The (Signless) Laplacian Spectral Radius with a Fixed Induced Subgraph of a Graph

Wen-Jun Li[a] Zhiwen Wang[b] Ji-Ming Guo[b]

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Abstract

For a graph G of size m, let $q(G)$ (resp. $\mu(G)$) denote the largest eigenvalue of its signless Laplacian matrix (resp. Laplacian matrix). In this paper, we investigate a complementary version of spectral Turán type problems on (signless) Laplacian matrix, concentrating on maximizing the largest (signless) Laplacian spectral radius of a graph with a fixed subgraph. For an arbitrary fixed graph H, we conjecture that if a graph G of size m contains H as an induced subgraph, then

$$q(G) \leqslant q(H_u^{m-m(H)}),$$

where u is some vertex of H and $H_u^{m-m(H)}$ is the graph obtained by attaching $m-m(H)$ pendant vertices to the vertex u. We confirm the conjecture for graphs of size at least $3(m(H)-\Delta(H))+3$. We also propose a similar conjecture on the Laplacian spectral radius that

$$\mu(G) \leqslant \mu(H_v^{m-m(H)})$$

for a graph G containing H as an induced subgraph. Moreover, we validate these two conjectures when H is a cycle, a path or a clique, which generalize some known results.

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1 Introduction and Inspiration

All graphs considered are simple and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Then its *order* is $|V(G)|$, denoted by $n(G)$, and its *size* is $|E(G)|$, denoted by $m(G)$. Given a vertex $v \in V(G)$, a vertex u is called a *neighbor* of v if

[a]School of Science, Changzhou Institute of Technology, Changzhou 213002, China (leewj1375@163.com).

[b]School of Mathematics, East China University of Science and Technology, Shanghai, P. R. China (walkerwzw@163.com (Z. Wang), Corresponding author: jimingguo@hotmail.com (J.-M. Guo)).

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$uv \in E(G)$. Let $N(v)$ be the set of neighbors of v. Denote by $d_i = |N(v_i)|$ the *degree* of v_i in G and by $\Delta(G)$ the *maximal degree* of G. The vertex v_i is called an *isolated vertex* if $d_i = 0$, and is called a *pendant vertex* if $d_i = 1$. The *girth* g of a graph G is the minimum length of cycles in G. Let $G + e$ be the graph by inserting an edge in G. For a subset S of $V(G)$, we use $G[S]$ to denote the induced subgraph of G induced by S. We say that S is a *clique* if $G[S]$ is a complete graph. The *clique number* of G is the cardinality of a maximal clique in G. The *eccentricity* of a vertex $u \in V$ is the maximum distance from u to other vertices of G. As usual, we denote the *cycle*, *path*, *star* and *complete graph* of order n by C_n, P_n, S_n and K_n, respectively.

Let $A(G)$ denote the *adjacency matrix* of a graph G, and $D(G) = \mathrm{diag}\{d_1, d_2, \ldots, d_n\}$ denote the *degree diagonal matrix* of G. The *Laplacian matrix* and *signless Laplacian matrix* are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively. Let $q(G)$ (resp. $\mu(G)$) denote the largest eigenvalue of $Q(G)$ (resp. $L(G)$), called signless Laplacian (resp. Laplacian) spectral radius.

A graph G is said to be H-free if G does not contain a copy of H as a subgraph. A spectral counterpart to Turán type problem has been studied by scholars. Very recently, many excellent works for various forbidden subgraphs have been obtained, such as triangles [15], cliques [2], friendship graphs [6] and $K_{s,t}$-minor [23], and references therein. The signless Laplacian spectral Turán type problem investigates the signless Laplacian spectral radius of a graph replacing the spectral radius for spectral Turán type problem. In a sense, the signless Laplacian matrix can significantly reveal the structure properties of graphs G since $Q(G)$ consists of the adjacency matrix and the diagonal matrix of degree sequence. So much literature studied the maximal signless Laplacian spectral radius among H-free graphs, including cycles [18] and linear forests [4]. In view of the size m of a graph G, Nosal and Nikiforov (see [17, 19]) showed G contains a triangle if G has spectral radius at least $\sqrt{m}$ unless G is a complete bipartite graph. Since then, many researchers pay attention to other subgraphs with restrictions on spectral conditions. Recently, a nice result obtained by Zhai, Xue and Lou [24] inferred if G does not contain the star S_{m+1} as a subgraph then $q(G) < m + 1$ for a graph G of size m. Wang and Guo [22] extended the result from S_{m+1} to a general star S_{m-k+1} with $k \leqslant O(\sqrt{m})$, and Hu, Lou and Huang [13] further gave a generalization. Motivated by these, we consider the maximal (signless) Laplacian spectral radius under edge-condition restriction with given induced subgraph, which can be viewed as a complementary version of spectral Turán type problems on (signless) Laplacian matrix.

Our topic also relates to a problem of considerable interest about characterizing the extremal graphs with the maximal or minimal (signless) Laplacian spectral radius for a class of graphs. Patra et al. [21] determined that among all the unicyclic graphs, the lollipop graph obtained by appending a pendent vertex of a path on $n-g\,(n > g)$ vertices to a vertex of a cycle on $g \geqslant 3$ vertices uniquely minimizes the Laplacian spectral radius for $n \geqslant 2g - 1$ when g is even and for $n \geqslant 3g - 1$ when g is odd. Zhai, Yu and Shu [25] determined the extremal graph with the maximal Laplacian spectral radius among all bicyclic graphs with a given girth. Zhai et al. [24] characterized the graph with maximal signless Laplacian spectral radius among all graphs with given size and clique number.

Lou et al. [16] determined the maximal (signless) Laplacian spectral radius with fixed size and diameter. Chen et al. [5] obtained the unique extremal graph with maximal $\mu(G)$ among all graphs with girth g (or circumference c) and size m.

One may note a graph of girth g (resp. diamter d or clique number ω) contains C_g (resp. P_{d+1} or K_w) as an induced subgraph. Considering the structures of extremal graphs in the above results, we naturally propose the following two conjectures.

For any graph H, let H_v^t be the graph obtained by attaching t pendant vertices to the vertex v in H.

Conjecture 1. Let G be a graph of size m and have H as an induced subgraph. Then $q(G) \leqslant q(H_v^{m-m(H)})$ for some $v \in V(H)$.

