Arc-Weighted Acyclic Orientation of Graphs

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Abstract

Let D be a digraph, and let $w : E(D) \to \{1, 2, \ldots\}$ be a positive integer weight assignment on the arcs of D. An arc $e = (u, v)$ is called dominating if $w(e) > d^+_{(D,w)}(v)$, where $d^+_{(D,w)}(v)$ denotes the weighted out-degree of v. We say (D, w) is acyclic if every non-empty sub-digraph (D', w) has a dominating arc. For a graph G and a mapping $f : V(G) \to \mathbb{N}$, we say G is arc-weighted f-degenerate if there is an arc-weighted orientation (D, w) of G (i.e., an orientation D of G together with a positive integer weight assignment w) such that (D, w) is acyclic and $d^+_{(D,w)}(v) \leqslant f(v)$ for each vertex v. The arc-weighted degeneracy $d_w(G)$ of G is the minimum d such that G is arc-weighted d-degenerate. Given an arc-weighted acyclic orientation (D, w) of G with $d^+_{(D,w)}(v) \leqslant f(v)$, there is not only an easy algorithm that constructs an (L, M)-colouring of G for any $(f + 1)$-DP-cover (L, M) of G, but also an easy winning strategy for the DP-$(f + 1)$-painting game on G. Moreover, arc-weighted f-degeneracy implies $(f + 1)$-Alon–Tarsi. As an arc-weighted acyclic orientation (D, w) of G with $w(e) = 1$ for all e is equivalent to a (non-weighted) acyclic orientation of G, $d_w(G)$ is bounded from above by the degeneracy $d(G)$ of G. We observe that the difference $d(G) - d_w(G)$ can be arbitrarily large. Thus, $d_w(G) + 1$ can provide a better upper bound than $d(G) + 1$ on its DP-paint number (and hence its DP-chromatic number, paint number and choice number), as well as its Alon–Tarsi number. Thomassen's proof of the 5-choosability of planar graphs can be easily adapted to prove that all planar graphs are arc-weighted 4-degenerate, implying that planar graphs are DP-5-paintable as well as 5-Alon–Tarsi. As an extension of the characterization of degree-choosable graphs, we prove that a connected graph G is arc-weighted $(d_G - 1)$-degenerate unless G is a GDP-tree. In particular, $d_w(G) \leqslant \Delta(G) - 1$ unless G is a complete graph or a cycle. Then we prove that 3-connected non-complete planar graphs are degree-truncated arc-weighted 14-degenerate, implying that such graphs have degree-truncated DP-paint number at most 15. The previously known upper bound for this parameter was 16.

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1 Introduction

For a graph G, an orientation D of G is a digraph obtained from G by replacing each edge uv with an arc (u,v) or (v,u). A digraph D is *acyclic* if it contains no directed cycles. Acyclic orientations of graphs are fundamental objects in graph theory and have numerous applications.

Given a digraph D and a vertex v of D, $N_D^+(v), N_D^-(v)$ are the sets of out-neighbours and in-neighbours of v, respectively. The *out-degree* of v is $d_D^+(v) = |N_D^+(v)|$, and the *in-degree* of v is $d_D^-(v) = |N_D^-(v)|$.

We denote by $\mathbb{N}^G$ the set of mappings $f : V(G) \to \mathbb{N} = \{0, 1, \ldots\}$. For $f, g \in \mathbb{N}^G$ and $a, b \in \mathbb{N}$, $af + bg \in \mathbb{N}^G$ is defined as $(af + bg)(v) = af(v) + bg(v)$. We write $f \leqslant g$ if $f(v) \leqslant g(v)$ for all $v \in V(G)$. For brevity, for an integer k, we denote by k the constant mapping in $\mathbb{N}^G$ defined as $k(v) = k$ for all vertices v. We denote by $d_G(v)$ the degree of v, and use d_G as a mapping in $\mathbb{N}^G$.

Definition 1. For $f \in \mathbb{N}^G$, we say G is *f-degenerate* if there is an acyclic orientation D of G such that $d_D^+(v) \leqslant f(v)$ for each vertex v. The *degeneracy* $d(G)$ is the minimum integer k such that G is k-degenerate.

The number $d(G)+1$ is called the *colouring number* of G (also called the *strict degeneracy* of G), and is an upper bound for many colouring parameters. We shall review the definitions of those parameters in Section 2. Here, for an illustration of these relations, we define list colouring and the choice number of a graph.

Definition 2. Let G be a graph. A *list assignment* of G is a mapping L which assigns to each vertex v of G a set $L(v)$ of permissible colours. Given a list assignment L of G, an *L-colouring* of G is a mapping $\phi : V(G) \to \cup_{v \in V(G)} L(v)$ such that for each vertex v, $\phi(v) \in L(v)$ and for any edge $e = uv$ of G, $\phi(u) \neq \phi(v)$. For $f \in \mathbb{N}^G$, an *f-list assignment* of G is a list assignment L of G with $|L(v)| \geqslant f(v)$ for each vertex v. We say G is *f-choosable* if G is L-colourable for every f-list assignment L. The *choice number* $\mathrm{ch}(G)$ of G is the minimum k for which G is k-choosable.

Assume G is f-degenerate, D is an acyclic orientation of G with $d_D^+(v) \leqslant f(v)$ and L is an $(f+1)$-list assignment of G. Then, D induces an ordering $v_1, v_2, \ldots, v_n$ of the vertices of G such that an edge v_iv_j of G is oriented as (v_i, v_j) if and only if $j < i$. Since v_i has only $d_D^+(v_i)$ neighbours v_j with $j < i$ and $|L(v_i)| \geqslant f(v_i) + 1 > d_D^+(v_i)$, we can properly colour the vertices of G greedily in the order $v_1, v_2, \ldots, v_n$ with colours from their lists. Therefore G is $(f+1)$-choosable.

Definition 3. An *arc-weighted orientation* of a graph G is a pair (D, w), where D is an orientation of G and $w : E(D) \to \{1, 2, \ldots\}$ is a mapping that assigns to each arc of D a positive integer as its weight.

Given an arc-weighted orientation (D, w) of a graph G, for each vertex v, the *weighted out-degree* and *weighted in-degree* of v are defined as

$$d^+_{(D,w)}(v) = \sum_{u \in N_D^+(v)} w((v,u)), \quad d^-_{(D,w)}(v) = \sum_{u \in N_D^-(v)} w((u,v)).$$

A natural problem is how to generalize the concept of acyclic orientation to arc-weighted orientations, and provide better upper bounds for various colouring parameters.

An orientation D of G can be viewed as an arc-weighted orientation (D, w) with $w(e) = 1$ for each arc e. A digraph D is acyclic if and only if every sub-digraph D' has a sink, i.e., a vertex v with $d^+_{D'}(v) = 0$. If D' has at least one arc, then one of the arcs $e = (u, v)$ has its head v being a sink, and hence $w(e) > d^+_{D'}(v)$. This motivates the following definition of a dominating arc in an arc-weighted digraph.

Definition 4. Assume (D, w) is an arc-weighted digraph. An arc $e = (u, v)$ is called *dominating* if $w(e) > d^+_{(D,w)}(v)$.

An arc $e = (u, v)$ with $w(e) = 1$ is dominating if and only if v is a sink. Thus one can alternatively define acyclic digraphs (viewed as arc-weighted digraphs with each arc of weight 1) as follows: A digraph D is acyclic if every non-empty sub-digraph D' of D has a dominating arc. This leads to the following generalization of acyclic digraphs to arc-weighted acyclic digraphs.

Definition 5. An arc-weighted digraph (D, w) is *acyclic* if, for every non-empty sub-digraph D' of D, (D', w) contains a dominating arc. We say that a graph G is *arc-weighted f-degenerate* if G has an arc-weighted acyclic orientation (D, w) with $d^+_{(D,w)}(v) \leqslant f(v)$ for each vertex v.

We interpret a directed cycle C in a digraph D as a "base" for "circulations of flow" in D. In this sense, a directed cycle C in D is not a directed cycle in (D, w) if it contains a dominating arc $e = (u, v)$: the flow of weight $w(e)$ is "blocked" at the vertex v. So we use the term acyclic for such an arc-weighted digraph (D, w). Some other generalizations of acyclic orientation to arc-weighted orientation are discussed in Section 4.

Definition 6. The *arc-weighted degeneracy* $d_w(G)$ of G is the minimum integer k such that G is arc-weighted k-degenerate.

Assume $|E(D)| = m$. Let $D_0 = D$, and for $i = 1, 2, \ldots, m$, let $e_i = (x_i, y_i)$ be a dominating arc in D_{i-1} and $D_i = D_{i-1} - e_i$. We obtain an ordering $e_1 = (x_1, y_1), e_2 = (x_2, y_2), \ldots, e_m = (x_m, y_m)$ of the arcs of D such that

$$w(e_i) > d^+_{(D-\{e_1,e_2,\ldots,e_{i-1}\},w)}(y_i) \qquad (1).$$

Conversely, if the arcs of D can be ordered this way, then every non-empty sub-digraph D' of D has a dominating arc (with respect to w). Thus, the existence of an ordering of the arcs of D satisfying (1) is equivalent to (D, w) being acyclic.

We say (D, w) is *minimal acyclic* if $w(e_i) = d^+_{(D-\{e_1,e_2,\ldots,e_{i-1}\},w)}(y_i)+1$ for $i = 1, 2, \ldots, m$ in (1). Note that if (D, w) is acyclic, then there is a weighting function w' such that $w' \leqslant w$ and (D, w') is minimal acyclic. For the purpose of deriving upper bounds for colouring parameters, it suffices to consider minimal arc-weighted acyclic digraphs.

It follows from the definition that $d_w(G) \leqslant d(G)$ for any graph G. Figure 1 shows that the Petersen graph P has $d_w(P) \leqslant 2$, while $d(P) = 3$. We shall observe later that the difference $d(G) - d_w(G)$ can be arbitrarily large.

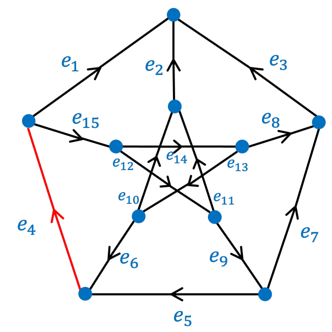

Figure 1: An arc-weighted acyclic orientation of the Petersen graph, where $w(e) = 1$ for a black arc and $w(e) = 2$ for a red arc. For each i, e_i is a dominating arc in $D - \{e_1, \ldots, e_{i-1}\}$.

The degeneracy $d(G)$ gives an upper bound $d(G) + 1$ for many colouring parameters of G, including the DP-paint number and the Alon–Tarsi number of G. We shall show that $d_w(G) + 1$ is also an upper bound for the DP-paint number and the Alon–Tarsi number of G. In the following, we prove that the arc-weighted f-degeneracy implies the $(f + 1)$-choosability, and the proof illustrates the idea of finding a proper colouring via a given arc-weighted acyclic orientation (D, w).

For a subset X of $V(G)$, 1_X is the characteristic function of X, defined as

$$1_X(v) = \begin{cases} 1, & \text{if } v \in X, \\ 0, & \text{otherwise.} \end{cases}$$

For simplicity, we write 1_x for $1_{\{x\}}$.

Lemma 7. *If G is arc-weighted f-degenerate, then G is $(f + 1)$-choosable.*

Proof. Assume G is arc-weighted f-degenerate, and L is an $(f + 1)$-list assignment of G. We prove that G is L-colourable. The proof is based on the induction of the number of edges of G.

