)-f_k(b)\left(l-2\right).$$

Since $p'>q'>0$, we have $\gcd(p',q')=1$. Note that

$$\alpha_1=p'\alpha \text{ and } \beta_1=p'\beta+q'\alpha.$$

For any prime factor p of α_1, we have either $p=p'$ or $p\mid\alpha$. If $p=p'$, then $p\nmid\beta_1$ because $\gcd(p',q')=\gcd(p',\alpha)=1$. If $p\mid\alpha$, then we also have $p\nmid\beta_1$ because $\gcd(\alpha,p')=\gcd(\alpha,\beta)=1$. Thus, $\gcd(\alpha_1,\beta_1)=1$. □

Remark 13. Lemma 8 for $e=1$ shows that if G satisfies the condition C1 with a gluable vertex v, then G^+, obtained by attaching a new vertex v_0 to v, also satisfies the condition C1. Lemma 12 generalizes this result by replacing G^+ with $G(B_{l,k})$, where G^+ corresponds to the special case $G(B_{1,0})$.

Next, we show that the matrix $M_{G(B_{l,k})}(x_1, \ldots, x_{n+l+k})$ is positive definite for some values x_i. Let α_1 and β_1 be the values determined in Lemma 12. Define the set

$$\mathcal{P}_0 = \{p \text{ is a prime} \mid p = \alpha_1 t - \beta_1, t \in \mathbb{Z}_{\geqslant 2}\}.$$

Using the symbols of Lemma 12, define the set

$$\mathcal{P} = \{p \in \mathcal{P}_0 \mid t > \beta/\alpha + 1 \text{ and } p \nmid f_k(b)\alpha\}\,. \tag{3}$$

It is clear that the complement of $\mathcal{P}$ in $\mathcal{P}_0$ is finite.

Lemma 14. *If $p_0 = \alpha_1 t_0 - \beta_1 \in \mathcal{P}$, then the matrix*

$$M_{G(B_{l,k})}\left(a_1, \ldots, a_{n-1}, t_0, 2, \ldots, 2, s_0, b_1, \ldots, b_k\right)$$

is positive definite.

Proof. Let $M^r_{G(B_{l,k})}$ denote the r-th leading principal submatrix of $M_{G(B_{l,k})}$, which is obtained by taking the first r rows and columns of $M_{G(B_{l,k})}$. Then, $\det M^r_{G(B_{l,k})}$ is the r-th leading principal minor of $M_{G(B_{l,k})}$. It suffices to prove that $\det M^r_{G(B_{l,k})} > 0$ for $r \geqslant 1$.

For $1 \leqslant r \leqslant n-1$, it is clearly $\det M^r_{G(B_{l,k})} > 0$ since the r-th leading principal minor of $M_G(a_1, \ldots, a_{n-1}, t)$ is positive. If $r = n$, the determinant $\det\ M^r_{G(B_{l,k})} = \alpha t_0 - \beta > 0$ as $t_0 > \beta/\alpha + 1$. For $n+1 \leqslant r \leqslant n+l-1$, the matrix $M^r_{G(B_{l,k})}$ can be viewed as

$$M_{G_{B_{r-n,0}}}(a_1, \ldots, a_{n-1}, t_0, 2, \ldots, 2).$$

Therefore, from Eqs.(1) and (2) we get

$$\det M^r_{G(B_{l,k})} = (r-n)\left(\alpha t_0 - \beta - \alpha\right) + \alpha t_0 - \beta > 0.$$

For $n+l \leqslant r \leqslant n+k+l$, the matrix $M^r_{G(B_{l,k})}$ can be viewed as matrix

$$M_{G(B_{l,r-n-l})}(a_1, \ldots, a_{n-1}, t_0, 2, \ldots, 2, s_0, b_1, \ldots, b_{r-n-l}).$$

Thus, by Eqs.(1) and (2) we get

$$\begin{aligned} &\det M^r_{G(B_{l,k})} \\ &\quad = f_{r-n-l}(b)\left((s_0 - g_{r-n-l}(b) - 1)\left((l-1)\left(\alpha t_0 - \beta - \alpha\right) + \alpha t_0 - \beta\right) + \alpha t_0 - \beta - \alpha\right). \end{aligned}$$

According to the proof Lemma 12, we get $f_{r-n-l}(b) > 0$ and $s_0 \geqslant g_k(b)+1 \geqslant g_{r-n-l}(b)+1$. Combining the fact $t_0 > \beta/\alpha + 1$, we get $\det M^r_{G(B_{l,k})} > 0$.

The proof is completed. □

If we relabel the vertices of $G(B_{l,k})$ such that v_n is listed last, then Lemma 12 implies that $G(B_{l,k})$ satisfies condition C1 with gluable vertex v_n. Furthermore, for $1 \leqslant r \leqslant l+k+n-1$, since the r-th leading principal minors of $M_{G(B_{l,k})}(a_1, \ldots, a_{n-1}, 2, \ldots, 2, s_0, b_1, \ldots, b_k, t_0)$ and $M_{G(B_{l,k})}(a_1, \ldots, a_{n-1}, 2, \ldots, 2, s_0, b_1, \ldots, b_k, t)$ are identical for any t, Lemma 14 implies that $G(B_{l,k})$ also satisfies condition C2. Hence, the following result holds.

Corollary 15. *If G is an initial graph with gluable vertex v_n, then so is $G(B_{l,k})$.*

Let $G^+(B_{l,k})$ denote the graph obtained from $G(B_{l,k})$ by adding a new vertex to the vertex corresponding to the value s_0, i.e., the vertex v_{n+l} in Fig. 2.

Lemma 16. *For any $p_0 = \alpha_1 t_0 - \beta_1 \in \mathcal{P}$ and $u \in \mathbb{Z}$, the $(n+l+k)$-th determinant divisor of the matrix*

$$M_{G^+(B_{l,k})}(a_1, \ldots, a_{n-1}, t_0, 2, \ldots, 2, s_0, b_1, \ldots, b_k, u),$$

equals to 1.

