Bounds on the Spectral Radii of Berge C_5-Free Linear r-Graphs

Bo Dong[a,b] Cunxiang Duan[c] Ligong Wang[a,b]

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Abstract

An r-uniform hypergraph (or r-graph) is called linear if any two edges intersect in at most one vertex. For a graph $F = \big(V(F), E(F)\big)$ and a hypergraph $\mathcal{B} = \big(V(\mathcal{B}), E(\mathcal{B})\big)$, $\mathcal{B}$ is called a Berge F if there exists a bijection $\phi : E(F) \to E(\mathcal{B})$ such that $e \subseteq \phi(e)$ for every $e \in E(F)$. A hypergraph H is Berge F-free if it contains no Berge F as a subhypergraph. Hou et al. [Electron. J. Combin. 28 (2021)] derived a upper bound for the spectral radius of Berge C_4-free linear r-graphs. In this paper, we establish upper bounds for the spectral radius of Berge C_5-free linear r-graphs for $r = 3$ and $r \geqslant 4$. Moreover, for $r > 4$, we propose a candidate extremal structure for the hypergraph with maximum spectral radius among all Berge C_5-free linear r-graphs.

Mathematics Subject Classifications: 05C35, 05C65

1 Introduction

A hypergraph $H = \big(V(H), E(H)\big)$ consists of a vertex set $V(H)$ and an edge set $E(H)$, where $E(H)$ is a family of nonempty subsets of $V(H)$. The hypergraph F is a subhypergraph of H, if $E(F) \subset E(H)$ and $V(F) \subset V(H)$. If every edge of H consists of r vertices, then H is called an r-uniform hypergraph, or simply an r-graph. In particular, a 2-graph is exactly an ordinary graph. A hypergraph is called linear if any two distinct edges intersect in at most one vertex.

Let $\mathcal{F}$ be a family of hypergraphs. A hypergraph H is called $\mathcal{F}$-free if it contains no member of $\mathcal{F}$ as a subhypergraph. If $\mathcal{F} = \{F\}$, we simply say that H is F-free. The Turán-type extremal problem is one of the central topics in extremal graph theory. It

[a] School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an, Shaanxi 710129, P.R. China (`dongbo98@163.com`; Corresponding author: `lgwangmath@163.com`).
[b] Xi'an-Budapest Joint Research Center for Combinatorics, Northwestern Polytechnical University, Xi'an, Shaanxi 710129, P.R. China.
[c] School of Mathematics, Taiyuan University of Technology, Taiyuan, Shanxi 030024, P.R. China (`cxduanmath@163.com`).

asks for the maximum number of edges in an n-vertex hypergraph avoiding a prescribed forbidden subhypergraph. More precisely, for a family $\mathcal{F}$ of r-graphs, the Turán number $ex_r(n, \mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-free r-graph on n vertices. Similarly, the linear Turán number $ex_r^{lin}(n, \mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-free linear r-graph on n vertices. For a positive integer m, let $[m] = \{1, 2, \ldots, m\}$.

A Berge cycle of length k is an alternating sequence of distinct vertices and distinct hyperedges of the form $v_1e_1v_2e_2 \cdots v_ke_kv_{k+1}$, where $v_i, v_{i+1} \in e_i$ for each $i \in [k]$ and $v_{k+1} = v_1$. A linear Berge cycle $v_1e_1v_2e_2 \cdots v_ke_kv_{k+1}$ is called a C_k^+ if $|e_i \cap e_{i+1}| = 1$ for every $i \in [k-1]$, $|e_1 \cap e_k| = 1$, and $e_i \cap e_j = \emptyset$ for all other pairs $\{i, j\}$ with $i \neq j$. As defined by Gerbner and Palmer [17], let $F = \big(V(F), E(F)\big)$ be a graph and let $\mathcal{B} = \big(V(\mathcal{B}), E(\mathcal{B})\big)$ be a hypergraph. Then $\mathcal{B}$ is called a Berge F if there exists a bijection $\phi : E(F) \to E(\mathcal{B})$ such that $e \subseteq \phi(e)$ for every $e \in E(F)$. For a given graph F, the corresponding Berge F is generally not unique. We write Berge-F for the family of all Berge F hypergraphs. More generally, for a family of graphs $\mathcal{F}$, a hypergraph H is called Berge $\mathcal{F}$-free if, for every $F \in \mathcal{F}$, it contains no member of Berge-F as a subhypergraph.

The Turán problem dates back at least to Mantel's theorem [22], and was later extended by Turán's theorem [32]. Since then, it has become one of the most influential topics in extremal combinatorics. For surveys on classical Turán-type problems, we refer to [11, 14, 19, 23]. For graphs, the problem has seen much progress in recent years, especially for nonbipartite forbidden subgraphs. For hypergraphs, however, much less is known. The linear Turán problem for hypergraphs is particularly challenging and has attracted considerable attention in recent years. For example, Gao et al. obtained in 2023 an asymptotic formula for the linear Turán number of C_5^+ in 3-uniform linear hypergraphs:

$$ex_3^{lin}(n, C_5^+) = \frac{1}{3\sqrt{3}} n^{3/2} + O(n).$$

Further recent results on linear Turán numbers can be found in [4, 9, 12, 16, 31].

Let $A(G)$ be the adjacency matrix of a graph G. Its largest eigenvalue is called the spectral radius of G and is denoted by $\rho(G)$. The spectral version of the Turán problem, also known as the Brualdi–Solheid–Turán problem, was posed by Nikiforov [28]: for a family of graphs $\mathcal{F}$, determine the maximum possible spectral radius of an $\mathcal{F}$-free graph on n vertices. We denote this maximum by $spex_2(n, \mathcal{F})$. For classical results in this topic, see, for example, the cases $F \cong K_r$ [34], $F \cong K_{s,t}$ [2], and $\mathcal{F} = \{C_\ell : \ell \text{ is odd}\}$ [25]. For further developments, we refer to the surveys [8, 20, 27].

The spectral analogue of the Turán problem has recently been extended to hypergraphs by Hou et al. [18] and Gao et al. [15]. For an r-graph H, let $\mathcal{A}(H)$ denote its adjacency tensor, introduced by Cooper and Dutle [3]. The notions of eigenvalues and eigenvectors of tensors were introduced independently by Qi [29] and Lim [21]. The spectral radius of H, denoted by $\rho(H)$, is defined to be the spectral radius of $\mathcal{A}(H)$. We write $spex_r(n, \mathcal{F})$ for the maximum spectral radius of an $\mathcal{F}$-free r-graph on n vertices, and $spex_r^{lin}(n, \mathcal{F})$ for the corresponding maximum spectral radius among $\mathcal{F}$-free linear r-graphs on n vertices.

For ordinary graphs, Nikiforov initiated the systematic study of spectral radius bounds for cycle-free graphs. In the case of even cycles, Nikiforov [26, 28] established asymptotic upper bounds for C_4-free and C_{2k+2}-free graphs ($k \geqslant 2$), namely

$$spex_2(n, C_4) = \frac{1}{2} + \sqrt{n - \frac{3}{4}} + O\left(\frac{1}{n}\right) \quad \text{and} \quad spex_2(n, C_{2k+2}) \leqslant \frac{k}{2} + \sqrt{kn} + o(n).$$

In contrast, the odd-cycle case behaves quite differently. Nikiforov [25] proved that for any odd integer ℓ, if n is sufficiently large, then the maximum spectral radius of a C_ℓ-free graph is attained by a complete bipartite graph, and hence

$$spex_2(n, C_\ell) = \sqrt{\lfloor n^2/4 \rfloor}.$$

These results have recently been extended to the setting of Berge cycles in linear r-graphs. For Berge C_4, Hou et al. [18] proved that the spectral radius of a Berge C_4-free linear r-graph satisfies

$$spex_r^{lin}(n, \text{Berge-}C_4) \leqslant \sqrt{\frac{3}{2} + \frac{1}{2(r-1)}}\,(n-1)^{1/2} + O\left(\frac{1}{\sqrt{n}}\right),$$

which may be viewed as a spectral extension of the even-cycle problem to hypergraphs. Under stronger topological restrictions, Hou et al. [18] also considered linear r-graphs of girth at least 5, equivalently Berge $\{C_3, C_4\}$-free linear r-graphs, and obtained

$$spex_r^{lin}(n, \text{Berge-}\{C_3, C_4\}) \leqslant \sqrt{\frac{n-1}{r-1}} + O\left(\frac{1}{n}\right).$$

Beyond Berge cycles, there has also been substantial progress on other forbidden subhypergraphs in hypergraphs. Recent work includes spectral results for Berge C_3 and Berge $K_{2,t}$ [30], Berge paths P_k [10, 33], the expanded clique K_{r+1}^+ [15], and cancellative hypergraphs [24].

Despite these developments, spectral radius bounds for longer Berge cycles remain largely open. In particular, Hou et al. [18] explicitly asked for bounds for Berge cycles of length $k > 4$. Motivated by this question, we study the spectral radius of Berge C_5-free linear r-graphs. Our main result is to establish upper bounds for the spectral radius in this setting for $r = 3$ and $r \geqslant 4$. In addition, for $r > 4$, we propose a candidate extremal structure that may maximize the spectral radius among all Berge C_5-free linear r-graphs.

The remainder of the paper is organized as follows. In Section 2, we collect several necessary lemmas together with the notation needed later. In Section 3, we prove our main results on spectral radius bounds for Berge C_5-free linear r-graphs.

2 Preliminaries

In this section, we recall several basic definitions and auxiliary results that will be used throughout the paper.

Let P_k denote the path with k vertices. A path (resp. Berge path) with endpoints v_1 and v_2 is denoted by $P(v_1, v_2)$ (resp. $\mathcal{P}(v_1, v_2)$). In particular, the trivial path consisting of a single vertex v_0 is denoted by $P(v_0, v_0) = P(v_0)$. Let S_{k+1} denote the star with k pendent edges, and let S^r_{k+1} denote the r-uniform hyperstar obtained by intersecting k pendent hyperedges in a single vertex. The unique vertex of degree at least 2 in S^r_{k+1} is called the center of the hyperstar. For two hypergraphs H_1 and H_2, we write $H_1 \cong H_2$ if they are isomorphic. For a hypergraph H, we write $F \subset H$ if F is a subhypergraph of H, and let $c(H)$ denote the number of connected components of H.

