Infinitesimal Bialgebra on Planar Binary Trees

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Abstract

We construct a weight-zero infinitesimal bialgebra structure on the $\mathbf{k}$-module spanned by planar binary trees, using the under product $\backslash$ of Aguiar–Sottile and a root-recursive coproduct $\Delta_{\mathrm{LR}}^{\epsilon}$. We prove that $\Delta_{\mathrm{LR}}^{\epsilon}$ is coassociative and satisfies the infinitesimal derivation rule with respect to $\backslash$, hence gives a unitary infinitesimal bialgebra distinct from the usual Loday–Ronco Hopf structure. We also obtain an elementary cut formula, establish freeness properties for unitary $(\backslash, \vee)$-algebras and unitary infinitesimal $(\backslash, \vee)$-bialgebras, and build an infinitesimal bialgebra isomorphism from the infinitesimal bialgebra of planar binary trees to the known infinitesimal bialgebra of planar rooted forests.

Mathematics Subject Classifications: 16T10, 16T30, 08B20, 05C05

1 Introduction

1.1 Motivation: infinitesimal compatibility on planar binary trees

Infinitesimal bialgebras were introduced by Joni and Rota as algebra–coalgebra structures in which the coproduct is a derivation with respect to the product [29]. In weight zero, this compatibility is expressed by

$$\Delta(ab) = a \cdot \Delta(b) + \Delta(a) \cdot b.$$

Aguiar further developed infinitesimal Hopf algebras and emphasized that they are not merely formal variants of ordinary Hopf algebras: they are related to associative Yang–Baxter equations, pre-Lie structures, dendriform structures, and combinatorial Hopf algebras [1]. The present paper takes this infinitesimal point of view in a concrete tree-combinatorial setting. Instead of asking for a Hopf coproduct on planar binary trees, we ask for a coproduct which is compatible with the associative operation $\backslash$ in the infinitesimal sense.

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Planar binary trees already carry rich Hopf-theoretic structures. The Loday–Ronco Hopf algebra [33] relates them to associahedra, bracketings, and the Tamari order [39, 40, 41]. The Aguiar–Sottile analysis of this Hopf algebra describes a distinguished associative operation $\backslash$, progressive trees, and a planar rooted-forest/planar binary-tree correspondence [2]. On the other hand, planar rooted forests form a classical source of combinatorial Hopf and infinitesimal bialgebras, including the Connes–Kreimer, Grossman–Larson, and related rooted-tree constructions [12, 13, 23, 31, 38]. Weighted infinitesimal bialgebras on planar rooted forests and their freeness properties were constructed in [42, 44]. The pre-Lie and operadic aspects of rooted trees are part of the background developed in [9, 10, 11]. Further Hopf-algebraic developments for rooted trees and decorated rooted trees were obtained by Foissy and Holtkamp [16, 17, 18, 19, 26, 27]. Related algebraic structures on or near planar binary trees include dialgebras, trialgebras, generalized bialgebras, magmatic bialgebras, and Hopf operads [28, 34, 35]. Planar binary trees also appear in QED and renormalization-inspired Hopf algebras, as well as in the Hopf-theoretic study of bi-leveled trees and multiplihedra [5, 6, 20]. Noncommutative symmetric functions, free quasi-symmetric functions, and the Malvenuto–Reutenauer Hopf algebra provide another structural background for the Loday–Ronco setting [14, 22, 25, 36, 37]. Operated-algebra methods, including free operated algebras, Rota–Baxter type constructions, rooted-forest cocycles, and cocycle $\vee$-Hopf algebras, give a parallel framework for the constructions considered here [4, 7, 21, 24, 43, 45].

These two developments suggest a natural problem. Since forest concatenation has a weight-zero infinitesimal coproduct, and since the Aguiar–Sottile correspondence sends forest concatenation to the product $\backslash$ on planar binary trees, one expects a corresponding infinitesimal coproduct on planar binary trees. However, this coproduct should not be confused with the Loday–Ronco Hopf coproduct, nor with the admissible-cut coproducts appearing in the Connes–Kreimer or Loday–Ronco Hopf settings. The purpose of this paper is to construct such an infinitesimal coproduct directly on planar binary trees and then to explain its universal and functorial meaning.

1.2 The guiding problem and the choice of operations

The operation used throughout the paper is the associative operation $\backslash$ from the Aguiar–Sottile framework [2], not the original Loday–Ronco Hopf product in the fundamental tree basis. For this operation, progressive planar binary trees are the free associative generators. The problem addressed here can therefore be stated as follows:

Guiding problem. Construct and characterize a coproduct $\Delta^{\epsilon}_{\mathrm{LR}}$ on the $\mathbf{k}$-module spanned by planar binary trees such that

$$\Delta^{\epsilon}_{\mathrm{LR}}(S\backslash T) = S\backslash\Delta^{\epsilon}_{\mathrm{LR}}(T) + \Delta^{\epsilon}_{\mathrm{LR}}(S)\backslash T,$$

and such that the construction is compatible with the binary grafting operation $\vee$ and with the rooted-forest infinitesimal coproduct.