Proof. Assume that d is the $(n+l+k)$-th determinant divisor of $M_{G^+(B_{l,k})}$. Since the matrix $M_{G(B_{l,k})}$ is an $(n+l+k)$-th leading principal submatrix of $M_{G^+(B_{l,k})}$ and $\det M_{G(B_{l,k})} = p_0$, we get $d \mid p_0$. On the other hand, consider the submatrix obtained by deleting the last row and the n-th column of $M_{G^+(B_{l,k})}$. The resulting matrix M' has the following form:

$$\begin{pmatrix}
G_{v_n} & & & & & & & & & & \\
* & -1 & & & & & & & & & \\
 & 2 & -1 & & & & & & & & \\
 & -1 & 2 & -1 & & & & & & & \\
 & & \ddots & \ddots & \ddots & & & & & & \\
 & & & -1 & 2 & -1 & & & & & \\
 & & & & -1 & t_l & -1 & -1 & \cdots & -1 & -1 \\
 & & & & & -1 & b_1 & 0 & & & \\
 & & & & & -1 & & b_2 & 0 & & \\
 & & & & & \vdots & & & \ddots & \ddots & \\
 & & & & & -1 & & & & b_k & 0
\end{pmatrix}.$$

By directed calculations, we get $\det M' = (-1)^{k+l} f_k(b)\alpha$, and thus $d \mid f_k(b)\alpha$. Since $\gcd(p_0, f_k(b)\alpha) = 1$ by the choice of p_0, we get $d = 1$. ☐

Lemmas 12, 14 and 16 imply that $G^+(B_{l,k})$ satisfies the conditions of Lemma 11. This yields the following result immediately.

Lemma 17. *The set $V_{G^+(B_{l,k})}(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$.*

According to the calculations in the proof of Lemma 12, we get the following result, which implies that the gluable vertex could be changed when $l = 1$ (see Fig. 3).

Corollary 18. *There exist integers $x_0, b_1, b_2, \ldots, b_k \in \mathbb{Z}_{\geqslant 2}$ satisfying $\alpha x_0 - \beta > \alpha$ such that*

$$d_{G(B_{1,k})}(a_1, a_2, \ldots, a_{n-1}, x_0, y, b_1, b_2, \ldots, b_k) = \alpha_2 y - \beta_2,$$

for some $\alpha_2 \in \mathbb{Z}_{\geqslant 1}$ and $\beta_2 \in \mathbb{Z}$ with $\gcd(\alpha_2, \beta_2) = 1$. It means that the gluable vertex turns to be the handle of $B_{1,k}$.

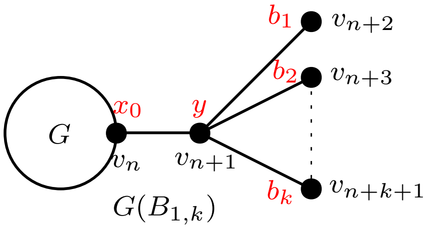

Figure 3: The graph $G(B_{1,k})$ with gluable vertex v_{n+1}

Proof. By taking $l = 1$ in the proof of Lemma 12, we get

$$d_{G(B_{1,k})}(a_1, \ldots, a_{n-1}, x_0, y, b_1, \ldots, b_k) = \alpha_2 y - \beta_2, \tag{4}$$

where $\alpha_2 = f_k(b)(\alpha x_0 - \beta)$ and $\beta_2 = f_k(b)g_k(b)(\alpha x_0 - \beta) + f_k(b)\alpha$. First, we choose $b_1, \ldots, b_k$ to be different primes. Next, since $\gcd(\alpha, \beta) = 1$, according to Lemma 9, we could choose $x_0 \in \mathbb{Z}_{\geqslant 2}$ such that $\alpha x_0 - \beta = p'$ is a prime satisfying $p' \nmid f_k(b)$ and $p' > \alpha$. Note that $\gcd(f_k(b), f_k(b)g_k(b)) = 1$. For any prime factor p of α_2, we get either $p = p'$ or $p = b_i$ for some $1 \leqslant i \leqslant k$. If $p = p'$, then $p \nmid \beta_2$ since $p' \nmid f_k(b)$ and $p' \nmid \alpha$. If $p = b_i$, then we also have $p \nmid \beta_2$ because $\gcd(b_i, f_k(b)g_k(b)) = \gcd(b_i, p') = 1$. This yields $\gcd(\alpha_2, \beta_2) = 1$. $\square$

Based on Corollary 18, we define the sets

$$\mathcal{Q}_0 := \{p \text{ is a prime} \mid p = \alpha_2 y - \beta_2, y \in \mathbb{Z}_{\geqslant 2}\},$$

and

$$\mathcal{Q} := \{p \in \mathcal{Q}_0 \mid \; y \geqslant g_k(b) + 1 \text{ and } p \nmid f_k(b)\alpha\}.$$

It is clear that the complement of $\mathcal{Q}$ in $\mathcal{Q}_0$ is finite. Similar to Lemma 14, we get the following result.

Corollary 19. *If $p = \alpha_2 y_0 - \beta_2 \in \mathcal{Q}$, then the matrix*

$$M_{G(B_{1,k})}(a_1, a_2, \ldots, a_{n-1}, x_0, y_0, b_1, b_2, \ldots, b_k)$$

is positive definite.