If $E(G) = \emptyset$, then choosing any colour $\phi(v) \in L(v)$ for each vertex v gives an L-colouring of G.

Assume $|E(G)| \geqslant 1$, and (D, w) is a minimal arc-weighted acyclic orientation of G with $d^+_{(D,w)} \leqslant f$. Without loss of generality, we may assume that $|L(v)| = f(v) + 1 = d^+_{(D,w)}(v) + 1$ for each vertex v.

By definition, (D, w) has a dominating arc $e = (x, y)$. Let $G' = G - xy$, let $f' = f - (f(y) + 1)1_x$, and let L' be the list assignment of G' obtained from L by deleting $L(y)$ from $L(x)$, i.e., $L'(x) = L(x) - L(y)$ and $L'(v) = L(v)$ for $v \neq x$. Since $|L(y)| = f(y) + 1$, we have $|L'(x)| \geqslant |L(x)| - |L(y)| = f(x) + 1 - (f(y) + 1) = f'(x) + 1$. Hence L' is an $(f' + 1)$-list assignment of G'.

Let $D' = D - \{(x, y)\}$, and let w' be the restriction of w to $E(D')$. Since e is dominating and $f(y) = d^+_{(D,w)}(y)$, we have $w(e) \geqslant f(y) + 1$. Therefore,

$$d^+_{(D',w')}(x) = d^+_{(D,w)}(x) - w(e) \leqslant f(x) - (f(y) + 1) = f'(x),$$

and for every $v \neq x$, $d^+_{(D',w')}(v) \leqslant f(v) = f'(v)$. Thus $d^+_{(D',w')} \leqslant f'$. Moreover, (D',w') is arc-weighted acyclic. According to the induction hypothesis, G' has a L'-colouring ϕ, which is also an L-colouring of G. $\square$

The proof of Lemma 7 gives an algorithm that finds an L-colouring of G via an arc-weighted acyclic orientation (D,w) with $d^+_{(D,w)}(v) < |L(v)|$ for each vertex v. The colouring procedure is to remove edges one by one, to obtain an empty graph G' on the same vertex set, and with each vertex having at least one colour available. Moreover, a proper colouring of G' using the available colours is also a proper colouring of G. It is interesting to compare this colouring process with the greedy colouring process that is used to derive a proper colouring through an acyclic orientation of G.

Let $\mathcal{L}$ be the set of pairs (G,f), where G is a graph, and $f \in \mathbb{N}^G$. For convenience, we let $(\emptyset,\emptyset)$ be a pair in $\mathcal{L}$, which stands for the non-vertex graph (the term "empty graph" is reserved for an edgeless graph). For $f \in \mathbb{Z}^G$ and a subset U of $V(G)$, let $f|U$ be the restriction of f to U, and let $f_{-U} : V(G)-U \to \mathbb{Z}$ be defined as $f_{-U}(x) = f(x)-|N_G(x)\cap U|$ for $x \in V(G)-U$ and $f_{-U} = \emptyset$ if $U = V(G)$. We write f_{-v} for $f_{-\{v\}}$, and when there is no confusion, we may use f for $f|U$. For any $u \in V(G)$, $E_G(u)$ is the set of edges incident to u in G.

Given a pair $(G,f) \in \mathcal{L}$ and a vertex x of G, the *vertex-deletion operation* Del_x on (G,f) is defined as

$$\mathsf{Del}_x(G,f) = (G-x, f_{-x}).$$

We say that operation Del_x is *legal* for (G,f) if $f_{-x}(v) \geqslant 0$ for $v \in V(G-x)$. It is easy to see that a graph G is f-degenerate if, starting from (G,f), there is a sequence of legal vertex-deletion operations that delete all vertices of G. We may view the move of deleting a vertex x as colouring the vertex x: Originally, each vertex x has $f(x)+1$ permissible colours. Once x is coloured, each of its neighbours loses one permissible colour. The legality condition ensures that no remaining vertex loses more colours than it can afford.

In the proof of Lemma 7, in removing the dominating arc $e = (x,y)$, we remove all the colours in $L(y)$ from $L(x)$. Formally, we define the operation of arc-deletion as follows:

Definition 8. Given $(G,f) \in \mathcal{L}$, if $xy \in E(G)$ and $s \geqslant 0$ is an integer, the *arc-deletion* operation $\mathsf{Del}_{(x,y),s}$ is defined as

$$\mathsf{Del}_{(x,y),s}(G,f) = (G',f'),$$

where $G' = G - xy$ and

$$f'(v) = \begin{cases} f(y) - s, & \text{if } v = y \\ f(x) - (f(y)+1-s), & \text{if } v = x, \\ f(v), & \text{otherwise.} \end{cases}$$

The operation $\mathsf{Del}_{(x,y),s}$ is legal for (G,f) if $f(x) \geqslant f(y)+1-s$ and $f(y) \geqslant s$.

For brevity, we write $\mathsf{Del}_{(x,y)}$ for $\mathsf{Del}_{(x,y),0}$.

The motivation for this definition is as follows: Assume L is an $(f+1)$-list assignment of G. To delete the edge xy, we modify the lists at its two ends so that they become disjoint. First we delete s colours from $L(y)$ to obtain the list $L'(y)$, and then delete all colours in $L'(y)$ from $L(x)$ to obtain the list $L'(x)$. For $v \neq x, y$, let $L'(v) = L(v)$. Since $L'(x) \cap L'(y) = \emptyset$, every L'-colouring of $G - xy$ is an L-colouring of G. The legality conditions mean precisely that the two endpoints can afford these losses: y loses s colours, while x is allowed to lose up to $f(y) + 1 - s$ colours. Thus if $G - xy$ is $(f' + 1)$-choosable, then G is $(f + 1)$-choosable.

We may recursively apply a sequence of arc-deletion operations to a pair $(G, f) \in \mathcal{L}$. Let

$$\Omega = \mathsf{Del}_{(x_1,y_1),s_1} \cdots \mathsf{Del}_{(x_k,y_k),s_k}$$

be a sequence of arc-deletion operations. If $k = 0$, then Ω is the identity operation. If $k \geqslant 1$, let

$$\Omega' = \mathsf{Del}_{(x_1,y_1),s_1} \cdots \mathsf{Del}_{(x_{k-1},y_{k-1}),s_{k-1}}.$$

We define

$$\Omega(G, f) = \Omega'\big(\mathsf{Del}_{(x_k,y_k),s_k}(G, f)\big).$$

We say that Ω is legal for (G, f) if $\mathsf{Del}_{(x_k,y_k),s_k}$ is legal for (G, f), and Ω' is legal for $\mathsf{Del}_{(x_k,y_k),s_k}(G, f)$.

The following lemma gives an alternate definition of arc-weighted f-degeneracy. Given a graph G, we denote by G_0 the edgeless graph with $V(G_0) = V(G)$.

Lemma 9. *A graph G is arc-weighted f-degenerate if and only if there is a sequence Ω of legal arc-deletion operations such that $\Omega(G, f) = (G_0, f_0)$ and $f_0 \geqslant 0$.*

Proof. Assume G is arc-weighted f-degenerate and (D, w) is an arc-weighted acyclic orientation of G with $d^+_{(D,w)} \leqslant f$. We prove by induction on the number of edges of G that there is a sequence Ω of legal arc-deletion operations such that $\Omega(G, f) = (G_0, f_0)$ and $f_0 \geqslant 0$. If G is edgeless, then there is nothing to prove. Otherwise, there is an arc $e = (u, v)$ such that $w(e) \geqslant d^+_{(D,w)}(v) + 1$.

Let $s = f(v) - d^+_{(D,w)}(v)$, and let $(G - e, f') = \mathsf{Del}_{(u,v),s}(G, f)$. It follows from the definition that $G - e$ is arc-weighted f'-degenerate. By the induction hypothesis, there is a sequence Ω' of arc-deletion operations such that $\Omega'(G - e, f') = (G_0, f_0)$ with $f_0 \geqslant 0$. Let $\Omega = \Omega'\mathsf{Del}_{(u,v),s}$. Then Ω is a sequence of legal moves such that $\Omega(G, f) = (G_0, f_0)$ with $f_0 \geqslant 0$.

Conversely, assume that there is a sequence Ω of legal moves such that $\Omega(G, f) = (G_0, f_0)$ with $f_0 \geqslant 0$. We show by induction on $\sum_{v \in V(G)} f(v)$ that there is an arc-weighted acyclic orientation (D, w) of G with $d^+_{(D,w)}(v) \leqslant f(v)$ for each vertex v. Again, if G is edgeless, then there is nothing to prove. Assume that the first move is $\mathsf{Del}_{(u,v),s}$, $\mathsf{Del}_{(u,v),s}(G, f) = (G - uv, f')$, and $\Omega(G, f) = \Omega'\mathsf{Del}_{(u,v),s}(G, f) = \Omega'(G - uv, f')$. By the induction hypothesis, there is an arc-weighted acyclic orientation (D', w') of $G - uv$ such that $d^+_{(D',w')}(x) \leqslant f'(x)$ for all $x \in V(G)$. We extend (D', w') to an arc-weighted orientation (D, w) by adding the arc $e = (u, v)$ with weight $w(e) = f'(v) + 1$. Then (D, w) is arc-weighted acyclic and $d^+_{(D,w)}(x) \leqslant f(x)$ for each vertex x. $\square$

The vertex-deletion operation has the same effect as a sequence of arc-deletion operations: Assume $N_G(x) = \{y_1, y_2, \ldots, y_k\}$. Let $s = f(x)$. Then Del_x has the same effect as

$$\mathsf{Del}_{(y_k,x)}\mathsf{Del}_{(y_{k-1},x)} \ldots \mathsf{Del}_{(y_2,x)}\mathsf{Del}_{(y_1,x),s}.$$

Note that after this sequence of operations, x becomes an isolated vertex, which can be viewed as deleted. On the other hand, the arc-deletion operation is more flexible and will lead to a better upper bound for many colouring parameters.

In Section 2, we define some colouring parameters, and prove that the arc-weighted degeneracy plus one provides an upper bound for all these colouring parameters.

In Section 3, we adapt Thomassen's proof to show that every planar graph is arc-weighted 4-degenerate, and hence 5-DP-paintable and 5-AT.

In Section 4, we study the degree-truncated arc-weighted strict degeneracy of a graph. We prove that every 3-connected non-complete planar graph has degree-truncated arc-weighted strict degeneracy at most 15. This implies that every 3-connected non-complete planar graph is degree-truncated 15-DP-colourable, and degree-truncated 15-AT, improving the previously known upper bounds on these parameters by 1.

In Section 5, we make some remarks and pose some open problems. In particular, we compare arc-weighted degeneracy with weak degeneracy introduced by Bernshteyn and Lee [3], and show that the difference $d(G) - d_w(G)$ can be arbitrarily large.

2 f-Alon–Tarsi and f-DP-paintable graphs

For a graph G, $d(G) + 1$ is an upper bound for many colouring parameters, including the DP-paint number and the Alon–Tarsi number of G. We shall show that $d_w(G) + 1$ is also an upper bound for its DP-paint number and Alon–Tarsi number.

In this section, we first review the definitions of some colouring parameters. Then we prove that if a graph G is arc-weighted f-degenerate, then G is $(f + 1)$-Alon–Tarsi and $(f + 1)$-DP-paintable.