The binary grafting operation $\vee$ is also essential. It gives the root decomposition

$T = T_l \vee T_r$ of every non-trivial planar binary tree. The coproduct constructed in (4) is root-recursive with respect to this decomposition, while the infinitesimal bialgebra property is proved with respect to the product $\backslash$ in Lemma 7 and Theorem 12. Thus the paper studies the interaction of two operations: $\vee$ controls the recursive construction, and $\backslash$ controls the infinitesimal compatibility.

1.3 Position among nearby tree bialgebras

The construction in this paper sits close to several familiar Hopf and infinitesimal frameworks, but it is not identical to any of them. The table below records the distinctions that will be used throughout the paper.

Framework	**Product**	**Coproduct / cuts**	**Role in this paper**
Planar rooted-forest Hopf algebras	Forest product in the Hopf setting	Hopf coproduct described by admissible cuts	Hopf-type contrast; not the coproduct studied here
Planar rooted-forest infinitesimal bialgebras	Forest concatenation	Recursive infinitesimal coproduct governed by B^+	Source of the transported infinitesimal structure in Theorem 27
Present planar binary-tree infinitesimal bialgebra	Under product $\backslash$ together with the grafting operation $\vee$	Root-recursive coproduct $\Delta^\epsilon_{\mathrm{LR}}$ and elementary vertex cuts	Main object constructed in (4) and Theorem 12

Table 1: Three nearby tree bialgebra frameworks.

Table 1 is meant to prevent two possible misreadings. First, $\Delta^\epsilon_{\mathrm{LR}}$ is not the Loday–Ronco Hopf coproduct in a new basis. It is an infinitesimal coproduct for the under product $\backslash$. Second, although Theorem 27 proves that the rooted-forest infinitesimal coproduct is transported to $\Delta^\epsilon_{\mathrm{LR}}$ by the map Θ, this bridge is an a posteriori identification. The coproduct $\Delta^\epsilon_{\mathrm{LR}}$ is constructed internally on planar binary trees in (4), and its main properties are proved before the bridge with rooted forests is established.

1.4 Main results and their significance

The main contributions of the paper are the following.

1. **A root-recursive infinitesimal coproduct.** We define a linear map $\Delta^\epsilon_{\mathrm{LR}} : H^\epsilon_{\mathrm{LR}} \to H^\epsilon_{\mathrm{LR}} \otimes H^\epsilon_{\mathrm{LR}}$ recursively by
$$\Delta^\epsilon_{\mathrm{LR}}(T_l \vee T_r) = T_l \otimes T_r + (\mathrm{id} \otimes \vee)(\Delta^\epsilon_{\mathrm{LR}}(T_l) \otimes T_r) + (\vee \otimes \mathrm{id})(T_l \otimes \Delta^\epsilon_{\mathrm{LR}}(T_r)),$$
with $\Delta^\epsilon_{\mathrm{LR}}(|) = 0$; see (4). Proposition 4 proves coassociativity, while Lemma 7 shows that this coproduct satisfies the weight-zero infinitesimal derivation rule with respect to $\backslash$. Consequently, $(H^\epsilon_{\mathrm{LR}}, \backslash, |, \Delta^\epsilon_{\mathrm{LR}})$ is a unitary infinitesimal bialgebra of weight zero by Theorem 12. This gives a natural infinitesimal bialgebra structure on planar binary trees which is different from the usual Hopf structures on the same combinatorial objects.

2. **An elementary vertex-cut formula.** For each internal vertex e of a planar binary tree T, we define the elementary vertex cut e and its associated pair $(P^e(T), R^e(T))$ in Definition 8. The main formula is

$$\Delta_{\mathrm{LR}}^{\epsilon}(T) = \sum_{e \in \mathrm{Int}(T)} P^e(T) \otimes R^e(T),$$

proved in Theorem 10. Its significance is that the infinitesimal coproduct is controlled by single-vertex cuts rather than by Hopf-type admissible cuts. This gives a direct combinatorial interpretation of every summand of $\Delta_{\mathrm{LR}}^{\epsilon}(T)$.

3. **One-sided grafting identities and comb examples.** The grafting operation $\vee$ gives two unary operators $B^r(T) = T \vee |$ and $B^{\ell}(T) = | \vee T$. Proposition 16 shows that these operators satisfy recursive identities with respect to $\Delta_{\mathrm{LR}}^{\epsilon}$. Corollary 17 then gives closed formulas for the coproducts of the left and right comb families. These formulas provide explicit test cases for the recursive definition and illustrate how the infinitesimal structure behaves on basic tree families. We also record the mirror symmetry of the coproduct and derive the corresponding infinitesimal compatibility for the over product $/$.