Proof. Let $M^r_{G(B_{1,k})}$ be the r-th leading principal submatrix of $M_{G(B_{1,k})}$, that is, $\det M^r_{G(B_{1,k})}$ is the r-th leading principal minor of $M_{G(B_{1,k})}$. It suffices to show $\det M^r_{G(B_{1,k})} > 0$ for $r \geqslant 1$. If $1 \leqslant r \leqslant n-1$, then $\det M^r_{G(B_{1,k})} > 0$ because $M^r_{G(B_{1,k})}$ is also the r-th leading principal submatrix of $M_G(a_1, \ldots, a_{n-1}, x_0)$ which is positive. If $r = n$, then

$$\det M^r_{G(B_{1,k})} = \det M_G(a_1, \ldots, a_{n-1}, x_0) = \alpha x_0 - \beta > 0.$$

If $n + 1 \leqslant r \leqslant n + 1 + k$, then the matrix $M^r_{G(B_{1,k})}$ can be viewed as the matrix

$$M_{G(B_{1,r-n-1})}(a_1, \ldots, a_{n-1}, x_0, y_0, b_1, \ldots, b_{r-n-1}).$$

Thus, from Eq.(4), we have

$$\begin{aligned}&\det M_{G(B_{1,r-n-1})}(a_1,\ldots,a_{n-1},x_0,y_0,b_1,\ldots,b_{r-n-1})\\&=f_{r-n-1}(b)\left((\alpha x_0-\beta)\left(y_0-g_{r-n-1}(b)\right)-\alpha\right).\end{aligned}$$

Since $f_{r-n-1}(b)>0$, $\alpha x_0-\beta>\alpha$ and $y_0\geqslant g_k(b)+1\geqslant g_{r-n-l}(b)+1$, we have

$$\det\ M_{G(B_{1,r-n-1})}>0.$$

The proof is completed. □

Analogous to Corollary 15, combining Corollaries 18 and 19, we get the following result.

Corollary 20. *If G is an initial graph with gluable vertex v_n, then $G(B_{1,k})$ is also an initial graph with gluable vertex v_{n+1}, where v_{n+1} is the handle (also the center) of $B_{1,k}$.*

A *rooted tree* $T(v_0)$ is a tree T with a designated vertex v_0 called the *root*. In the rooted tree $T(v_0)$, for a vertex $v\in V(T(v_0))$, denote $P_T(v_0,v)$ as the unique path from v_0 to v. The length of $P_T(v_0,v)$ is just the distance $d_T(v_0,v)$ from v_0 to v, which is also called the *depth* of v. Clearly, $d_T(v_0,v_0)=0$. The *height* $h(T(v_0))$ of $T(v_0)$ is defined as the largest depth in $T(v_0)$, that is

$$h(T(v_0))=\max_{v\in V(T(v_0))}\{d_T(v_0,v)\}.$$

Lemma 21. *If T is a tree on n vertex with $n\geqslant 3$, then there exists a vertex $v_0\in V(T)$ such that the rooted tree $T(v_0)$ contains two vertices v_1,v_2 satisfying*

$$d_T(v_0,v_1),d_T(v_0,v_2)\geqslant h(T(v_0))-1 \text{ and } V(P_T(v_0,v_1))\cap V(P_T(v_0,v_2))=\{v_0\}.$$

Proof. We proceed by induction on n. If $n=3$, then T is the star S_2, and the statement holds by taking v_0 to be the center and v_1,v_2 to be the two pendant vertices. Assume that the statement holds for $n-1$. If $\mid V(T)\mid=n$, then let T' be the tree obtained from T by deleting a pendant vertex v. By inductive assumption, there exists a vertex v_0' such that the rooted tree $T'(v_0')$ containing two vertices v_1', v_2' satisfying

$$d_{T'}(v_0',v_1'),d_{T'}(v_0',v_2')\geqslant h(T'(v_0'))-1 \text{ and } V(P_{T'}(v_0',v_1'))\cap V(P_{T'}(v_0',v_2'))=\{v_0'\}.$$

Now, consider the tree T. Assume that the vertex v' is the neighbor of v in T. If $d_{T'}(v_0',v')\leqslant h(T'(v_0'))-1$, then $h(T(v_0'))=h(T'(v_0'))$, and the result follows by taking $v_0=v_0'$ and $v_1=v_1'$, $v_2=v_2'$. If $d_{T'}(v_0',v')=h(T'(v_0'))$, we claim that $V(P_{T'}(v_0',v_i'))\cap V(P_{T'}(v_0',v'))=\{v_0'\}$ for some $i\in\{1,2\}$. Otherwise, assume that $u_i\in V(P_{T'}(v_0',v_i'))\cap V(P_{T'}(v_0',v'))\setminus\{v_0'\}$ for $1\leqslant i\leqslant 2$. Clearly, $u_1\neq u_2$ because $V(P_{T'}(v_0',v_1'))\cap V(P_{T'}(v_0',v_2'))=\{v_0'\}$. Since v_0' is an internal vertex of the unique path between u_1 and u_2, the vertex v_0' has degree 2 in the path $P_{T'}(v_0',v')$, which is impossible. Thus, without loss of generality, assume that $V(P_{T'}(v_0',v_1'))\cap V(P_{T'}(v_0',v'))=\{v_0'\}$. Taking v_0 to be the neighbor of v_0' in the path $P_{T'}(v'_0, v')$, we consider the rooted tree $T(v_0)$. Since v_0 is a neighbor of v'_0, for any $x \in V(T')$, we get $d_T(v_0, x) \leqslant d_{T'}(v'_0, x) + 1$, that is $d_{T'}(v'_0, x) \geqslant d_T(v_0, x) - 1$. This yields $h(T'(v'_0)) \geqslant h(T(v_0)) - 1$. Note that

$$\begin{cases} d_T(v_0, v) = d_{T'}(v'_0, v') + 1 - 1 = h(T'(v'_0)) \geqslant h(T(v_0)) - 1, \\ d_T(v_0, v'_1) = d_{T'}(v'_0, v'_1) + 1 \geqslant h(T'(v'_0)) - 1 + 1 \geqslant h(T(v_0)) - 1. \end{cases}$$

Also, we have

$$V(P_T(v_0, v)) \cap V(P_T(v_0, v'_1)) = \{v_0\}.$$

The result follows by taking $v_1 = v'_1$ and $v_2 = v$. □

Let T be a tree on n vertices. Since $1 \in V_{E_8}(2)$ (see [9, Page 239]), Theorem 1 implies that $V_T(2) \supset \mathbb{Z}_{>0}$ whenever E_8 is an induced subgraph of T. Therefore, it suffices to consider the case where T does not contain E_8. If T is a tree on n vertices that does not contain E_8, then by Lemma 21, there exists a vertex v_0 such that the rooted tree $T(v_0)$ contains two vertices v_1 and v_2 satisfying

$$d_T(v_0, v_1), d_T(v_0, v_2) \geqslant h(T(v_0)) - 1 \quad \text{and} \quad V(P_T(v_0, v_1)) \cap V(P_T(v_0, v_2)) = \{v_0\}.$$

Lemma 22. *Let T be a tree on n vertices containing no E_8, and let v_0, v_1, and v_2 be the vertices described above. Then the followings hold.*

(i) If $h(T(v_0)) = 3$, then T is either of the form T_1 or of the form T_2 shown in Fig.4.