An r-order n-dimensional real tensor $\mathcal{T}$ is a multidimensional array with entries $\mathcal{T}_{i_1 i_2 \cdots i_r} \in \mathbb{R}$, where $i_j \in [n]$ for each $j \in [r]$. For a vector $x = (x_1, x_2, \ldots, x_n)^\top \in \mathbb{C}^n$, let $\mathcal{T}x^{r-1}$ denote the n-dimensional vector whose i-th component is defined by [29]

$$(\mathcal{T}x^{r-1})_i = \sum_{i_2,\ldots,i_r \in [n]} \mathcal{T}_{i i_2 \cdots i_r} x_{i_2} \cdots x_{i_r}.$$

A number $\lambda \in \mathbb{C}$ is called an eigenvalue of $\mathcal{T}$ if there exists a nonzero vector $x \in \mathbb{C}^n$ such that

$$\mathcal{T}x^{r-1} = \lambda x^{[r-1]}, \qquad \text{where } x^{[r-1]} = (x_1^{r-1}, x_2^{r-1}, \ldots, x_n^{r-1})^\top.$$

In this case, x is called an eigenvector corresponding to λ. The maximum modulus of the eigenvalues of $\mathcal{T}$ is called the spectral radius of $\mathcal{T}$, and is denoted by $\rho(\mathcal{T})$.

Let H be an r-graph with n vertices. The adjacency tensor $\mathcal{A} = \mathcal{A}(H)$, introduced by Cooper and Dutle [3], is the r-order n-dimensional tensor defined by

$$\mathcal{A}_{i_1 \cdots i_r} = \begin{cases} \frac{1}{(r-1)!}, & \text{if } \{i_1, \ldots, i_r\} \in E(H), \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalues of $\mathcal{A}(H)$ are called the eigenvalues of H, and the spectral radius $\rho(\mathcal{A}(H))$ is called the spectral radius of H, denoted by $\rho(H)$.

The following generalized Perron–Frobenius theorem for tensors was established by Chang, Pearson and Zhang, see [5, 7].

Theorem 1. *([7]) For an r-order n-dimensional nonnegative irreducible tensor $\mathcal{T} = (\mathcal{T}_{i_1 i_2 \cdots i_r})$, let $\lambda_0 \in \mathbb{C}$ and $x_0 \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ satisfy $\mathcal{T}x_0^{r-1} = \lambda_0 x_0^{[r-1]}$. Then the pair (λ_0, x_0) has the following properties:*

(1) $\lambda_0 > 0$ is an eigenvalue of $\mathcal{T}$;

(2) $x_0 > 0$, that is, all components of x_0 are positive;

(3) if λ is an eigenvalue with a nonnegative eigenvector, then $\lambda = \lambda_0$; in particular, the nonnegative eigenvector is unique up to a positive scalar multiple;

(4) if λ is any eigenvalue of $\mathcal{T}$, then $|\lambda| \leqslant \lambda_0 = \rho(\mathcal{T})$.

By the Perron–Frobenius theorem, the eigenvector corresponding to the spectral radius $\rho(\mathcal{T})$ is unique up to a multiplicative constant. Such an eigenvector is called the principal eigenvector of $\mathcal{T}$. Following [15, 18], we normalize the principal eigenvector so that its maximum component is equal to 1. For the adjacency tensor of an r-graph H, it is known that $\mathcal{A}(H)$ is weakly irreducible if and only if H is connected, see [13, 35].

Let H be a connected r-graph with n vertices. Then $\rho = \rho(H) > 0$ is an eigenvalue of H, and the principal eigenvector $x = (x_1, x_2, \dots, x_n)^\top$ corresponding to $\rho(H)$ is strictly positive. A vertex $u \in V(H)$ is called an extremal vertex of H if $x_u = \max_{v \in V(H)}\{x_v\}$.

Throughout the paper, we always normalize the principal eigenvector x so that $x_u = \max_{v \in V(H)}\{x_v\} = 1$.

For any $v \in V(H)$, let $d_H(v)$ denote the degree of v in H, namely the number of edges containing v. Unless otherwise stated, we simply write $d(v)$ for $d_H(v)$. Similarly, let $N_H(v)$ denote the neighborhood of v in H, that is,

$$N_H(v) = \{v' \in V(H) : \{v, v'\} \text{ is contained in some edge of } H\}.$$

Unless otherwise stated, we write $N(v)$ for $N_H(v)$.

A walk of length k (or simply a k-walk) in H is an alternating sequence of vertices and edges of the form $v_1 e_1 v_2 e_2 \cdots v_k e_k v_{k+1}$, where $v_i \neq v_{i+1}$ and $v_i, v_{i+1} \in e_i$ for each $i \in [k]$. The vertices v_1 and v_{k+1} are called the starting vertex and terminal vertex of the walk, respectively. For a vertex $v \in V(H)$, let $\omega_k(v)$ denote the number of k-walks in H starting at v.

Lemma 2. *([15]) Let H be a connected r-graph with an extremal vertex u, and let $x \in \mathbb{R}^n$ be the normalized principal eigenvector whose maximum component is 1. Then*

$$\rho^k(H) \leqslant \frac{\omega_k(u)}{(r-1)^k}.$$

Corollary 3. *([18]) Let H be a connected r-graph with an extremal vertex u. Then*

$$\rho(H) \leqslant \sqrt{\frac{\sum_{v \in N(u)} d(v)}{r-1}}. \tag{1}$$

Corollary 3 follows from the identity $\omega_2(u) = (r-1)\sum_{v \in N(u)} d(v)$.

Inequality (1) will be our basic tool for deriving upper bounds on the spectral radius of Berge C_5-free linear r-graphs. To estimate the quantity $\sum_{v \in N(u)} d(v)$, we decompose it into three parts and bound each part separately. We first introduce some notation.

Let H be a connected linear r-graph with n vertices. For any two distinct vertices $v_1, v_2 \in V(H)$, if they lie in a common edge, then that edge is unique by linearity. Define

$$e_{v_1 v_2} = \begin{cases} e, & \text{if } \{v_1, v_2\} \subseteq e \in E(H), \\ \{v_1, v_2\}, & \text{otherwise.} \end{cases}$$

For a vertex $v \in V(H)$, let $E(v)$ denote the set of edges of H containing v. Clearly, $|E(v)| = d(v)$. For a set $X \subseteq V(H)$ and an integer $i \in [r]$, let

$$E_i(X) = \{e \in E(H) : |e \cap X| = i\}.$$

In addition, for $v \in V(H)$, let

$$E_i^v(X) = \{e \in E_i(X) : v \in e\}.$$

The notions of partial hypergraphs and induced sub-hypergraphs were introduced by Berge in [1]. To distinguish Berge's induced notion from the standard notion of subhypergraph, we deliberately use the hyphenated term *sub-hypergraph* for the former.

Definition 4. Let H be a connected linear r-graph with n vertices, and let $u \in V(H)$ be an extremal vertex. Define $\mathcal{B}_1$ to be the partial hypergraph of H corresponding to u such that

$$E(\mathcal{B}_1) = \left[\bigcup_{i=2}^{r} E_i(N(u))\right] \setminus E(u) \quad \text{and} \quad V(\mathcal{B}_1) = \bigcup_{e \in E(\mathcal{B}_1)} e.$$

Definition 5. Let H' be the sub-hypergraph of $\mathcal{B}_1$ induced by $N(u)$, where H, u, and $\mathcal{B}_1$ are as in Definition 4. That is,

$$E(H') = \{e \cap N(u) : e \in E(\mathcal{B}_1)\} \quad \text{and} \quad V(H') = \bigcup_{e \in E(H')} e.$$

Note that H' is linear but not necessarily uniform, and $V(H') \subseteq N(u)$. Moreover, $V(H') = N(u)$ if and only if for every $v \in N(u)$ there exists an edge $e \in E(\mathcal{B}_1)$ such that $v \in e$.

Remark 6. For each $e' \in E(H')$, there exists an edge $e \in E(\mathcal{B}_1) \subseteq E(H)$ such that $e' = e \cap N(u)$. Since H is linear and $|e'| \geqslant 2$, the edge e is uniquely determined by e', and conversely. Hence there exists a bijection $\varphi : E(H') \to E(\mathcal{B}_1)$ such that $\varphi(e') \cap N(u) = e'$ for every $e' \in E(H')$.

For each $v \in V(H)$, define

$$d_1(v) = |E_1^v(N(u))| \qquad \text{and} \qquad d_2(v) = d(v) - d_1(v).$$

In particular, if $v \in N(u)$, then $d_2(v) = \sum_{i=2}^{r} |E_i^v(N(u))|$.

Lemma 7. *Let H be a connected linear r-graph with extremal vertex u, and let $\mathcal{B}_1$ and H' be defined in Definitions 4 and 5. Then*

$$\sum_{v \in N(u)} d(v) = \sum_{v \in N(u)} d_1(v) + \sum_{v \in V(H')} d_{H'}(v) + d_H(u)(r-1). \tag{2}$$

Proof. For any vertex $v \in N(u)$, if $v \notin V(H')$, then $d_{\mathcal{B}_1}(v) = d_{H'}(v) = 0$. If $v \in V(H')$, then by Definitions 4 and 5 we have $d_{\mathcal{B}_1}(v) = d_{H'}(v)$. Since H is linear, for each $v \in N(u)$

the edge containing both u and v is unique. Therefore,

$$\begin{aligned}\sum_{v\in N(u)} d(v) &= \sum_{v\in N(u)} d_1(v) + \sum_{v\in N(u)} d_2(v) \\ &= \sum_{v\in N(u)} d_1(v) + \sum_{v\in N(u)} \left[\sum_{i=2}^{r} |E_i^v(N(u))|\right] \\ &= \sum_{v\in N(u)} d_1(v) + \sum_{v\in N(u)} \left[\sum_{i=2}^{r} |E_i^v(N(u)) \setminus \{e_{uv}\}| + 1\right] \\ &= \sum_{v\in N(u)} d_1(v) + \sum_{v\in N(u)} d_{\mathcal{B}_1}(v) + d(u)(r-1) \\ &= \sum_{v\in N(u)} d_1(v) + \sum_{v\in V(H')} d_{H'}(v) + d(u)(r-1).\end{aligned}$$

This completes the proof. $\square$

3 Main results

Let $\mathcal{H}$ be the family of Berge C_5-free linear r-graphs on n vertices. In this section, we establish several structural properties of H' and determine the maximum value of $\sum_{v\in V(H')} d_{H'}(v)$, where u is an extremal vertex of a hypergraph $H \in \mathcal{H}$ and H' is the corresponding induced sub-hypergraph defined in Definition 5. Based on these results together with Inequality (1), we then derive upper bounds on the spectral radii of Berge C_5-free linear r-graphs for $r = 3$ and $r \geqslant 4$.

