On Sets of Subspaces with Restricted Hyperplane Intersection Numbers

Tim Alderson[a] Simeon Ball[b]

Submitted: Apr 3, 2026; Accepted: Aug 23, 2026; Published: Sep 25, 2026

© The authors. Released under the CC BY-ND license (International 4.0).

Abstract

Let $\mathcal{X}$ be a set of $(h-1)$-dimensional subspaces of $\mathrm{PG}(kh-1,q)$ with the property that every hyperplane contains at most t elements of $\mathcal{X}$. We prove the upper bound $|\mathcal{X}| \leqslant (t-k+2)q^h+t$, and characterise the structure of $\mathcal{X}$ in the case of equality. We call spanning sets attaining this bound *length-maximal*. For $k=3$, these sets are higher-dimensional analogues of maximal arcs; when $h=1$ they are precisely the maximal arcs of $\mathrm{PG}(2,q)$, which for $t<q$ exist if and only if q is even and t divides q. For $k=4$ and $q^h>2$, we show that any length-maximal set must satisfy $t=q^h+1$ and that every hyperplane is either a t-secant or a 1-secant. Such sets exist for all q and h, arising as field reductions of ovoids of $\mathrm{PG}(3,q^h)$. For $k \geqslant 5$ and $q^h>3$, no length-maximal set exists. The case $q^h=3$, that is $(q,h)=(3,1)$, is exceptional: there, examples arising from the ternary Golay code exist for $k=5$ and $k=6$, and none exist for $k \geqslant 7$. In the language of additive codes, these results assert that additive codes over $\mathbb{F}_{q^h}$ attaining the natural Griesmer-type bound do not exist when the code dimension is 5 or more and $q^h>3$, apart from these two sporadic $\mathbb{F}_3$ examples. We also determine the strongly regular graphs associated with length-maximal sets.

Mathematics Subject Classifications: 51E20, 94B05, 05E30

1 Introduction

A *maximal arc* $\mathcal{X}$ in $\mathrm{PG}(2,q)$ is a set of points with the property that every line is incident with 0 or t points of $\mathcal{X}$. It is straightforward to prove that $|\mathcal{X}|=(t-1)q+t$ and that t divides q. Denniston [10] provided examples for q even and all possible values of t. Further examples were discovered by Thas [13, 15], Mathon [12], and Hamilton and Mathon [11]. When q is odd, maximal arcs were shown not to exist by Ball, Blokhuis and Mazzocca

[a]Department of Mathematics and Statistics, University of New Brunswick Saint John, Saint John, NB, Canada (`tim.alderson@unb.ca`).
[b]Departament de Matemàtiques, Universitat Politècnica de Catalunya, Barcelona, Spain (`simeon.michael.ball@upc.edu`).

[2]; see also [3]. The parameter set $(t, q) = (3, 9)$ was ruled out earlier by Cossu [7] and the parameter set $(t, q) = (3, 3^h)$ by Thas [14].

In this article we consider a natural generalisation of maximal arcs to higher dimensions. A maximal arc in $\mathrm{PG}(2, q^h)$ yields, by field reduction, a set of $(h-1)$-dimensional subspaces in $\mathrm{PG}(3h-1, q)$ with the property that every hyperplane contains either 0 or t subspaces. This leads us to study sets $\mathcal{X}$ of $(h-1)$-dimensional subspaces in $\mathrm{PG}(kh-1, q)$ with the property that every hyperplane contains at most t subspaces of $\mathcal{X}$.

There is an equivalent formulation in terms of additive codes. An additive $[n, k, d]_q^h$ code C is an $\mathbb{F}_q$-linear subspace of $\mathbb{F}_{q^h}^n$ of size q^{kh} in which every non-zero vector has at least d non-zero coordinates. Let G be a $kh \times n$ matrix whose row span is C. By fixing a basis $\{e_1, \ldots, e_h\}$ for $\mathbb{F}_{q^h}$ over $\mathbb{F}_q$ we can write

$$G = \sum_{j=1}^{h} e_j G_j,$$

where G_j is a $kh \times n$ matrix with entries from $\mathbb{F}_q$. For each coordinate $i \in \{1, \ldots, n\}$ we define a subspace π_i as the span of the i-th column of G_j, $j \in \{1, \ldots, h\}$. Thus π_i is a subspace of $\mathrm{PG}(kh-1, q)$ of projective dimension at most $h-1$. The property that every non-zero vector of C has at least d non-zero coordinates is equivalent to the property that every hyperplane of $\mathrm{PG}(kh-1, q)$ contains at most $t = n - d$ elements of the multi-set $\mathcal{X} = \{\pi_1, \ldots, \pi_n\}$.

In this setting the length bound $|\mathcal{X}| \leqslant (t - k + 2)q^h + t$ is a Griesmer-type bound for additive codes. Codes attaining this bound are additive few-weight codes; for $k = 3$ they are two-weight codes whose non-zero weights are n and $n - t$. An additive permutation on each coordinate produces an equivalent code containing the all-one vector.

Although maximal arcs do not exist in planes of odd characteristic, Mathon [8] discovered an example of 21 lines in $\mathrm{PG}(5, 3)$ (equivalently, $k = 3$, $h = 2$, $q = 3$, $t = 3$) with the property that every hyperplane contains 0 or 3 lines. This example shows that a direct generalisation of the Ball–Blokhuis–Mazzocca nonexistence theorem to higher dimensions is not possible. To our knowledge it is the only known example in odd characteristic, and the uniqueness question remains open.

The main results of this paper are as follows. Theorem 1 establishes the bound $|\mathcal{X}| \leqslant (t - k + 2)q^h + t$ and characterises equality. The divisibility condition $t - k + 3 \mid q^h$ (Theorem 5) provides a further necessary condition for length-maximal sets. Theorem 7 shows that for $k = 4$ and $q^h > 2$, any length-maximal set must satisfy $t = q^h + 1$ and have no zero-secant hyperplanes. Such sets exist for all q and h, as field reductions of ovoids of $\mathrm{PG}(3, q^h)$ (Example 17). Theorem 8 shows that no length-maximal set exists for $k = 5$ when $q^h > 3$, and by induction this extends to all $k \geqslant 5$. The borderline case $(q, h) = (3, 1)$, is genuinely exceptional in that length-maximal sets exist for $k = 5$ and $k = 6$ but not for $k \geqslant 7$ (Example 10 and Proposition 11). In Section 4 we determine the strongly regular graphs associated with length-maximal sets. The case $q = 2$ is discussed in Section 5.