4. **A universal property for the pair $(\backslash, \vee)$.** We introduce unitary $(\backslash, \vee)$-algebras and prove that $(H_{\mathrm{LR}}^{\epsilon}, \backslash, |, \vee)$ is the free such algebra on the empty set in Theorem 20. We then introduce unitary infinitesimal $(\backslash, \vee)$-bialgebras in Definition 21. Proposition 22 shows that the planar binary tree object constructed here is an object of this category, and Theorem 23 proves that it is free. This shows that $\Delta_{\mathrm{LR}}^{\epsilon}$ is not an auxiliary coproduct introduced only for the present construction; rather, it is determined by the unit, the under product $\backslash$, the grafting operation $\vee$, and the recursive infinitesimal compatibility.

5. **A bridge with planar rooted forests.** The final part of the paper compares the internal construction with the known infinitesimal bialgebra on planar rooted forests. Proposition 25 first proves a uniqueness criterion for $\Delta_{\mathrm{LR}}^{\epsilon}$ in terms of $\Delta_{\mathrm{LR}}^{\epsilon}(|) = 0$, the B^r-recursion, and the infinitesimal derivation rule. Theorem 27 shows that there is an isomorphism $\overline{\Theta} : (H_{\mathrm{LR}}^{\epsilon}, \backslash, |, \Delta_{\mathrm{LR}}^{\epsilon}) \longrightarrow (\mathbf{k}\mathcal{F}, \cdot, \mathbf{1}, \Delta_{\mathrm{RT}}^{\epsilon})$ of unitary infinitesimal bialgebras, which is indeed the linear extension of Knuth's rotation correspondence. Hence $\overline{\Theta}$ identifies the planar binary tree coproduct with the rooted-forest infinitesimal coproduct, although $\Delta_{\mathrm{LR}}^{\epsilon}$ and its properties are first constructed intrinsically on planar binary trees.

Taken together, these results clarify the infinitesimal counterpart of the Aguiar–Sottile rooted-forest/binary-tree correspondence. They also separate three different notions which are often close in tree combinatorics: Hopf admissible cuts, planar rooted-forest infinitesimal cocycles, and the elementary cut coproduct on planar binary trees.

Convention Throughout this paper, $\mathbf{k}$ is a unitary commutative ring of characteristic zero. All modules, algebras, coalgebras, bialgebras, tensor products, and linear maps are taken over $\mathbf{k}$. By an algebra we mean an associative algebra, not necessarily unitary. For any set X, $\mathbf{k}X$ denotes the free $\mathbf{k}$-module with basis X. We use Sweedler's notation for the coproduct:

$$\Delta(X) = \sum_{(X)} X_{(1)} \otimes X_{(2)}.$$

2 The infinitesimal bialgebra of planar binary trees

This section constructs the infinitesimal bialgebra structure on the $\mathbf{k}$-module spanned by planar binary trees. We begin by fixing the basic tree operations used throughout the construction.

2.1 Planar binary trees and the grafting operation

We first recall the operations $\vee$ and $\backslash$ on planar binary trees and the corresponding bimodule structure on $H_{\mathrm{LR}}^{\epsilon} \otimes H_{\mathrm{LR}}^{\epsilon}$.

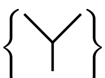
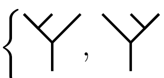
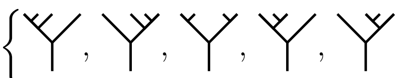
Definition 1. [1, 44] A **unitary infinitesimal bialgebra (of weight zero)** is a quadruple $(A, \cdot, 1_A, \Delta)$, where $(A, \cdot, 1_A)$ is a unitary associative algebra and

$$\Delta : A \longrightarrow A \otimes A$$

is a coassociative coproduct, not necessarily counitary, such that

$$\Delta(ab) = a \cdot \Delta(b) + \Delta(a) \cdot b, \quad \forall\, a, b \in A. \tag{1}$$

A **planar binary tree** [33] is a finite rooted planar tree in which every internal vertex has exactly two ordered incoming branches, called its left and right branches. The trivial binary tree, denoted by $|$, has no internal vertices. Let $\mathcal{Y}$ be the set of planar binary trees. For $T \in \mathcal{Y}$, we denote by $\mathrm{Int}(T)$ the set of internal vertices of T; in particular, $\mathrm{Int}(|) = \emptyset$. For $n \geqslant 0$, let $\mathcal{Y}_n$ denote the set of planar binary trees satisfying $|\mathrm{Int}(T)| = n$; thus $\mathcal{Y}_0 = \{|\,\}$. For example,

$\mathcal{Y}_1 = \{$  $\}$, $\mathcal{Y}_2 = \{$  $\}$, $\mathcal{Y}_3 = \{$  $\}$.

Set

$$H_{\mathrm{LR}}^{\epsilon} := \mathbf{k}\mathcal{Y} = \bigoplus_{n \geqslant 0} \mathbf{k}\mathcal{Y}_n.$$

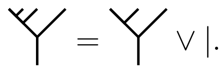
Definition 2. [33, 43] The **binary grafting operation** is the bi-linear map

$$\vee : H_{\mathrm{LR}}^{\epsilon} \otimes H_{\mathrm{LR}}^{\epsilon} \longrightarrow H_{\mathrm{LR}}^{\epsilon}, \quad T_l \otimes T_r \longmapsto T_l \vee T_r,$$

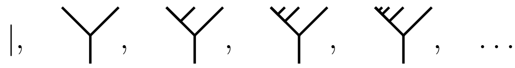
where $T_l \vee T_r$ is obtained by creating a new root and attaching T_l and T_r as its left and right branches, respectively.