3.1 Structural properties of H'

We begin with several basic properties of a hypergraph $H \in \mathcal{H}$.

Lemma 8. *Let u be an extremal vertex of $H \in \mathcal{H}$, and let H' be the sub-hypergraph corresponding to u defined in Definition 5. Then H' contains no Berge P_4.*

Proof. Suppose to the contrary that there exists a Berge P_4, say $v_1e_1v_2e_2v_3e_3v_4$, in H'.

If $e_{v_1v_4} \notin E(u)$, then $ue_{uv_1}v_1\varphi(e_1)v_2\varphi(e_2)v_3\varphi(e_3)v_4e_{uv_4}u$ is a Berge C_5 in H, a contradiction.

Now assume that $e_{v_1v_4} \in E(u)$. Since H is linear, we have

$$\{v_2\} \subseteq e_2 \cap e_{uv_2} \subseteq \varphi(e_2) \cap e_{uv_2} \subseteq \{v_2\}. \tag{3}$$

Hence $\varphi(e_2) \cap e_{uv_2} = \{v_2\}$. Similarly, $\varphi(e_2) \cap e_{uv_3} = \{v_3\}$. Since $v_2 \neq v_3$ and both lie in $\varphi(e_2)$, it follows that $e_{uv_2} \neq e_{uv_3}$ and both belong to $E(u)$. Therefore, $ue_{uv_3}v_3\varphi(e_3)v_4e_{v_1v_4}v_1$ $\varphi(e_1)v_2e_{uv_2}u$ is a Berge C_5 in H, a contradiction. This completes the proof. $\square$

We next generalize the notion of k-partite r-graphs introduced by Gao et al. [15] to nonuniform hypergraphs. Recall that the rank of a hypergraph is the maximum cardinality of its edges. A hypergraph (not necessarily uniform) is called k-partite if its rank is at most k and its vertex set can be partitioned into k parts such that each edge intersects each part in at most one vertex.

Lemma 9. *Let u be an extremal vertex of $H \in \mathcal{H}$, and let $H' = \bigcup_{i=1}^{c(H')} A_i$ be the sub-hypergraph defined in Definition 5, where A_i is the i-th component of H'. Then the following hold:*

(1) Let $k = \big|\{e \cap V(H') \neq \emptyset : e \in E(u)\}\big|$. Then H' is k-partite with k-partition $\{e \cap V(H') \neq \emptyset : e \in E(u)\}$, where $k \leqslant d(u)$.

(2) For each $i \in [c(H')]$, the component A_i is either a hyperstar, an isolated hyperedge, or a cycle C_3.

(3) For any $e \in E(H')$ and any $v \in e$, if $|e| \geqslant 3$, then $e_{uv} \cap V(H') = \{v\}$.

(4) If $A_i \cong C_3$ for some $i \in [c(H')]$, then $e_{uv} \cap V(H') = \{v\}$ for every vertex $v \in V(A_i)$.

Proof. (1). Let $e \in E(H')$ and $v \in e$. For any part $V(H') \cap e_{uv'} \neq \emptyset$ with $v' \in N(u)$, we have

$$e \cap \big(V(H') \cap e_{uv'}\big) = e \cap e_{uv'} = \begin{cases} \{v\}, & \text{if } e_{uv'} = e_{uv}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Hence each edge of H' intersects each part in at most one vertex.

(2). This follows immediately from Lemma 8.

(3). Let $e \in E(H')$ contain at least three distinct vertices $v_1, v_2, v_3 \in V(H')$. Suppose, for a contradiction, that there exists a vertex $v_4 \in e_{uv_1} \cap V(H')$ with $v_4 \neq v_1$. Let $e' \in E(H')$ be an edge containing v_4 and another vertex $v_5 \in V(H')$. By part (1), we have $\varphi(e') \neq e_{uv_1}$, and hence $v_5 \neq v_1$.

If $v_5 \in e \setminus \{v_1\}$, then without loss of generality assume $v_5 = v_2$. Then $v_2\varphi(e')v_4e_{uv_1}v_1\varphi(e)v_3e_{uv_3}ue_{uv_2}v_2$ is a Berge C_5 in H, a contradiction.

Now suppose $v_5 \notin e$. Since v_2 and v_3 are distinct, at least one of them is different from v_5. Without loss of generality assume $v_3 \neq v_5$. Then $e_{uv_3} \neq e_{uv_5}$ and both belong to $E(u)$. Hence $ue_{uv_3}v_3\varphi(e)v_1e_{v_1v_4}v_4\varphi(e')v_5e_{uv_5}u$ is a Berge C_5 in H, a contradiction.

(4). Let A_i be the cycle $v_1e_1v_2e_2v_3e_3v_1$, where $e_1 = \{v_1, v_2\}$, $e_2 = \{v_2, v_3\}$, and $e_3 = \{v_1, v_3\}$. Suppose, for a contradiction, that there exists a vertex $v_4 \in e_{uv_1} \cap V(H')$ with $v_4 \neq v_1$. Let $e_4 \in E(H')$ be an edge containing v_4 and another vertex $v_5 \neq v_4$. Then $v_5 \notin \{v_1, v_2, v_3\}$. Otherwise the component would not be isomorphic to C_3.

Among v_2 and v_3, at least one differs from v_5. Without loss of generality assume $v_3 \neq v_5$. Then $e_{uv_3} \neq e_{uv_5}$ and both belong to $E(u)$. Therefore, $ue_{uv_3}v_3\varphi(e_3)v_1e_{v_1v_4}v_4\varphi(e_4)v_5 e_{uv_5}u$ is a Berge C_5 in H, a contradiction. This completes the proof. ☐

3.2 The maximum value of $\sum_{v\in N(u)} d_1(v)$

Let H be a connected linear r-graph with n vertices, and let $u \in V(H)$ be an extremal vertex of H. For any vertex $v \in N(u)$, define $S(v) = \{v' \in V(H) : e_{vv'} \in E_1^v(N(u))\}$, $S = \bigcup_{v\in N(u)} S(v)$, and $S' = \{v \in S : d_1(v) \geqslant 2\}$.

For any edge $e \in E_1(N(u))$, the unique vertex in $e \cap N(u)$, denoted by $\gamma(e)$, is called the root of e. Define $f(e) = |\{v \in e : v \neq \gamma(e),\, d_1(v) \geqslant 2\}| = |e \cap S'|$. We partition the edge set $E_1(N(u))$ into three parts:

$$
\begin{aligned}
E_0 &= \{e \in E_1(N(u)) : f(e) = 0\},\\
E_1 &= \{e \in E_1(N(u)) : f(e) = 1\},\\
E_2 &= \{e \in E_1(N(u)) : f(e) \geqslant 2\}.
\end{aligned}
$$

Thus, $E_1(N(u)) = E_0 \cup E_1 \cup E_2$.

Definition 10. Let H be a connected linear r-graph with n vertices and extremal vertex $u \in V(H)$. Let E_2 and S' be defined as above. Define $\mathcal{B}_2$ to be the sub-hypergraph of H induced by S' such that

$$E(\mathcal{B}_2) = \{e \cap S' : e \in E_2\} \qquad \text{and} \qquad V(\mathcal{B}_2) = S'.$$

Remark 11. For every edge $e \in E_0$, we have $e \cap S' = \emptyset$. For every edge $e \in E_1$, the set $e \cap S'$ is a loop, that is, a singleton. Moreover, $\mathcal{B}_2$ may contain isolated vertices, and this happens precisely when such a vertex lies only in edges from E_1.

Remark 12. There exists a bijection $\psi : E(\mathcal{B}_2) \to E_1 \cup E_2$ such that $\psi(e') \cap S' = e'$ for every $e' \in E(\mathcal{B}_2)$. In particular, $|E(\mathcal{B}_2)| = |E_1 \cup E_2|$. An example is shown in Figure 1.

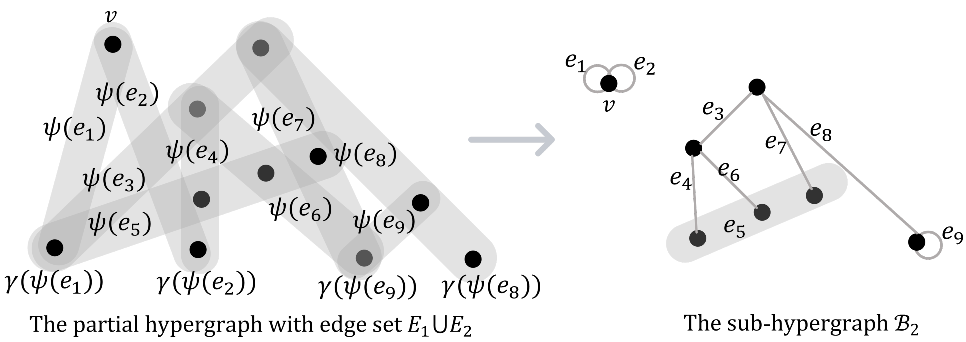

Figure 1: An example of the bijection ψ.

Let $\epsilon = |E(\mathcal{B}_1)|$ and $\bar{\nu} = \frac{\sum_{e\in E(\mathcal{B}_1)} |e\cap N(u)|}{\epsilon}$, so that $2 \leqslant \bar{\nu} \leqslant r$. Since $\{u\} \cup N(u) \cup V(\mathcal{B}_1) \cup S \subseteq V(H)$, we have

$$1 + d(u)(r-1) + (r-\bar{\nu})\epsilon + |S| \leqslant n. \tag{4}$$

Since $\sum_{v\in N(u)} d_1(v) = |E_1(N(u))|$, we may write $\sum_{v\in N(u)} d_1(v) = \frac{|E_1(N(u))|}{|S|}|S|$. In view of the upper bound on $|S|$ given by Inequality (4), maximizing $\sum_{v\in N(u)} d_1(v)$ essentially reduces to minimizing the ratio $\frac{|S|}{|E_1(N(u))|}$. We estimate this quantity below.