Throughout, $k \geqslant 2$, and dimension refers to projective dimension.

2 An upper bound on the size of $\mathcal{X}$

Theorem 1. *Let $\mathcal{X}$ be a multi-set of $(h-1)$-dimensional subspaces in $\mathrm{PG}(kh-1,q)$ with the property that every hyperplane contains at most t subspaces of $\mathcal{X}$. Then*

$$|\mathcal{X}| \leqslant (t-k+2)q^h + t, \tag{1}$$

with equality if and only if

1. *every set of $(k-1)$ elements of $\mathcal{X}$ spans a $((k-1)h-1)$-dimensional subspace, and*

2. *every hyperplane contains either $0, 1, \ldots, k-3$ or t elements of $\mathcal{X}$.*

Proof. Write $n = |\mathcal{X}|$. Let $x_1, \ldots, x_{k-2}$ be elements of $\mathcal{X}$ and let $\pi = \langle x_1, \ldots, x_{k-2} \rangle$. The dimension of π is $(k-2)h-1-e_1$ for some $e_1 \geqslant 0$. Let $(k-1)h-1-e_1-e_2$ denote the maximum dimension of $\langle \pi, x \rangle$ over $x \in \mathcal{X} \setminus \{x_1, \ldots, x_{k-2}\}$.

We count pairs (x, H) where $x \in \mathcal{X} \setminus \{x_1, \ldots, x_{k-2}\}$ and H is a hyperplane containing both x and π. By hypothesis each such hyperplane contains at most $t-k+2$ elements of $\mathcal{X} \setminus \{x_1, \ldots, x_{k-2}\}$, so this count is at most

$$\frac{q^{2h+e_1}-1}{q-1}(t-k+2).$$

Since $(k-1)h-1-e_1-e_2$ is the maximum dimension of $\langle \pi, x \rangle$, each element of $\mathcal{X} \setminus \{x_1, \ldots, x_{k-2}\}$ lies in at least

$$\frac{q^{h+e_1+e_2}-1}{q-1}$$

hyperplanes through π, so the count is at least

$$\frac{q^{h+e_1+e_2}-1}{q-1}(n-k+2).$$

Combining these inequalities,

$$n \leqslant \frac{q^{2h+e_1}-1}{q^{h+e_1+e_2}-1}(t-k+2)+k-2.$$

This bound is maximised when $e_1 = e_2 = 0$, giving $n \leqslant (t-k+2)q^h + t$.

Equality requires $e_1 = e_2 = 0$, which means every $(k-1)$ elements of $\mathcal{X}$ span a $((k-1)h-1)$-flat (part 1 of Theorem 1), and every hyperplane containing $k-2$ elements of $\mathcal{X}$ contains exactly t elements of $\mathcal{X}$ (part 2 of Theorem 1). □

The bound (1) is called the *maximal arc bound*, and a set $\mathcal{X}$ attaining it which spans $\mathrm{PG}(kh-1,q)$ is called *length-maximal*. The spanning condition excludes the degenerate case $|\mathcal{X}| = t = k-2$, in which both conditions of Theorem 1 are vacuous, and corresponds to the requirement that the associated additive code have size q^{kh}. A length-maximal $\mathcal{X}$ corresponds to an additive few-weight code over $\mathbb{F}_{q^h}$; by part 2 of Theorem 1 its non-zero weights lie in $\{\, n-t,\ n,\ n-1,\ \ldots,\ n-(k-3) \,\}$, and for $k=3$ it is a two-weight code with weights n and $n-t$.

Lemma 2. *If $\mathcal{X}$ is a length-maximal set of $(h-1)$-dimensional subspaces in* $\mathrm{PG}(kh-1,q)$ *with $q^h > 2$ and $k \geqslant 4$, then $t \geqslant k$.*

Proof. Since $\mathcal{X}$ spans $\mathrm{PG}(kh-1,q)$, we have $|\mathcal{X}| \geqslant k$. If $t \leqslant k-2$ then $t-k+2 \leqslant 0$, so $|\mathcal{X}| = (t-k+2)q^h + t \leqslant t \leqslant k-2$, a contradiction; hence $t \geqslant k-1$. If $t = k-1$ then $|\mathcal{X}| = q^h + k - 1$ and the corresponding additive code is a $[q^h+k-1, k, q^h]$ MDS code over $\mathbb{F}_{q^h}$. Since $k \geqslant 4$ and $q^h > 2$, a classical result of Bruen and Silverman [5, Theorem 6] requires $36 \mid q^h$, whence $t \geqslant k$. $\square$

Lemma 3. *Let $\mathcal{X}$ be a set of $(h-1)$-dimensional subspaces in* $\mathrm{PG}(3h-1,q)$ *with the property that every hyperplane contains at most t elements of $\mathcal{X}$. If $|\mathcal{X}| = (t-1)q^h + t$ then $t \leqslant q^h + 1$. If $t < q^h + 1$ then there exists a hyperplane disjoint from $\mathcal{X}$. If $t = q^h + 1$ then every hyperplane contains exactly t elements of $\mathcal{X}$.*

Proof. Counting pairs (x, H) where $x \in \mathcal{X}$ and H is a hyperplane containing x,

$$|\mathcal{X}| \cdot \frac{q^{2h}-1}{q-1} \leqslant t \cdot \frac{q^{3h}-1}{q-1},$$

with equality if and only if every hyperplane contains exactly t elements of $\mathcal{X}$. Substituting $|\mathcal{X}| = (t-1)q^h + t$ and simplifying,

$$(t-1)q^h + t \leqslant t\,\frac{q^{3h}-1}{q^{2h}-1} = t\left(q^h + \frac{1}{q^h+1}\right).$$

This rearranges to $q^h(q^h - t + 1) \geqslant 0$, which holds if and only if $t \leqslant q^h + 1$, with equality precisely when $t = q^h + 1$ and every hyperplane is a t-secant. For $t < q^h - 1$, apply Theorem 1, part 2. $\square$

We next establish a divisibility condition satisfied by t in the extremal case.