(ii) If $h(T(v_0)) \geqslant 4$, then T is of the form shown in Fig.5.

Proof. First, we consider the case for $h(T(v_0)) = 3$. For any vertex $u \in V(T(v_0))$, we have $d_T(v_0, u) \leqslant 3$, and there exists at least one vertex u_3 such that $d_T(v_0, u_3) = 3$. Let $P_T(v_0, u_3) = v_0u_1u_2u_3$. If $d(u_1) \geqslant 3$, we claim that each vertex v with depth 3 must satisfy $u_1 \in V(P_T(v_0, v))$. Otherwise, $V(P_T(v_0, u_3)) \cap V(P_T(v_0, v)) = \{v_0\}$, and thus $V(P_T(v_0, u_3)) \cup V(P_T(v_0, v)) \cup \{x\}$ forms an E_8 for $x \in N(u_1) \setminus \{v_0, u_2\}$ (see Fig. 4), a contradiction. Hence, T is of the form T_1 show in Fig. 4. If $d(u_1) = 2$, we claim that, for each v with depth 3 must satisfy that $d(x) = 2$ for any $x \in N(v_0) \cap V(P_T(v_0, v))$. Otherwise, let v be a vertex of depth 3, say $P_T(v_0, v) = v_0xx_1v$, and $y \in N(x) \setminus \{v_0, x_1\}$. Since $d(u_1) = 2$, we get $V(P_T(v_0, u_3)) \cap V(P_T(v_0, v)) = \{v_0\}$. Therefore, $V(P_T(v_0, u_3)) \cup V(P_T(v_0, v)) \cup \{y\}$ forms an E_8 (see Fig. 4), a contradiction. Hence, T has the form T_2 shown in Fig. 4.

Next, we consider the case for $h(T(v_0) \geqslant 4$. Since $h(T(v_0)) \geqslant 4$, there exists at least one vertex u such that $d_T(v_0, u) = h(T(v_0)) \geqslant 4$. By Lemma 21, there exists a vertex $w \in \{v_1, v_2\}$ such that $d_T(v_0, w) \geqslant h(T(v_0)) - 1 \geqslant 3$, and

$$V(P_T(v_0, u)) \cap V(P_T(v_0, w)) = \{v_0\}.$$

Let u' and w' denote the neighbors of u and w in the path $P_T(u, w)$, respectively. If $d(w) \geqslant 2$, let $x \in N(w) \setminus V(P_T(v_0, w))$. We claim that $d(v) = 2$ for every vertex

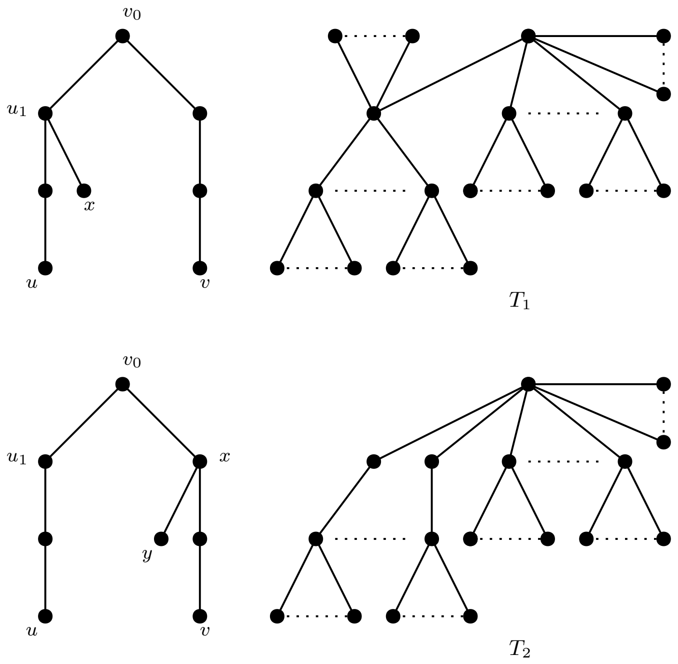

Figure 4: The form of T when $h(T(v_0)) = 3$

$v \in V(P_T(u', w')) \setminus \{u'\}$. Otherwise, assume that $v' \in V(P_T(u', w')) \setminus \{u'\}$ with $d(v') \geqslant 3$. Let $v'' \in N(v') \setminus V(P_T(u, x))$. If $v' \in P_T(v_0, u') \setminus \{u'\}$, then this would lead to an E_8 since $d(v', u) \geqslant 2$ and $d(v, x) \geqslant d(v_0, x) \geqslant 4$; similarly, if $v' \in P_T(v_0, w')$, an E_8 would also arise since $d(v', x) \geqslant d(w', x) \geqslant 2$ and $d(v', u) \geqslant d(v_0, u) \geqslant 4$ (see Fig. 5). Both cases are impossible. Similarly, if $d(w) = 1$, then $d(v) = 2$ for any vertex $v \in V(P_T(u', w')) \setminus \{u', w'\}$. Hence, T has the form shown in Fig. 5. $\square$

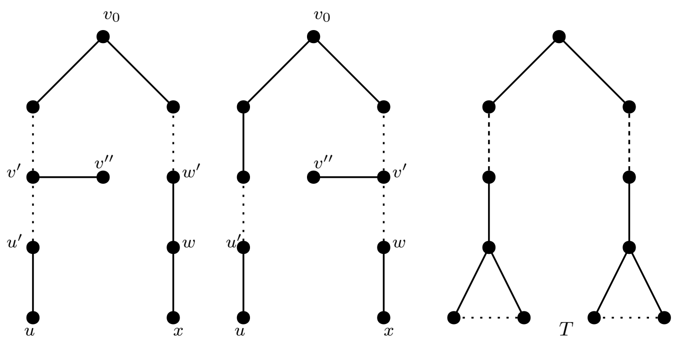

Figure 5: The form of T when $h(T(v_0)) \geqslant 4$

Now we are ready to prove Theorem 3.