For a hypergraph G, define $g(G) = \sum_{e\in E(G)}(|e|-2)$.

Lemma 13. *Let H be a connected linear r-graph with extremal vertex $u \in V(H)$, and suppose that $E_1(N(u)) \neq \emptyset$. Let E_i ($i \in [3]$), S, and S' be defined as above. Then*

$$\frac{|S|}{|E_1(N(u))|} \geqslant r-3+\frac{|S'|-g(\mathcal{B}_2)}{|E_1(N(u))|}. \tag{5}$$

Equality holds if and only if $E_0 = E_1 = \emptyset$ and $E_2 = E_1(N(u))$.

Proof. By the definitions of E_i ($i \in [3]$), S', and $\mathcal{B}_2$, we have

$$\begin{aligned}\frac{|S|}{|E_1(N(u))|} &= \frac{|S\setminus S'|+|S'|}{|E_0|+|E_1|+|E_2|}\\ &= \frac{(r-1)|E_0|+(r-2)|E_1|+(r-3)|E_2|-g(\mathcal{B}_2)+|S'|}{|E_0|+|E_1|+|E_2|}\\ &\geqslant r-3+\frac{|S'|-g(\mathcal{B}_2)}{|E_1(N(u))|}.\end{aligned}$$

Equality holds if and only if $E_0 = E_1 = \emptyset$ and $E_2 = E_1(N(u))$. This completes the proof. □

Remark 14. If $E_0 = E_1 = \emptyset$ and $E_2 = E_1(N(u)) \neq \emptyset$, then $d_{\mathcal{B}_2}(v) \geqslant 2$ for every vertex $v \in V(\mathcal{B}_2)$. Hence $\mathcal{B}_2$ contains no loops, no isolated edges, no isolated vertices, and no pendent edges.

Remark 15. To minimize the ratio $\frac{|S|}{|E_1(N(u))|}$, we assume that $H \in \mathcal{H}$ has an extremal vertex $u \in V(H)$ such that $E_0 = E_1 = \emptyset$ and $E_2 = E_1(N(u)) \neq \emptyset$.

Definition 16. Let $col : E(\mathcal{B}_2) \to [\kappa]$ be a coloring function, such that $col(e_1) = col(e_2)$ if and only if $e_{u\gamma(\psi(e_1))} = e_{u\gamma(\psi(e_2))}$, where $\kappa = \left|\{e_{u\gamma(\psi(e))} : e \in E(\mathcal{B}_2)\}\right|$.

Lemma 17. *Let $\mathcal{B}_2$ be the induced sub-hypergraph of H fixed in Remark 15. Let $\mathcal{P}(v,w)$ be a nontrivial Berge path in $\mathcal{B}_2$ with endpoints v and w. For any edge e (resp. e') incident to v (resp. w) and not contained in $\mathcal{P}(v,w)$, whenever such edges exist, the following hold:*

- *if the length $\mathbb{L}(\mathcal{P}(v,w))$ is odd, then $col(e) = col(e')$;*
- *if $\mathbb{L}(\mathcal{P}(v,w))$ is even, then $col(e) \neq col(e')$.*

Proof. Let $u_1 = \gamma(\psi(e))$ and $u_2 = \gamma(\psi(e'))$. We consider two cases.
Case 1. $\mathbb{L}(\mathcal{P}(v,w))$ is odd.

We argue by induction on $\mathbb{L}(\mathcal{P}(v,w))$. If $\mathbb{L}(\mathcal{P}(v,w)) = 1$, say $\mathcal{P}(v,w) = ve_1w$, suppose for a contradiction that $col(e) \neq col(e')$, see Figure 2. Then $u_1 \neq u_2$ and $e_{uu_1} \neq e_{uu_2}$ by Definition 16. Hence $ue_{uu_1}u_1\psi(e)v\psi(e_1)w\psi(e')u_2e_{uu_2}u$ is a Berge C_5 in H, a contradiction.

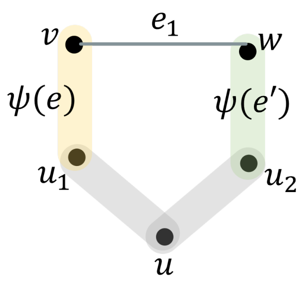

Figure 2: $\mathbb{L}(\mathcal{P}(v,w)) = 1$.

Assume now that the statement holds for all odd lengths at most $2m-1$. Let $\mathbb{L}(\mathcal{P}(v,w)) = 2m+1$, so that $\mathcal{P}(v,w) = ve_1v_1e_2v_2\cdots v_{2m}e_{2m+1}w$. By the induction hypothesis, $col(e) = col(e_{2m})$ and $col(e') = col(e_{2m})$. Therefore $col(e) = col(e')$.

Case 2. $\mathbb{L}(\mathcal{P}(v,w))$ is even.

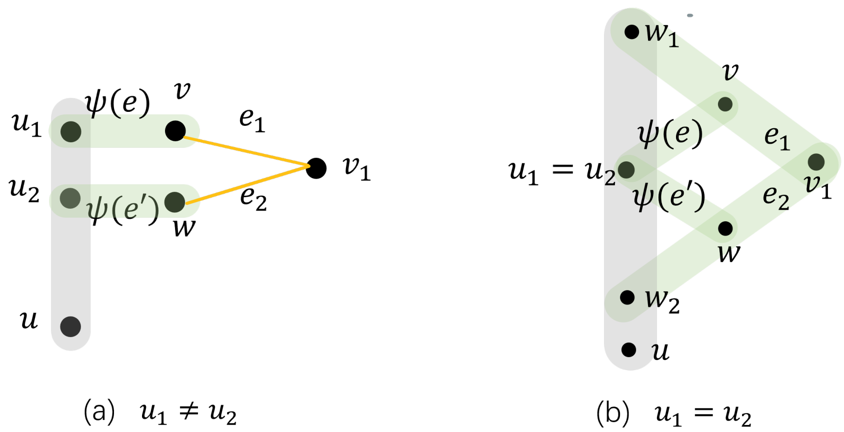

Figure 3: $\mathbb{L}(\mathcal{P}(v,w)) = 2$.

First consider $\mathbb{L}(\mathcal{P}(v,w)) = 2$, say $\mathcal{P}(v,w) = ve_1v_1e_2w$. Suppose for a contradiction that $col(e) = col(e')$. Then either $u_1 = u_2$ or $e_{u_1u_2} \in E(u)$.

If $u_1 \neq u_2$ and $e_{u_1u_2} \in E(u)$, then $u_1\psi(e)v\psi(e_1)v_1\psi(e_2)w\psi(e')u_2e_{u_1u_2}u_1$ is a Berge C_5 in H, a contradiction, see Figure 3(a).

If $u_1 = u_2$, then both subpaths ve_1v_1 and v_1e_2w have length 1. Thus, e, e_1, e_2, e' all receive the same color. Since $e_1 \cap e_2 = \{v_1\} \notin N(u)$ and H is linear, the edges e_1 and e_2 have different roots, say $w_1, w_2 \in N(u)$, respectively. Because $col(e_1) = col(e_2)$, we have $e_{w_1w_2} \in E(u)$. Then $u_1\psi(e)v\psi(e_1)w_1e_{w_1w_2}w_2\psi(e_2)w\psi(e')u_1$ is a Berge C_5 in H, a contradiction, see Figure 3(b). Thus $col(e) \neq col(e')$ when $\mathbb{L}(\mathcal{P}(v,w)) = 2$.

Now let $\mathbb{L}(\mathcal{P}(v,w)) = 2m > 2$, say $\mathcal{P}(v,w) = ve_1v_1e_2v_2\cdots v_{2m-1}e_{2m}w$. By repeated application of Case 1 to suitable odd subpaths and the case of length 2 just proved, we conclude that the parity of the path forces $col(e) \neq col(e')$. This completes the proof. □

We now characterize $\mathcal{B}_2$ using Lemma 17. Let B_i denote the i-th component of $\mathcal{B}_2$, where $\mathcal{B}_2 = \bigcup_{i=1}^{c(\mathcal{B}_2)} B_i$.

Given a Berge cycle $\mathcal{C}$, a chord of $\mathcal{C}$ is an edge $e \notin E(\mathcal{C})$ containing at least two vertices of degree greater than 2, each lying in some edge of $\mathcal{C}$. Given a C_k^+ together with some chords, a *type-1 chord* is a chord containing exactly one intersection vertex of C_k^+, while all its other vertices are not intersection vertices of C_k^+.

Lemma 18. *Let $\mathcal{B}_2 = \bigcup_{i=1}^{c(\mathcal{B}_2)} B_i$ be the induced sub-hypergraph of H fixed in Remark 15. Then, for each $i \in [c(\mathcal{B}_2)]$, the component B_i is isomorphic either to a C_3^+ with possible chords or to a Berge-odd-cycle-free hypergraph. Moreover, the following hold:*

(1) If B_i is Berge-odd-cycle-free, then B_i is properly colored, that is, adjacent edges receive distinct colors. In addition, B_i is 2-regular.

(2) If B_i is isomorphic to a C_3^+ with possible chords, then every chord is of type 1 and contains exactly two distinguished vertices, one of degree 2 and the other of degree greater than 2.

(3) If B_i is Berge-odd-cycle-free, then the edges of B_i receive exactly two colors under the κ-edge-coloring of Definition 16.

Proof. By Remark 14, the hypergraph $\mathcal{B}_2$ contains no pendent edges. Hence each component contains a Berge cycle. Fix $i \in [c(\mathcal{B}_2)]$, and let $v_1e_1v_2e_2 \cdots v_\ell e_\ell v_1$ be any Berge cycle contained in B_i.

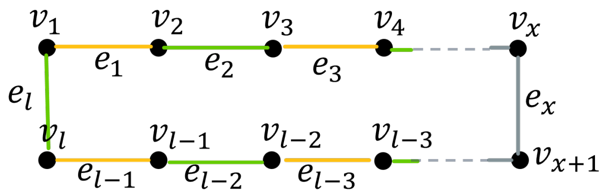

Figure 4: A Berge cycle of odd length $\ell > 3$.