For a graph G, let $<$ be an arbitrary ordering of $V(G)$ and let

$$P_G(x_v : v \in V(G)) = \prod_{uv \in E(G), u<v} (x_u - x_v).$$

For $t \in \mathbb{N}^G$, let $c_{G,t}$ be the coefficient of the monomial $\prod_{v \in V(G)} x_v^{t(v)}$ in the expansion of P_G. For $f \in \mathbb{N}^G$, we say that G is *f-Alon–Tarsi* (f-AT for short) if for some $t \in \mathbb{N}^G$ with $t + 1 \leqslant f$, $c_{G,t} \neq 0$. The *Alon–Tarsi number* of G is defined as

$$\mathrm{AT}(G) = \min\{k : G \text{ is } k\text{-AT}\}.$$

For a mapping $\phi : V(G) \to \mathbb{N}$, let $P_G(\phi)$ be the evaluation of P_G at $x_v = \phi(v)$ for $v \in V(G)$. Then $P_G(\phi) \neq 0$ if and only if ϕ is a proper colouring of G, i.e., $\phi(u) \neq \phi(v)$

for each edge uv of G. Given a list assignment L of G, finding a proper L-colouring of G is equivalent to finding a non-zero point of P_G in the grid $\bigtimes_{v \in V(G)} L(v)$. Combinatorial Nullstellensatz [1] gives a sufficient condition for the existence of a non-zero point of a polynomial $f(x_1, x_2, \ldots, x_n)$ in a grid $S_1 \times S_2 \times \ldots \times S_n$: If $\prod_{i=1}^n x_i^{t_i}$ is a highest degree monomial of f, the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in the expansion of f is non-zero and $t_i \leqslant |S_i|-1$, then f has a non-zero point in $S_1 \times S_2 \times \ldots \times S_n$. From this it follows that $\mathrm{ch}(G) \leqslant \mathrm{AT}(G)$ for every graph G.

Recently, Kozik and Podkanowicz introduced a polynomial for edge-weighted graphs. Using this polynomial, they gave an alternative definition of the Alon–Tarsi number [17].

Let $w \in \mathbb{N}^{E(G)}$ be an integer weighting of the edges of G such that $w(e) \geqslant 1$ for each edge e. Let

$$P_{G,w}(x_v : v \in V(G)) = \prod_{e=uv \in E, u<v} (x_u^{w(e)} - x_v^{w(e)}).$$

For $t \in \mathbb{N}^G$, let $c_{(G,w),t}$ be the coefficient of the monomial $\prod_{v \in V(G)} x_v^{t(v)}$ in the expansion of $P_{G,w}$.

It is easy to see that $P_{G,w} = P_G \cdot g$ for some polynomial g. Hence, if $c_{(G,w),t} \neq 0$, then there exists $t' \in \mathbb{N}^G$ such that $t' \leqslant t$ and $c_{G,t'} \neq 0$. Thus we have the following lemma.

Lemma 10. *A graph G is f-AT if and only if there exist a weighting $w \in \mathbb{N}^{E(G)}$ of the edges of G so that $w(e) \geqslant 1$ for each edge e, and $t \in \mathbb{N}^G$ such that $c_{(G,w),t} \neq 0$ and $t+1 \leqslant f$.*

Definition 11. An arc-weighted digraph (D, w) is called *Eulerian* if $d^+_{(D,w)}(v) = d^-_{(D,w)}(v)$ for each vertex v.

Assume (D, w) is an arc-weighted orientation of G. Let

$$\begin{aligned}
\mathcal{E}(D,w) &= \{H \subseteq E(D) : (D[H], w) \text{ is an Eulerian weighted digraph}\},\\
\mathcal{E}_e(D,w) &= \{H \in \mathcal{E}(D,w) : |H| \text{ is even}\},\\
\mathcal{E}_o(D,w) &= \{H \in \mathcal{E}(D,w) : |H| \text{ is odd}\},\\
\mathrm{diff}(D,w) &= |\mathcal{E}_e(D,w)| - |\mathcal{E}_o(D,w)|.
\end{aligned}$$

It is known [17] that

$$c_{(G,w),d^+_{(D,w)}} = \pm\, \mathrm{diff}(D,w).$$

We say (D, w) is an *arc-weighted AT-orientation* of (G, w) if $\mathrm{diff}(D, w) \neq 0$. If (D, w) is an arc-weighted AT-orientation of (G, w) and $w(e) = 1$ for each edge e, then we say D is an AT-orientation of G. Thus, we have the following lemma.

Lemma 12. *For a graph G and $f \in \mathbb{N}^G$, the following are equivalent:*

1. *G is f-AT.*

2. *G has an AT-orientation D with $d^+_D + 1 \leqslant f$.*

3. *G has an arc-weighted AT-orientation (D,w) with $d^+_{(D,w)} + 1 \leqslant f$.*

Theorem 13. *If G has an arc-weighted acyclic orientation (D,w), then G is $(d^+_{(D,w)}+1)$-AT. So if G is arc-weighted f-degenerate, then G is $(f+1)$-AT.*

Proof. For any non-empty sub-digraph D' of D, (D',w) has a dominating arc. Hence (D',w) is not an arc-weighted Eulerian sub-digraph. Thus, (D,w) has only one arc-weighted Eulerian sub-digraph, namely, the empty sub-digraph. Thus, $\mathrm{diff}(D,w) \neq 0$, and (D,w) is an arc-weighted AT-orientation of G. $\square$

Next, we show that if G is arc-weighted f-degenerate, then G is $(f+1)$-DP-paintable. First we review the definition of DP-colouring, painting game, and DP-painting game.

Definition 14. A *DP-cover* of a graph G is a pair (L,M), where L is a mapping that assigns to each vertex v a set $L(v)$ of permissible colours such that $L(v) \cap L(u) = \emptyset$ for $u \neq v$, and $M = \{M_e : e \in E(G)\}$, where for each edge $e = uv$, M_e is a matching between $L(u)$ and $L(v)$. For $f \in \mathbb{N}^G$, we say (L,M) is an f-DP-cover of G if $|L(v)| \geqslant f(v)$ for each vertex $v \in V(G)$.

Definition 15. Given a DP-cover (L,M) of a graph G, an (L,M)-colouring of G is a mapping $\phi : V(G) \to \bigcup_{v \in V(G)} L(v)$ such that for each vertex $v \in V(G)$, $\phi(v) \in L(v)$, and for each edge $e = uv \in E(G)$, $\phi(u)\phi(v) \notin M_e$. We say G is (L,M)-*colourable* if it has an (L,M)-colouring.

For an (L,M)-colouring ϕ of G, we say the vertex v of G is coloured by the colour $\phi(v)$.

Definition 16. Let G be a graph and $f \in \mathbb{N}^G$. We say G is *f-DP-colourable* if for every f-DP-cover (L,M), G has an (L,M)-colouring. The *DP-chromatic number* of G is defined as

$$\chi_{\mathrm{DP}}(G) = \min\{k : G \text{ is } k\text{-DP-colourable}\}.$$

Given an f-list assignment L of a graph G, let (L',M) be the f-DP-cover of G induced by L, where $L'(v) = \{(i,v) : i \in L(v)\}$ for each vertex v of G, and $M = \{M_{uv} : uv \in E(G)\}$ is defined as

$$M_{uv} = \{\{(i,u)(i,v)\} : i \in L(u) \cap L(v)\}$$

for each edge uv of G. It follows directly that G is L-colourable if and only if G is (L',M)-colourable. Therefore if G is f-DP-colourable, then it is f-choosable, and hence $\mathrm{ch}(G) \leqslant \chi_{\mathrm{DP}}(G)$. The proof of Lemma 7 can be easily adapted to prove the following result:

Lemma 17. *If G is arc-weighted f-degenerate, then G is $(f+1)$-DP-colourable.*

The DP-paint number $\chi_{\mathrm{DPP}}(G)$ is the online version of the DP-chromatic number, and is defined through a two-player game [15].

Definition 18. Given a DP-cover (L, M) of a graph G, a *partial* (L, M)*-colouring* of G is a mapping $\phi : W \to \bigcup_{v \in V(G)} L(v)$ for a subset W of $V(G)$ such that for each vertex $v \in W$, $\phi(v) \in L(v)$, and for each edge $e = uv \in E(G[W])$, $\phi(u)\phi(v) \notin M_e$. We say W is the set of coloured vertices in this partial (L, M)-colouring.

Definition 19. For $f \in \mathbb{N}^G$, the f*-DP-painting game* on G is played by two players: Lister and Painter. Initially each vertex v has $f(v)$ tokens and is uncoloured.

In the ith round, Lister chooses a cover (L_i, M_i) of G, and for each vertex v, removes $|L_i(v)|$ tokens from v (thus $|L_i(v)|$ is at most the number of remaining tokens of v). Painter constructs a partial (L_i, M_i)-colouring h_i of G that colours a subset X_i of vertices of G. The game ends either if all vertices of G are coloured, in which case Painter wins the game, or some uncoloured vertex has no token left, in which case Lister wins. We say G is f*-DP-paintable* if Painter has a winning strategy for this game. The *DP-paint number* $\chi_{\mathrm{DPP}}(G)$ of G is the minimum k such that G is k-DP-paintable.

If Lister chooses an f-DP-cover (L, M) in the first move, then Painter needs to respond with an (L, M)-colouring of G (instead of a partial (L, M)-colouring of G), for otherwise, Painter loses the game. So if G is f-DP-paintable, then G is f-DP-colourable, and the converse does not hold.

In the definition of f-DP-painting game, if each cover (L_i, M_i) chosen by Lister is required to satisfy the additional constraint that $|L_i(v)| \leqslant 1$ for each vertex v, then the game is called the f*-painting game*. This definition of the f-painting game is slightly different from (but equivalent to) the standard formulation in the literature (see, e.g., [20]). We say G is f*-paintable* if Painter has a winning strategy in the f-painting game on G, and the *paint number* $\chi_P(G)$ is the minimum k such that G is k-paintable. Equivalently, in the ith round, Lister reveals a set $U_i = \{v : |L_i(v)| = 1\}$ of uncoloured vertices, which can be understood to be the set of vertices v whose lists contain colour i. Painter chooses an independent set X_i of U_i, and colours vertices of X_i by the colour i. In this sense, the f-painting game on G is an online version of f-list colouring of G.

It follows from the definitions that for every graph G, $\mathrm{ch}(G) \leqslant \chi_P(G) \leqslant \chi_{\mathrm{DPP}}(G)$ and $\chi_{\mathrm{DP}}(G) \leqslant \chi_{\mathrm{DPP}}(G)$.

The following lemma from [15] can be viewed as an alternative definition of f-DP-paintability and f-paintability.

Lemma 20. *Let G be a graph and $f \in \mathbb{N}^G$. Then G is f-DP-paintable (respectively, f-paintable) if and only if one of the following holds:*

(i) $V(G) = \emptyset$.

(ii) For every g-DP-cover (L, M) of G with $0 \neq g \leqslant f$ (respectively, for every g-DP-cover (L, M) of G with $g(v) \leqslant \min\{1, f(v)\}$ for each vertex v), there is a subset X of $V(G)$, so that $G[X]$ is (L, M)-colourable, and $G - X$ is $(f - g)$-DP-paintable (respectively, $(f - g)$-paintable).

Note that for a graph G to be f-DP-colourable, each vertex v has at least one colour, i.e., $f(v) \geqslant 1$. In the DP-painting game, we consider DP-colouring of subgraphs of G.

The mappings g, f involved in the description of the game may take the value 0 on some vertices.