Proof of the Theorem 3. Let T be a tree on n vertices. Since $1 \in V_{E_8}(2)$ (see [9, Page 239]), by Theorem 1, we conclude that $V_T(2) \supset \mathbb{Z}_{>0}$ whenever E_8 is an induced subgraph of T. Therefore, it only needs to consider the case that T does not contain E_8 as an induced subgraph. If $n=1$, the result is trivial. If $n \in \{2,3\}$, then $T=P_n$ and thus $V_T(2)=V_{d_T}(2)$ (see [9, Page 241]). Therefore, Theorem 2(i) implies $V_T(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$. If $n \geqslant 4$, let v_0, v_1, v_2 be the vertices determined by Lemma 21. We consider the hight of the rooted tree $T(v_0)$.

Case 1. $h(T(v_0))=1$.

In this case, the tree $T(v_0)$ is a star. Let $T_0=u_1$ be an isolate vertex, and T_1 the graph obtained by attaching the center u_2 of $B_{1,n-3}$ to u_1. Then, T is obtained from T_1 by attaching a new vertex u_n to u_2, that is $T=T_0^+(B_{1,n-3})$ (see Fig. 6). Since T_0 is an initial graph, Lemma 17 indicates $V_T(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$.

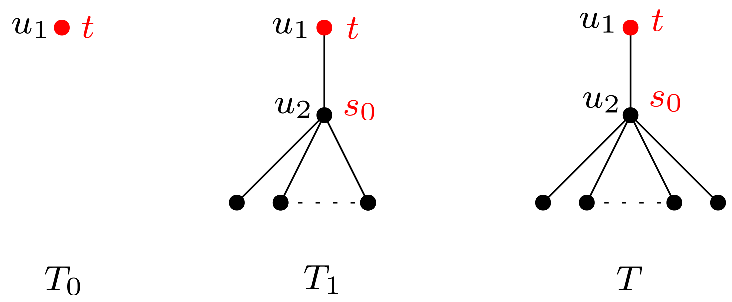

Figure 6: The form of $T(v_0)$ for $h(T(v_0))=1$

Case 2. $h(T(v_0))=2$.

In this case, there exist integers $n_0, n_1, \ldots, n_m \geqslant 0$ with $\sum_{i=0}^{m} n_i = n-1$ such that T is formed by the following way. Let T_0 be the star S_{n_0} with center u_0, and let T_{i+1} be the tree obtained from T_i by attaching the center u_{i+1} of $S_{n_{i+1}}$ to u_0 for $0 \leqslant i \leqslant m-1$. The graph T is the tree obtained from T_m by attaching a new vertex u_n to u_m (see Fig. 7).

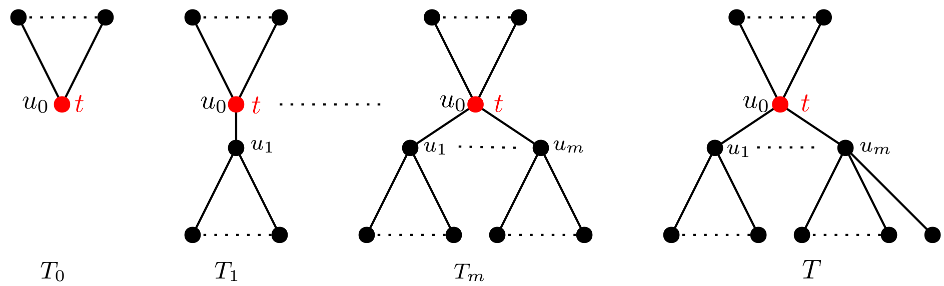

Figure 7: The form of $T(v_0)$ for $h(T(v_0))=2$

Taking $a_1, \ldots, a_{n_0}$ to be different primes, we get $d_{T_0}(a_1, \ldots, a_{n_0}, t) = \alpha t - \beta$ with $\gcd(\alpha, \beta)=1$, where $\alpha = \prod_{i=1}^{n_0} a_i$, $\beta = \sum_{i=1}^{n_0} \prod_{j \neq i} a_j$, and the variable t corresponds to u_0. Furthermore, by directed calculations, one could easily verify that the r-th leading principal minor of $M_{T_0}(a_1, \ldots, a_{n_0}, t)$ is positive for $1 \leqslant r \leqslant n_0$. Hence, T_0 is an initial graph with gluable vertex u_0. Corollary 15 implies that, if T_i is an initial graph with gluable vertex u_0, then so is T_{i+1} for $0 \leqslant i \leqslant m-1$. Hence, T_{m-1} is an initial graph with gluable vertex u_0. Therefore, Lemma 17 indicates $V_T(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$ because $T = T_{m-1}^{+}(B_{1,n_m})$.

Case 3. $h(T(v_0)) = 3$.

In this case, according to Lemma 22, $T(v_0)$ either has the form H_m^+ or has the form F_s^+ as shown in Fig. 8, where the rooted vertex is v_0 and the gluable vertex is colored red.

If $T = H_m^+$, then it could be constructed by the following way. Let H_0 be the graph shown in Fig. 8, with rooted vertex v_0, and H_1 is obtained from H_0 by attaching the center u_0 of S_k to v_0 for some k. For $1 \leqslant i \leqslant m-1$, the graph H_{i+1} is obtained from H_i by attaching the center of B_{1,n_i} to u_0 for some n_i. The graph T is obtained from H_m by attaching a new vertex to the center of $B_{1,n_{m-1}}$. In Case 2, we have already proved that H_0 is an initial graph with gluable vertex v_0. Hence, Corollary 20 implies that H_1 is an initial graph with gluable vertex u_0 because $H_1 = H_0(B_{1,k})$. This indicates by Corollary 15 that H_{m-1} is an initial graph with gluable vertex u_0. Since $T = H_m^+ = H_{m-1}^+(B_{1,n_{m-1}})$, Lemma 17 indicates $V_T(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$.