Suppose first that $\ell > 3$ is odd. Choose any vertex v_x on the cycle with $2 \leqslant x \leqslant \ell - 1$, see Figure 4. Then one of the two subpaths from v_1 to v_x and from v_{x+1} to v_ℓ has odd length, and the other has even length. By Lemma 17, the colors forced on e_x and e_ℓ are incompatible, regardless of how these edges are colored. This contradiction shows that B_i contains no Berge odd cycle of length greater than 3.

Since $\mathcal{B}_2$ is linear, any Berge C_3 is in fact a C_3^+. Suppose that B_i contains a C_3^+, say $v_1e_1v_2e_2v_3e_3v_1$, but is not isomorphic to a C_3^+ with possible chords. Then there exists an edge $e_4 \notin \{e_1, e_2, e_3\}$ intersecting some e_x in a vertex v_4. Without loss of generality, assume $x = 3$. By Lemma 17, we have $col(e_1) = col(e_2) = col(e_3)$. Choose a vertex $v_5 \in e_4 \setminus \bigcup_{j=1}^{3} e_j$. By Remark 14, $d_{\mathcal{B}_2}(v_5) \geqslant 2$, so there exists an edge $e_5 \notin \{e_1, e_2, e_3, e_4\}$ containing v_5.

If $v_4 \in \{v_1, v_3\}$, assume without loss of generality that $v_4 = v_3$, see Figure 5(a). Since $\mathbb{L}(v_5e_4v_3) = 1$, we obtain $col(e_2) = col(e_5)$. On the other hand, the path $v_5e_4v_3e_2v_2$ has length 2, so $col(e_1) \neq col(e_5)$. This contradicts $col(e_1) = col(e_2)$.

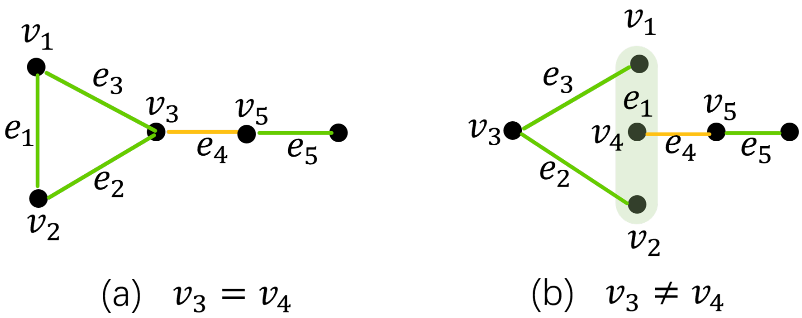

Figure 5: B_i is isomorphic to C_3^+ with a non-chord edge e_4.

If instead $v_4 \in e_3 \setminus \{v_1, v_3\}$, see Figure 5(b). Since $\mathbb{L}(v_5e_4v_4) = 1$, we have $col(e_3) = col(e_5)$. But the path $v_5e_4v_4e_3v_3$ has length 2, so $col(e_2) \neq col(e_5)$, contradicting $col(e_2) = col(e_3)$. Therefore, whenever B_i contains a C_3^+, it must be isomorphic to a C_3^+ with possible chords.

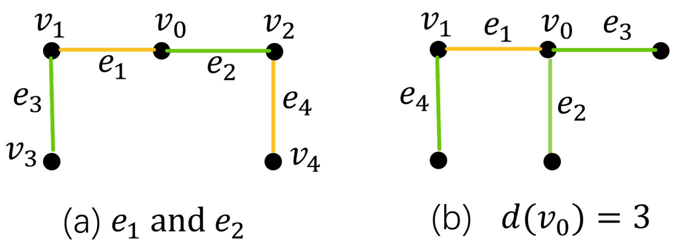

Figure 6: B_i is Berge-odd-cycle-free.

(1). Assume that B_i is Berge-odd-cycle-free. Let e_1 and e_2 be adjacent edges with $e_1 \cap e_2 = \{v_0\}$, and let $v_1 \in e_1 \setminus \{v_0\}$ and $v_2 \in e_2 \setminus \{v_0\}$, see Figure 6(a). By Remark 14, we have $d_{\mathcal{B}_2}(v_1) \geqslant 2$ and $d_{\mathcal{B}_2}(v_2) \geqslant 2$. Choose edges $e_3 \neq e_1$ and $e_4 \neq e_2$ incident to v_1 and v_2, respectively. Then $e_3 \neq e_4$, for otherwise B_i would contain a C_3^+. By Lemma 17, the edges e_3 and e_4 receive different colors, while e_1 and e_4 share a color and e_2 and e_3 share a color. Thus adjacent edges in B_i receive distinct colors.

Next suppose that some vertex $v_0 \in V(B_i)$ satisfies $d_{\mathcal{B}_2}(v_0) > 2$, see Figure 6(b). Let e_1, e_2, e_3 be three distinct edges incident to v_0, and let $v_1 \in e_1 \setminus \{v_0\}$. Since $d_{\mathcal{B}_2}(v_1) \geqslant 2$, choose another edge $e_4 \neq e_1$ incident to v_1. The path $v_0e_1v_1$ has length 1. Thus by Lemma 17, the edges e_2, e_3, e_4 all receive the same color. This contradicts the fact that adjacent edges must receive distinct colors. Hence every vertex of B_i has degree 2.

(2). Assume that B_i is isomorphic to a C_3^+, say $v_1e_1v_2e_2v_3e_3v_1$, together with a chord e, see Figure 7. Let $\gamma(\psi(e_j)) = u_j$ for $j \in [3]$. By Lemma 17 and linearity, u_1, u_2, u_3 are pairwise distinct and $e_{uu_1} = e_{uu_2} = e_{uu_3} \in E(u)$. Since $\mathcal{B}_2$ is linear, any chord of this C_3^+ contains at most one vertex of $\{v_1, v_2, v_3\}$. Thus it suffices to show that e contains at most one vertex in $\bigcup_{j=1}^{3} e_j \setminus \{v_1, v_2, v_3\}$. Suppose, for a contradiction, that

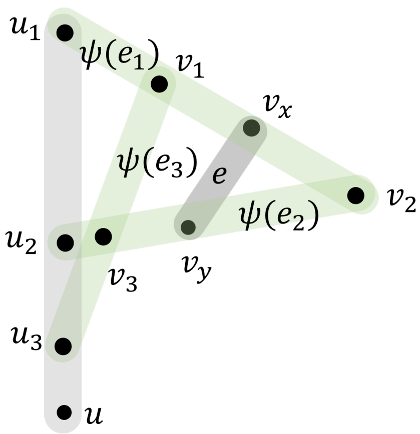

Figure 7: C_3^+ with a chord e.

there are distinct vertices $v_x, v_y \in e$ with $v_x \in e_1 \setminus \{v_1, v_2\}$ and $v_y \in e_2 \setminus \{v_2, v_3\}$. Then $u_1\psi(e_1)v_x\psi(e)v_y\psi(e_2)v_3\psi(e_3)u_3e_{u_1u_3}u_1$ forms a Berge C_5 in H, a contradiction.

(3). Assume that B_i is Berge-odd-cycle-free. Fix an edge $e \in E(B_i)$ and suppose $col(e) = x \in [\kappa]$. By part (1), all adjacent edges of e receive a common color $y \in [\kappa]$ distinct from x. Now let $e' \neq e$ be any other edge of B_i. Since B_i is Berge-odd-cycle-free, there exists an even Berge path connecting either e' to e or e' to an edge adjacent to e. By Lemma 17, it follows that $col(e') \in \{x, y\}$. Hence exactly two colors are used on $E(B_i)$.

This completes the proof. $\square$

Note that if $g(\mathcal{B}_2) = 0$, then $\mathcal{B}_2$ is a 2-graph with no isolated vertices, isolated edges, or pendent edges.

Corollary 19. *Let $\mathcal{B}_2 = \bigcup_{i=1}^{c(\mathcal{B}_2)} B_i$ be the induced sub-hypergraph of H fixed in Remark 15. If $g(\mathcal{B}_2) = 0$, then $B_i \cong C_\ell$ for each $i \in [c(\mathcal{B}_2)]$, where $\ell = 2m$ for some $m > 1$ or $\ell = 3$. Moreover, $\frac{|S'|}{|E_2|} = 1$.*

We next show that for a Berge-odd-cycle-free component B_i, the edges of the same color are either root-uniform or root-distinct:

- *root-uniform*: all such edges have the same root;
- *root-distinct*: all such edges have distinct roots.

Lemma 20. *Let $\mathcal{B}_2 = \bigcup_{i=1}^{c(\mathcal{B}_2)} B_i$ be the induced sub-hypergraph of H fixed in Remark 15, and let B_i be any Berge-odd-cycle-free component of $\mathcal{B}_2$. Then the following hold:*

(1) If there exist two edges $e_0, e_0' \in E(B_i)$ of the same color such that $\gamma(\psi(e_0)) = \gamma(\psi(e_0')) \in N(u)$, then all edges of color $col(e_0)$ in B_i are root-uniform.

(2) If there exist two edges $e_0, e_0' \in E(B_i)$ of the same color such that $\gamma(\psi(e_0)) \neq \gamma(\psi(e_0'))$ and $e_{u\gamma(\psi(e_0))} = e_{u\gamma(\psi(e_0'))} \in E(u)$, then all edges of the same color in B_i are root-distinct.

(3) If all same-colored edges of B_i are root-uniform, then $B_i \cong C_{2\ell}$ for some $\ell > 1$.

Proof. First consider any two same-colored edges $e_1, e_2 \in E(B_i)$, and let e_4 (resp. e_3) be an edge adjacent to e_1 (resp. e_2).

Claim 1. If $\gamma(\psi(e_1)) = \gamma(\psi(e_2)) = v_0 \in N(u)$, then e_3 and e_4 have the same root in $N(u)$. On the other hand, if $\gamma(\psi(e_1)) \neq \gamma(\psi(e_2))$ and $e_{u\gamma(\psi(e_1))} = e_{u\gamma(\psi(e_2))} \in E(u)$, then $\gamma(\psi(e_3)) \neq \gamma(\psi(e_4))$ and $e_{u\gamma(\psi(e_3))} = e_{u\gamma(\psi(e_4))} \in E(u)$.