Lemma 4. *Let $\mathcal{X}$ be a multi-set of $(h-1)$-dimensional subspaces in* $\mathrm{PG}(kh-1,q)$ *of size $(t-k+2)q^h + t$ with the property that every hyperplane contains either 0 or t subspaces of $\mathcal{X}$. Let π be an $(s-1)$-dimensional subspace skew to $\mathcal{X}$. Then π is contained in exactly*

$$\frac{q^h - q^{kh-s-h}}{q-1} + \frac{(k-2)(q^{kh-s} - q^h)}{t(q-1)}$$

zero-secant hyperplanes. In particular, if there exists a point skew to $\mathcal{X}$, then t divides $(k-2)q^h$.

Proof. Let a_0 and a_t denote the number of zero-secant and t-secant hyperplanes containing π, respectively. Counting hyperplanes through π,

$$a_0 + a_t = \frac{q^{kh-s}-1}{q-1},$$

and counting pairs (x, H) with $x \in \mathcal{X}$ and H a hyperplane through both π and x,

$$ta_t = \big((t-k+2)q^h + t\big)\frac{q^{kh-s-h}-1}{q-1}.$$

Solving for a_0 gives the stated formula. Applying this for $s = 0$ and $s = 1$ yields

$$t \mid (k-2)q^h(1+q+\cdots+q^{kh-h-1}) \quad \text{and} \quad t \mid (k-2)q^h(1+q+\cdots+q^{kh-h-2}),$$

from which t divides $(k-2)q^h$. $\square$

Theorem 5. *Let $\mathcal{X}$ be a length-maximal set of $(h-1)$-dimensional subspaces in* $\mathrm{PG}(kh-1, q)$ *with $k \geqslant 3$ and $t < q^h + k - 2$. Then $t - k + 3$ divides q^h.*

Proof. For $k = 3$, part 2 of Theorem 1 gives that every hyperplane contains 0 or t elements of $\mathcal{X}$. Since $t < q^h + 1$, we have $|\mathcal{X}| = (t-1)q^h + t \leqslant q^{2h}$. The elements of $\mathcal{X}$ are pairwise disjoint by part 1 of Theorem 1, so they cover at most $q^{2h}\frac{q^h-1}{q-1} < \frac{q^{3h}-1}{q-1}$ points of $\mathrm{PG}(3h-1, q)$. Hence there is a point skew to $\mathcal{X}$, and Lemma 4 gives that t divides q^h.

For $k \geqslant 4$, we argue inductively. By part 1 of Theorem 1, taking the quotient at any element $x \in \mathcal{X}$ yields a set $\mathcal{X}'$ of $(h-1)$-dimensional subspaces in $\mathrm{PG}((k-1)h-1, q)$ of size $((t-1)-(k-1)+2)q^h + (t-1)$, with every hyperplane containing at most $t-1$ elements of $\mathcal{X}'$. Thus $\mathcal{X}'$ is length-maximal with parameter $t-1$ in $\mathrm{PG}((k-1)h-1, q)$. Since $t - 1 < q^h + k - 3$, the induction hypothesis gives $(t-1)-(k-1)+3 = t-k+3$ divides q^h. $\square$

3 Length-maximal sets for $k \geqslant 4$

We enumerate secant hyperplanes of a length-maximal set $\mathcal{X}$. A hyperplane containing precisely s elements of $\mathcal{X}$ is called an *s-secant.* For $\mathcal{X}$ length-maximal we write $N(t, h, k, s)$ for the number of s-secants of $\mathcal{X}$ in $\mathrm{PG}(kh-1, q)$.

Lemma 6. *Let $\mathcal{X}$ be a length-maximal set of $(h-1)$-dimensional subspaces in* $\mathrm{PG}(kh-1, q)$. *Set $n = |\mathcal{X}| = (t-k+2)q^h + t$. Then*

1. $N(t, h, 3, t) = \dfrac{q^{3h}-1}{q-1} - \dfrac{q^h-1}{q-1} \cdot \left(\dfrac{q^h(q^h+1)}{t} - q^h\right)$
2. $N(t, h, 3, 0) = \dfrac{q^h-1}{q-1} \cdot \left(\dfrac{q^h(q^h+1)}{t} - q^h\right)$
3. $N(t, h, 4, t) = \left(q^h + 1 - \dfrac{2q^h}{t}\right)\left(q^h + 1 - \dfrac{q^h}{t-1}\right)\dfrac{q^{2h}-1}{q-1}$
4. $N(t, h, 4, 1) = n \cdot N(t-1, h, 3, 0) = \big((t-2)q^h + t\big) \cdot \dfrac{q^h-1}{q-1} \cdot \left(\dfrac{q^h(q^h+1)}{t-1} - q^h\right)$

5. $N(q^h+2,h,5,2) = \dfrac{n}{2}\cdot N(q^h+1,h,4,1) = \dfrac{q^{2h}+2}{2}\cdot(q^{2h}+1)\cdot\dfrac{q^h-1}{q-1}$

6. $N(q^h+2,h,5,q^h+2) = \dfrac{(q^{2h}+2)(q^{2h}+1)q^h(q^h-1)}{(q^h+2)(q-1)}$

Proof. Set $n=(t-k+2)q^h+t$ and we use the term $(h-1)$-flat for an $(h-1)$-dimensional subspace.

Part 1. By part 2 of Theorem 1, every hyperplane is a 0-secant or a t-secant when $k=3$. Counting pairs (x,H) with $x\in\mathcal{X}$ and H a t-secant through x, and using the fact that each $(h-1)$-flat in $\mathrm{PG}(3h-1,q)$ lies in $\frac{q^{2h}-1}{q-1}$ hyperplanes,

$$N(t,h,3,t)\cdot t = n\cdot\frac{q^{2h}-1}{q-1}.$$

Substituting $n=(t-1)q^h+t$ and rearranging gives the formula. Part 2 follows since $N(t,h,3,0)+N(t,h,3,t)=|\mathrm{PG}(3h-1,q)|$.