Conjecture 2. Let G be a graph of size m and have H as an induced subgraph. Then $\mu(G) \leqslant \mu(H_v^{m-m(H)})$ for some $v \in V(H)$.

In this paper, we confirm Conjectures 1 and 2 when H is a cycle, a path or a clique. We also show that Conjecture 1 is true when $m \geqslant 3(m(H) - \Delta(H)) + 3$.

2 Preliminaries

We need some necessary upper bounds of (signless) Laplacian spectral radius. Let $\overline{G}$ be the complement of a graph G.

Lemma 3 ([11], Corollary 13.1.4). *Let G be a graph with n vertices. Then $\mu(G) \leqslant n$, with equality if and only if $\overline{G}$ is disconnected.*

Lemma 4 ([1], Theorem 2). *Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$ and vertex degrees $d_i \, (1 \leqslant i \leqslant n)$. Then*

$$\mu(G) \leqslant \max\{d_i + d_j : v_iv_j \in E\}.$$

If G is connected, then equality holds if and only if G is a semiregular bipartite graph.

For each $1 \leqslant i \leqslant n$, we denote by m_i the *average 2-degrees* of v_i, that is, $m_i = \frac{\sum_{v_iv_j \in E(G)} d_j}{d_i}$.

Lemma 5 ([20], Theorem 2.4). *Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, vertex degrees d_i and average 2-degrees $m_i \, (1 \leqslant i \leqslant n)$. Then*

$$\mu(G) \leqslant \max\{d_i + m_i : v_i \in V(G)\}.$$

If G is connected, then equality holds if and only if G is a semiregular bipartite graph.

The following bound is always no more than n.

Lemma 6 ([9], Theorem 2.1). *Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$ and vertex degrees $d_i \, (1 \leqslant i \leqslant n)$. Then*

$$\mu(G) \leqslant \max\{d_i + d_j - |N(v_i) \cap N(v_j)| : v_iv_j \in E\}.$$

For the signless Laplacian spectral radius, the similar results hold.

Lemma 7 ([8], Theorem 4.7)**.** *Let G be a graph with $V(G) = \{v_1, v_2, \dots, v_n\}$ and vertex degrees $d_i (1 \leqslant i \leqslant n)$. Then*

$$q(G) \leqslant \max\{d_i + d_j : v_i v_j \in E(G)\},$$

with equality if and only if G is regular or semiregular bipartite.

Lemma 8 ([10], Lemma 2.3)**.** *Let G be a graph with $V(G) = \{v_1, v_2, \dots, v_n\}$, vertex degrees d_i and average 2-degrees $m_i (1 \leqslant i \leqslant n)$. Then*

$$q(G) \leqslant \max\{d_i + m_i : v_i \in V(G)\},$$

with equality if and only if G is either semiregular bipartite or regular.

The following lemmas give the lower bound of (signless) Laplacian spectral radius in terms of the maximal degree.

Lemma 9 ([3], Proposition 3.9.3)**.** *Let G be a connected graph with $n \geqslant 2$ vertices and maximal degree Δ. Then $\mu(G) \geqslant \Delta + 1$, with equality if and only if $\Delta = n - 1$.*

Lemma 10 ([7], p.23)**.** *Let G be a connected graph with $n \geqslant 2$ vertices and maximal degree Δ. Then $q(G) \geqslant \Delta + 1$, with equality if and only if G is a star.*

The following lemma illustrates the relation between Laplace spectrum and signless Laplace spectrum.

Lemma 11 ([26], Lemma 2.1)**.** *Let G be a graph. Then $\mu(G) \leqslant q(G)$. Moreover, if G is connected, then equality holds if and only if G is a bipartite graph.*

3 The effect on (signless) Laplacian spectral radius by deleting an edge

Let $\lambda_1(M)$ be the largest eigenvalue of a Hermitian matrix M.

Lemma 12 ([12], Theorem 4.3.1)**.** *Let A, B be Hermitian matrices of order n. Then*

$$\lambda_1(A + B) \leqslant \lambda_1(A) + \lambda_1(B),$$

with equality if and only if there is a nonzero vector x such that $Ax = \lambda_1(A)x$, $Bx = \lambda_1(B)x$, and $(A + B)x = \lambda_1(A + B)x$.

Lemma 13. *Let G be a graph of size m. If $m \geqslant 2$, then there exists an edge $e \in E(G)$ such that $\mu(G) \leqslant \mu(G - e) + 1$.*

Proof. First suppose G has $2K_2$ as a subgraph. Let $E(2K_2) = \{v_1v_2, v_3v_4\}$. Then

$$2L(G) = L(G - v_1v_2) + L(G - v_3v_4) + L(2K_2 \cup (n-4)K_1).$$

By Lemma 12, we have

$$2\mu(G) \leqslant \mu(G - v_1v_2) + \mu(G - v_3v_4) + 2.$$

If $\mu(G) > \mu(G - v_iv_{i+1}) + 1$ for $i = 1, 3$, then

$$2\mu(G) > \mu(G - v_1v_2) + \mu(G - v_3v_4) + 2,$$

a contradiction. Therefore, there exists an edge $e \in E(2K_2)$ such that $\mu(G) \leqslant \mu(G-e)+1$.

Next suppose G has no $2K_2$ as a subgraph. Then G is either S_{m+1} or K_3 with some isolated vertices. If G is S_{m+1} with some isolated vertices, then for any edge $e \in G$, by Lemma 9, we have $\mu(G) = \mu(S_{m+1}) = m+1 = \mu(G-e)+1$. If G is K_3 with some isolated vertices, then for any edge $e \in G$, we have $\mu(G) = 3 < 4 = \mu(G-e)+1$. This completes the proof. □

Note that for any edge $e \in K_3$, we have $q(K_3) = 4 = q(K_3 - e) + 1$. Similar as the proof of Lemma 13, by Lemma 11, we have the following result.