For example,

Every non-trivial planar binary tree $T \in \mathcal{Y}$, $T \neq |$, admits a unique root decomposition $T = T_l \vee T_r$, where T_l and T_r are called respectively the left and right branches of the root. For instance,



The trees



are called the **right comb trees** [2]. Denote by F_n^r the right comb tree with n internal vertices:

$$F_0^r := |, \quad F_n^r := F_{n-1}^r \vee |, \quad \forall\, n \geqslant 1. \tag{2}$$

Similarly, the left comb trees are given by

$$F_0^\ell := |, \quad F_n^\ell := | \vee F_{n-1}^\ell, \quad \forall\, n \geqslant 1. \tag{3}$$

2.2 Recursive definition of the infinitesimal coproduct

We now define a coproduct on H_{LR}^ϵ by recursion on the root decomposition $T = T_l \vee T_r$. This construction only uses the grafting operation $\vee$ at this stage. Its coassociativity will be proved first; the under product $\backslash$ will then be introduced in order to formulate and prove the infinitesimal compatibility relation.

Define the **infinitesimal coproduct** to be the linear map

$$\Delta_{\mathrm{LR}}^\epsilon : H_{\mathrm{LR}}^\epsilon \longrightarrow H_{\mathrm{LR}}^\epsilon \otimes H_{\mathrm{LR}}^\epsilon$$

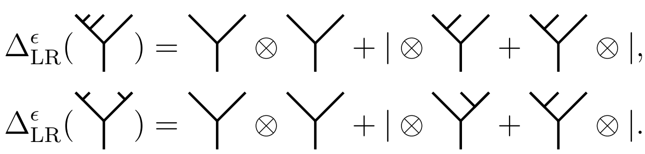
by setting

$$\Delta_{\mathrm{LR}}^\epsilon(T) := \begin{cases} 0, & \text{if}\, T = |, \\ T_l \otimes T_r + (\mathrm{id} \otimes \vee)\big(\Delta_{\mathrm{LR}}^\epsilon(T_l) \otimes T_r\big) & \\ \quad + (\vee \otimes \mathrm{id})\big(T_l \otimes \Delta_{\mathrm{LR}}^\epsilon(T_r)\big) & \text{if}\, T = T_l \vee T_r \neq |. \end{cases} \tag{4}$$

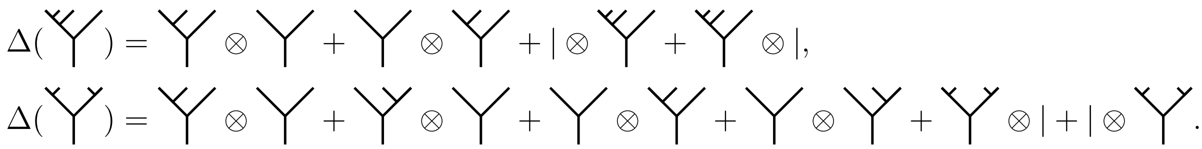
Equivalently, for $T = T_l \vee T_r \neq |$,

$$\Delta_{\mathrm{LR}}^\epsilon(T) := T_l \otimes T_r + \sum_{(T_l)} T_{l,(1)} \otimes \big(T_{l,(2)} \vee T_r\big) + \sum_{(T_r)} \big(T_l \vee T_{r,(1)}\big) \otimes T_{r,(2)}. \tag{5}$$

Example 3. For some low-degree planar binary trees, we have



For comparison, let Δ denote the Loday–Ronco Hopf coproduct on undecorated planar binary trees [33]. For the same two trees, one has



Proposition 4. $\Delta^{\epsilon}_{\mathrm{LR}}$ *is coassociative:*

$$(\Delta^{\epsilon}_{\mathrm{LR}} \otimes \mathrm{id})\Delta^{\epsilon}_{\mathrm{LR}} = (\mathrm{id} \otimes \Delta^{\epsilon}_{\mathrm{LR}})\Delta^{\epsilon}_{\mathrm{LR}}.$$

Proof. Let $T \in \mathcal{Y}$. We prove the identity by induction on $|\mathrm{Int}(T)| \geqslant 0$. For the initial step of $|\mathrm{Int}(T)| = 0$, we have $T = |$. Hence

$$(\Delta^{\epsilon}_{\mathrm{LR}} \otimes \mathrm{id})\Delta^{\epsilon}_{\mathrm{LR}}(|) = 0 = (\mathrm{id} \otimes \Delta^{\epsilon}_{\mathrm{LR}})\Delta^{\epsilon}_{\mathrm{LR}}(|).$$