Part 3. By part 1 of Theorem 1, any two elements of $\mathcal{X}$ span a $(2h-1)$-flat; each such flat lies in $\frac{q^{2h}-1}{q-1}$ hyperplanes of $\mathrm{PG}(4h-1,q)$. By part 2 of Theorem 1, each hyperplane is a 0-, 1-, or t-secant when $k=4$. Counting triples (x,y,H) with $x\neq y$ in $\mathcal{X}$ and H a t-secant through both,

$$N(t,h,4,t)\cdot\binom{t}{2}=\binom{n}{2}\cdot\frac{q^{2h}-1}{q-1}.$$

Substituting $n=(t-2)q^h+t$ and simplifying gives the formula.

Part 4. For $k=4$, a 1-secant hyperplane contains exactly one element $x_i\in\mathcal{X}$. The quotient of $\mathcal{X}$ at any $x_i\in\mathcal{X}$ is a length-maximal set $\mathcal{X}_i^*$ in $\mathrm{PG}(3h-1,q)$ with parameter $t-1$. Each 0-secant of $\mathcal{X}_i^*$ corresponds to a 1-secant of $\mathcal{X}$ through x_i, so summing over all x_i,

$$N(t,h,4,1)=n\cdot N(t-1,h,3,0).$$

Substituting the formula from part 2 of Lemma 6 with t replaced by $t-1$ gives the result.

Part 5. Now let $k=5$ and $t=q^h+2$, so $n=q^{2h}+2$. By part 2 of Theorem 1, every hyperplane is a 0-, 1-, 2-, or t-secant. Since any hyperplane containing 3 or more elements of $\mathcal{X}$ must contain t elements (as any 3 elements span a $(3h-1)$-flat, and any hyperplane through a $(3h-1)$-flat that contains a fourth element of $\mathcal{X}$ must contain all t by part 2 of Theorem 1), the secant types are restricted to 0, 1, 2, and t.

For $k=5$, each element $x_i\in\mathcal{X}$ lies in a 2-secant hyperplane H if and only if H contains exactly one further element $x_j\neq x_i$. The number of 2-secant hyperplanes through x_i equals the number of 1-secant hyperplanes of the quotient set $\mathcal{X}_i^*$ in $\mathrm{PG}(4h-1,q)$, which has parameter $t-1=q^h+1$. Summing over all x_i and dividing by 2 (since each 2-secant is counted twice),

$$N(q^h+2,h,5,2)=\frac{n}{2}\cdot N(q^h+1,h,4,1).$$

By Theorem 7 (proved below), the quotient set $\mathcal{X}_i^*$ has $t-1 = q^h+1$ and has no 0-secants, so $N(q^h+1, h, 4, 1) = |\mathrm{PG}(4h-1, q)| - N(q^h+1, h, 4, q^h+1)$. Alternatively, substituting $t-1 = q^h+1$ directly into part 4 of Lemma 6 and simplifying gives

$$N(q^h+2, h, 5, 2) = \frac{q^{2h}+2}{2} \cdot (q^{2h}+1) \cdot \frac{q^h-1}{q-1}.$$

Part 6. With $k=5$, $t = q^h+2$, $n = q^{2h}+2$, any three elements of $\mathcal{X}$ span a $(3h-1)$-flat (since any $k-1=4$ elements span a $(4h-1)$-flat by part 1 of Theorem 1, which implies any three also span a $(3h-1)$-flat). Each $(3h-1)$-flat in $\mathrm{PG}(5h-1,q)$ lies in $\frac{q^{2h}-1}{q-1}$ hyperplanes. Counting quadruples (x_1, x_2, x_3, H) with distinct $x_i \in \mathcal{X}$ and H a t-secant through all three,

$$N(q^h+2, h, 5, q^h+2) \cdot \binom{t}{3} = \binom{n}{3} \cdot \frac{q^{2h}-1}{q-1}.$$

Substituting $n = q^{2h}+2$ and $t = q^h+2$ and simplifying,

$$\begin{aligned} N(q^h+2, h, 5, q^h+2) &= \frac{(q^{2h}+2)(q^{2h}+1)q^{2h}}{(q^h+2)(q^h+1)q^h} \cdot \frac{q^{2h}-1}{q-1} \\ &= \frac{(q^{2h}+2)(q^{2h}+1)q^h(q^h-1)}{(q^h+2)(q-1)}. \end{aligned}$$

□

Theorem 7. *Let $\mathcal{X}$ be a length-maximal set of $(h-1)$-dimensional subspaces in $\mathrm{PG}(4h-1, q)$ with $q^h > 2$. Then $t = q^h+1$ and every hyperplane is either a t-secant or a 1-secant of $\mathcal{X}$.*

Proof. By part 2 of Theorem 1, every hyperplane is a 0-, 1-, or t-secant. The counts $N(t,h,4,0)$, $N(t,h,4,1)$, $N(t,h,4,t)$ are non-negative integers summing to $|\mathrm{PG}(4h-1,q)|$. Using parts 3 and 4 of Lemma 6,

$$\begin{aligned} N(t,h,4,0) &= |\mathrm{PG}(4h-1,q)| - N(t,h,4,t) - N(t,h,4,1) \\ &= \frac{q^h(q^h-1)(t-q^h-1)\big(t(q^h+1)-4q^h-2\big)}{t(q-1)}. \end{aligned} \tag{2}$$

By Lemma 2, $t \geqslant 4$. Taking the quotient of $\mathcal{X}$ at any of its elements yields a length-maximal set in $\mathrm{PG}(3h-1,q)$ with parameter $t-1$, so Lemma 3 gives $t-1 \leqslant q^h+1$, that is, $t \leqslant q^h+2$. We rule out every value other than $t = q^h+1$.

For $4 \leqslant t \leqslant q^h$, $(t-q^h-1) \leqslant -1$, and $t(q^h+1)-(4q^h+2) \geqslant 4(q^h+1)-(4q^h+2) = 2 > 0$, so from (2) $N(t,h,4,0) < 0$, a contradiction.