Lemma 14. *Let G be a graph size m. If $m \geqslant 2$, then there exists an edge $e \in E(G)$ such that $q(G) \leqslant q(G-e)+1$.*

Theorem 15. *Let H be a subgraph of G of size $t (t \geqslant 2)$. If $H \neq S_{t+1}$, then there exists an edge $e \in E(H)$ such that $\mu(G) \leqslant \mu(G-e)+1$.*

Proof. If H has $2K_2$ as a subgraph, then by the proof of Lemma 13, there exists an edge $e \in E(H)$ such that $\mu(G) \leqslant \mu(G-e)+1$. If H has no $2K_2$ as a subgraph, then $H = K_3$. Let $E(K_3) = \{e_1, e_2, e_3\}$. Then

$$3L(G) = L(G - e_1) + L(G - e_2) + L(G - e_3) + L(K_3 \cup (n-3)K_1).$$

By Lemma 12, we have

$$3\mu(G) \leqslant \sum_{i=1}^{3} \mu(G - e_i) + 3.$$

If $\mu(G) > \mu(G - e_i) + 1$ for $i = 1, 2, 3$, then

$$3\mu(G) > \sum_{i=1}^{3} \mu(G - e_i) + 3,$$

a contradiction. Therefore, there exists an edge $e \in E(H)$ such that $\mu(G) \leqslant \mu(G-e)+1$. This completes the proof. □

Note that $\mu(K_{1,t}) = q(K_{1,t}) = t+1 > e(K_{1,t})$ and $q(K_3) = 4 > e(K_3)$. By Lemma 14 and similar reason of Theorem 15, we have the following result.

Theorem 16. *Let H be a subgraph of G of size $t (t \geqslant 2)$. If H is neither S_{t+1} nor K_3, then there exists an edge $e \in E(H)$ such that $q(G) \leqslant q(G-e)+1$.*

4 Contribution to Conjectures 1 and 2

4.1 Proof of Conjecture 1 when $m \geqslant 3(m(H) - \Delta(H)) + 3$

Lemma 17 ([13])**.** *Let $k \geqslant 0$ be an integer and G be a graph of size $m \geqslant 3k+3$. If $q(G) > m-k+1$, then $\Delta(G) \geqslant m-k$.*

Theorem 18. *Let G be a graph of size m and have H as an induced subgraph. If $m \geqslant 3(m(H) - \Delta(H)) + 3$, then $q(G) \leqslant q(H_v^{m-m(H)})$ for some $v \in V(H)$.*

Proof. Let $G^\star$ be the graph of size m having H as an induced subgraph with maximum signless Laplacian spectral radius. We only need to prove $G^\star \in \{H_v^{m-m(H)} : v \in V(H)\}$. If H is a star, let u be the center of H, then $H_u^{m-m(H)} \cong K_{1,m}$. The result holds for $q(G) \leqslant q(K_{1,m})$. Next suppose H is not a star. Let U be the set of maximal degree vertices of H and $u \in U$. Then by Lemma 10, we have

$$q(G^\star) \geqslant q(H_u^{m-m(H)}) > m - m(H) + \Delta(H) + 1.$$

First we claim that $\Delta(G^\star) \geqslant m - m(H) + \Delta(H)$. Otherwise, by Lemma 17, we have $q(G^\star) \leqslant m - m(H) + \Delta(H) + 1$, a contradiction. Let w be a maximal degree vertex of $G^\star$. Then we claim that $w \in U$ and $d_{V(G^\star)\setminus V(H)}(w) = m - m(H)$. On the contrary, if $w \in V(G^\star) \setminus V(H)$, then $\Delta(G^\star) \leqslant m - m(H)$, a contradiction. If $w \in V(H) \setminus U$, then $\Delta(G^\star) \leqslant m - m(H) + \Delta(H) - 1$, a contradiction. Thus, $w \in U$. Note that $d_{V(G^\star)\setminus V(H)}(w) \leqslant m - m(H)$. If $d_{V(G^\star)\setminus V(H)}(w) \leqslant m - m(H) - 1$, then $\Delta(G^\star) \leqslant m - m(H) + \Delta(H) - 1$, a contradiction. Thus, $d_{V(G^\star)\setminus V(H)}(w) = m - m(H)$. Hence, $G^\star$ is the graph of size m obtained by attaching $m - m(H)$ pendant vertices to a maximal degree vertex of H. This completes the proof. □

4.2 Proof of Conjectures 1 and 2 when H is a cycle C_ℓ

If $H_{v_1}^{m-m(H)} \cong H_{v_2}^{m-m(H)}$ for any two vertices $v_1, v_2 \in V(H)$, then we write $H^{m-m(H)}$ for $H_v^{m-m(H)}$, where $v \in V(H)$.

Lemma 19. *Let $H = C_\ell^{m-\ell} \cup 2K_1$ with $m \geqslant \ell + 2$. Let $\mathcal{G} = \{H + e : H + e$ has C_ℓ as an induced subgraph$\}$. For $G \in \mathcal{G}$, if $G \neq C_\ell^{m-\ell+1} \cup 2K_1$, then $q(G) < m(G) - \ell + 3$ and $\mu(G) < m(G) - \ell + 3$.*

Proof. Let $G \in \mathcal{G}$ and $G \neq C_\ell^{m-\ell+1} \cup 2K_1$. We need to prove that $q(G) < m - \ell + 4$. In H, let $C_\ell = v_1 v_2 \cdots v_\ell v_1$, $V(C_\ell^{m-\ell}) \setminus V(C_\ell) = \{u_1, \ldots, u_{m-\ell}\}$ and v_t is the neighbour of $\{u_1, \ldots, u_{m-\ell}\}$, $2K_1 = \{w_1, w_2\}$. If $G = C_\ell^{m-\ell} \cup K_2$, then by Lemma 7, we have $q(G) < \max\limits_{v_i v_j \in E(G)} \{d_i + d_j\} = m - \ell + 4$. Next assume that $G = (C_\ell^{m-\ell} + e) \cup 2K_1$, where $C_\ell^{m-\ell} + e \neq C_\ell^{m-\ell+1}$. Since G has C_ℓ as an induced subgraph, we have $e = u_i u_j$, $1 \leqslant i < j \leqslant m - \ell$ or $e = v_i u_j$, $1 \leqslant i \leqslant \ell (i \neq t)$, $1 \leqslant j \leqslant m - \ell$. By Lemma 8, we check that

$$\begin{aligned} q(G) &< \max\{m - \ell + 2 + \frac{m-\ell+6}{m-\ell+2}, 3 + \frac{m-\ell+6}{3}, 2 + \frac{m-\ell+5}{2}\} \\ &\leqslant m - \ell + 4. \end{aligned}$$