For the induction step of $|\mathrm{Int}(T)| \geqslant 1$,

$$\begin{aligned}
(\Delta^{\epsilon}_{\mathrm{LR}} \otimes \mathrm{id})\Delta^{\epsilon}_{\mathrm{LR}}(T) &\overset{(5)}{=} \sum_{(T_l)} T_{l,(1)} \otimes T_{l,(2)} \otimes T_r + \sum_{(T_l)} \Delta^{\epsilon}_{\mathrm{LR}}(T_{l,(1)}) \otimes \big(T_{l,(2)} \vee T_r\big) \\
&\quad + \sum_{(T_r)} \Delta^{\epsilon}_{\mathrm{LR}}\big(T_l \vee T_{r,(1)}\big) \otimes T_{r,(2)} \\
&\overset{(5)}{=} \sum_{(T_l)} T_{l,(1)} \otimes T_{l,(2)} \otimes T_r + \sum_{(T_l)} \Delta^{\epsilon}_{\mathrm{LR}}(T_{l,(1)}) \otimes \big(T_{l,(2)} \vee T_r\big) \\
&\quad + \sum_{(T_r)} T_l \otimes T_{r,(1)} \otimes T_{r,(2)} \\
&\quad + \sum_{(T_l),(T_r)} T_{l,(1)} \otimes \big(T_{l,(2)} \vee T_{r,(1)}\big) \otimes T_{r,(2)} \\
&\quad + \sum_{(T_r),(T_{r,(1)})} \big(T_l \vee T_{r,(1)(1)}\big) \otimes T_{r,(1)(2)} \otimes T_{r,(2)}.
\end{aligned} \tag{6}$$

Similarly,

$$\begin{aligned}
(\mathrm{id} \otimes \Delta^{\epsilon}_{\mathrm{LR}})\Delta^{\epsilon}_{\mathrm{LR}}(T) &\overset{(5)}{=} \sum_{(T_r)} T_l \otimes T_{r,(1)} \otimes T_{r,(2)} + \sum_{(T_l)} T_{l,(1)} \otimes \Delta^{\epsilon}_{\mathrm{LR}}\big(T_{l,(2)} \vee T_r\big) \\
&\quad + \sum_{(T_r)} \big(T_l \vee T_{r,(1)}\big) \otimes \Delta^{\epsilon}_{\mathrm{LR}}(T_{r,(2)}) \\
&\overset{(5)}{=} \sum_{(T_r)} T_l \otimes T_{r,(1)} \otimes T_{r,(2)} + \sum_{(T_l)} T_{l,(1)} \otimes T_{l,(2)} \otimes T_r
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{(T_l),(T_{l,(2)})} T_{l,(1)} \otimes T_{l,(2)(1)} \otimes \big(T_{l,(2)(2)} \vee T_r\big) \\
&+ \sum_{(T_l),(T_r)} T_{l,(1)} \otimes \big(T_{l,(2)} \vee T_{r,(1)}\big) \otimes T_{r,(2)} \\
&+ \sum_{(T_r)} \big(T_l \vee T_{r,(1)}\big) \otimes \Delta^{\epsilon}_{\mathrm{LR}}(T_{r,(2)}). \qquad (7)
\end{aligned}
$$

We now compare (6) and (7). Write $T = T_l \vee T_r$, applying the induction hypothesis to T_l gives

$$
(\Delta^{\epsilon}_{\mathrm{LR}} \otimes \mathrm{id})\Delta^{\epsilon}_{\mathrm{LR}}(T_l) = (\mathrm{id} \otimes \Delta^{\epsilon}_{\mathrm{LR}})\Delta^{\epsilon}_{\mathrm{LR}}(T_l),
$$

that is,

$$
\sum_{(T_l),(T_{l,(1)})} T_{l,(1)(1)} \otimes T_{l,(1)(2)} \otimes T_{l,(2)} = \sum_{(T_l),(T_{l,(2)})} T_{l,(1)} \otimes T_{l,(2)(1)} \otimes T_{l,(2)(2)}.
$$

Hence,

$$
\sum_{(T_l)} \Delta^{\epsilon}_{\mathrm{LR}}(T_{l,(1)}) \otimes \big(T_{l,(2)} \vee T_r\big) = \sum_{(T_l),(T_{l,(2)})} T_{l,(1)} \otimes T_{l,(2)(1)} \otimes \big(T_{l,(2)(2)} \vee T_r\big). \qquad (8)
$$

Similarly,

$$
\sum_{(T_r),(T_{r,(1)})} \big(T_l \vee T_{r,(1)(1)}\big) \otimes T_{r,(1)(2)} \otimes T_{r,(2)} = \sum_{(T_r)} \big(T_l \vee T_{r,(1)}\big) \otimes \Delta^{\epsilon}_{\mathrm{LR}}(T_{r,(2)}). \qquad (9)
$$

Comparing (6) with (7), the common terms are identical, while the remaining terms are identified by (8) and (9). This proves the induction step and completes the proof. □

We now introduce the under product. This product will be used to express the infinitesimal compatibility of $\Delta^{\epsilon}_{\mathrm{LR}}$.