Lemma 21. *Let G be a graph, $f, g \in \mathbb{N}^G$ and $0 \neq g \leqslant f+1$. If G is arc-weighted f-degenerate, then there is a non-empty subset X of $V(G)$ such that $G[X]$ is arc-weighted $(g-1)$-degenerate and $G-X$ is arc-weighted $(f-g)$-degenerate.*

Proof. The proof is by induction on $\sum_{v \in V(G)} f(v)$.

Assume that G is arc-weighted f-degenerate. If $E(G) = \emptyset$, then let $X = \{v : g(v) \geqslant 1\}$, and the conclusion holds.

Assume $|E(G)| \geqslant 1$, and (D, w) is a minimal arc-weighted acyclic orientation of G. If there is a vertex x for which $f(x) > d^+_{(D,w)}(x)$, then let $f' = f - 1_{\{x\}}$ and $g' = \max\{0, g - 1_{\{x\}}\}$. Then $g' \leqslant f' + 1$ and G is arc-weighted f'-degenerate. If $g' = 0$, then $g = 1_{\{x\}}$. Let $X = \{x\}$. From the definition it follows that $G[X]$ is arc-weighted $(g-1)$-degenerate and $G - X$ is arc-weighted $(f-g)$-degenerate.

Otherwise, assume $g' \neq 0$. By the induction hypothesis, there is a non-empty subset X of $V(G)$, such that $G[X]$ is arc-weighted $(g'-1)$-degenerate, and $G-X$ is arc-weighted $(f'-g')$-degenerate. Then $G[X]$ is arc-weighted $(g-1)$-degenerate, and $G-X$ is arc-weighted $(f-g)$-degenerate.

Assume $f(x) = d^+_{(D,w)}(x)$ for each vertex x. By definition, (D, w) has a dominating arc $e = (x, y)$. As (D, w) is a minimal arc-weighted acyclic orientation, $w(e) = d^+_{(D,w)}(y) + 1 = f(y) + 1$. Then $G' = G - e$ is arc-weighted f'-degenerate, where $f' = f - (f(y) + 1)1_{\{x\}}$.

Let $g' \in \mathbb{N}^{G'}$ be defined as $g'(v) = g(v)$, except that

$$g'(x) = \min\{f'(x) + 1, \max\{0, g(x) - g(y)\}\}.$$

If $g' = 0$, then $g(x) \leqslant g(y)$ and $g(v) = 0$ for $v \neq x$. But $g(x) \leqslant g(y) = 0$ implies that $g = 0$, a contradiction. Thus $g' \neq 0$. By the induction hypothesis, there is a non-empty subset X of $V(G')$, so that $G'[X]$ is arc-weighted $(g'-1)$-degenerate, and $G' - X$ is arc-weighted $(f' - g')$-degenerate.

Case 1 $x \in X$.

As $G - X = G' - X$ is arc-weighted $(f' - g')$-degenerate, and $(f-g)(v) = (f'-g')(v)$ for $v \in V(G-X)$, we conclude that $G - X$ is arc-weighted $(f-g)$-degenerate. It remains to show that $G[X]$ is arc-weighted $(g-1)$-degenerate.

If $y \notin X$, then $G[X] = G'[X]$ is arc-weighted $(g'-1)$-degenerate, and hence arc-weighted $(g-1)$-degenerate, since $g'(v) \leqslant g(v)$ for every $v \in X$.

Assume that $y \in X$. As $G'[X]$ is arc-weighted $(g'-1)$-degenerate, and $x \in X$, we have $g'(x) \geqslant 1$. Hence $g'(x) = \min\{f'(x) + 1, g(x) - g(y)\} \leqslant g(x) - g(y)$. Let (D', w') be an arc-weighted acyclic orientation of $G'[X]$ with $d^+_{(D',w')} \leqslant g' - 1$. Let (D'', w'') be obtained from (D', w') by adding the arc $e = (x, y)$ of weight $w''(e) = g(y) = g'(y)$. Then

$$w''(e) = g'(y) \geqslant d^+_{(D',w')}(y) + 1 = d^+_{(D'',w'')}(y) + 1,$$

and

$$d^+_{(D'',w'')}(x) = d^+_{(D',w')}(x) + w''(e) < g'(x) + g(y) \leqslant g(x).$$

Hence, e is a dominating arc in (D'', w'') and $(D'' - e, w'') = (D', w')$. Therefore (D'', w'') is an arc-weighted acyclic orientation of $G[X]$ with $d^+_{(D'',w'')} \leqslant g - 1$. Hence $G[X]$ is arc-weighted $(g - 1)$-degenerate.

Case 2 $x \notin X$.

Then $G[X] = G'[X]$ is arc-weighted $(g' - 1)$-degenerate, and $g'(v) = g(v)$ for all $v \in X$. Hence $G[X]$ is arc-weighted $(g - 1)$-degenerate. It remains to show that $G - X$ is arc-weighted $(f - g)$-degenerate.

As $G' - X$ is arc-weighted $(f' - g')$-degenerate, we know that $(f' - g')(x) \geqslant 0$. So $g'(x) \neq f'(x) + 1$ and hence

$$g'(x) = \max\{0, g(x) - g(y)\} \geqslant g(x) - g(y).$$

This implies that $(f' - g')(x) \leqslant (f(x) - (f(y) + 1)) - (g(x) - g(y)) \leqslant (f - g)(x)$ (note that $f(y) + 1 \geqslant g(y)$). If $y \in X$, then $G - X = G' - X$ is arc-weighted $(f - g)$-degenerate. Therefore, we assume that $y \notin X$.

Let (D', w') be an arc-weighted acyclic orientation of $G' - X$ with $d^+_{(D',w')} \leqslant (f' - g')$. Let (D'', w'') be obtained from (D', w') by adding the arc $e = (x, y)$ of weight $w''(e) = f'(y) - g'(y) + 1 = f(y) - g(y) + 1$. Thus, (D'', w'') is an arc-weighted acyclic orientation of $G - X$. As $d^+_{(D'',w'')}(x) \leqslant (f' - g')(x) + w''(e) \leqslant (f - g)(x)$, we have $d^+_{(D'',w'')} \leqslant f - g$, and hence $G - X$ is arc-weighted $(f - g)$-degenerate.

This completes the proof of Lemma 21. $\square$

Theorem 22. *If G is arc-weighted f-degenerate, then G is $(f + 1)$-DP-paintable.*

Proof. We prove this by induction on the number of vertices. If $V(G) = \emptyset$, then this follows from the definition. Assume $V(G) \neq \emptyset$. Assume $g \in \mathbb{N}^G$ and $0 \neq g \leqslant f + 1$ and (L, M) is a g-DP-cover of G. We shall show that there is a non-empty subset X of $V(G)$ such that $G[X]$ is g-DP-colourable, and $G - X$ is $(f + 1 - g)$-DP-paintable. By Lemma 21, there is a non-empty subset X of $V(G)$ such that $G[X]$ is arc-weighted $(g - 1)$-degenerate, and $G - X$ is arc-weighted $(f - g)$-degenerate. According to Lemma 17, $G[X]$ is g-DP-colourable, and according to the induction hypothesis, $G - X$ is $(f + 1 - g)$-DP-paintable. By Lemma 20, G is $(f + 1)$-DP-paintable. $\square$

3 Arc-weighted acyclic orientations of planar graphs

Thomassen [22] proved that every planar graph is 5-choosable. This result was strengthened in [9] to show that every planar graph is 5-DP-colourable. Hefetz [12] asked whether every planar graph is 5-AT. The question remained open for a decade before being answered affirmatively in [27]; two alternative proofs were subsequently given in [11] and [17]. The proofs in [9, 11, 17, 27] are all parallel to Thomassen's proof. This is not a coincidence. The following result shows that Thomassen's proof can be easily adapted to show that every planar graph is arc-weighted 4-degenerate and hence 5-DP-paintable, as well as 5-AT.

Theorem 23. *Assume G is a plane graph with boundary $B(G) = [v_1v_2 \dots v_n]$. Let*

$$f_{G,v_1v_2}(v) = \begin{cases} 0, & \text{if } v \in \{v_1, v_2\}, \\ 2, & \text{if } v \in B(G) - \{v_1, v_2\}, \\ 4, & \text{if } v \text{ is an interior vertex.} \end{cases}$$

Then $G - v_1v_2$ is arc-weighted f_{G,v_1v_2}-degenerate.

Proof. The proof is by induction on $|V(G)|$. The case $|V(G)| \leqslant 3$ is trivial. Assume $|V(G)| \geqslant 4$, and the theorem holds for graphs with fewer vertices. If $B(G)$ has a chord $e = v_iv_j$ that separates G into two parts G_1, G_2, with $v_1, v_2 \in V(G_1)$, then we apply the induction hypothesis to obtain an arc-weighted acyclic orientation (D_1, w_1) of $G_1 - v_1v_2$ with $d^+_{(D_1,w_1)}(v) \leqslant f_{G_1,v_1v_2}(v)$ and an arc-weighted acyclic orientation (D_2, w_2) of $G_2 - v_iv_j$ with $d^+_{(D_2,w_2)}(v) \leqslant f_{G_2,v_iv_j}(v)$.

Now we show that $(D, w) = (D_1 \cup D_2, w_1 \cup w_2)$ is an arc-weighted acyclic orientation of $G - v_1v_2$ with $d^+_{(D,w)}(v) \leqslant f_{G,v_1v_2}(v)$. We extend f_{G_1,v_1v_2} to $V(G)$ by letting $f_{G_1,v_1v_2}(v) = 0$ for $v \in V(G) - V(G_1)$, and extend f_{G_2,v_iv_j} to $V(G)$ similarly. Then $f_{G_1,v_1v_2} + f_{G_2,v_iv_j} \leqslant f_{G,v_1v_2}$. So $d^+_{(D,w)}(v) \leqslant f_{G,v_1v_2}(v)$ for each vertex v of G.

For any non-empty sub-digraph D' of D, if $E(D') \cap E(D_1) \neq \emptyset$, then since (D_1, w_1) is acyclic, there is an arc $e = (x, y) \in E(D') \cap E(D_1)$ that is dominating in $(D' \cap D_1, w)$. As the arcs in D_2 do not increase the out-degree of y, e is also a dominating arc for D'. If $E(D') \cap E(D_1) = \emptyset$, then there is an arc $e = (x, y) \in E(D') \cap E(D_2)$ that is a dominating arc in $(D' \cap D_2, w)$, which is also a dominating arc in (D', w).

Assume $B(G)$ has no chord. Let $G' = G - v_n$. Let $u_1, u_2, \dots, u_k$ be the interior neighbours of v_n. By the induction hypothesis, there is an arc-weighted acyclic orientation (D', w') of $G' - v_1v_2$ with $d^+_{(D',w')}(v) \leqslant f_{G',v_1v_2}(v)$ for all $v \in V(G')$. We extend D' to an arc-weighted orientation (D, w) of G, by adding arcs $(v_n, v_1), (v_n, v_{n-1})$ of weight 1, and arcs (u_i, v_n) for $i = 1, 2, \dots, k$ of weight 2.

It follows from the definition that $d^+_{(D,w)}(v) \leqslant f_{G,v_1v_2}(v)$ for all $v \in V(G)$. Now we show that (D, w) is an arc-weighted acyclic orientation of $G - v_1v_2$. Let D' be a non-empty sub-digraph of D. We show that (D', w) has a dominating arc.