For $t = q^h+2$, substituting into part 4 of Lemma 6 gives $N(t,h,4,1) = 0$, so every hyperplane is a 0- or t-secant. Since $|\mathcal{X}| = q^{2h}+q^h+2$ and each element is an $(h-1)$-flat with $\frac{q^h-1}{q-1}$ points, the elements of $\mathcal{X}$ cover at most $(q^{2h}+q^h+2) \cdot \frac{q^h-1}{q-1} < \frac{q^{4h}-1}{q-1}$ points of $\mathrm{PG}(4h-1,q)$, so some point is skew to $\mathcal{X}$. Lemma 4 then gives $t \mid 2q^h$; but $t = q^h+2$ divides $2q^h = 2(q^h+2)-4$ only if $q^h+2 \mid 4$, giving $q^h \leqslant 2$. Therefore $t = q^h+1$, and substituting gives $N(t,h,4,0) = 0$. □

Theorem 8. *There is no length-maximal set of* $(h-1)$*-dimensional subspaces in* $\mathrm{PG}(5h-1,q)$ *when* $q^h > 3$.

Proof. Suppose for a contradiction that $\mathcal{X}$ is such a set, with $|\mathcal{X}| = (t-3)q^h + t$. The quotient of $\mathcal{X}$ at any element is a length-maximal set in $\mathrm{PG}(4h-1,q)$ with parameter $t-1$. By Theorem 7, we obtain $t-1 = q^h+1$, that is $t = q^h+2$. Hence $n = |\mathcal{X}| = q^{2h}+2$.

The secant types are 0, 1, 2, and $t = q^h+2$. Consider the quantity

$$\delta = N(q^h+2, h, 5, q^h+2) + N(q^h+2, h, 5, 2) - |\mathrm{PG}(5h-1,q)|, \tag{3}$$

which satisfies $\delta \leqslant 0$ since $N(q^h+2,h,5,0)$ and $N(q^h+2,h,5,1)$ are non-negative. Substituting parts 5 and 6 of Lemma 6 and expanding, we obtain

$$2(q^h+2)(q-1)\,\delta = (q^h)^6 - 5(q^h)^5 + 7(q^h)^4 - 3(q^h)^3 = (q^h)^3(q^h-1)^2(q^h-3). \tag{4}$$

Since $q^h > 3$, this gives $\delta > 0$, contradicting $\delta \leqslant 0$. (When $q^h = 3$ the right-hand side of (4) vanishes and there are two secant types. This case is genuinely exceptional, see Example 10.) $\square$

Remark 9. By induction on the quotient construction, Theorem 8 extends to all $k \geqslant 5$ when $q^h > 3$.

Example 10. For $(q,h) = (3,1)$, length-maximal sets exist both for $k=5$ and for $k=6$. In each case $t = k$ and $|\mathcal{X}| = k+6$, so $\mathcal{X}$ is a set of $k+6$ points of $\mathrm{PG}(k-1,3)$ giving a $[k+6,k,6]_3$ code. In each case the dual code has minimum distance k, so every $k-1$ of the points are linearly independent and part 1 of Theorem 1 holds.

For $k=5$, let $\mathcal{X}$ be the 11 points of $\mathrm{PG}(4,3)$ given by the columns of a generator matrix of the $[11,5,6]_3$ code dual to the perfect ternary Golay code. This is a projective two-weight code with weights 6 and 9, so every hyperplane meets $\mathcal{X}$ in $11-6=5$ or $11-9=2$ points; thus $t = 5 = q^h+2$ and the secant types are 2 and t. Here the right-hand side of (4) vanishes.

For $k=6$, let $\mathcal{X}$ be the 12 points of $\mathrm{PG}(5,3)$ given by the columns of a generator matrix of the self-dual extended ternary Golay code $[12,6,6]_3$. Its weights are 6, 9 and 12, so every hyperplane meets $\mathcal{X}$ in 6, 3 or 0 points; thus $t=6$ and the secant types are 0, $3 = k-3$ and t. In particular this length-maximal set is a three-weight code.

Proposition 11. *For* $q^h > 2$ *there is no length-maximal set with* $k \geqslant 7$.

Proof. From Theorem 8 it suffices to consider $(q,h) = (3,1)$. Iterating the quotient construction down to dimension 5 and applying Theorem 7, any length-maximal set with $(q,h) = (3,1)$ and $k \geqslant 5$ has $t = k$ and $|\mathcal{X}| = (t-k+2)\cdot 3 + t = k+6$. Suppose such a set $\mathcal{X}$ exists with $k=7$, so that $t=7$ and $n = |\mathcal{X}| = 13$ in $\mathrm{PG}(6,3)$. By part 1 of Theorem 1, any six points of $\mathcal{X}$ span a unique hyperplane. By part 2 of Theorem 1 such a hyperplane must be a t-secant. Counting the pairs (S,H) in which S is a six-element subset of $\mathcal{X}$ and H is the hyperplane spanned by S gives

$$\binom{t}{6} N(7,1,7,7) = \binom{n}{6}, \qquad \text{so} \qquad N(7,1,7,7) = \frac{\binom{13}{6}}{\binom{7}{6}} = \frac{1716}{7},$$

which is not an integer. This rules out $k = 7$, and the cases $k \geqslant 8$ are ruled out inductively. $\square$

Remark 12. Restricting to the case $h = 1$, let $\kappa(s, q)$ denote, as in [1], the largest k for which a length-maximal linear code of Singleton defect s over $\mathbb{F}_q$ exists. Conjecture 5.1 of [1] asserts that $\kappa(s, q) \leqslant 4$ for $q > 3$, so Theorem 8, together with the quotient induction in Remark 9, establishes it. Furthermore, Theorem 7 shows that a length-maximal set with $k = 4$ has $t = q + 1$, that is $s = q - 2$, while Example 17 shows that such sets exist for every q. Hence $\kappa(q - 2, q) = 4$ (for $q > 3$), which is the third of the three cases of Conjecture 5.2 of [1].

4 The associated strongly regular graphs

The classical correspondence among projective two-weight codes, two-intersection sets and strongly regular graphs is well known; see Delsarte [9] and Calderbank and Kantor [6]. In this section we interpret the preceding results from the graph-theoretic side.