Definition 5. [2] The **under product**[1] is the bi-linear map

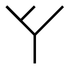
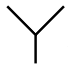
$$
\backslash : H^{\epsilon}_{\mathrm{LR}} \otimes H^{\epsilon}_{\mathrm{LR}} \longrightarrow H^{\epsilon}_{\mathrm{LR}}, \quad S \otimes T \longmapsto S\backslash T,
$$

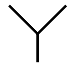
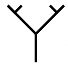
where $S\backslash T$ is obtained by adjoining the root of T to the rightmost branch of S.

The trivial tree $|$ is the unit, satisfying $|\backslash T = T = T\backslash|$. Moreover, if $S \neq |$ and $S = S_l \vee S_r$ is its root decomposition, then, for every $T \in \mathcal{Y}$, one has

$$
S\backslash T = S_l \vee (S_r\backslash T). \qquad (10)
$$

[1]We use the term "under product" for \, following Aguiar and Sottile [2], rather than the more generic phrase "associative product". This avoids confusing \ with other associative products on planar binary trees, in particular the Loday–Ronco Hopf product.

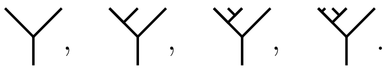
Example 6. Let $S =$  $=$  $\vee \mid$ and $T =$ . Then,

$$S\backslash T = S_l \vee (S_r\backslash T) =$$  $\vee\ (\mid\backslash$  $) =$  $\vee$  $=$ .

A **right decomposition** [2] of a planar binary tree T is a way of writing T as $R\backslash S$. The trivial right decompositions are

$$T = T\backslash|, \quad T = |\backslash T.$$

A non-trivial tree T is called **progressive** [2] if the trivial right decompositions are its only right decompositions. Equivalently, if $T = T_l \vee T_r$ is the root decomposition of T, then T is progressive if and only if $T_r = |$. Thus every non-trivial progressive tree is of the form $T_l \vee |$. For instance,



Moreover, every non-trivial planar binary tree admits a unique factorization

$$T = U_1\backslash \cdots \backslash U_m, \quad U_i \in \mathrm{P}(\mathcal{Y}), \quad m \geqslant 1,$$

where $\mathrm{P}(\mathcal{Y})$ denotes the set of progressive planar binary trees. In particular, Aguiar and Sottile [2] showed that $(H^{\epsilon}_{\mathrm{LR}}, \backslash, |)$ is the free unitary associative algebra on $\mathrm{P}(\mathcal{Y})$.

We equip $H^{\epsilon}_{\mathrm{LR}} \otimes H^{\epsilon}_{\mathrm{LR}}$ with a bimodule structure induced by the under product $\backslash$:

$$\begin{aligned} H^{\epsilon}_{\mathrm{LR}} \otimes (H^{\epsilon}_{\mathrm{LR}} \otimes H^{\epsilon}_{\mathrm{LR}}) &\longrightarrow H^{\epsilon}_{\mathrm{LR}} \otimes H^{\epsilon}_{\mathrm{LR}}, \quad U \otimes (V \otimes W) \longmapsto (U\backslash V) \otimes W, \\ (H^{\epsilon}_{\mathrm{LR}} \otimes H^{\epsilon}_{\mathrm{LR}}) \otimes H^{\epsilon}_{\mathrm{LR}} &\longrightarrow H^{\epsilon}_{\mathrm{LR}} \otimes H^{\epsilon}_{\mathrm{LR}}, \quad (U \otimes V) \otimes W \longmapsto U \otimes (V\backslash W). \end{aligned} \tag{11}$$

The mirror counterpart of the under product $\backslash$ is the **over product** [2]:

$$/ : H^{\epsilon}_{\mathrm{LR}} \otimes H^{\epsilon}_{\mathrm{LR}} \to H^{\epsilon}_{\mathrm{LR}}, \quad S \otimes T \longmapsto S/T,$$

where S/T is obtained by grafting the root of S to the leftmost branch of T.

Let $\tau : \mathcal{Y} \to \mathcal{Y}$ be the mirror symmetry [32] of planar binary trees, defined recursively by

$$\tau(|) := |, \quad \tau(T_l \vee T_r) := \tau(T_r) \vee \tau(T_l),$$

and extended linearly to $H^{\epsilon}_{\mathrm{LR}}$. This symmetry satisfies $\tau^2 = \mathrm{id}$ and

$$\tau(S\backslash T) = \tau(T)/\tau(S), \quad \forall\, S, T \in H^{\epsilon}_{\mathrm{LR}}, \tag{12}$$

equivalently,

$$S/T = \tau\big(\tau(T)\backslash\tau(S)\big) \quad \forall\, S, T \in H^{\epsilon}_{\mathrm{LR}}. \tag{13}$$

Furthermore, with the opposite coproduct

$$(\Delta^{\epsilon}_{\mathrm{LR}})^{\mathrm{op}} := \sigma \circ \Delta^{\epsilon}_{\mathrm{LR}}, \ \text{ where } \sigma(U \otimes V) := V \otimes U,$$

one has

$$\Delta_{\mathrm{LR}}^{\epsilon}(\tau(T)) = (\tau\otimes\tau)(\Delta_{\mathrm{LR}}^{\epsilon})^{\mathrm{op}}(T), \quad \forall\, T\in H_{\mathrm{LR}}^{\epsilon}. \tag{14}$$