1. If $(v_n, v_1) \in E(D')$, then (v_n, v_1) is a dominating arc in (D', w).
2. Otherwise, if $(u_i, v_n) \in E(D')$ for some i, then (u_i, v_n) is a dominating arc in (D', w).
3. Otherwise, if $E(D') - \{(v_n, v_{n-1})\} \neq \emptyset$, then $(D' - \{(v_n, v_{n-1})\}, w)$ has a dominating arc by the induction hypothesis, which is also a dominating arc in (D', w).
4. Otherwise, (v_n, v_{n-1}) is a dominating arc of D'. □

4 Degree-truncated arc-weighted strict degeneracy

A graph G is called *degree-choosable* (respectively, *degree-DP-colourable*) if G is d_G-choosable (respectively, d_G-DP-colourable).

The class of degree-choosable graphs was characterized in [5, 10]: A connected graph G is not degree-choosable if and only if it is a Gallai-tree, i.e., each block of G is either a clique or an odd cycle (a block of G is a maximal 2-connected subgraph of G). The class of degree-DP-colourable graphs was characterized in [2, 16]: A connected graph G is not degree-DP-colourable if and only if it is a GDP-tree, i.e., each block of G is either a clique or a cycle.

Lemma 24 below characterizes arc-weighted $(d_G - 1)$-degenerate graphs.

Lemma 24. *A connected graph G is not arc-weighted $(d_G - 1)$-degenerate if and only if G is a GDP-tree.*

Proof. If G is a GDP-tree, then G is not degree-DP-colourable, and hence it is not arc-weighted $(d_G - 1)$-degenerate.

Assume that G is a connected graph and is not a GDP-tree. Then G has a block B that is neither a cycle nor a complete graph. We claim that B has an induced theta subgraph H, i.e., three internally disjoint paths connecting two vertices. Indeed, if B is chordal, since B is not complete, B contains at least two maximal cliques. As B is 2-connected, the two maximal cliques have at least two common vertices. Choosing two vertices in their intersection and one vertex from each clique outside the intersection gives H. If B is not chordal, then B has an induced cycle C of length at least 4. If there is a vertex $v \in V(B) - V(C)$ adjacent to at least three vertices of C, then v together with a segment of C induces a theta graph. Otherwise, there is an induced path P connecting two vertices of C, whose interior vertices are disjoint from C and not adjacent to the vertices in C. Then $P \cup C$ is again an induced theta graph.

Let H be an induced theta graph, and let $e = uv$ be an edge of H with $d_H(u) = 3$ and $d_H(v) = 2$. Then H has an acyclic orientation D' in which v is the unique source and u is the unique sink. Reverse the direction of the edge uv as an arc (u, v) of weight 2, we obtain an arc-weighted digraph (D, w) (other arcs of D are of weight 1). For any non-empty subdigraph D'' of D, either $(u, v) \in E(D'')$ and hence is a dominating arc in (D'', w) or D'' has an arc (x, y) with y being a sink and hence is a dominating arc. So (D, w) is acyclic and $d^+_{(D,w)}(x) \leqslant d_H(x) - 1$ for each vertex x. Let D'' be an acyclic orientation of $G - E(H)$ such that the vertices of H are the only sources. Such an orientation exists, for example, by ordering the vertices so that every vertex in $V(G) - V(H)$ has a neighbor preceding it and all vertices of H come first and then orienting every edge from the earlier endpoint to the later endpoint. Then $(D \cup D'', w)$ (with all other arcs of weight 1) is acyclic and each vertex x has $d^+_{(D \cup D'',w)}(x) \leqslant d_G(x) - 1$. Thus, G is arc-weighted $(d_G - 1)$-degenerate. $\square$

Definition 25. Let G be a graph, and k be a positive integer. Let

$$\phi_k(v) = \min\{k, d_G(v)\}.$$

If G is ϕ_k-choosable, then we say G is degree-truncated k-choosable. The degree-truncated choice number of G is

$$\mathrm{ch}^{\mathrm{dt}}(G) = \min\{k : \ G \text{ is degree-truncated } k\text{-choosable}\},$$

and for a family $\mathcal{G}$ of graphs,

$$\mathrm{ch}^{\mathrm{d}}(\mathcal{G}) = \max\{\mathrm{ch}^{\mathrm{d}}(G) : G \in \mathcal{G}\}.$$

If G is not degree-choosable, then G is not degree-truncated k-choosable for any k. Thus, the degree-truncated choice number of Gallai-trees is not defined. For connected graphs G other than Gallai-trees, $\mathrm{ch}^{\mathrm{d}}(G)$ is well-defined and is at most the maximum degree of G.

Problems concerning the degree-truncated choice number of planar graphs were first considered by Bruce Richter (see [13]). For any positive integer k, the complete bipartite graph K_{2,k^2} is planar, not a Gallai-tree, and not degree-truncated k-choosable. Thus there is no constant upper bound on the degree-truncated choice number of planar graphs that are not Gallai-trees.

Let $\mathcal{P}$ be the family of 3-connected non-complete planar graphs. Richter [13] asked whether $\mathrm{ch}^{\mathrm{d}}(\mathcal{P}) \leqslant 6$.

Richter's question has attracted some attention [7, 13] and has remained open for a decade. It also remained open whether there is a constant upper bound for $\mathrm{ch}^{\mathrm{d}}(\mathcal{P})$. Recently, a negative answer was given in [26], where a graph $G \in \mathcal{P}$ with $\mathrm{ch}^{\mathrm{d}}(G) = 8$ was constructed. In [26] it was also proved that $\mathrm{ch}^{\mathrm{d}}(\mathcal{P}) \leqslant 16$. These results were improved in [14] and [23]. In [23], a graph $G \in \mathcal{P}$ with $\mathrm{ch}^{\mathrm{d}}(G) = 10$ was given, and it was also proved in [23] that $\mathrm{ch}^{\mathrm{d}}(\mathcal{P}) \leqslant 11$. So $\mathrm{ch}^{\mathrm{d}}(\mathcal{P}) = 10$ or 11, and it was conjectured in [23] that $\mathrm{ch}^{\mathrm{d}}(\mathcal{P}) = 10$.

We generalize this concept to the degree-truncated arc-weighted strict degeneracy of graphs.

Definition 26. Let G be a graph, and k be a positive integer. Let $\phi_k(v) = \min\{k, d_G(v)\}$. If G is arc-weighted $(\phi_k - 1)$-degenerate, then we say G is *degree-truncated arc-weighted strict k-degenerate*. The degree-truncated arc-weighted strict degeneracy of G is defined as

$$\mathrm{sd}^{\mathrm{d}}_w(G) = \min\{k : G \text{ is arc-weighted } (\phi_k - 1)\text{-degenerate}\}.$$

For a family $\mathcal{G}$ of graphs,

$$\mathrm{sd}^{\mathrm{d}}_w(\mathcal{G}) = \max\{\mathrm{sd}^{\mathrm{d}}_w(G) : G \in \mathcal{G}\}.$$

Given a graph G, the degree-truncated DP-chromatic number $\chi^{\mathrm{d}}_{\mathrm{DP}}(G)$ of G, the degree-truncated DP-paint number $\chi^{\mathrm{d}}_{\mathrm{DPP}}(G)$ of G, the degree-truncated paint number $\chi^{\mathrm{d}}_{P}(G)$ of G and the degree-truncated AT number $\mathrm{AT}^{\mathrm{d}}(G)$ of G are defined similarly. For a family $\mathcal{G}$ of graphs, $\chi^{\mathrm{d}}_{\mathrm{DP}}(\mathcal{G})$, $\chi^{\mathrm{d}}_{\mathrm{DPP}}(\mathcal{G})$, $\chi^{\mathrm{d}}_{P}(\mathcal{G})$, and $\mathrm{AT}^{\mathrm{d}}(\mathcal{G})$ are the maximum of the corresponding parameters for the graphs in $\mathcal{G}$. It follows from Theorems 13 and 22 that for any family $\mathcal{G}$ of graphs,

$$\chi^{\mathrm{d}}_{\mathrm{DP}}(\mathcal{G}) \leqslant \chi^{\mathrm{d}}_{\mathrm{DPP}}(\mathcal{G}) \leqslant \mathrm{sd}^{\mathrm{d}}_w(\mathcal{G}), \text{ and } \chi^{\mathrm{d}}_{P}(\mathcal{G}) \leqslant \mathrm{AT}^{\mathrm{d}}(\mathcal{G}) \leqslant \mathrm{sd}^{\mathrm{d}}_w(\mathcal{G}).$$

In [26] it was proved that $\chi^{\mathrm{d}}_{\mathrm{DP}}(\mathcal{P}) \leqslant 16$. Theorem 27 below implies that

$$\chi^{\mathrm{d}}_{\mathrm{DP}}(\mathcal{P}),\ \chi^{\mathrm{d}}_{\mathrm{DPP}}(\mathcal{P}),\ \chi^{\mathrm{d}}_{P}(\mathcal{P}),\ \mathrm{AT}^{\mathrm{d}}(\mathcal{P}) \leqslant 15.$$

Theorem 27. *For the family $\mathcal{P}$ of 3-connected non-complete planar graphs, we have* $\mathrm{sd}_w^{\boldsymbol{d}}(\mathcal{P}) \leqslant 15$.

Proof. Assume that the theorem is not true and G is a counterexample with minimal number of vertices and subject to this, with maximum number of edges.

Let

$$V_1 = \{v \in V(G) : d_G(v) < 15\} \text{ and } V_2 = \{v \in V(G) : d_G(v) \geqslant 15\}.$$

Let $\phi(v) = \min\{15, d_G(v)\}$.

If $V_2 = \emptyset$, then $\phi = d_G$. Since a 3-connected non-complete graph is not a GDP-tree, Lemma 24 implies that G is arc-weighted $(\phi-1)$-degenerate. Hence, we may assume that $V_2 \neq \emptyset$.

Let $\mathcal{Q}$ be the set of connected components of $G[V_1]$. For each $Q \in \mathcal{Q}$, let θ_Q be the face of $G[V_2]$ that contains Q, and let $V(\theta_Q)$ denote the set of vertices on the boundary of θ_Q. Note that $G[V_2]$ need not be connected, and hence $G[V(\theta_Q)]$ need not be connected.

Note that any two vertices u, v of V_2 on the boundary of a same face of G are adjacent, for otherwise adding the edge uv is a counterexample to Theorem 27 with more edges than G, in contradiction to the choice of G.

As G is 3-connected, this implies that each face θ of $G[V_2]$ contains at most one connected component of $G[V_1]$. So the faces θ_Q, $Q \in \mathcal{Q}$, are pairwise distinct. Also every vertex v of $V(\theta_Q)$ has a neighbor in Q, for otherwise, by adding an edge joining the two neighbors of v in $V(\theta_Q)$ results in a counterexample with more edges.

Given a non-empty plane graph X, let $\Theta(X)$ be the bipartite graph with partite sets $V(X)$ and $F(X)$ (the set of faces of X) in which $v\theta$ is an edge if and only if $v \in V(\theta)$, i.e., v is on the boundary of θ.

An acyclic orientation of a graph H with exactly one source and one sink is called a *bipolar orientation* of H. It is well-known [8, 18] that for a 2-connected plane graph H and two adjacent vertices u, v on the boundary of the infinite face of H, there is a bipolar orientation D such that

1. u is the unique source, v is the unique sink, and
2. for each face θ of H, the induced sub-digraph $D[V(\theta)]$ also has a unique source s_θ and a unique sink t_θ, and
3. for each vertex $w \in V(H) - \{u, v\}$, there are exactly two faces θ incident to w such that $w \notin \{s_\theta, t_\theta\}$.