Finally, suppose that $G = \Gamma \cup K_1$, where Γ is the graph obtained by joining an isolated vertex to some vertex of $C_\ell^{m-\ell}$. Without loss of generality, assume w_1 is adjacent to some vertex of $C_\ell^{m-\ell}$. Then $e = w_1u_i$, $1 \leqslant i \leqslant m-\ell$ or $e = w_1v_i$, $1 \leqslant i \leqslant \ell$, $i \neq t$. In both cases, by Lemma 8, we check that

$$\begin{aligned} q(G) &< \max\{m-\ell+2+\frac{m-\ell+5}{m-\ell+2}, 3+\frac{m-\ell+5}{3}, 2+\frac{m-\ell+4}{2}\} \\ &< m-\ell+4. \end{aligned}$$

Similarly, by Lemmas 4 and 5, the result holds for $\mu(G)$. This completes the proof. $\square$

Let $\mathcal{F}_1$ be the set of graphs obtained by joining two edges between C_ℓ and K_1.

Lemma 20. *For each graph $F \in \mathcal{F}_1$, $q(F) < q(C_\ell^2)$ and $\mu(F) < \mu(C_\ell^2)$.*

Proof. Let $q = q(C_\ell^2)$. By Lemma 10, we have $q > 5$. We may assume that $q(F) > 5$. Then F is the graph obtained by joining K_1 to two adjacent vertices in C_ℓ. Otherwise, by Lemma 7, we have $q(F) < \max\limits_{v_iv_j \in E(G)}\{d_i + d_j\} = 5$, a contradiction. In C_ℓ^2, let $C_\ell = v_1v_2\cdots v_\ell v_1$ and $V(C_\ell^2) \setminus V(C_\ell) = \{v_{\ell+1}, v_{\ell+2}\}$, where $v_{\ell+1}$ and $v_{\ell+2}$ are adjacent to v_1. We construct the new graph F' by removing the edge $v_1v_{\ell+2}$ in C_ℓ and adding the new edge $v_{l+1}v_2$. Then $F' = F \cup K_1$ and $q(F') = q(F) > 5$.

Suppose Let X (resp. Y) be the Perron vector of $Q(F')$ (resp. $Q(C_\ell^2)$) with coordinate x_i (resp. y_i) corresponding to the vertex v_i. By the symmetry of the graph, we have $x_1 = x_2$.

First we claim that $y_3 \leqslant y_2$. Note that $y_1 > y_i$ for $i = 2, 3, \ldots, \ell$. Let $Z = (y_1, y_3, y_2, y_4, y_5, \ldots, y_\ell, y_{\ell+1}, y_{\ell+2})^\top$. Then

$$\begin{aligned} 0 = q - q &\geqslant Z^\top Q(C_\ell^2)Z - Y^\top Q(C_\ell^2)Y \\ &= (y_1+y_3)^2 + (y_2+y_4)^2 - (y_1+y_2)^2 - (y_3+y_4)^2 \\ &= 2(y_3-y_2)(y_1-y_4). \end{aligned}$$

Since $y_1 > y_4$, we have $y_3 \leqslant y_2$.

We have the following equation

$$\begin{aligned} X^\top Y(q - q(F')) &= X^\top(Q(C_\ell^2) - Q(F'))Y \\ &= \sum_{uv \in C_\ell^2}(x_u+x_v)(y_u+y_v) - \sum_{uv \in F'}(x_u+x_v)(y_u+y_v) \\ &= (x_{\ell+2}+x_1)(y_{\ell+2}+y_1) - (x_{\ell+1}+x_2)(y_{\ell+1}+y_2). \end{aligned} \tag{1}$$

By the characteristic equation of $Q(F')$, we have $(q(F')-2)x_{\ell+1} = x_1+x_2$. Since $q(F') > 5$, we have

$$x_{\ell+1} \leqslant \frac{2}{q(F')-2}x_1 < \frac{2}{3}x_1. \tag{2}$$

By the characteristic equation of $Q(C_\ell^2)$, we have $(q-2)y_2 = y_1 + y_3$. Since $q > 5$ and $y_3 \leqslant y_2$, we have

$$y_2 \leqslant \frac{1}{q-3} y_1. \tag{3}$$

Note that $x_{\ell+2} = 0$, $x_1 = x_2$ and $y_{\ell+1} = y_{\ell+2} = \frac{y_1}{q-1}$. Then by (1), (2) and (3), we have

$$\begin{aligned} X^\top Y(q - q(F')) &> \frac{q}{q-1} x_1 y_1 - \frac{5}{3}\left(\frac{1}{q-1} + \frac{1}{q-3}\right) x_1 y_1 \\ &= \frac{(3q-4)(q-5)}{3(q-1)(q-3)} x_1 y_1 \\ &> 0. \end{aligned}$$

Therefore, we have $q(F) = q(F') < q(C_\ell^2)$. As for $\mu(F)$ for $F \in \mathcal{F}_1$, by Lemmas 6 and 9, we have $\mu(F) < 5 < \mu(C_\ell^2)$. This completes the proof. □

For a subgraph H of G, if $G - E(H)$ has isolated vertices, we ignore them.

Theorem 21. *Let G be a connected graph of size m having C_ℓ as an induced subgraph. Then $q(G) \leqslant q(C_\ell^{m-\ell})$, with equality if and only if $G = C_\ell^{m-\ell}$.*

Proof. By Lemma 10, we have $q(C_\ell^{m-\ell}) > m - \ell + 3$. Consider the graph $G - E(C_\ell)$. Let $t = |V(G - E(C_\ell)) \cap V(C_\ell)|$. We distinguish the following three cases:

Case 1: $G - E(C_\ell) = S_{m-\ell+1}$.