Let $\mathcal{X}$ be a length-maximal set of $(h-1)$-dimensional subspaces of $\mathrm{PG}(kh-1, q)$ with $k \geqslant 3$ and $n = |\mathcal{X}|$. By part 1 of Theorem 1 the elements of $\mathcal{X}$ are pairwise disjoint, so the h-dimensional $\mathbb{F}_q$-subspaces $V_1, \ldots, V_n$ of $\mathbb{F}_q^{kh}$ that they determine meet pairwise in the zero vector. Put

$$D = \Big(\bigcup_{i=1}^{n} V_i\Big) \setminus \{0\}, \qquad |D| = n(q^h - 1),$$

and let $\Gamma(\mathcal{X})$ denote the Cayley graph on $(\mathbb{F}_q^{kh}, +)$ with connection set D. Thus $\Gamma(\mathcal{X})$ has q^{kh} vertices and is regular of degree $n(q^h - 1)$.

Let S denote the set of points of $\mathrm{PG}(kh-1, q)$ covered by the members of $\mathcal{X}$. Since these members are pairwise disjoint,

$$|S| = N := n\,\frac{q^h - 1}{q - 1},$$

and S spans $\mathrm{PG}(kh-1, q)$ because $\mathcal{X}$ does. Thus S is a spanning set of N points of $\mathrm{PG}(kh-1, q)$, giving rise to a projective $[N, kh]_q$ code in the sense of Calderbank and Kantor [6, Section 2], and $D = \{v \in \mathbb{F}_q^{kh} \setminus \{0\} : \langle v\rangle \in S\}$. Consequently $\Gamma(\mathcal{X})$ is the graph that [6, Section 3] associates with S, and the results of that section apply verbatim.

Proposition 13. *Let $\mathcal{X}$ be a length-maximal set of $(h-1)$-dimensional subspaces of $\mathrm{PG}(kh-1, q)$ with $k \geqslant 3$, and let $n = |\mathcal{X}|$. If H is an s-secant hyperplane of $\mathcal{X}$ then the eigenvalue of $\Gamma(\mathcal{X})$ afforded by the corresponding character is*

$$\theta_H = sq^h - n.$$

Consequently $\Gamma(\mathcal{X})$ is strongly regular if and only if $\mathcal{X}$ has exactly two secant types $s_1 < s_2$, in which case its parameters are

$$v = q^{kh}, \qquad K = n(q^h - 1), \qquad \lambda = K + r + \sigma + r\sigma, \qquad \mu = K + r\sigma,$$

where $r = s_2 q^h - n$ and $\sigma = s_1 q^h - n$ are the restricted eigenvalues.

Proof. Let H be an s-secant hyperplane of $\mathcal{X}$. Each of the $n-s$ members of $\mathcal{X}$ not contained in H meets H in a hyperplane of itself, so

$$|S \cap H| = s\,\frac{q^h-1}{q-1} + (n-s)\,\frac{q^{h-1}-1}{q-1},$$

and the codeword of the projective code determined by H has weight

$$w_H = |S| - |S \cap H| = (n-s)q^{h-1}.$$

By [6, Lemma 3.4] the eigenvalue of $\Gamma(\mathcal{X})$ afforded by the corresponding character is $K - qw_H$, where $K = N(q-1) = n(q^h-1)$ is the valency; hence

$$\theta_H = n(q^h-1) - q(n-s)q^{h-1} = sq^h - n,$$

as claimed. The map $s \mapsto sq^h - n$ is injective, so the restricted eigenvalues of $\Gamma(\mathcal{X})$ correspond bijectively to the secant types of $\mathcal{X}$.

If S is the whole of $\mathrm{PG}(kh-1,q)$ then $D = \mathbb{F}_q^{kh} \setminus \{0\}$, the graph $\Gamma(\mathcal{X})$ is complete and there is a single secant type. Otherwise S is a proper, non-empty, spanning set of points, and by [6, Theorem 3.2] the graph $\Gamma(\mathcal{X})$ is strongly regular if and only if S is a two-intersection set, that is, if and only if $\mathcal{X}$ has exactly two secant types $s_1 < s_2$. In that case the associated code is a projective two-weight code with weights $w_1 = (n-s_2)q^{h-1}$ and $w_2 = (n-s_1)q^{h-1}$ by [6, Theorem 3.1], and the parameters of $\Gamma(\mathcal{X})$ given in [6, Corollary 3.7] are, in terms of the restricted eigenvalues $r = K - qw_1 = s_2q^h - n$ and $\sigma = K - qw_2 = s_1q^h - n$, precisely $\mu = K + r\sigma$ and $\lambda = K + r + \sigma + r\sigma$. $\square$

Corollary 14. *Let $\mathcal{X}$ be a length-maximal set of $(h-1)$-dimensional subspaces of $\mathrm{PG}(kh-1,q)$.*

1. *If $k=3$ and $t < q^h+1$ then $\Gamma(\mathcal{X})$ is strongly regular with parameters*

$$\bigl(q^{3h},\ n(q^h-1),\ n(t-2)+q^h-t,\ n(t-1)\bigr), \qquad n = (t-1)q^h + t,$$

   the restricted eigenvalues being $r = q^h - t$ and $\sigma = -n$.

2. *If $k=4$ and $q^h > 2$ then $\Gamma(\mathcal{X})$ is strongly regular with parameters*

$$\bigl(q^{4h},\ (q^{2h}+1)(q^h-1),\ q^h-2,\ q^h(q^h-1)\bigr),$$

   the restricted eigenvalues being $r = q^h - 1$ and $\sigma = q^h - q^{2h} - 1$.