Lemma 28. *Assume H is a non-empty plane graph and uv is an edge on the boundary of the infinite face of H. There exists, for each face θ, a subset $P_H(\theta)$ of $V(\theta)$ such that the following hold:*

1. *For each finite face $\theta \in F(H)$, $|P_H(\theta)| = |V(\theta)| - 2$, and for the infinite face θ^*, $P_H(\theta^*) = V(\theta^*)$.*
2. *Each vertex in $\{u, v\}$ is contained only in $P_H(\theta^*)$, and each other vertex of H is contained in at most two of the sets $\{P_H(\theta) : \theta \in F(H)\}$.*

3. *If $v_1, v_2 \in V(\theta) - P_H(\theta)$, then there is a cycle C in H containing v_1 and v_2 such that θ is contained in the interior of C.*

Proof. If H is 2-connected, then let D be a bipolar orientation of H with u be the unique source and v be the unique sink. For each interior face θ of H, let $P_H(\theta) = V(\theta) - \{s_\theta, t_\theta\}$ and for the infinite face θ^* of H, let $P_H(\theta^*) = V(\theta^*)$. Then each of u, v is contained only in $P_H(\theta^*)$, and each vertex $w \neq u, v$ of H is contained in exactly two $P_H(\theta)$ for which $w \in V(\theta)$ and $w \notin \{s_\theta, t_\theta\}$.

If H is connected but not 2-connected, then let $B_1, B_2, \ldots, B_k$ be the blocks of H. Assume uv is a boundary edge of B_1. For each $i \geqslant 2$, let $u_i v_i$ be a boundary edge of B_i such that u_i is the cut-vertex of H contained in B_i along the shortest path from B_i to B_1. For each i, for each face $\theta \in F(B_i)$, let $P_{B_i}(\theta)$ be defined as above.

Then for each face θ of H, there is a subset I_θ of $\{1, 2, \ldots, k\}$ such that θ is the intersection $\cap_{i \in I_\theta} \theta_i$, where θ_i is a face of B_i. Hence $V(\theta) = \cup_{i \in I_\theta} V(\theta_i)$. If θ is the infinite face of H, then each θ_i is the infinite face of B_i, and we define $P_H(\theta) = V(\theta)$. Otherwise, there is a unique index $j \in I_\theta$ such that θ_j is a finite face of B_j. In this case, define

$$P_H(\theta) = V(\theta) \setminus (V(\theta_j) \setminus P_{B_j}(\theta_j)) = V(\theta) \setminus \{s_{\theta_j}, t_{\theta_j}\}.$$

We verify that these sets have the desired properties. Property (1) follows immediately from the definition. For property (2), for a vertex $x \notin \{u, v\}$, let B_x be the block containing x that is closest to B_1. If x is a non-cut vertex, then x is contained in no other block. If x is a cut-vertex, then in every other block B_i containing x, the vertex x is the unique source and is not contained in $P_{B_i}(\theta')$ for any finite face θ' of B_i. Hence

$$|\{P_H(\theta) : \theta \in F(H), x \in P_H(\theta)\}| = |\{P_{B_x}(\theta) : \theta \in F(B_x), x \in P_{B_x}(\theta)\}| \leqslant 2.$$

If u or v is a cut vertex, then it is respectively the source or sink in B_1 and is the unique source in every other block containing it. Thus u and v belong only to $P_H(\theta^*)$. Finally, for each finite face θ, $V(\theta) \setminus P_H(\theta) = \{s_{\theta_j}, t_{\theta_j}\}$. The boundary cycle of θ_j contains these two vertices and encloses θ, proving property (3).

If H is not connected, then let $J_1, \ldots, J_k$ be the connected components of H. Similarly, for each face θ of H, there is a subset I_θ of $\{1, 2, \ldots, k\}$ such that $\theta = \cap_{i \in I_\theta} \theta_i$ and $V(\theta) = \cup_{i \in I_\theta} V(\theta_i)$, where θ_i is a face of J_i. If θ is the infinite face of H, then θ_i is the infinite face of J_i for each $i \in I_\theta$. Otherwise, there is exactly one index $j \in I_\theta$ such that θ_j is a finite face of J_j. Let $P_H(\theta) = \cup_{i \in I_\theta} P_{J_i}(\theta_i)$. It is easy to verify that $P_H(\theta)$ satisfies (1), (2) and (3). $\square$

We apply Lemma 28 to $G[V_2]$. For $Q \in \mathcal{Q}$, vertices in $P_{G[V_2]}(\theta_Q)$ are *protectors* of Q, and vertices in $V(\theta_Q) - P_{G[V_2]}(\theta_Q)$ are *non-protectors* of Q. Thus each component $Q \in \mathcal{Q}$ has at most two non-protectors, and each vertex $v \in V_2$ is a protector of at most two components of $\mathcal{Q}$.

Lemma 29. *For each component $Q \in \mathcal{Q}$, there is a leaf-block B of Q such that B contains a non-root vertex adjacent to a protector of Q.*

Proof. Assume to the contrary that no protector of Q is adjacent to a non-root vertex of a leaf-block of Q.

First, note that Q has at least two leaf-blocks. Otherwise, Q itself is a leaf-block, and every vertex of Q is a non-root vertex. Recall that every vertex of $V(\theta_Q)$ is adjacent to some vertex of B. Thus at least one protector in $V(\theta_Q)$ is adjacent to Q, which contradicts the assumption.

Let B_1 and B_2 be two leaf-blocks of Q. By the assumption, we have $N_{G[V_2]}(U(B_i)) \subseteq \{v_1, v_2\}$ for $i = 1, 2$, where $U(B_i)$ is the set $V(B_i)$ with its unique cut vertex removed, and v_1, v_2 are the two non-protectors of Q (if they exist).

If $N_{G[V_2]}(U(B_i)) = \{v_1, v_2\}$ for $i = 1, 2$, then there is a cycle C contained in $\{v_1, v_2\} \cup U(B_1) \cup U(B_2)$ such that all vertices of $Q - C$ lie in the interior of C, while all vertices of $V(\theta_Q) - \{v_1, v_2\}$ lie in the exterior of C. Hence no vertex of $V(\theta_Q) - \{v_1, v_2\}$ can be adjacent to $U(B_1) \cup U(B_2)$, contradicting the fact that every vertex of $V(\theta_Q)$ is adjacent to some vertex of Q.

Therefore, without loss of generality, assume that $v_2 \notin N_G(U(B_1))$. Let x be the root of B_1. Since no protector of Q is adjacent to a vertex of $U(B_1)$ and v_1, v_2 are the only non-protectors of Q, we have $N_G(U(B_1)) \setminus U(B_1) \subseteq \{x, v_1\}$. Since Q has another leaf block, the set $\{x, v_1\}$ separates $U(B_1)$ from the rest of G, contradicting the 3-connectivity of G. $\square$

For each $Q \in \mathcal{Q}$, choose a non-root vertex v_Q of a leaf block of Q such that

1. v_Q is adjacent to a protector of Q.
2. Subject to (1), the number of non-protectors adjacent to v_Q is minimum.

For each $Q \in \mathcal{Q}$, we now describe how to orient the edges inside Q and the edges between Q and V_2.

Assume that Q is not a GDP-tree. Take an arc-weighted acyclic orientation (D_Q, w_Q) of Q given by Lemma 24, such that $d^+_{(D_Q,w_Q)}(v) \leqslant d_Q(v) - 1$ for every vertex $v \in V(Q)$. Orient all edges between Q and V_2 from Q to V_2 with weight 1.

Assume that Q is a GDP-tree. Note that $d_Q(v_Q) \leqslant 3$ and if $d_Q(v_Q) = 3$, then the leaf-block of Q containing v_Q is a copy of K_4. We consider two cases. Arcs defined below have weight 1, unless it is specified otherwise.

Case 1 $d_Q(v_Q) \leqslant 2$, or $d_Q(v_Q) = 3$ and v_Q is adjacent to at most one non-protector of Q.

In this case, we orient edges in Q so that it is acyclic with v_Q as the unique source, orient all edges between Q and V_2 as arcs from Q to V_2, except that if v is a protector of Q adjacent to v_Q, then the edge vv_Q is oriented as an arc (v, v_Q) of weight 5.

Case 2 $d_Q(v_Q) = 3$ and v_Q is adjacent to two non-protectors of Q.

In this case, the leaf-block B of Q containing v_Q is a copy of K_4, and by Lemma 28, there is a cycle C contained in $V(\theta_Q)$ containing the two non-protectors of Q, and Q is contained in the interior of C. Then B has one vertex u_Q that is contained in the interior of the triangle T induced by the other three vertices of B. Note that since v_Q is adjacent to the two non-protectors in C, $v_Q \neq u_Q$.

If u_Q is the root of B (and hence a cut-vertex of Q), then there is another leaf-block B' contained in the interior of T. Then a non-root vertex v' of B' is adjacent to a vertex of V_2. As the two non-protectors of Q lie in the exterior of T, v' is not adjacent to any non-protector of Q. Hence v' is adjacent to a protector of Q. This contradicts the choice of v_Q. Thus u_Q is a non-root vertex of B. If u_Q is adjacent to some vertex of V_2, then u_Q is adjacent to a protector of Q, but not adjacent to any non-protector of Q. Again this contradicts the choice of v_Q. Thus u_Q is not adjacent to any vertex of V_2. We orient all edges between Q and V_2 as arcs from Q to V_2, except that if v is a protector of Q adjacent to v_Q, then the edge vv_Q is oriented as an arc (v, v_Q) of weight 5. Orient edges in Q so that it is acyclic with v_Q as the unique source, and with u_Q be a sink.

After this process has been completed for all $Q \in \mathcal{Q}$, we obtain an arc-weighted orientation (D', w') of $G - E(G[V_2])$. Let (D'', w'') be an arc-weighted acyclic orientation of $G[V_2]$ with $d^+_{(D'',w'')}(v) \leqslant 4$ for each vertex $v \in V_2$. By Theorem 23, such an orientation exists. Let $(D, w) = (D', w') \cup (D'', w'')$. Now we show that (D, w) is an arc-weighted acyclic orientation and $d^+_{(D,w)}(v) \leqslant \phi(v) - 1$ for each vertex v.

First, we check the out-degree of the vertices of D. If Q is not a GDP-tree, then for every $v \in V(Q)$, $d^+_{(D,w)}(v) \leqslant d^+_{(D_Q,w_Q)}(v) + d_G(v) - d_Q(v) \leqslant d_G(v) - 1$. If Q is a GDP-tree, then all outgoing arcs of a vertex $v \in V(Q)$ have weight 1, and every such vertex has an incoming arc. Hence $d^+_{(D,w)}(v) \leqslant d_G(v) - 1$. For $v \in V_2$, an edge vw with $w \in V_1$ is an out-going arc of v if and only if $w = v_Q$ for some $Q \in \mathcal{Q}$ such that v is a protector of Q. As v is a protector of at most two components $Q \in \mathcal{Q}$, and each such arc (v, v_Q) has weight 5, we know that such arcs contribute at most 10 to $d^+_{(D,w)}(v)$. As $d^+_{(D'',w'')}(v) \leqslant 4$, we conclude that $d^+_{(D,w)}(v) \leqslant 14 = \phi(v) - 1$.

We now show that (D, w) is arc-weighted acyclic. Let H be an arbitrary non-empty subdigraph of D. For each $Q \in \mathcal{Q}$, let E_Q be the set of arcs of D incident with a vertex of Q, and let

$$\mathcal{Q}_H = \{Q \in \mathcal{Q} : E(H) \cap E_Q \neq \emptyset\}.$$

We prove by induction on $|\mathcal{Q}_H|$ that H contains a dominating arc.