Let w be the center of $S_{m-\ell+1}$. If $w \in V(C_\ell)$, then $G = C_\ell^{m-\ell}$. Next assume $w \notin V(C_\ell)$. Then $1 \leqslant t \leqslant m - \ell$. When $m - \ell = 2$, if $t = 1$, then G is a graph obtained by identifying an end vertex of P_3 and a vertex of C_ℓ. Then by Lemma 7, we have $q(G) < \max\limits_{v_iv_j \in E(G)} \{d_i + d_j\} = 5 = m - \ell + 3$. If $t = 2$, then $G \in \mathcal{F}_1$. By Lemma 20, we have $q(G) < q(C_\ell^2)$. When $m - \ell \geqslant 3$, by Lemma 8, we have

$$\begin{aligned} q(G) &< \max\{m - \ell + \frac{m - \ell + 2t}{m - \ell}, 3 + \frac{m - \ell + 6}{3}\} \\ &\leqslant m - \ell + 3. \end{aligned}$$

Case 2: $G - E(C_\ell) = K_3$.

Then $t = 1$. By Lemma 7, we have $q(G) < \max\limits_{v_iv_j \in E(G)} \{d_i + d_j\} = 6 = m - \ell + 3$.

Case 3: $G - E(C_\ell) \notin \{S_{m-\ell+1}, K_3\}$.

Note that the situations $m = \ell$ and $m = \ell + 1$ are trivial. Next assume that $m \geqslant \ell + 2$. Let $G_0 = G$ and $m(G_i) = m_i$. By Theorem 16, we give the following procedure. Let the procedure stop at step k, we get some $G_k (0 \leqslant k \leqslant m - \ell - 2)$. Then $m_k = m - k$.

Subcase 3.1: $k = m - \ell - 2$.

Then $m_k = \ell + 2$. First assume that G_k has at least two nontrival connected components. Then $G_k = C_\ell^1 \cup K_2$, $C_\ell \cup 2K_2$ or $C_\ell \cup S_3$. By Lemma 7, we check that

$$q(G_k) < \max_{v_iv_j \in E(G)} \{d_i + d_j\} \leqslant 5 = m_k - \ell + 3.$$

Algorithm 1 Algorithm of cycle

$i \leftarrow 0$
if $i = m - \ell - 2$ **then**
  print G_i
else if $G_i - E(C_\ell) \in \{S_{m_i-\ell+1}, K_3\}$ **then**
  print G_i
else
  find $e_i \notin E(C_\ell)$ such that $q(G_i) \leqslant q(G_i - e_i) + 1$
  $i \leftarrow i + 1$
  $G_i \leftarrow G_{i-1} - e_{i-1}$
end if

If $k = 0$, the result holds. Next assume $k \geqslant 1$. Using the procedure, for any $i = 0, 1, \ldots, k-1$, $G_i - E(C_\ell) \notin \{S_{m_i-\ell+1}, K_3\}$. Then by Theorem 16, we have

$$\begin{aligned} q(G) &\leqslant q(G_1) + 1 \\ &\leqslant q(G_2) + 2 \\ &\vdots \\ &\leqslant q(G_k) + k \\ &< m - \ell + 3. \end{aligned}$$

Next assume that G_k has exactly one connected component. Then $G_k = C_\ell^{m_k-\ell}$, $G_k \in \mathcal{F}_1$, G_k is a graph obtained by adding two pendant edges to two distinct vertices of C_ℓ or G_k is a graph obtained by identifying a pendant vertex of P_3 and a vertex of C_ℓ. First assume that $G_k = C_\ell^{m_k-\ell}$. We can assume that $k \geqslant 1$. Using the procedure, we have $G_{k-1} \neq C_\ell^{m_{k-1}-\ell}$, then by Lemma 19, we have

$$q(G_{k-1}) < m_{k-1} - \ell + 3.$$

Using the procedure, for any $i = 0, 1, \ldots, k-2$, $G_i - E(C_\ell) \notin \{S_{m_i-\ell+1}, K_3\}$. Then by Theorem 16, we have

$$q(G) \leqslant q(G_{k-1}) + k - 1 < m - \ell + 3.$$

Next assume that $G_k \neq C_\ell^{m_k-\ell}$. Let F_1 be the graph obtained by merging an edge from $C_\ell (\ell \geqslant 3)$ and K_3. If $G_k \neq F_1$, then by Lemma 8, we check that

$$q(G_k) < \max_{v_i \in V(G)} \{d_i + m_i\} = 5 = m_k - \ell + 3.$$

Similar as the above proof, by Theorem 16, we have

$$q(G) \leqslant q(G_k) + k < m - \ell + 3.$$

If $G_k = F_1$, by Lemma 20, we may assume that $k \geqslant 1$. Then we consider G_{k-1}. Using the procedure, we have $G_{k-1} - E(C_\ell) \notin \{S_4, K_3\}$. Then we have $G_{k-1} = F_1 \cup K_2$ or G_{k-1} is a graph obtained by adding a pendant edge to the vertex of degree 2 on C_ℓ in F_1. By Lemma 7, we check that

$$q(G_{k-1}) < \max_{v_i v_j \in E(G)} \{d_i + d_j\} = 6 = m_{k-1} - \ell + 3.$$

Similar as the above proof, by Theorem 16, we have

$$q(G) \leqslant q(G_{k-1}) + k - 1 < m - \ell + 3.$$

Subcase 3.2: $G_k - E(C_\ell) \in \{S_{m_k - \ell + 1}, K_3\}$.

By the proof of Subcase 3.1, we can assume $G_k \neq C_\ell^{m_k - \ell}$ and $0 \leqslant k \leqslant m - \ell - 3$. Then by the proof of Cases 1 and 2, we have $q(G_k) < m_k - \ell + 3$. Using the procedure, for any $i = 0, 1, \ldots, k-1$, $G_i - E(C_\ell) \notin \{S_{m_i - \ell + 1}, K_3\}$. Then by Theorem 16, we have

$$q(G) \leqslant q(G_k) + k < m - \ell + 3.$$

This completes the proof. □

Note that for any graph G with girth g, G has C_g as an induced subgraph. By Theorem 21, we have the following corollary, which has been proved by Chen et al. [5].

Corollary 22. *Let G be a graph of size m and girth g. Then $q(G) \leqslant q(C_g^{m-g})$, with equality if and only if $G = C_g^{m-g}$.*

Similar as the proof of Theorem 21, by Theorem 15 and Lemmas 4, 5, 9, 19, 20, we have the following result for $\mu(G)$.

Theorem 23. *Let G be a connected graph of size m having C_ℓ as an induced subgraph. Then $\mu(G) \leqslant \mu(C_\ell^{m-\ell})$, with equality if and only if $G = C_\ell^{m-\ell}$.*

Similarly, by Theorem 23, we have the following result.