If $T = F_s^+$, then it could be constructed by the following way. Let F_0 be the graph shown in Fig. 8. For $0 \leqslant i \leqslant s-1$, the graph F_{i+1} is obtained from F_i by attaching the handle of B_{2,n_i} to v_0 for some n_i. The graph T is obtained from F_s by attaching a new vertex to the center of $B_{2,n_{s-1}}$. Similarly, the graph F_0 is an initial graph with gluable vertex v_0. Corollary 15 implies that F_{s-1} is also an initial graph with gluable vertex v_0. Since $T = F_s^+ = F_{s-1}^+(B_{2,n_{s-1}})$, Lemma 17 indicates $V_T(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$.

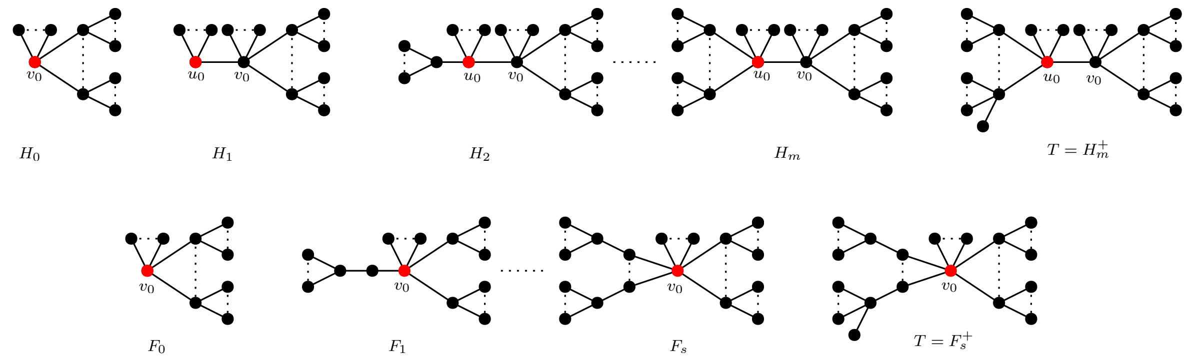

Figure 8: The forms of $h(T(v_0))$ for $h(T(v_0)) = 3$

Case 4. $h(T(v_0)) \geqslant 4$.

In this case, according to Lemma 22, the tree T is of the form given in Fig. 9, which could be formed by the following way. Let T_0 be a star S_k for some k with center u_0, and T_1 the graph obtained from T_0 by attaching the handle of $B_{l,m}$ to the vertex u_0. The tree T is obtained from T_1 by attaching a new vertex to the center of $B_{l,m}$. Since T_0 is an initial graph with gluable vertex u_0 and $T = T_0^+(B_{l,m})$, Lemma 17 indicates that the set $V_T(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$. □

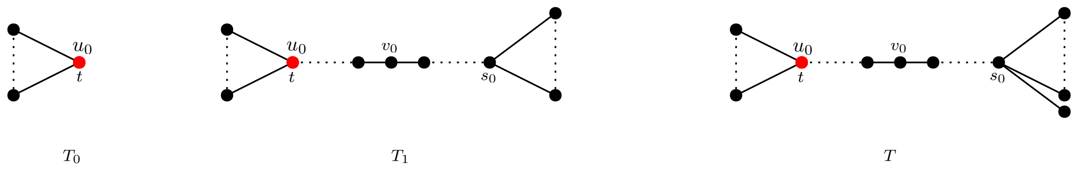

Figure 9: The form of $T(v_0)$ for $h(T(v_0)) \geqslant 4$

4 Proof of Theorem 4

We consider all graphs in Theorem 4 one by one. First, we consider the cycle C_n and the extended cycle C_{n-1}^{+}.

Lemma 23. *If $G = C_n$ or C_{n-1}^{+}, then $V_G(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$.*

Proof. If $G = C_n = v_1 v_2 \cdots v_n$, let G_v denote the subgraph obtained from G by deleting the vertex v_n; if $G = C_{n-1}^{+}$, let G_v denote the graph obtained from G by deleting one of the non-pendent neighbors, say v_n of the unique vertex v_{n-2} of degree 3 (see Fig.10). Clearly, $G_v = P_{n-1} = v_1 v_2 \cdots v_{n-1}$. By directed calculations, we get

$$d_{P_{n-1}}(2, 2, \ldots, t) = (n-1)t - (n-2),$$

and $\gcd(n-1, n-2) = 1$. Let

$$\mathcal{P}_1 := \{p \text{ is a prime} \mid p = (n-1)t - (n-2), t \geqslant 2 \text{ and } p \nmid -n(-n+1)\}.$$

The complement of $\mathcal{P}_1$ in $\{p \text{ is a prime} \mid p = (n-1)t - (n-2),\ t \geqslant 2\}$ is finite. Deleting the last row and the $(n-1)$-th column of the matrix $M_G(2, \ldots, 2, t, u)$ where $u \in \mathbb{Z}_{\geqslant 2}$, the determinant of the resulting matrix is either $-n$ or $-n+1$. Thus, for any $t_0 \geqslant 2$ with $(n-1)t_0 - (n-2) = p_0 \in \mathcal{P}_1$, the $(n-1)$-th determinant divisor of the matrix $M_G(2, \ldots, 2, t_0, u)$ is equal to 1. Besides, by directed calculations, for $1 \leqslant r \leqslant n-2$, the r-th leading principal minor of the matrix $M_{P_{n-1}}(2, \ldots, 2, t_0)$ is equal to $r+1 > 0$; for $r = n-1$, the $(n-1)$-th leading principal minor is equal to $p_0 > 0$. Thus, the matrix $M_{P_{n-1}}(2, \ldots, 2, t_0)$ is positive definite. By Lemma 11, the set $V_G(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$. $\square$

Next, we consider the complete bipartite graph $K_{k,l}$.