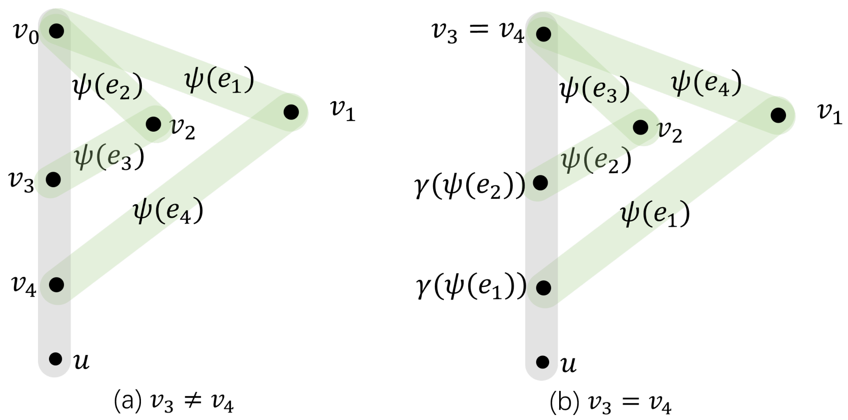

Figure 8: Proof of Claim 1.

Proof of Claim 1. Let $\gamma(\psi(e_3)) = v_3 \in N(u)$, $\gamma(\psi(e_4)) = v_4 \in N(u)$, and suppose that $e_2 \cap e_3 = \{v_2\}$ and $e_1 \cap e_4 = \{v_1\}$. By parts (1) and (3) of Lemma 18, we have $e_{uv_3} = e_{uv_4}$ and $e_{uv_4} \neq e_{uv_1}$.

Suppose first that $\gamma(\psi(e_1)) = \gamma(\psi(e_2)) = v_0$ but $v_3 \neq v_4$. Then

$$v_0\psi(e_1)v_1\psi(e_4)v_4e_{v_4v_3}v_3\psi(e_3)v_2\psi(e_2)v_0$$

is a Berge C_5 in H, a contradiction, see Figure 8(a).

Similarly, if $\gamma(\psi(e_1)) \neq \gamma(\psi(e_2))$ but $v_3 = v_4$, then

$$v_3\psi(e_3)v_2\psi(e_2)\gamma(\psi(e_2))e_{\gamma(\psi(e_2))\gamma(\psi(e_1))}\gamma(\psi(e_1))\psi(e_1)v_1\psi(e_4)v_4$$

is a Berge C_5 in H, a contradiction, see Figure 8(b). This proves Claim 1. $\square$

(1). Assume that there exist two same-colored edges $e_0, e_0' \in E(B_i)$ with $\gamma(\psi(e_0)) = \gamma(\psi(e_0')) \in N(u)$. Let $e_x, e_y' \in E(B_i)$ be any other two same-colored edges. We show that $\gamma(\psi(e_x)) = \gamma(\psi(e_y'))$. Since B_i is connected, there exist Berge paths connecting e_0 to e_x and e_0' to e_y', whose edge sequences we denote by $e_1 \cdots e_{x-1}$ and $e_1' \cdots e_{y-1}'$, respectively. By Lemma 17, the integers x and y have the same parity. Repeatedly applying Claim 1 along these paths and using transitivity, we conclude that whenever i and j have the same parity, the edges e_i and e_j' have the same root. In particular, e_x and e_y' have the same root.

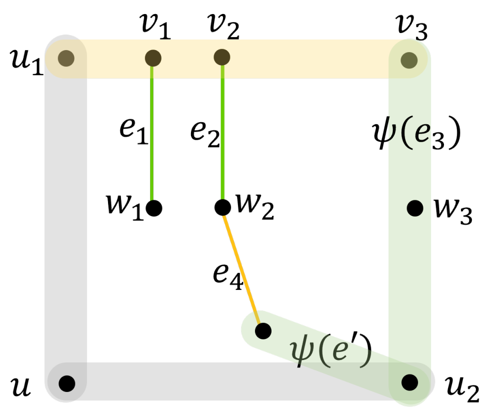

Figure 9: Same-colored edges having the same root.

(2). The proof is analogous. Using Claim 1 repeatedly, one sees that if two same-colored edges begin with distinct roots lying in the same edge of $E(u)$, then the same holds for any other pair of same-colored edges.

(3). Assume that every two same-colored edges of B_i have the same root. By parts (1) and (3) of Lemma 18, all edges of B_i correspond to exactly two roots $u_1, u_2 \in N(u)$ with $e_{uu_1} \neq e_{uu_2}$. Since B_i contains a Berge even cycle, say a Berge $C_{2\ell}$ with $\ell > 1$, and B_i is 2-regular by Lemma 18(1), it remains only to show that every edge of B_i has size 2.

Suppose, for a contradiction, that there exists an edge $e \in E(B_i)$ with $\gamma(\psi(e)) = u_1$ such that e contains at least three distinct vertices v_1, v_2, v_3, see Figure 9. By Lemma 18(1) and (3), each vertex of B_i lies in exactly two incident edges whose corresponding roots are u_1 and u_2, respectively. Let e_j be the edge other than e incident to v_j, and let $w_j \in e_j \setminus \{v_j\}$ for $j = 1, 2, 3$. Let e_4 be the other edge incident to w_2, and choose $w_4 \in e_4 \setminus \{w_2\}$. Since $e_4 \neq e$, at least one of w_1, w_3 differs from w_4. Without loss of generality assume $w_3 \neq w_4$. Let e' be the other edge incident to w_4. Then $\gamma(\psi(e_4)) = u_1$ and $\gamma(\psi(e')) = u_2$, $e_{u_2w_4} = \psi(e') \in E(H)$. As $w_4 \neq w_3$, we have $e_{u_2w_4} \neq \psi(e_3)$. Hence $u_2\psi(e_3)v_3\psi(e)v_2\psi(e_2)w_2\psi(e_4)w_4e_{u_2w_4}u_2$ is a Berge C_5 in H, a contradiction. Therefore every edge of B_i has size 2, and so $B_i \cong C_{2\ell}$. This completes the proof. $\square$

An *internal path* is a 2-uniform path whose internal vertices all have degree 2. Let G be a hypergraph. An *ear* contained in G is an internal nontrivial path in G. A *trivial ear* is a trivial path $P(v_0, v_0) = P(v_0) = v_0$ that lies in two distinct edges $e, e' \in E(G)$ with $|e| > 2$ and $|e'| > 2$.

Lemma 21. *Let G be a linear, not necessarily uniform, 2-regular hypergraph such that $|e| \geqslant 2$ for every $e \in E(G)$. Given any edge $e \in E(G)$ with $|e| > 2$, there exists at least one ear $P(v_0, v_s) = v_0e_1 \cdots e_sv_s$ ($s \geqslant 0$) such that:*

(1) $v_0 \in e$, $v_s \in e_{s+1} \in E(G)$, and $|e_{s+1}| > 2$;

(2) $|e \setminus \{v_0, v_s\}| > 1$ and $|e_{s+1} \setminus \{v_0, v_s\}| > 1$.

Proof. Choose any vertex $v_0 \in e$. We perform the following procedure $\mathscr{P}_1$ to obtain an ear $P(v_0, v_s)$ such that $|e_{s+1}| > 2$.

Algorithm 1: $\mathscr{P}_1$

Input: $e \in E(G)$ with $|e| > 2$ and $v_0 \in e$

Output: an ear $P(v_0, v_s)$ with endpoints $v_0, v_s \in V(G)$ such that $v_s \in e_{s+1}$ and $|e_{s+1}| > 2$

set $s = 0$, $e_0 = e$, and $P(v_0, v_s) = v_0$;
while $|e_{s+1}| = 2$ **do**
  $P(v_0, v_{s+1}) = P(v_0, v_s)e_{s+1}v_{s+1}$, where $e_s \neq e_{s+1}$, $e_s \cap e_{s+1} = \{v_s\}$, and $v_{s+1} \in e_{s+1}$;
  $s = s + 1$;
end

We claim that $e_0, e_1, \ldots, e_s$ are pairwise distinct when $s > 0$. This is proved by induction. Clearly, $e_1 \neq e_0$. Assume that $e_0, \ldots, e_k$ are pairwise distinct for some $k \leqslant s-1$. Then $v_k \notin \{v_0, v_1, \ldots, v_{k-1}\}$. Otherwise $v_k = v_\ell$ for some $\ell < k$, which would imply $d_G(v_\ell) > 2$, a contradiction. Since $v_k \in e_{k+1}$ and v_k has not appeared earlier, we obtain $e_{k+1} \notin \{e_0, \ldots, e_k\}$. Because $|E(G)|$ is finite, the procedure $\mathscr{P}_1$ terminates after finitely many steps.

Suppose now, for a contradiction, that for every choice of $v_0 \in e$, if $v_s \neq v_0$ is the other endpoint of the ear obtained from $\mathscr{P}_1$, then $|e \setminus \{v_0, v_s\}| \leqslant 1$ or $|e_{s+1} \setminus \{v_0, v_s\}| \leqslant 1$. Since $|e|, |e_{s+1}| > 2$, this can happen only if $\{v_0, v_s\} \subseteq e$ or $\{v_0, v_s\} \subseteq e_{s+1}$, forcing $e = e_{s+1}$. Because each internal vertex has degree 2 and each internal edge has size 2, the endpoint v_s is uniquely determined by v_0 and vice versa. Thus the endpoints in the common edge $e = e_{s+1}$ occur in pairs, and so $|e| = |e_{s+1}|$ must be even. But then the only way to have $|e \setminus \{v_0, v_s\}| \leqslant 1$ would be $|e| = 2$, contradicting $|e| > 2$. Therefore there exists an ear $P(v_0, v_s)$ satisfying both (1) and (2). This completes the proof. □

Let B_i be any Berge-odd-cycle-free component of $\mathcal{B}_2$. We construct an ear set $W_k = \{P^1, \ldots, P^k\}$ and a 2-graph G_k by the following procedure $\mathscr{P}_2$.