Corollary 24. *Let G be a graph of size m and girth g. Then $\mu(G) \leqslant \mu(C_g^{m-g})$, with equality if and only if $G = C_g^{m-g}$.*

4.3 Proof of Conjectures 1 and 2 when H is a path P_{d+1}

Lemma 25. *Let $H = (P_{d+1})_v^{m-d} \cup 2K_1$ with $m \geqslant d+2$ for some $v \in V(P_{d+1})$. Let $\mathcal{G} = \{H + e : H + e \text{ has } P_{d+1} \text{ as an induced subgraph}\}$. For $G \in \mathcal{G}$, if $G \neq (P_{d+1})_v^{m-d+1} \cup 2K_1$, then $q(G) < m(G) - d + 3$ and $\mu(G) < m(G) - d + 3$.*

Proof. Let $G \in \mathcal{G}$ and $G \neq (P_{d+1})_v^{m-d+1} \cup 2K_1$. We need to prove that $q(G) < m - d + 4$. In H, let $P_{d+1} = v_1 v_2 \cdots v_{d+1}$, $V((P_{d+1})_v^{m-d+1}) \setminus V(P_{d+1}) = \{u_1, \ldots, u_{m-d}\}$ and v_t is the neighbour of $\{u_1, \ldots, u_{m-d}\}$, $2K_1 = \{w_1, w_2\}$. If $G = (P_{d+1})_v^{m-d} \cup K_2$, by Lemma 7, we have $q(G) < \max_{v_i v_j \in E(G)} \{d_i + d_j\} \leqslant m - d + 4$. Next suppose that $G = ((P_{d+1})_v^{m-d} + e) \cup 2K_1$, where $(P_{d+1})_v^{m-d} + e \neq (P_{d+1})_v^{m-d+1}$. Since G has P_{d+1} as an induced subgraph, we have $e = u_iu_j$, $1 \leqslant i < j \leqslant m-d$ or $e = v_iu_j$, $1 \leqslant i \leqslant d+1 (i \neq t)$, $1 \leqslant j \leqslant m-d$. Since $m-d \geqslant 2$, by Lemma 8, we check that

$$\begin{aligned} q(G) &< \max\{m-d+2+\frac{m-d+6}{m-d+2}, 3+\frac{m-d+6}{3}, 2+\frac{m-d+5}{2}\} \\ &\leqslant m-d+4. \end{aligned}$$

Finally, suppose that $G = \Gamma \cup K_1$, where Γ is the graph obtained by joining an isolated vertex to some vertex of $(P_{d+1})_v^{m-d}$. Without loss of generality, assume w_1 is adjacent to some vertex of $(P_{d+1})_v^{m-d}$ in G. Then $e = w_1u_i$, $1 \leqslant i \leqslant m-d$ or $e = w_1v_i$, $1 \leqslant i \leqslant d+1$, $i \neq t$. In both cases, by Lemma 8, we check that

$$\begin{aligned} q(G) &< \max\{m-d+2+\frac{m-d+5}{m-d+2}, 3+\frac{m-d+5}{3}, 2+\frac{m-d+4}{2}\} \\ &< m-d+4. \end{aligned}$$

Similarly, by Lemmas 4 and 5, the result holds for $\mu(G)$. This completes the proof. □

Let $v^\star$ be the vertex with minimum eccentricity of the path P_{d+1}. For simplicity, we denote $(P_{d+1})_{v^\star}^{m-d}$ by $P^\star_{m,d}$. Let $diam(G)$ denote the diameter of graph G.

Lemma 26 ([14], Theorem 4)**.** *Let T be a tree with diameter D of size m. Then $q(T) \leqslant q(P^\star_{m,D})$, with equality if and only if $T = P^\star_{m,D}$.*

Let $\mathcal{F}_2$ be the set of graphs obtained by joining two edges between P_{d+1} and K_1.

Lemma 27. *For each graph $F \in \mathcal{F}_2$, $q(F) < q(P^\star_{d+2,d})$ and $\mu(F) < \mu(P^\star_{d+2,d})$.*

Proof. First we prove that $q(F) < q(P^\star_{d+2,d})$. By Lemma 10, $q(P^\star_{d+2,d}) > 5$. Let F_2 be the graph obtained by merging an edge from $P_{d+1} (d \geqslant 2)$ and K_3. For any graph $F \in \mathcal{F}_2$, if $F \neq F_2$, then by Lemma 7, we have $q(F) < \max_{v_iv_j \in E(G)} \{d_i + d_j\} \leqslant 5$. If $F = F_2$, then $diam(F) = diam(P^\star_{d+2,d})$. By Theorem 1.1 in [16], we have $q(F) < q(P^\star_{d+2,d})$. For $\mu(F)$, by Lemma 11, we have $\mu(F) \leqslant q(F) < q(P^\star_{d+2,d}) = \mu(P^\star_{d+2,d})$. This completes the proof. □

Theorem 28. *Let G be a connected graph of size m having P_{d+1} as an induced subgraph. Then $q(G) \leqslant q(P^\star_{m,d})$, with equality if and only if $G = P^\star_{m,d}$.*

Proof. Let $P_{d+1} = v_1v_2 \cdots v_dv_{d+1}$. By Lemma 10, we have $q(P^\star_{m,d}) > m-d+3$. We claim that $q((P_{d+1})_v^{m-d}) \leqslant q(P^\star_{m,d})$ for any vertex $v \in V(P_{d+1})$, with equality if and only if $(P_{d+1})_v^{m-d} = P^\star_{m,d}$. By Lemma 7, we have $q((P_{d+1})_{v_1}^{m-d}) = q((P_{d+1})_{v_{d+1}}^{m-d}) < \max_{v_iv_j \in E(G)} \{d_i + d_j\} = m-d+3$. For $v \in V(P_{d+1}) \setminus \{v_1, v_{d+1}\}$, since $diam((P_{d+1})_v^{m-d}) = diam(P^\star_{m,d})$, by Lemma 26, the claim holds. Consider the graph $G - E(P_{d+1})$. Let $t = |V(G - E(P_{d+1})) \cap V(P_{d+1})|$. We distinguish the following three cases:
Case 1: $G - E(P_{d+1}) = S_{m-d+1}$.