Lemma 24. *If G is a complete bipartite graph $K_{k,l}$, then $V_{d_G}(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$.*

Proof. When $k = 1$, the graph $K_{1,l}$ is a star, then by Theorem 3, we get the result. For $k \geqslant 2$, it suffices to show that $K_{k-1,l}$ satisfies condition C1, and then the result follows by Lemma 7 (ii) since $G_v = K_{k-1,l}$ is just the graph obtained from $G = K_{k,l}$ by deleting a vertex of degree l. Without loss of generality, we show that $K_{k,l}$ satisfies condition C1 for any $k \geqslant 1$ and proceed by induction on k. If $k = 1$, then $K_{1,l} = S_l$ is a star. By the proof of Case 2 of Theorem 3, we get $K_{1,l} = S_l$ satisfies condition C1. Assume that the statement holds for $k-1$, i.e., $K_{k-1,l}$ satisfies condition C1. We now consider the graph $G = K_{k,l}$, whose vertex set can be partitioned into two disjoint parts

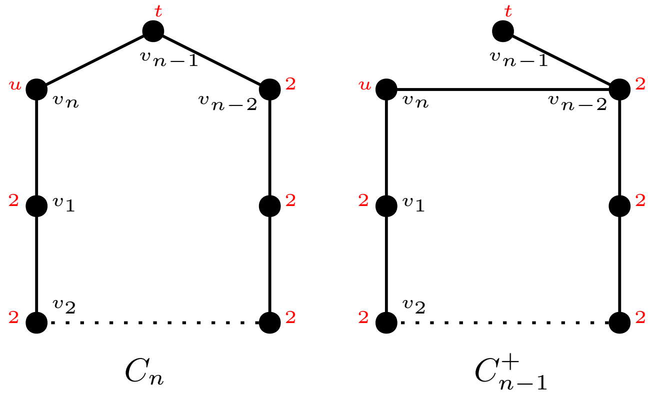

Figure 10: The forms of C_n and C^+_{n-1}

$U = \{u_1, \ldots, u_k\}$ and $V = \{v_1, \ldots, v_l\}$ where $K_{k,l} = U \vee V$ (see Fig. 11). By induction hypothesis, for the graph $K_{k-1,l}$ there exist integers $a_1, \ldots, a_l, q_1, \ldots, q_{k-2} \in \mathbb{Z}_{\geqslant 2}$ such that $d_{K_{k-1,l}}(q_{k-2}, \ldots, q_1, t, a_1, \ldots, a_l) = \alpha t - \beta$ where $\alpha \in \mathbb{Z}_{\geqslant 1}$, $\beta \in \mathbb{Z}$ and $\gcd(\alpha, \beta) = 1$. Then, for the graph $K_{k-1,l}$, we choose $q_{k-1} \in \mathbb{Z}_{\geqslant 2}$ and $q_{k-1} \geqslant \beta/\alpha + 1$ which is coprime with β, then

$$d_{K_{k,l}}(q_{k-1}, \ldots, q_1, t, a_1, \ldots, a_l) = (q_{k-1}\alpha - \beta)t - q_{k-1}\beta.$$

As $q_{k-1} \in \mathbb{Z}_{\geqslant 2}$ and $q_{k-1} \geqslant \beta/\alpha + 1$, we have $(q_{k-1}\alpha - \beta) \in \mathbb{Z}_{\geqslant 1}$ and $q_{k-1}\beta \in \mathbb{Z}$. Moreover, since $\gcd(q_{k-1}, \beta) = 1$ and $\gcd(\alpha, \beta) = 1$, we get $\gcd(q_{k-1}\alpha - \beta, q_{k-1}\beta) = 1$. Therefore, the graph $K_{k,l}$ satisfies the condition C1. $\square$

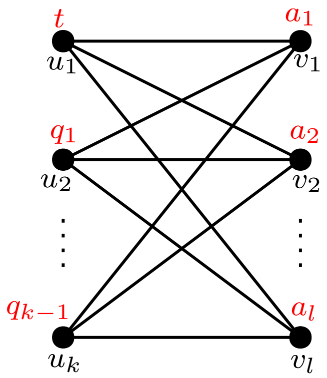

Figure 11: The form of the graph $K_{k,l}$

Lemma 25. *If $G = K_{k,l}$ is a complete bipartite graph, then $V_G(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$.*

Proof. If $k = 1$, the $K_{k,l}$ is the star $K_{1,l}$, and the result follows from Theorem 3. If $k \geqslant 2$, by the proof of Lemma 24 there exist $q_{k-2}, \ldots, q_1, a_1, \ldots, a_l$ such that

$$d_{K_{i-1,l}}(q_{i-2}, q_{i-3}, \ldots, q_1, t, a_1, \ldots, a_l) = \alpha_{i-2}t - \beta_{i-2}$$

where $\gcd(\alpha_{i-2}, \beta_{i-2}) = 1$ for $2 \leqslant i \leqslant k$. Let G_v denote the subgraph by deleting the vertex v with degree l of G. For $t > \max\{2,\ \sum_{j=1}^{l} \frac{1}{a_j},\ \beta_{i-2}/\alpha_{i-2},\ i = 2, \ldots, k\}$, define

$$\mathcal{P}_2 := \left\{ p \text{ is a prime} \mid p = \alpha_{k-2}t - \beta_{k-2}, \text{ and } p \nmid \prod_{i=1}^{k-2} q_i \prod_{i=1}^{l} a_i \left(\sum_{i=1}^{l} \frac{1}{a_i} \right) \right\}.$$

Clearly, the complement of the set $\mathcal{P}_2$ in $\{p$ is a prime $\mid p = \alpha_{k-2}t - \beta_{k-2}$ and $t \geqslant 2\}$ is finite. Moreover, for $\alpha_{k-2}t_0 - \beta_{k-2} = p_0 \in \mathcal{P}_2$, taking $u \in \mathbb{Z}_{\geqslant 2}$ and deleting the first row and the k-th column of the matrix $M_G(u, q_{k-2}, \ldots, q_1, t_0, a_1, \ldots, a_l)$, then the determinant of the resulting matrix is equal to $(-1)^{k-1} \prod_{i=1}^{k-2} q_i \prod_{i=1}^{l} a_i (\sum_{i=1}^{l} \frac{1}{a_i})$. Thus, for $p_0 \in \mathcal{P}_2$, the $(n-1)$-th determinant divisor of $M_G(u, q_{k-2}, \ldots, q_1, t_0, a_1, \ldots, a_l)$ is equal to 1. Since the r-th leading principal minor of the transpose of matrix $M_{G_v}(q_{k-2}, \ldots, q_1, t_0, a_1, \ldots, a_l)$ is greater than 0 for $1 \leqslant r \leqslant n-1$, the matrix $M_{G_v}(q_{k-2}, \ldots, q_1, t_0, a_1, \ldots, a_l)$ is positive definite. The Lemma 11 indicates the result. $\square$