Remark 22. Since $|e \setminus \{v_l, w_l\}| > 1$, there is no loop in $E(G_j)$ for any $0 \leqslant j \leqslant k$. Thus each G_j is simple and 2-uniform. Moreover, each G_j is 2-regular, and therefore G_k is a disjoint union of even cycles. Also, $|V(G_j)| = |V(G_{j+1})| + |V(P^{j+1})|$ and $|E(G_j)| = |E(G_{j+1})| + |E(P^{j+1})|$. Thus

$$|V(B_i)| = |V(G_k)| + \sum_{j=1}^{k} |V(P^j)| \qquad \text{and} \qquad |E(B_i)| = |E(G_k)| + \sum_{j=1}^{k} |E(P^j)|. \tag{6}$$

Remark 23. Let $p(B_i)$ denote the number of ears of B_i obtained from $\mathscr{P}_2$. Since each ear contributes 2 to $g(B_i)$, we have $2p(B_i) \leqslant g(B_i)$. If $g(B_i) > 2p(B_i)$, then some edge of G_k would still have size greater than 2, contradicting the termination of $\mathscr{P}_2$. Hence $k = p(B_i) = \frac{1}{2}g(B_i)$.

Algorithm 2: $\mathscr{P}_2$

Input: a Berge-odd-cycle-free component B_i of H
Output: a set of ears W_k and a 2-graph G_k

set $k = 0$, $G_0 = B_i$, and $W_0 = \emptyset$;
while *there exists an edge of* G_k *containing at least* 3 *vertices* **do**
  select an edge $e = \{v_1, \ldots, v_{|e|}\} \in E(G_k)$ such that $|e| > 2$;
  set $l = 1$, and let w_1 be the other endpoint of the ear $P(v_1, w_1)$ obtained by $\mathscr{P}_1$;
  while $|e \setminus \{v_l, w_l\}| \leqslant 1$ **do**
    $l = l + 1$;
    obtain the ear $P(v_l, w_l)$ in G_k by $\mathscr{P}_1$;
  end
  $P^{k+1} = P(v_l, w_l)$ and $W_{k+1} = W_k \cup \{P^{k+1}\}$;
  $V(G_{k+1}) = V(G_k) \setminus V(P^{k+1})$;
  $E(G_{k+1}) = \{e \cap V(G_{k+1}) : e \in E(G_k)\}$;
  $k = k + 1$;
end

Lemma 24. *Let $\mathcal{B}_2 = \bigcup_{i=1}^{c(\mathcal{B}_2)} B_i$ be the induced sub-hypergraph of H fixed in Remark 15, and let B_i be any Berge-odd-cycle-free component of $\mathcal{B}_2$. Then $|V(B_i)| - g(B_i) = |E(B_i)| - p(B_i)$.*

Proof. Let $W_k = \{P^1, \ldots, P^k\}$ be the ear set of B_i obtained from $\mathscr{P}_2$. Then

$$\begin{aligned}
|V(B_i)| - g(B_i) &= 2|E(B_i)| - |V(B_i)| \\
&= 2|E(B_i)| - |V(G_k)| - \sum_{j=1}^{k} |V(P^j)| \\
&= 2|E(B_i)| - |E(G_k)| - \sum_{j=1}^{k} |E(P^j)| - k \\
&= |E(B_i)| - p(B_i), \qquad (7)
\end{aligned}$$

where we use the fact that B_i is 2-regular and the identities in Equation (6). This completes the proof. □

Definition 25. Define B to be the connected Berge-odd-cycle-free linear $(r-1)$-graph with edge set $E(B) = \{e_j : j \in [2(r-1)]\}$, where $e_j = \{v_j^x : x \in [r-1]\}$ and $v_{2y-1}^x = v_{2x}^y$ for each $x, y \in [r-1]$.

We now determine the minimum value of $\frac{|S|}{|E_1(N(u))|}$ in Inequality (5).

Theorem 26. *Let $\mathcal{B}_2 = \bigcup_{i=1}^{c(\mathcal{B}_2)} B_i$ be the induced sub-hypergraph of H fixed in Remark 15. Then $\frac{|S|}{|E_1(N(u))|} \geqslant \frac{r-1}{2}$. Equality holds if $B_i \cong B$ for every $i \in [c(\mathcal{B}_2)]$, where B is defined in Definition 25.*

Proof. By Inequality (5), it suffices to minimize $\frac{|S'|-g(\mathcal{B}_2)}{|E_2|}$ under the condition $E_0 = E_1 = \emptyset$. Since

$$\frac{|S'| - g(\mathcal{B}_2)}{|E_2|} = \frac{\sum_{j=1}^{c(\mathcal{B}_2)}\big(|V(B_j)| - g(B_j)\big)}{\sum_{j=1}^{c(\mathcal{B}_2)} |E(B_j)|} \geqslant \min_{j\in[c(\mathcal{B}_2)]} \left\{\frac{|V(B_j)| - g(B_j)}{|E(B_j)|}\right\},$$

it is enough to consider a component B_i for which $\frac{|V(B_i)|-g(B_i)}{|E(B_i)|}$ is minimum.

By Lemma 18, each component B_i is either isomorphic to a C_3^+ with possible chords or Berge-odd-cycle-free.

Case 1. B_i is isomorphic to a C_3^+ with possible chords.

By Lemma 17 and Lemma 18(2), all edges of B_i receive the same color, and every two edges of B_i have distinct roots in $N(u)$. Since an edge of $E(u)$ can provide at most $r-1$ different roots, we have $|E(B_i)| \leqslant r-1$. Hence

$$\frac{|V(B_i)| - g(B_i)}{|E(B_i)|} \geqslant \frac{3}{r-1}.$$

Case 2. B_i is Berge-odd-cycle-free.

By Lemma 20, either all same-colored edges of B_i are root-uniform or all same-colored edges of B_i are root-distinct.

Subcase 2.1. All same-colored edges of B_i are root-uniform.

By Lemma 20(3), we have $B_i \cong C_{2\ell}$ for some $\ell > 1$. Therefore,

$$\frac{|V(B_i)| - g(B_i)}{|E(B_i)|} = 1.$$

Subcase 2.2. Any two same-colored edges of B_i are root-distinct.

Let G_k and $W_k = \{P^1, \ldots, P^k\}$ be the 2-graph and ear set of B_i obtained from $\mathscr{P}_2$. Since G_k is a disjoint union of even cycles, write $|E(G_k)| = 2m$. By Lemma 24,

$$\frac{|V(B_i)| - g(B_i)}{|E(B_i)|} = \frac{|E(B_i)| - p(B_i)}{|E(B_i)|} = \frac{2m + \sum_{j=1}^{k}\big(|E(P^j)| - 1\big)}{2m + \sum_{j=1}^{k} |E(P^j)|} \geqslant \frac{2m - p(B_i)}{2m}.$$

Equality holds only if $|E(P^j)| = 0$ for all $j \in [k]$, that is, each P^j is trivial.

If each P^j is trivial, then $|E(B_i)| = |E(G_k)| = 2m$, and $k = p(B_i)$ is the number of trivial ears obtained from $\mathscr{P}_2$. Each edge $e \in E(B_i)$ contains exactly two distinct vertices belonging to $V(G_k)$, and $|e| \leqslant r-1$. Thus each edge contributes at most $r-3$ trivial ears. Since every trivial ear lies in exactly two distinct edges of B_i, we obtain $p(B_i) \leqslant (r-3)m$. Therefore,

$$\frac{|V(B_i)| - g(B_i)}{|E(B_i)|} \geqslant \frac{2m - (r-3)m}{2m} = \frac{5-r}{2}. \tag{8}$$

Combining the above cases, we obtain $\frac{|S|}{|E_1(N(u))|} \geqslant r - 3 + \frac{5-r}{2} = \frac{r-1}{2}$.

The extremal component B_i that attains equality in Inequality (8) is not unique.

We now verify that equality in Inequality (8) holds when $B_i \cong B$, where B is defined in Definition 25 and satisfies parts (1) and (3) of Lemma 18. Construct a 2-graph G with vertex set $V(G) = E(B)$, where for two vertices $e_i, e_j \in V(G)$, we put $e_ie_j \in E(G)$ if and only if $e_i \cap e_j \neq \emptyset$ in B. Clearly, $N_G(e_{2y-1}) = \{e_{2x} : x \in [r-1]\}$ for each $y \in [r-1]$, and $N_G(e_{2x}) = \{e_{2y-1} : y \in [r-1]\}$ for each $x \in [r-1]$. Hence $G \cong K_{r-1,r-1}$, the balanced complete bipartite graph on $2(r-1)$ vertices. The edge sequence of any Berge cycle in B yields a cycle in G. Since G contains no odd cycle, B is Berge-odd-cycle-free.

Moreover, for every vertex $v^x_{2y-1} \in V(B)$ (equivalently $v^y_{2x} \in V(B)$), we have that $d_B(v^x_{2y-1}) = d_B(v^y_{2x}) = 2$, because $E_B(v^x_{2y-1}) = E_B(v^y_{2x}) = \{e_{2y-1}, e_{2x}\}$. If every two adjacent edges of B are assigned different colors, then B is 2-edge-colorable. Therefore, B satisfies parts (1) and (3) of Lemma 18. This completes the proof. $\square$

3.3 Spectral radius of a Berge C_5-free linear r-graph

Theorem 27. *Let H be a Berge C_5-free linear r-graph on n vertices. Then*

(1) if $r = 3$, then $\rho(H) \leqslant \sqrt{n}$;

(2) if $r \geqslant 4$, then $\rho(H) \leqslant \sqrt{\frac{r}{(r-1)^2}n}$.

Proof. Let $\epsilon = |E(\mathcal{B}_1)|$ and $\bar{\nu} = \frac{\sum_{e\in E(\mathcal{B}_1)} |e\cap N(u)|}{\epsilon}$. By Inequality (1), Equation (2), and Theorem 26, we have

$$\begin{aligned}\rho(H) &\leqslant \sqrt{\frac{\sum_{v\in N(u)} d_1(v) + \sum_{v\in V(H')} d_{H'}(v) + d_H(u)(r-1)}{r-1}}\\ &\leqslant \sqrt{\frac{2|S|/(r-2) + \bar{\nu}\epsilon + d(u)(r-1)}{r-1}}. \qquad (9)\end{aligned}$$

Here Inequality (9) follows from $\sum_{v\in N(u)} d_1(v) = \frac{|E_1(N(u))|}{|S|}|S| \leqslant \frac{2|S|}{r-1}$ and $\sum_{v\in V(H')} d_{H'}(v) = \bar{\nu}\epsilon$.