Let w be the center of S_{m-d+1}. If $w \in V(P_{d+1})$, then $G = (P_{d+1})_v^{m-d}$ for some vertex $v \in V(P_{d+1})$. By the above proof, the result holds. Next assume $w \notin V(P_{d+1})$. Then $1 \leqslant t \leqslant m-d$. When $m-d=2$, if $t=1$, then G is a graph obtained by identifying an end vertex of P_3 and a vertex of P_{d+1}. Then by Lemma 7, we check that $q(G) < 5$. If $t=2$, then $G \in \mathcal{F}_2$. By Lemma 27, we have $q(G) < q(P^{\star}_{d+2,d})$. When $m-d \geqslant 3$, by Lemma 8, we check that

$$\begin{aligned} q(G) &< \max\{m-d+\frac{m-d+2t}{m-d}, 3+\frac{m-d+6}{3}\} \\ &\leqslant m-d+3. \end{aligned}$$

Case 2: $G - E(P_{d+1}) = K_3$.

Then $t=1$. By Lemma 7, we check that $q(G) < \max\limits_{v_iv_j \in E(G)}\{d_i+d_j\} \leqslant 6 = m-d+3$.

Case 3: $G - E(P_{d+1}) \notin \{S_{m-d+1}, K_3\}$.

Note that the situations $m=d$ is trivial and by the above proof the situations $m=d+1$ is done. Next assume that $m \geqslant d+2$. Let $G_0 = G$ and $m(G_i) = m_i$. By Theorem 16, we give the following procedure. Let the procedure stop at step k, we get some $G_k (0 \leqslant k \leqslant m-d-2)$. Then $m_k = m-k$.

Algorithm 2 Algorithm of path

$i \leftarrow 0$
if $i = m-d-2$ **then**
    print G_i
else if $G_i - E(P_{d+1}) \in \{S_{m_i-d+1}, K_3\}$ **then**
    print G_i
else
    find $e_i \in E(G_i) \setminus E(P_{d+1})$ such that $q(G_i) \leqslant q(G_i - e_i) + 1$
    $i \leftarrow i+1$
    $G_i \leftarrow G_{i-1} - e_{i-1}$
end if

Subcase 3.1: $k = m-d-2$.

Then $m_k = d+2$. First assume that G_k has at least two nontrival connected components. Then $G_k = (P_{d+1})_v^1 \cup K_2$, $P_{d+1} \cup 2K_2$ or $P_{d+1} \cup S_3$. By Lemma 7,

$$q(G_k) < \max_{v_iv_j \in E(G)}\{d_i+d_j\} \leqslant 5 = m_k - d + 3.$$

If $k=0$, the result holds. Next assume $k \geqslant 1$. Using the procedure, for any $i = 1, 2, \ldots, k-1$, $G_i - E(P_{d+1}) \notin \{S_{m_i-d+1}, K_3\}$. Then by Theorem 16, we have

$$q(G) \leqslant q(G_k) + k < m-d+3.$$

Next assume that G_k has exactly one connected component. Then $G_k = (P_{d+1})_v^2$, $G_k \in \mathcal{F}_2$, G is a graph obtained by adding two pendant edges to two distinct vertices of P_{d+1}, or

G is a graph obtained by identify a pendant vertex of P_3 and a vertex of P_{d+1}. First assume that $G_k = (P_{d+1})_v^2$. We can assume that $k \geqslant 1$. Using the procedure, we have $G_{k-1} \neq (P_{d+1})_v^{m_{k-1}-d}$, then by Lemma 25, we have

$$q(G_{k-1}) < m_{k-1} - d + 3.$$

Using the procedure, for any $i = 0, 1, \ldots, k-2$, $G_i - E(P_{d+1}) \notin \{S_{m_i-d+1}, K_3\}$. Then by Theorem 16, we have

$$q(G) \leqslant q(G_{k-1}) + k - 1 < m - d + 3.$$

Next assume $G_k \neq (P_{d+1})_v^2$. Let F_2 be the graph obtained by merging an edge from $P_{d+1}(d \geqslant 2)$ and K_3. If $G_k \neq F_2$, then by Lemma 8, we have

$$q(G_k) < \max_{v_i \in V(G)} \{d_i + m_i\} \leqslant 5 = m_k - d + 3.$$

Similar as the above proof, by Theorem 16, we have

$$q(G) \leqslant q(G_k) + k < m - d + 3.$$

If $G_k = F_2$, then consider G_{k-1}. Using the procedure, we have $G_{k-1} - E(P_{d+1}) \notin \{S_4, K_3\}$. Then $G_{k-1} = F_2 \cup K_2$ or G_{k-1} is a graph obtained by adding a pendant edge to the vertex on P_{d+1} whose degree is not 3 in F_2. By Lemma 8, we check that

$$q(G_{k-1}) < \max_{v_i v_j \in E(G)} \{d_i + d_j\} = 6 = m_{k-1} - d + 3.$$

Similar as the above proof, by Theorem 16, we have

$$q(G) \leqslant q(G_{k-1}) + k - 1 < m - d + 3.$$

Subcase 3.2: $G_k - E(P_{d+1}) \in \{S_{m_k-d+1}, K_3\}$.

We can assume $G_k \neq (P_{d+1})_v^{m_k-d}$ for some $v \in V(P_{d+1})$ and $0 \leqslant k \leqslant m - d - 3$. Then by the proof of Cases 1 and 2, we have $q(G_k) < m_k - d + 3$. Using the procedure, for any $i = 0, 1, \ldots, k-1$, $G_i - E(P_{d+1}) \notin \{S_{m_i-d+1}, K_3\}$. Similar as the above proof, by Theorem 16, we have

$$q(G) \leqslant q(G_k) + k < m - d + 3.$$

This completes the proof. $\square$

Similar as the proof of Theorem 28, by Theorem 15 and Lemmas 4, 5, 9, 25, 26, 27, we have the following result for $\mu(G)$.

Theorem 29. *Let G be a connected graph of size m having P_{d+1} as an induced subgraph. Then $\mu(G) \leqslant \mu(P^{\star}_{m,d})$, with equality if and only if $G = P^{\star}_{m,d}$.*

Note that for any graph G with diameter D, G has P_{D+1} as an induced graph. Then by Theorems 28 and 29, we have the following Corollary which has proved by Lou et al. [16].