3. *If $k \geqslant 5$ and $q^h > 3$ then no such $\mathcal{X}$ exists. For $(q,h) = (3,1)$, length-maximal sets exist precisely for $k \in \{5,6\}$: for $k=5$ the graph $\Gamma(\mathcal{X})$ is strongly regular with parameters $(243, 22, 1, 2)$; for $k=6$ the secant types are 0, 3 and 6, so $\Gamma(\mathcal{X})$ is not strongly regular.*

Proof. In each case we determine the secant types and appeal to Proposition 13. For $k=3$, part 2 of Theorem 1 gives the types 0 and t, and by Lemma 3 both occur when $t<q^h+1$; substituting $s_1=0$ and $s_2=t$ gives $r=tq^h-n=q^h-t$ and $\sigma=-n$, whence $\mu=K+r\sigma=n(t-1)$ and $\lambda=\mu+r+\sigma=n(t-2)+q^h-t$. For $k=4$ and $q^h>2$, Theorem 7 gives $t=q^h+1$, $n=q^{2h}+1$ and the two secant types 1 and t; substituting yields $r=q^h-1$, $\sigma=q^h-q^{2h}-1$, $\mu=q^h(q^h-1)$ and $\lambda=q^h-2$. For $k\geqslant 5$ the existence statement is Theorem 8 together with Proposition 11, and the secant types for $k=5$ and $k=6$ are read off from Example 10: for $k=5$ they are 2 and 5, so that $n=11$, $K=22$, $r=4$ and $\sigma=-5$, giving $(243,22,1,2)$; for $k=6$ there are three types. □

Remark 15. For $h=1$ the graphs of part 1 of Corollary 14 are the strongly regular graphs classically associated with maximal arcs of $\mathrm{PG}(2,q)$; a hyperoval of $\mathrm{PG}(2,4)$, for instance, gives $\mathrm{SRG}(64,18,2,6)$. For $h=1$ the sets of part 2 are the ovoids of $\mathrm{PG}(3,q)$, and for an elliptic quadric the graph is the affine polar graph $\mathrm{VO}^-(4,q)$; inequivalent ovoids may give non-isomorphic graphs with the same parameters, cf. [6]. For $h>1$, the examples in part 2 furnished by Example 17 arise by restriction of scalars from ovoids of $\mathrm{PG}(3,q^h)$. The graph with parameters $(243,22,1,2)$ in part 3 is the Berlekamp–van Lint–Seidel graph [4], the coset graph of the perfect ternary Golay code.

Remark 16. When $k=3$ and $t=q^h+1$ every hyperplane is a t-secant by Lemma 3, so D is all of $\mathbb{F}_q^{3h}\setminus\{0\}$ and $\Gamma(\mathcal{X})$ is complete; this is the reason for the hypothesis $t<q^h+1$ in part 1 of Corollary 14. With this boundary case set aside, Corollary 14 gives the complete list: the associated graph is non-trivially strongly regular when $k=3$ and $t<q^h+1$, when $k=4$ and $q^h>2$, and in the single exceptional case $k=5$, $(q,h)=(3,1)$. In the case $k=4$ it is Theorem 7 that reduces the possible secant types from three to two.

5 Remarks on small cases and open questions

For $k=2$, a length-maximal set is a set of $(h-1)$-dimensional subspaces of $\mathrm{PG}(2h-1,q)$ such that every hyperplane contains at most t elements, with $|\mathcal{X}|=tq^h+t=t(q^h+1)$. This corresponds to a partition of $\mathrm{PG}(2h-1,q)$ into $(h-1)$-flats (a spread), which exists for all q and h.

For $k=3$, length-maximal sets correspond to the higher-dimensional analogues of maximal arcs. The field reduction of a maximal arc in $\mathrm{PG}(2,q^h)$ yields an example for every parameter $t\mid q^h$ and all q even. Mathon's example [8] provides an instance with $k=3$, $h=2$, $q=3$, $t=3$ in odd characteristic, and to our knowledge is the only such example in odd characteristic. Whether this example is unique, and whether further examples exist in odd characteristic for larger parameters, are questions that remain open.

Example 17. For $k=4$ the bound of Theorem 7 is attained for every prime power q and every $h\geqslant 1$, by field reduction of an ovoid. Let $\mathcal{O}$ be an ovoid of $\mathrm{PG}(3,q^h)$: a set of $q^{2h}+1$ points meeting every plane in 1 or q^h+1 points. Elliptic quadrics are ovoids and exist for every q^h. Regard $\mathbb{F}_q^{4h}$ as a $4h$-dimensional vector space over $\mathbb{F}_q$, and to each point $\langle v\rangle_{\mathbb{F}_{q^h}}$ of $\mathrm{PG}(3,q^h)$ associate the $(h-1)$-flat $\mathrm{PG}_{\mathbb{F}_q}(\mathbb{F}_{q^h}v)$ of $\mathrm{PG}(4h-1,q)$. Set $\mathcal{X}=\{\,\mathrm{PG}_{\mathbb{F}_q}(\mathbb{F}_{q^h}v):$

$\langle v\rangle_{\mathbb{F}_{q^h}} \in \mathcal{O}\}$. An $\mathbb{F}_q$-hyperplane of $\mathbb{F}_q^{4h}$ has the form $\ker\big(x \mapsto \mathrm{Tr}_{\mathbb{F}_{q^h}/\mathbb{F}_q}(a\cdot x)\big)$ for some nonzero $a \in \mathbb{F}_q^{4h}$, and by nondegeneracy of the trace form $\mathrm{PG}_{\mathbb{F}_q}(\mathbb{F}_{q^h}v)$ is contained in it if and only if $a \cdot v = 0$. Thus the $\mathbb{F}_q$-hyperplanes correspond to the planes $a \cdot X = 0$ of $\mathrm{PG}(3, q^h)$, an element of $\mathcal{X}$ lying in a hyperplane exactly when the corresponding point lies in the corresponding plane. Hence every hyperplane meets $\mathcal{X}$ in 1 or $q^h + 1$ members, so $t = q^h + 1$ and

$$|\mathcal{X}| = q^{2h} + 1 = (t-2)q^h + t.$$

Therefore $\mathcal{X}$ is length-maximal, with every hyperplane a 1-secant or a t-secant, and the associated two-weight additive code has weights q^{2h} and $q^{2h} - q^h$. When $h = 1$ these length-maximal sets are precisely the ovoids of $\mathrm{PG}(3, q)$.