If $\mathcal{Q}_H = \emptyset$, then H is a non-empty subdigraph of D'', and hence contains a dominating arc. Now assume that $\mathcal{Q}_H \neq \emptyset$, and choose $Q \in \mathcal{Q}_H$.

First suppose that Q is not a GDP-tree. If $H - E_Q$ is non-empty, then, by induction, $H - E_Q$ contains a dominating arc. Since all arcs between Q and V_2 are directed from Q to V_2, restoring the arcs in E_Q does not increase the out-degree of the head of this arc. Hence it remains dominating in H. If $H - E_Q$ is empty, then either H contains an arc from Q to V_2, which is dominating since its head is a sink in H, or H is a non-empty subdigraph of (D_Q, w_Q) and hence contains a dominating arc.

It remains to consider the case that Q is a GDP-tree. If Q is oriented as in Case 2 and H contains an arc incident with u_Q, then this arc is dominating, since every such arc is directed into u_Q and u_Q has no neighbour in V_2.

Otherwise, suppose that H contains an arc (v, v_Q), where v is a protector of Q. In Case 1, we have $d^+_{(H,w)}(v_Q) \leqslant 4$. The same inequality holds in Case 2, since the arc between v_Q and u_Q is not present in H. Therefore, (v, v_Q) is dominating, as $w((v, v_Q)) =$

$5 > d^+_{(H,w)}(v_Q)$.

Finally, suppose that neither of the preceding cases applies. If $H - E_Q$ is non-empty, then it contains a dominating arc by induction. Since H contains no arc from a protector of Q to v_Q, every arc of $H \cap E_Q$ between Q and V_2 is directed from Q to V_2. Thus restoring E_Q does not increase the out-degree of the head of the dominating arc, so it remains dominating in H.

If $H - E_Q$ is empty, then either H contains an arc from Q to V_2, whose head is a sink in H, or H is contained in the acyclic orientation of Q. In the latter case, an arc incident with a sink of H is dominating. Thus, in every case, H contains a dominating arc.

This completes the proof of Theorem 27. □

A graph H is a *minor* of a graph G if a copy of H can be obtained from G by deleting vertices and edges and contracting edges. A family $\mathcal{G}$ of graphs is *minor-closed* if $G \in \mathcal{G}$ and H is a minor of G implies that $H \in \mathcal{G}$. If $\mathcal{G}$ is minor-closed and does not contain all the graphs, then $\mathcal{G}$ is *proper minor-closed*. It was proved in [26] that if $\mathcal{G}$ is a proper minor-closed family of graphs, and $G \in \mathcal{G}$ is highly connected and non-complete, then $\mathrm{ch}^{\mathrm{d}}(G) \leqslant k$ for some constant k. Theorem 30 strengthens this result to arc-weighted strict degeneracy. The proof uses a similar idea as in the proof of Theorem 27, and is parallel to the proof of Theorem 5 in [26].

Theorem 30. *Assume $\mathcal{G}$ is a proper minor-closed family of graphs. Let s be the minimum integer such that $K_{s,t} \notin \mathcal{G}$ for some positive integer t. Then there is a constant $k = k(s)$ such that every s-connected graph in $\mathcal{G}$ which is not a GDP-tree is degree-truncated arc-weighted strict k-degenerate. In particular, for any surface Σ, there is a constant k such that every 3-connected non-complete graph embedded in Σ is degree-truncated arc-weighted strict k-degenerate.*

Proof. First we note that for any proper minor-closed family $\mathcal{G}$ of graphs, there exist positive integers s, t such that the complete bipartite graph $K_{s,t} \notin \mathcal{G}$. This is so, because there is a complete graph $K_s \notin \mathcal{G}$ (for otherwise $K_s \in \mathcal{G}$ for all s and hence every graph is in $\mathcal{G}$ and $\mathcal{G}$ is not proper), and K_s is a minor of $K_{s,s}$. So the integer s as described in the theorem is well-defined.

Let t be the minimum integer such that $K_{s,t} \notin \mathcal{G}$. Let

$$q = 4^{s+1} s! st(s+t-1) + 1, \text{ and } k = 2^{s+2} tq.$$

Let

$$f(v) = \min\{k-1, d_G(v) - 1\}.$$

We prove that G is arc-weighted f-degenerate by using the alternate definition of arc-weighted f-degeneracy given in Lemma 9. For simplicity, we also use the vertex-deletion operation (recall that Del_v can be realized by a sequence of arc-deletion operations, as described in Section 1). Thus we shall apply the two operations: arc-deletion and vertex-deletion to (G, f) to obtain a no-vertex graph.

If $s = 1$, then graphs in $\mathcal{G}$ have maximum degree at most $t-1$. For $G \in \mathcal{G}$ which is not a GDP-tree, G is arc-weighted $(d_G - 1)$-degenerate, and hence degree-truncated arc-weighted $(k-1)$-degenerate. Assume $s \geqslant 2$.

Let $V_1 = \{v \in V(G) : d(v) < k\}$ and $V_2 = V(G) - V_1$. Let $p = |V_2|$. Using the fact that G is $K_{s,t}$-minor-free, we can show that G is $(2^{s+2}t - 1)$-degenerate.

The following two lemmas, proved in [21], are needed in the proof.

Lemma 31. *Every non-empty graph G with at least $2^{s+1}t|V(G)|$ edges has a $K_{s,t}$-minor.*

Lemma 32. *If G is a bipartite graph with partite sets A and B, and with at least $(s-1)|A| + 4^{s+1}s!t|B|$ edges, then G has a $K_{s,t}$-minor.*

The following is a consequence of Lemma 32.

Corollary 33. *Assume G is a bipartite graph with partite sets A and B, and each vertex in A has degree at least s. If G has no $K_{s,t}$-minor, then $|E(G)| < 4^{s+1}s!st|B|$ and consequently, B has a vertex v of degree at most $4^{s+1}s!st$.*

Proof. Assume G has no $K_{s,t}$-minor. By Lemma 32, $|E(G)| < (s-1)|A| + 4^{s+1}s!t|B|$. So $|E(G)| - (s-1)|A| < 4^{s+1}s!t|B|$.

As each vertex in A has degree at least s, we have $|E(G)| \geqslant s|A|$, which implies that $|E(G)| - (s-1)|A| \geqslant |E(G)| - \frac{s-1}{s}|E(G)| = |E(G)|/s$. Therefore $|E(G)| \leqslant s(|E(G)| - (s-1)|A|) < 4^{s+1}s!st|B|$. □

It follows from Lemma 31 that each subgraph of G has minimum degree at most $2^{s+2}t - 1$, that is, G is $(2^{s+2}t-1)$-degenerate. Hence the vertices of V_2 can be ordered as $w_1, w_2, \dots, w_p$ so that each w_i has at most $2^{s+2}t - 1$ neighbors w_j with $j < i$.

We process the vertices of V_2 in the order $w_1, w_2, \dots, w_p$. When processing w_i, for each neighbor w_j of w_i with $j > i$, we apply the arc-deletion operation

$$\mathsf{Del}_{(w_j, w_i),\, f(w_i) - q + 1},$$

where $f(w_i)$ denotes the current value of f at the time the operation is applied. Proceeding in this way for $i = 1, 2, \dots, p$, all edges of $G[V_2]$ are deleted. The choice of k and the fact that G is $(2^{s+2}t - 1)$-degenerate guarantee that all these operations are legal. Let (G', f') be the resulting pair. Then V_2 is an independent set in G' and $f'(w_i) \geqslant q - 1$ for every $w_i \in V_2$.

We contract each connected component Q of $G'[V_1]$ into a vertex v_Q. Denote the resulting bipartite graph by H. By Corollary 33, there exists an ordering $u_1, u_2, \dots, u_p$ of the vertices of V_2 such that, for each i, the vertex u_i has degree at most $4^{s+1}s!st$ in $H - \{u_{i+1}, \dots, u_p\}$.

We process the vertices of V_2 one by one in the order $u_1, u_2, \dots, u_p$. Denote by X the set of deleted vertices. Initially $X = \emptyset$. In the process, deleted vertices are added to X. So X is a dynamic set, whose size increases in the process.

To process u_i, we do the following:

- For each connected component Q of $G[V_1]$ that is a GDP-tree, delete vertices of Q that are not adjacent to any vertex of V_2 in $G - X$ and that are not cut-vertices of $Q - X$.
- For each connected component Q of $G[V_1]$ that is a GDP-tree and for which u_i is the last vertex in the ordering $u_1, u_2, \ldots, u_p$ having a neighbour in Q, choose a non-root vertex w of a leaf block of $Q - X$. As w is not a cut-vertex of $Q - X$, w is adjacent to u_i. Apply the operation $\mathsf{Del}_{(u_i,w)}$.
- Delete vertex u_i.

It can be shown that all these operations are legal. Denote by (G'', f'') the resulting pair after all vertices of V_2 are processed (note that all vertices of V_2 are deleted).

Assume that Q is a connected component of $G[V_1]$ which is not a GDP-tree. No vertex of Q is deleted during the processing of V_2. Hence, $f''(x) = d_Q(x) - 1$ for every $x \in V(Q)$.

Since Q is connected and is not a GDP-tree, Lemma 24 implies that Q is arc-weighted $(d_Q - 1)$-degenerate. Therefore, Q is arc-weighted $f''|_{V(Q)}$-degenerate. By Lemma 9, there is a sequence of legal arc-deletion operations which transforms $(Q, f''|_{V(Q)})$ into an edgeless pair (Q_0, g) with $g \geqslant 0$. All vertices of Q_0 are isolated and hence can subsequently be deleted legally by vertex-deletion operations.

Assume Q is a connected component of $G[V_1]$ and a GDP-tree. The graph $Q - X$ is connected and contains a vertex w such that $f''(w) \geqslant d_{Q-X}(w)$, while $f''(x) \geqslant d_{Q-X}(x) - 1$ for every other vertex $x \in V(Q - X)$. By repeatedly deleting a leaf of a spanning tree of $Q - X$ rooted at w, and deleting w last, all vertices can be deleted legally, as the required inequalities are preserved after each deletion. □

5 Some remarks and questions

As a generalization of degeneracy, Bernshteyn and Lee [3] introduced the concept of weak degeneracy.

A graph G is f-degenerate if starting with the pair (G, f), we can apply legal vertex-deletion operations recursively to delete all vertices. Assume that each vertex x has a set $L(x)$ of $f(x)+1$ acceptable colours. The vertex-deletion operation Del_x is interpreted as x being coloured and each of its neighbours y losing one permissible colour. Thus applying Del_x to (G, f) results in $(G - x, f')$, where $f' = f - 1_{N_G(x)}$. The operation Del_x is legal if $f' \geqslant 0$.

For the definition of weak degeneracy, another operation is used, called delete-save operation. This operation is defined as follows:

Definition 34. Assume $(G, f) \in \mathcal{L}$, $x \in V(G)$ and $y \in N_G(x)$. The *delete-save operation* $\mathsf{Del\text{-}Save}_{(x,y)}$ on (G, f) is defined as

$$\mathsf{Del\text{-}Save}_{(x,y)}(G, f) = (G - x, f_{-x} + 1_y).$$

The operation $\mathsf{Del\text{-}Save}_{(x,y)}$ is *legal* for (G, f) if $f_{-x} + 1_y \geqslant 0$ and $f(x) > f(y)$.