Corollary 30. *Let G be a graph of size m and diameter D. Then $q(G) \leqslant q(P^{\star}_{m,D})$ and $\mu(G) \leqslant \mu(P^{\star}_{m,D})$, with equality if and only if $G = P^{\star}_{m,D}$.*

4.4 Proof of Conjectures 1 and 2 when H is a clique

Zhai et al. [24] proved that Conjecture 1 is true when H is a maximal clique of G.

Lemma 31 ([24], Theorem 1.2). *Let G be a graph of clique number $\omega \geqslant 2$ and size $m \geqslant \binom{\omega}{2}$. If G is a graph without isolated vertices, then $q(G) \leqslant q(K_{\omega}^{m-\binom{\omega}{2}})$, with equality if and only if $G \cong K_{\omega}^{m-\binom{\omega}{2}}$.*

Lemma 32 ([24], Lemma 2.6). *If $s \geqslant 3$, $k \geqslant 0$ and $k+s \geqslant 4$, then $q(K_s^k) < q(K_{s-1}^{k+s-1})$.*

Theorem 33. *Let G be a graph having $K_{\omega}(\omega \geqslant 2)$ as an induced subgraph of size $m \geqslant \binom{\omega}{2}$. Then*

$$q(G) \leqslant q(K_{\omega}^{m-\binom{\omega}{2}}),$$

with equality if and only if $G \cong K_{\omega}^{m-\binom{\omega}{2}}$, with the only exception that $m(G) = 3$ and $\omega = 2$, in which case $q(G) \leqslant q(S_4) = q(K_3)$.

Proof. Let $\omega^{\star}$ be the clique number of G. Then $\omega \leqslant \omega^{\star}$. If $\omega = \omega^{\star}$, by Lemma 31, the result holds. Next suppose $\omega < \omega^{\star}$. Then $\omega^{\star} \geqslant 3$. Let $f(x) = x + m - \binom{x}{2}$. Then $f(x)$ is decreasing when $x \in [\frac{3}{2}, +\infty)$. It follows that for $2 \leqslant \omega \leqslant \omega^{\star}$, $\omega + m - \binom{\omega}{2} \geqslant \omega^{\star} + m - \binom{\omega^{\star}}{2}$. First assume that $\omega^{\star} + m - \binom{\omega^{\star}}{2} \geqslant 4$. By Lemmas 31 and 32, we have

$$q(G) \leqslant q(K_{\omega^{\star}}^{m-\binom{\omega^{\star}}{2}}) < q(K_{\omega^{\star}-1}^{m-\binom{\omega^{\star}-1}{2}}) < \cdots < q(K_{\omega+1}^{m-\binom{\omega+1}{2}}) < q(K_{\omega}^{m-\binom{\omega}{2}}).$$

Next assume that $\omega^{\star} + m - \binom{\omega^{\star}}{2} \leqslant 3$. Note that $m \geqslant \binom{\omega^{\star}}{2}$, then $\omega^{\star} \leqslant 3$, which implies that $\omega^{\star} = 3$. Then we have $(\omega^{\star}, m) = (3, 3)$, that is $G = K_3$. Then $q(K_3) = 4 = q(S_4)$. This completes the proof. $\square$

We now prove that Conjecture 2 is true when H is a clique of G.

Theorem 34. *Let G be a connected graph having K_{ω} as an induced subgraph of size $m \geqslant \binom{\omega}{2}$. Then $\mu(G) \leqslant \mu(K_{\omega}^{m-\binom{\omega}{2}})$, with equality if and only if $G \cong K_{\omega}^{m-\binom{\omega}{2}}$.*

Proof. Let $s = m - \binom{\omega}{2}$. By Lemma 3, we have

$$\mu(K_{\omega}^{m-\binom{\omega}{2}}) = \Delta(K_{\omega}^{m-\binom{\omega}{2}}) + 1 = n(K_{\omega}^{m-\binom{\omega}{2}}) = \omega + s.$$

Suppose $G \neq K_{\omega}^{m-\binom{\omega}{2}}$. Let G_1 and K_{ω} be the edge disjoint subgraphs of G and $E(K_{\omega}) \cup E(G_1) = E(G)$. Suppose G_1 has t connected components $F_1, F_2, \ldots, F_t$. Clearly, $n(F_i) \leqslant m(F_i) + 1$. If there exists a connected component, say F_1, such that $n(F_1) \leqslant m(F_1)$, then

$$n(G_1) \leqslant \sum_{i=1}^{t} m(F_i) + t - 1 = s + t - 1.$$

Therefore,

$$\begin{aligned} n(G) &= |V(K_w)| + |V(G_1)| - |V(K_w) \cap V(G_1)| \\ &\leqslant \omega + n(G_1) - t \leqslant \omega + s - 1. \end{aligned}$$

By Lemma 3, we have

$$\mu(G) \leqslant n(G) \leqslant \omega + s - 1 < \mu(K_\omega^{m-\binom{\omega}{2}}).$$

Next assume that $n(F_i) = m(F_i) + 1$ for $i = 1, 2, \ldots, t$. If $|V(K_w) \cap V(G_1)| \geqslant t + 1$, then

$$\begin{aligned} n(G) &= |V(K_w)| + |V(G_1)| - |V(K_w) \cap V(G_1)| \\ &\leqslant \omega + \sum_{i=1}^{t} m(F_i) + t - t - 1 \\ &\leqslant \omega + s - 1. \end{aligned}$$

By Lemma 3, we have

$$\mu(G) \leqslant n(G) \leqslant \omega + s - 1 < \mu(K_\omega^{m-\binom{\omega}{2}}).$$

If $|V(K_w) \cap V(G_1)| = t$, then

$$\begin{aligned} n(G) &= n(G_1) + \omega - t \\ &= \sum_{i=1}^{t} m(F_i) + t + \omega - t \\ &= \omega + s. \end{aligned}$$

Since $G \neq K_\omega^{m-\binom{\omega}{2}}$, we have $\overline{G}$ is connected. Then by Lemma 3, we have

$$\mu(G) < n(G) = \omega + s = \mu(K_\omega^{m-\binom{\omega}{2}}).$$

This completes the proof. □

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