As in (11), the over product $/$ induces a bimodule structure on $H_{\mathrm{LR}}^{\epsilon}\otimes H_{\mathrm{LR}}^{\epsilon}$ given by

$$\begin{aligned} H_{\mathrm{LR}}^{\epsilon}\otimes(H_{\mathrm{LR}}^{\epsilon}\otimes H_{\mathrm{LR}}^{\epsilon}) &\longrightarrow H_{\mathrm{LR}}^{\epsilon}\otimes H_{\mathrm{LR}}^{\epsilon}, \quad U\otimes(V\otimes W)\longmapsto (U/V)\otimes W,\\ (H_{\mathrm{LR}}^{\epsilon}\otimes H_{\mathrm{LR}}^{\epsilon})\otimes H_{\mathrm{LR}}^{\epsilon} &\longrightarrow H_{\mathrm{LR}}^{\epsilon}\otimes H_{\mathrm{LR}}^{\epsilon}, \quad (U\otimes V)\otimes W\longmapsto U\otimes(V/W). \end{aligned}$$

The following result collects the properties for both the under and over products.

Lemma 7. *For all $S,T\in H_{\mathrm{LR}}^{\epsilon}$, one has*

$$\Delta_{\mathrm{LR}}^{\epsilon}(S\backslash T) = S\backslash\Delta_{\mathrm{LR}}^{\epsilon}(T) + \Delta_{\mathrm{LR}}^{\epsilon}(S)\backslash T, \tag{15}$$

$$\Delta_{\mathrm{LR}}^{\epsilon}(S/T) = S/\Delta_{\mathrm{LR}}^{\epsilon}(T) + \Delta_{\mathrm{LR}}^{\epsilon}(S)/T. \tag{16}$$

Proof. We first prove (15) by induction on $|\operatorname{Int}(S)|\geqslant 0$ for $S\in\mathcal{Y}$. For the initial step of $|\operatorname{Int}(S)|=0$, we have $S=|$. Hence

$$\Delta_{\mathrm{LR}}^{\epsilon}(|\backslash T) = \Delta_{\mathrm{LR}}^{\epsilon}(T) = |\backslash\Delta_{\mathrm{LR}}^{\epsilon}(T) + \Delta_{\mathrm{LR}}^{\epsilon}(|)\backslash T.$$

For the induction step of $|\operatorname{Int}(S)|\geqslant 1$, on one hand,

$$\begin{aligned} \Delta_{\mathrm{LR}}^{\epsilon}(S\backslash T) &\overset{(10)}{=} \Delta_{\mathrm{LR}}^{\epsilon}\big(S_l\vee(S_r\backslash T)\big)\\ &\overset{(4)}{=} S_l\otimes(S_r\backslash T) + \sum_{(S_l)} S_{l,(1)}\otimes\big(S_{l,(2)}\vee(S_r\backslash T)\big)\\ &\quad + (\vee\otimes\mathrm{id})\big(S_l\otimes\Delta_{\mathrm{LR}}^{\epsilon}(S_r\backslash T)\big)\\ &= S_l\otimes(S_r\backslash T) + \sum_{(S_l)} S_{l,(1)}\otimes\big(S_{l,(2)}\vee(S_r\backslash T)\big)\\ &\quad + (\vee\otimes\mathrm{id})\big(S_l\otimes(S_r\backslash\Delta_{\mathrm{LR}}^{\epsilon}(T) + \Delta_{\mathrm{LR}}^{\epsilon}(S_r)\backslash T)\big)\\ &\quad \text{(by the inductive hypothesis)}\\ &= S_l\otimes(S_r\backslash T) + \sum_{(S_l)} S_{l,(1)}\otimes\big(S_{l,(2)}\vee(S_r\backslash T)\big)\\ &\quad + (\vee\otimes\mathrm{id})\big(S_l\otimes(S_r\backslash\Delta_{\mathrm{LR}}^{\epsilon}(T))\big) + (\vee\otimes\mathrm{id})\big(S_l\otimes(\Delta_{\mathrm{LR}}^{\epsilon}(S_r)\backslash T)\big). \end{aligned} \tag{17}$$

On the other hand,

$$\begin{aligned} S\backslash\Delta_{\mathrm{LR}}^{\epsilon}(T) &= \sum_{(T)}(S\backslash T_{(1)})\otimes T_{(2)}\\ &\overset{(10)}{=} \sum_{(T)}\big(S_l\vee(S_r\backslash T_{(1)})\big)\otimes T_{(2)}\\ &= (\vee\otimes\mathrm{id})\big(S_l\otimes(S_r\backslash\Delta_{\mathrm{LR}}^{\epsilon}(T))\big), \end{aligned} \tag{18}$$