Assume that L is a list assignment of G with $|L(v)| = f(v) + 1$ for each vertex v. The delete-save operation $\mathsf{Del\text{-}Save}_{(x,y)}$ means that we colour vertex x with a colour in $L(x) - L(y)$, and hence x is deleted, and each uncoloured neighbour of x loses one permissible colour, however, y does not lose any colour. For x to be coloured in this way, the set $L(x) - L(y)$ needs to be non-empty. Thus, the operation is legal if $f(x) > f(y)$.

We say that G is *weak f-degenerate* if there is a sequence Ω of legal vertex-deletion and delete-save operations such that $\Omega(G, f) = (\emptyset, f')$, where $\emptyset$ is a non-vertex graph and $f' \geqslant 0$.

Observe that $\mathsf{Del\text{-}Save}_{(x,y)}$ applied to (G, f) has the same effect as $\mathsf{Del}_x\mathsf{Del}_{(x,y)}$ applied to (G, f). Therefore, if G is weak f-degenerate, then it is arc-weighted f-degenerate. The following lemma shows that weak f-degeneracy corresponds to a special type of arc-weighted acyclic orientation.

Lemma 35. *A graph G is weak f-degenerate if and only if there is an arc-weighted orientation (D, w) such that $d^+_{(D,w)}(v) \leqslant f(v)$ for each vertex v, and for each sub-digraph D' of D, either D' has a sink, or (D', w) has a dominating arc (u, v) such that u is a sink in $D' - (u, v)$.*

Proof. We prove the lemma by induction on the number of vertices of G. The case $|V(G)| = 1$ is trivial. Assume $|V(G)| \geqslant 2$.

"$\Rightarrow$": Assume that G is weak f-degenerate. Then G has a vertex x such that $\mathsf{Del}_x(G, f) = (G - x, f')$ and $G - x$ is weak f'-degenerate, or there is an arc (x, y) such that $\mathsf{Del\text{-}Save}_{(x,y)}(G, f) = (G - x, f')$ and $G - x$ is weak f'-degenerate.

According to the induction hypothesis, $G - x$ has an arc-weighted orientation (D', w') that satisfies the required conditions. If $\mathsf{Del}_x(G, f) = (G - x, f')$, then extend (D', w') to an orientation (D, w) of G by orienting every edge xz as the arc (z, x) of weight 1. Then $d^+_{(D,w)}(z) = d^+_{(D',w')}(z) + 1 \leqslant f'(z) + 1 = f(z)$ for every $z \in N_G(x)$, while $d^+_{(D,w)}(x) = 0 \leqslant f(x)$, and $d^+_{(D,w)}(v) = d^+_{(D',w')}(v) \leqslant f'(v) = f(v)$ for every other vertex v. Every sub-digraph of D containing x has x as a sink, while the condition for every other sub-digraph follows from the induction hypothesis.

If $\mathsf{Del\text{-}Save}_{(x,y)}(G, f) = (G - x, f')$, then extend (D', w') to (D, w) by orienting xy as (x, y) of weight $d^+_{(D',w')}(y) + 1$, and every other edge xz as (z, x) of weight 1. Since $\mathsf{Del\text{-}Save}_{(x,y)}$ is legal and $f'(y) = f(y)$, $d^+_{(D,w)}(x) = d^+_{(D',w')}(y) + 1 \leqslant f(y) + 1 \leqslant f(x)$. For $z \in N_G(x) \setminus \{y\}$, $d^+_{(D,w)}(z) = d^+_{(D',w')}(z) + 1 \leqslant f'(z) + 1 = f(z)$, and $d^+_{(D,w)}(v) = d^+_{(D',w')}(v) \leqslant f'(v) = f(v)$ for every other vertex v. Thus $d^+_{(D,w)} \leqslant f$. For any sub-digraph D' containing (x, y), (x, y) is dominating, and x is a sink in $D' - (x, y)$. If D' contains x but not (x, y), then x is a sink; otherwise, the result follows from the induction hypothesis.

"$\Leftarrow$": Assume that (D, w) is an arc-weighted orientation of G satisfying the conditions in the lemma. Assume that D has a sink x. Then Del_x is legal. Let $\mathsf{Del}_x(G, f) = (G-x, f')$. Since every edge incident to x is oriented towards x, the restriction $(D - x, w)$ satisfies the conditions of the lemma with respect to f'. By the induction hypothesis, $G - x$ is weak f'-degenerate. Hence G is weak f-degenerate.

Suppose that D has no sink. By assumption, D has a dominating arc (x, y) such that x is a sink in $D - (x, y)$. Thus (x, y) is the only out-arc of x. If $f(y) = d^+_{(D,w)}(y)$, then

$f(x) \geqslant w(x,y) > f(y)$, so $\mathsf{Del\text{-}Save}_{(x,y)}$ is legal. Let $(G-x,f') = \mathsf{Del\text{-}Save}_{(x,y)}(G,f)$. Otherwise, $f(y) \geqslant d^+_{(D,w)}(y)+1$, let $(G-x,f') = \mathsf{Del}_x(G,f)$. In either case, $(D-x,w)$ satisfies the conditions of the lemma with respect to f'. By the induction hypothesis, $G-x$ is weak f'-degenerate. Hence G is weak f-degenerate. $\square$

Assume L is an $(f+1)$-list assignment of G. If G is f-degenerate (respectively, weak f-degenerate or arc-weighted f-degenerate), then there is an algorithm, which we call the D-Algorithm (respectively, the WD-Algorithm or AWD-algorithm), that finds an L-colouring of G.

Both the D-Algorithm and the WD-Algorithm delete one vertex in each move. The difference is that in the WD-Algorithm, a move that deletes vertex v may "protect" one neighbour of v from losing a colour.

The AWD-Algorithm deletes one edge in each move. The advantage of the operation $\mathsf{Del}_{(x,y),s}$ lies in its flexibility: it prohibits the vertex x from using any colour assigned to the vertex y without immediately colouring x. This flexibility is useful in many cases, for example in Thomassen's proof of the 5-choosability of planar graphs. In [3], a proof was given for the weak 4-degeneracy of planar graphs. However, that proof has a gap. A correct and more complicated proof of this result was given in [4]. Our paper has gone through a series of revisions. The first version of this paper appeared on arXiv in August 2023. At that time, there was no correct proof of the weak 4-degeneracy of planar graphs. Also, the proof of arc-weighted 4-degeneracy of planar graphs is shorter and more transparent than the correct proof of their weak 4-degeneracy.

The weak degeneracy $d^*(G)$ of a graph G is the minimum k such that G is weak k-degenerate. From the definition it follows that $d_w(G) \leqslant d^*(G)$. In [25] it was proved that for any integer r, there are r-regular graphs G with $d^*(G) \leqslant \lfloor \frac{r}{2} \rfloor + 2$. As r-regular graphs have degeneracy r, we know that the difference $d(G) - d^*(G)$ can be arbitrarily large, and hence the difference $d(G) - d_w(G)$ can be arbitrarily large. The difference between $d^*(G)$ and $d_w(G)$ can also be arbitrarily large. The argument of Bradshaw and Zeng [6] can be interpreted in our framework to give, for some absolute constant $c > 0$, $d_w(K_{n,n}) \leqslant n - c\sqrt{n \log n}$. On the other hand, Bernshteyn and Lee [3] showed that $d^*(K_{n,n}) \geqslant n - \sqrt{2n}$. Consequently,

$$d^*(K_{n,n}) - d_w(K_{n,n}) \geqslant c\sqrt{n \log n} - \sqrt{2n},$$

which tends to infinity as $n \to \infty$.

In earlier versions of this paper, different generalizations of acyclic orientation were studied, and the arc-weighted acyclic orientation (D,w) in the current paper was called Type-2 acyclic in one of the earlier versions. The Type-1 acyclic arc-weighted orientation is defined as follows:

Definition 36. We say that an arc-weighted digraph (D,w) is *Type-1* acyclic if (D,w) contains no nonempty Eulerian sub-digraph, i.e. a sub-digraph (D',w) with $d^+_{(D',w)}(v) = d^-_{(D',w)}(v)$ for each vertex v.

The Type-1 acyclic arc-weighted digraph (D, w) is also a natural generalization of an acyclic digraph to the arc-weighted setting. Compared to the Type-2 arc-weighted acyclic digraph (which we simply call arc-weighted acyclic digraph in this paper), there are two undesirable features of Type-1 acyclic arc-weighted digraphs:

(1) It is co-NP-complete to determine whether a given arc-weighted digraph is Type-1 acyclic. Equivalently, determining whether a given arc-weighted digraph contains a non-empty Eulerian sub-digraph is NP-complete. This is through a reduction from the zero-sum subset problem: given a set S of integers, the zero-sum subset problem asks if S has a non-empty subset S' with $\Sigma_{a \in S'} a = 0$. It is well-known that the zero-sum subset problem is NP-complete. Let $S^+ = \{a \in S : a > 0\}$ and $S^- = \{a \in S : a < 0\}$. Let (D, w) be the arc-weighted digraph consisting of two vertices u, v, $|S^+|$ parallel arcs (u, v) of weights S^+ and $|S^-|$ parallel arcs (v, u) of weights $\{|a| : a \in S^-\}$. Then D has a non-empty sub-digraph D' with $d^+_{(D',w)}(u) = d^-_{(D',w)}(u)$ if and only if S has a non-empty subset S' with sum $\Sigma_{a \in S'} a = 0$.

(2) Given a Type-1 acyclic arc-weighted digraph (D, w), let $f(v) = d^+_{(D,w)}(v) + 1$ for each vertex v. We know that G is f-AT and therefore f-paintable and f-choosable. However, given an f-list assignment L of G, we do not know of an efficient algorithm to find an L-colouring of G. It is an interesting and challenging problem to find an efficient algorithm (even a sub-exponential algorithm) that constructs an L-colouring from such an orientation.

Also, the following question remains open.

Question 37. Given a Type-1 acyclic arc-weighted digraph (D, w), let $f(v) = d^+_{(D,w)}(v)+1$ for each vertex v. Is G f-DP-colourable? Or even f-DP-paintable?

As shown in Lemma 35, weak degeneracy corresponds to a special family of arc-weighted acyclic orientations (D, w). This was called Type-4 acyclic orientation in an earlier version of this paper. Type-3 acyclic orientation corresponds to a generalization of weak degeneracy, where the operation $\mathsf{Del\text{-}Save}_{(x,y)}$ is replaced by $\mathsf{Del\text{-}Save}_{(x,U)}$ where U is a subset of $N_G(x)$: When colouring a vertex x, instead of protecting one neighbour y of x (by colouring x with a colour from $L(x) - L(y)$), we may protect a subset U of $N_G(x)$ (by colouring x with a colour from $L(x) - \cup_{y \in U} L(y)$, provided that $|L(x)| > \sum_{y \in U} |L(y)|$). In this language, the vertex deletion operation Del_x is the same as $\mathsf{Del\text{-}Save}_{x,\emptyset}$. For the reasons mentioned above and for brevity, we have decided to concentrate on Type-2 arc-weighted acyclic orientations in the current version of the paper. However, the other types of arc-weighted acyclic orientation are also useful and have been studied in some published papers [6, 19, 24], where the earlier version of this paper is cited.

For a given graph G and $f \in \mathbb{N}^G$, it can be determined in linear time whether G is f-degenerate. Given an arc-weighted orientation (D, w) of a graph G, a naive peeling procedure determines in time $O(m^2)$ whether (D, w) is acyclic. The following question remains open:

Question 38. What is the complexity of the following problem:

Instance: A graph G and a function $f \in \mathbb{N}^G$.

Question: Is G arc-weighted f-degenerate?

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