By Inequality (4), we have $1 + d(u)(r-1) + (r-\bar{\nu})\epsilon + |S| \leqslant n$, which implies

$$d(u)(r-1) + (r-\bar{\nu})\epsilon + |S| < n. \qquad (10)$$

To bound $\rho(H)$, let $F = \frac{2|S|}{r-2} + \bar{\nu}\epsilon + d(u)(r-1)$ denote the numerator on the right-hand side of Inequality (9).

Case 1. $r = 3$.

In this case, $F = 2|S| + \bar{\nu}\epsilon + 2d(u)$, and Inequality (10) becomes $2d(u) + (3-\bar{\nu})\epsilon + |S| < n$. Thus $|S| < n - 2d(u) - (3-\bar{\nu})\epsilon$. Substituting this into F, we obtain

$$F < 2[n - 2d(u) - (3-\bar{\nu})\epsilon] + \bar{\nu}\epsilon + 2d(u) = 2n - 2d(u) + (3\bar{\nu} - 6)\epsilon.$$

By Lemma 9, if $\bar{\nu} > 2$, then $\epsilon \leqslant \left\lfloor \frac{d(u)}{\bar{\nu}} \right\rfloor$. Since $\bar{\nu} \leqslant 3$, we have $0 \leqslant (3\bar{\nu} - 6)\epsilon \leqslant 3\left\lfloor \frac{d(u)}{3} \right\rfloor \leqslant d(u)$. Hence $F < 2n - d(u) < 2n$. Therefore, $\rho(H)^2 \leqslant \frac{F}{3-1} < \frac{2n}{2} = n$, which implies $\rho(H) \leqslant \sqrt{n}$.

Case 2. $r \geqslant 4$.

For $r \geqslant 4$, the coefficient of $|S|$ in F is at most 1, so it is advantageous to take $|S| = 0$. Then Inequality (10) becomes $(r - \bar{\nu})\epsilon + d(u)(r-1) < n$, and $F = \bar{\nu}\epsilon + d(u)(r-1)$. Since $d(u)(r-1) < n - (r-\bar{\nu})\epsilon$, substituting into F gives $F < n + \epsilon(2\bar{\nu} - r)$.

By Lemma 9, if $\bar{\nu} > 2$, then $\epsilon \leqslant \left\lfloor \frac{d(u)}{\bar{\nu}} \right\rfloor$, and hence $d(u) \geqslant \bar{\nu}\epsilon$. Substituting this into $d(u)(r-1) + (r-\bar{\nu})\epsilon < n$, we obtain $[(r-1)\bar{\nu} + r - \bar{\nu}]\epsilon < n$, that is, $[r + (r-2)\bar{\nu}]\epsilon < n$. Therefore,

$$F < n + \epsilon(2\bar{\nu} - r) \leqslant n + \frac{2\bar{\nu} - r}{r + (r-2)\bar{\nu}}\, n = \left(\frac{r}{r-2} - \frac{r^2}{(r-1)[r + (r-2)\bar{\nu}]} \right) n \leqslant \frac{r}{r-1} n.$$

Consequently, $\rho(H)^2 \leqslant \frac{F}{r-1} \leqslant \frac{r}{(r-1)^2} n$, and hence $\rho(H) \leqslant \sqrt{\frac{r}{(r-1)^2} n}$. This completes the proof. $\square$

Note that when $r \geqslant 4$, attaining the above upper bound requires $\bar{\nu} = r$ and $|S| = 0$.

We now define the r-fan hypergraph Fan^r. For $r \geqslant 2$, the r-fan Fan^r is the linear r-graph consisting of r edges $f_1, \ldots, f_r$, all containing a common vertex u (called the center), together with an additional edge g that intersects each f_i in a vertex distinct from u.

Motivated by the extremal structure suggested by Theorem 27, we conclude with the following problem.

Problem 28. For fixed $r \geqslant 4$ and sufficiently large n satisfying $n-1 \equiv 0 \pmod{r-1}$, does the coalescence of $\left\lfloor \frac{n-1}{r(r-1)} \right\rfloor$ copies of Fan^r and the hyperstar $S^r_{\frac{n-1}{r-1} - r\left\lfloor \frac{n-1}{r(r-1)} \right\rfloor + 1}$, obtained by identifying their centers, maximize the spectral radius among all C_5-free linear n-vertex r-graphs?

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References

[1] C. Berge. Hypergraphs: Combinatorics of Finite Sets. North-Holland, Amsterdam, 1989.

[2] L. Babai, B. Guiduli. Spectral extrema for graphs: the Zarankiewicz problem. Electron. J. Combin., 16 (2009), #R123.

[3] J. Cooper, A. Dutle. Spectra of uniform hypergraphs. Linear Algebra Appl., 436 (2012) 3268-3292.

[4] C. Collier-Cartaino, N. Graber, T. Jiang. Linear Turán numbers of r-uniform linear cycles and related Ramsey numbers. Combin. Probab. Comput., 27 (2018) 358-386.

[5] K. Chang, K. Pearson, T. Zhang. Errata for Perron-Frobenius theorem for nonnegative tensors. Commun. Math. Sci., 10 (2012) 1025-1025.

[6] K. Chang, K. Pearson, T. Zhang. On eigenvalue problems of real symmetric tensors. J. Math. Anal. Appl., 350 (2009) 416-422.

[7] K. Chang, K. Pearson, T. Zhang. Perron-Frobenius theorem for nonnegative tensors. Commun. Math. Sci., 6 (2008) 507-520.

[8] M. Chen, X. Zhang. Some new results and problems in spectral extremal graph theory. (Chinese) J. Anhui Univ. Nat. Sci., 42 (2018) 12-25.

[9] B. Ergemlidze, E. Győri, A. Methuku. Asymptotics for Turán numbers of cycles in 3-uniform linear hypergraphs. J. Combin. Theory, Ser. A, 163 (2019) 163-181.

[10] L. Fang, A. Chang, W. Xu, G. Gao. The spectral radius of 3-graphs without Berge paths of given length. Discrete Appl. Math., 377 (2025) 414-428.

[11] Z. Füredi. Turán type problems. Surveys in Combinatorics, Cambridge University Press, (1991) 253-300.

[12] Z. Füredi, A. Gyárfás. An extension of Mantel's Theorem to k-graphs. Amer. Math. Monthly, 127 (2020) 263-268.

[13] S. Friedland, A. Gaubert, L. Han. Perron-Frobenius theorems for nonnegative multilinear forms and extensions. Linear Algebra Appl., 438 (2013) 738-749.

[14] Z. Füredi, M. Simonovits. The history of degenerate (bipartite) extremal graph problems. In Erdős Centennial, Springer, (2013) 169-264.

[15] G. Gao, A. Chang, Y. Hou. Spectral radius on linear r-graphs without expanded K_{r+1}. SIAM J. Discrete Math., 36 (2022) 1000-1011.

[16] D. Gerbner, A. Methuku. M. Vizer, Asymptotics for the Turán number of Berge $K_{2,t}$. J. Comb. Theory, Ser. B, 137 (2019) 264-290.

[17] D. Gerbner, C. Palmer. Extremal results for Berge hypergraphs. SIAM J. Discrete Math., 31 (2017) 2314-2327.

[18] Y. Hou, A. Chang, J. Cooper. Spectral extremal results for hypergraphs. Electron. J. Combin., 28 (2021) #P3.46.

[19] P. Keevash. Hypergraph Turán problems. Surveys in Combinatorics, Cambridge University Press, 392 (2011) 83-139.

[20] Y. Li, W. Liu, L. Feng. A survey on spectral conditions for some extremal graph problems. Advance in Math. (China), 51 (2022) 193-258.

[21] L. Lim. Singular values and eigenvalues of tensors: a variational approach. In: Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP 05), 1 (2005) 129-132.

[22] M. Mantel. Problem 28. Wiskundige Opgaven, 10 (1907) 60-61.

[23] D. Mubayi, J. Verstraëte. A survey of Turán problems for expansions. In Recent Trends in Combinatorics, Springer International Publishing, (2016) 117-143.

[24] Z. Ni, L. Liu, L. Kang. Spectral Turán-type problems on cancellative hypergraphs. Electron. J. Comb., 31 (2) (2024) #P2.32.

[25] V. Nikiforov. A spectral condition for odd cycles in graphs. Linear Algebra Appl., 428 (2008) 1492-1498.

[26] V. Nikiforov. Bounds on graph eigenvalues II. Linear Algebra Appl., 427 (2007) 183-189.

[27] V. Nikiforov. Some new results in extremal graph theory. In R. Chapman (Ed.), Surveys in Combinatorics, London Mathematical Society Lecture Note Series, (2011) 141-182.

[28] V. Nikiforov. The spectral radius of graphs without paths and cycles of specified length. Linear Algebra Appl., 432 (2010) 2243-2256.

[29] L. Qi. Eigenvalues of a real supersymmetric tensor. J. Symbolic Comput., 40 (2005) 1302-1324.

[30] C. She, Y. Fan, L. Kang. Spectral bipartite Turán problems on linear hypergraphs. Discrete Math., 348 (2025) 114435.

[31] C. Timmons. On r-uniform linear hypergraphs with no Berge $K_{2,t}$. Electron. J. Combin., 24 (2017) #P4.34.

[32] P. Turán. On an extremal problem in graph theorem. Mat. Fiz. Lapok, 48 (1941) 436-452.

[33] W. Wang, Z. Huang, H. Shi. The spectral radius of the hypergraphs without the expansion of P_3 or Berge-P_k. Discrete Appl. Math., 380 (2026) 588-598.

[34] H. Wilf. Spectral bounds for the clique and independence numbers of graphs. J. Combin. Theory, Ser. B, 40 (1986) 113-117.

[35] Y. Yang, Q. Yang. Further results for Perron-Frobenius theorem for nonnegative tensors. SIAM J. Matrix Anal. Appl., 31 (2010) 2517-2530.