The results of Section 3 are constrained to $q^h > 2$, and Theorem 8 requires $q^h > 3$. In particular the binary case $q = 2$ is excluded only when $h = 1$, so these results do apply to $\mathbb{F}_2$-linear codes over $\mathbb{F}_{2^h}$ for every $h \geqslant 2$. For instance, taking $q = 2$ and $h = 2$, Theorem 7 shows that a length-maximal set in $\mathrm{PG}(7, 2)$ has $t = 5$ and consists of 17 lines, as realised by Example 17 from an ovoid of $\mathrm{PG}(3, 4)$, while Theorem 8 shows that none exists in $\mathrm{PG}(9, 2)$. For $(q, h) = (2, 1)$, the trivial binary $[k+1, k, 2]$ MDS codes, the frames $\{\langle e_1\rangle, \ldots, \langle e_k\rangle, \langle e_1 + \cdots + e_k\rangle\}$ of $\mathrm{PG}(k-1, 2)$, with $t = k-1$, provide length-maximal sets for every k. The existence question for $(q, h) = (2, 1)$ is completely settled by results in the literature. A bound of Faldum and Willems in [1], gives $t \leqslant q + k - 2$ for every length-maximal set with $h = 1$ and $k \geqslant 3$, so for $q = 2$ only $t = k-1$ and $t = k$ can occur. The MDS examples above realise $t = k-1$ for every k, and Alderson and Zhang [1] show that length-maximal sets with $t = k$ exist precisely for $k \leqslant 4$. For $k = 4$ such a set is the complement of a plane in $\mathrm{PG}(3, 2)$, an 8-cap, with secant types 0 and 4.

In the language of additive codes, our results assert that additive $[n, k, d]_q^h$ codes meeting the maximal arc bound do not exist for $k \geqslant 5$ when $q^h > 3$. The case $q^h = 3$, namely $(q, h) = (3, 1)$, is exceptional in that the ternary Golay code yields length-maximal sets for $k = 5$ and $k = 6$, while none exist for $k \geqslant 7$ (Example 10 and Proposition 11). For $k = 4$ and $q^h > 2$, any such code must satisfy $t = q^h + 1$ and have exactly two nonzero weights $n-1$ and $n-t$; such codes exist for all q and h, arising by field reduction of an ovoid of $\mathrm{PG}(3, q^h)$ (Example 17).

Acknowledgements

We thank Adrián Zámečník, Václav Rozhoň and Robert Šámal of Charles University for sending us a note generated using `bolzano.app` that drew our attention to the construction in Example 17. We also thank the referee for a careful reading and for several suggestions which improved the paper, in particular for prompting the discussion of Section 4.

Tim Alderson acknowledges the support of the Natural Sciences and Engineering Research Council of Canada (NSERC), [funding reference number 2019-04103]. Cette recherche a été financée par le Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG), [numéro de référence 2019-04103].

Simeon Ball acknowledges the support of the Spanish Ministry of Science, Innovation and Universities grant PID2023-147202NB-I00.

References

[1] T.L. Alderson and Z. Zhang, Projective systems and bounds on the length of codes of non-zero defect, *J. Algebra Combin. Discrete Struct. Appl.*, to appear. arXiv:2504.19325.

[2] S. Ball, A. Blokhuis and F. Mazzocca, Maximal arcs in Desarguesian planes of odd order do not exist, *Combinatorica*, **17** (1997) 31–41. doi:10.1007/BF01196129.

[3] S. Ball and A. Blokhuis, An easier proof of the maximal arcs conjecture, *Proc. Amer. Math. Soc.*, **126** (1998) 3377–3380. doi:10.1090/S0002-9939-98-04653-X.

[4] E.R. Berlekamp, J.H. van Lint and J.J. Seidel, A strongly regular graph derived from the perfect ternary Golay code, in: *A Survey of Combinatorial Theory*, North-Holland, Amsterdam, 1973, pp. 25–30. doi:10.1016/B978-0-7204-2262-7.50008-9.

[5] A.A. Bruen and R. Silverman, On the non-existence of certain M.D.S. codes and projective planes, *Math. Z.*, **183** (1983) 171–175. doi:10.1007/BF01214819.

[6] R. Calderbank and W.M. Kantor, The geometry of two-weight codes, *Bull. London Math. Soc.*, **18** (1986) 97–122. doi:10.1112/blms/18.2.97.

[7] A. Cossu, Su alcune proprietà dei $\{k;n\}$-archi di un piano proiettivo sopra un corpo finito, *Rend. Mat. e Appl.*, **20** (1961) 271–277.

[8] F. De Clerck, M. Delanote, N. Hamilton and R. Mathon, Perp-systems and partial geometries, *Adv. Geom.*, **2** (2002) 1–12. doi:10.1515/advg.2001.025.

[9] P. Delsarte, Weights of linear codes and strongly regular normed spaces, *Discrete Math.*, **3** (1972) 47–64. doi:10.1016/0012-365X(72)90024-6.

[10] R.H.F. Denniston, Some maximal arcs in finite projective planes, *J. Combin. Theory*, **6** (1969) 317–319. doi:10.1016/S0021-9800(69)80095-5.

[11] N. Hamilton and R. Mathon, More maximal arcs in Desarguesian projective planes and their geometric structure, *Adv. Geom.*, **3** (2003) 251–261. doi:10.1515/advg.2003.015.

[12] R. Mathon, New maximal arcs in Desarguesian planes, *J. Combin. Theory Ser. A*, **97** (2002) 353–368. doi:10.1006/jcta.2001.3218.

[13] J.A. Thas, Construction of maximal arcs and partial geometries, *Geom. Dedicata*, **3** (1974) 61–64. doi:10.1007/BF00181361.

[14] J.A. Thas, Some results concerning $\{(q+1)(n-1);n\}$-arcs and $\{(q+1)(n-1)+1;n\}$-arcs in finite projective planes of order q, *J. Combin. Theory Ser. A*, **19** (1975) 228–232. doi:10.1016/S0097-3165(75)80012-4.

[15] J.A. Thas, Construction of maximal arcs and dual ovals in translation planes, *Europ. J. Combinatorics*, **1** (1980) 189–192. doi:10.1016/S0195-6698(80)80052-7.