Lemma 26. *If $G = K_n$ is a complete graph, then $V_G(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$.*

Proof. Let G_v denote a subgraph obtained from G by deleting a vertex v_n. It is clear that G_v is an complete graph K_{n-1}. Set $a_i \geqslant 2$ and choose $b_i = a_i + 1$ to be different primes such that $\sum_{i=1}^{n-2} \frac{1}{b_i} < \frac{2}{3}$. Consider the polynomial

$$\begin{aligned} d_{G_v}(a_1, \ldots, a_{n-2}, t) &= \textstyle\prod_{i=1}^{n-2} b_i (1 - \sum_{i=1}^{n-2} \frac{1}{b_i}) t - \prod_{i=1}^{n-2} b_i (\sum_{i=1}^{n-2} \frac{1}{b_i}) \\ &= \alpha t - \beta. \end{aligned}$$

Since $\{b_1, \ldots, b_{n-2}\}$ are different primes, we get $\gcd(\alpha, \beta) = 1$. Let

$$\mathcal{P}_3 := \left\{ p \text{ is a prime} \mid p = \alpha t - \beta, t \geqslant 2 \text{ and } p \nmid \prod_{i=1}^{n-2} b_i \right\}.$$

Clearly, the complement of $\mathcal{P}_3$ in $\{p$ is a prime $\mid p = \alpha t - \beta$ and $t \geqslant 2\}$ is finite. Deleting the last row and the $(n-1)$-th column of the matrix $M_G(a_1, \ldots, a_{n-2}, t, u)$ where $u \in \mathbb{Z}_{\geqslant 2}$, the determinant of the resulting matrix is $-\prod_{i=1}^{n-2} b_i$. Thus, for any t_0 with $\alpha t_0 - \beta = p_0 \in \mathcal{P}_3$, the $(n-1)$-th determinant divisor of the matrix $M_G(a_1, \ldots, a_{n-2}, t_0, u)$ is equal to 1. Besides, for $1 \leqslant r \leqslant n-1$, the r-th leading principal minor of the matrix $M_{G_v}(a_1, \ldots, a_{n-2}, t_0)$ is equal to $\prod_{i=1}^{r-1} b_i (a_r - (a_r + 1) \sum_{i=1}^{r-1} \frac{1}{b_i}) > 0$. This means $M_{G_v}(a_1, \ldots, a_{n-2}, t_0)$ is positive definite. Thus, Lemma 11 indicates that the set $V_G(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$. $\square$

Lemma 27. *If $G = C(P_3)$ is the cone of P_3, then $V_G(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$.*

Proof. Let v be one of the vertices with degree 3 in $C(P_3)$, and let G_v denote the subgraph obtained from G by deleting v. Clearly, $G_v = P_3 = v_1 v_2 v_3$ where v_3 has degree 1. By directed calculations, we get

$$d_{G_v}(2, 2, t) = 3t - 2.$$

Define

$$\mathcal{P}_4 := \{p \text{ is a prime} \mid p = 3t - 2, t \geqslant 2 \text{ and } p \nmid 6\}.$$

The complement of the set $\mathcal{P}_4$ in $\{p \text{ is a prime } \mid p = 3t-2 \text{ and } t \geqslant 2\}$ is finite. Moreover, for $3t_0 - 2 \in \mathcal{P}_4$, the 3-th determinant divisor of the matrix $M_G(2, 2, t_0, u)$ where $u \in \mathbb{Z}_{\geqslant 2}$ is equal to 1. Moreover, it is easy to check that $M_{G_v}(2, 2, t_0)$ is positive definite. Thus, Lemma 11 indicates that the set $V_G(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$. □

Proof of Theorem 4. The result follows by combining Lemmas 23, 25, 26, and 27. □

5 Concluding Remarks

In his earlier work, Lorenzini showed that $V_G(2)$ contains all positive integers whenever a connected graph G contains an induced subgraph H satisfying $1 \in V_H(2)$. However, this criterion does not apply to several special classes of connected simple graphs. Lorenzini subsequently studied some of these remaining cases and proved that, for graphs in those classes, the set $V_{d_G}(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$.

In this paper, we resolve all of the remaining cases. Our main result (Theorem 5) establishes that $V_G(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$ for every connected simple graph G; in other words, $V_G(2)$ contains almost all non-negative integers. Since $V_G(2) \subseteq V_{d_G}(2)$, it follows that $V_{d_G}(2)$ is also dense in $\mathbb{Z}_{\geqslant 0}$. Thus, our results strengthen and generalize Lorenzini's earlier work to all connected simple graphs.

Our approach highlights a close interaction between graph-theoretic constructions and analytic number theory. The main ingredient is Dirichlet's theorem on primes in arithmetic progressions. Using suitable families of primes, we construct subsets of $V_G(2)$ whose union is dense in $\mathbb{Z}_{\geqslant 0}$. For trees, the proof begins with carefully chosen initial trees and extends them inductively in a controlled manner. For the remaining graph classes, we apply these density arguments together with suitable graph reductions to deduce the desired conclusion.

Although our results show that $V_{d_G}(2)$ is dense in $\mathbb{Z}_{\geqslant 0}$ for every connected simple graph G, it remains an open problem whether the complement of $V_{d_G}(2)$ in $\mathbb{Z}_{\geqslant 0}$ is finite. Our results indicate that this complement is extremely sparse, and provide further evidence that it may indeed be finite.

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