and

$$
\begin{aligned}
\Delta_{\mathrm{LR}}^{\epsilon}(S)\backslash T &\overset{(4)}{=} \Big(S_l \otimes S_r + \sum_{(S_l)} S_{l,(1)} \otimes (S_{l,(2)} \vee S_r) \\
&\qquad + \sum_{(S_r)} (S_l \vee S_{r,(1)}) \otimes S_{r,(2)}\Big)\backslash T \\
&\overset{(11)}{=} S_l \otimes (S_r\backslash T) + \sum_{(S_l)} S_{l,(1)} \otimes \big((S_{l,(2)} \vee S_r)\backslash T\big) \\
&\qquad + \sum_{(S_r)} (S_l \vee S_{r,(1)}) \otimes (S_{r,(2)}\backslash T) \\
&\overset{(10)}{=} S_l \otimes (S_r\backslash T) + \sum_{(S_l)} S_{l,(1)} \otimes \big(S_{l,(2)} \vee (S_r\backslash T)\big) \\
&\qquad + \sum_{(S_r)} (S_l \vee S_{r,(1)}) \otimes (S_{r,(2)}\backslash T) \\
&= S_l \otimes (S_r\backslash T) + \sum_{(S_l)} S_{l,(1)} \otimes \big(S_{l,(2)} \vee (S_r\backslash T)\big) \qquad (19) \\
&\qquad + (\vee \otimes \mathrm{id})\big(S_l \otimes (\Delta_{\mathrm{LR}}^{\epsilon}(S_r)\backslash T)\big).
\end{aligned}
$$

Substituting (18) and (19) into (17), we conclude (15). Equation (16) follows from

$$
\begin{aligned}
\Delta_{\mathrm{LR}}^{\epsilon}(S/T) &\overset{(13)}{=} \Delta_{\mathrm{LR}}^{\epsilon}\big(\tau(\tau(T)\backslash\tau(S))\big) \\
&\overset{(14)}{=} (\tau\otimes\tau)\big(\Delta_{\mathrm{LR}}^{\epsilon}\big)^{\mathrm{op}}\big(\tau(T)\backslash\tau(S)\big) \\
&\overset{(15)}{=} (\tau\otimes\tau)\left[\big(\Delta_{\mathrm{LR}}^{\epsilon}(\tau(T))\backslash\tau(S)\big)^{\mathrm{op}} + \big(\tau(T)\backslash\Delta_{\mathrm{LR}}^{\epsilon}(\tau(S))\big)^{\mathrm{op}}\right] \\
&\overset{(14)}{=} (\tau\otimes\tau)\left[\big((\tau\otimes\tau)\big(\Delta_{\mathrm{LR}}^{\epsilon}\big)^{\mathrm{op}}(T)\backslash\tau(S)\big)^{\mathrm{op}} + \big(\tau(T)\backslash(\tau\otimes\tau)\big(\Delta_{\mathrm{LR}}^{\epsilon}\big)^{\mathrm{op}}(S)\big)^{\mathrm{op}}\right] \\
&\overset{(13)}{=} S/\Delta_{\mathrm{LR}}^{\epsilon}(T) + \Delta_{\mathrm{LR}}^{\epsilon}(S)/T.
\end{aligned}
$$

This completes the proof. □

2.3 An elementary cut formula for the infinitesimal coproduct

We now give a combinatorial interpretation of the coproduct $\Delta_{\mathrm{LR}}^{\epsilon}$ in (4). For each $T \in \mathcal{Y}$, the planar embedding induces a strict total order $<_T$ on $\mathrm{Int}(T)$ from left to right. In more detail, this order is defined recursively on the number of internal vertices: if $\mathrm{Int}(|) = \emptyset$, then there is nothing to do; if $T = T_l \vee T_r$, then the restrictions of $<_T$ to $\mathrm{Int}(T_l)$ and $\mathrm{Int}(T_r)$ coincide with $<_{T_l}$ and $<_{T_r}$, respectively, and

$$u <_T \rho_T <_T v, \quad \forall\, u \in \mathrm{Int}(T_l), v \in \mathrm{Int}(T_r).$$

Thus every internal vertex $e \in \mathrm{Int}(T)$ separates the internal vertices of T into left and right parts.

Definition 8. Let $T \in \mathcal{Y}$. An **elementary cut** of T is the choice of an internal vertex $e \in \mathrm{Int}(T)$. Cutting T at e removes the vertex e and determines an associated pair

$$\big(P^e(T), R^e(T)\big) \in \mathcal{Y} \times \mathcal{Y},$$

where $P^e(T)$ (resp. $R^e(T)$) is the induced planar binary tree from T whose internal vertices are

$$\mathrm{Int}\big(P^e(T)\big) = \{u \in \mathrm{Int}(T) \mid u <_T e\}, \qquad (\text{resp. } \mathrm{Int}\big(R^e(T)\big) = \{v \in \mathrm{Int}(T) \mid e <_T v\}).$$

Recursively, the pair $\big(P^e(T), R^e(T)\big)$ can be obtained as follows. The trivial tree $|$ has no elementary cuts. If $T = T_l \vee T_r \neq |$ and $e \in \mathrm{Int}(T)$, then

$$\big(P^e(T), R^e(T)\big) := \begin{cases} (T_l, T_r), & \text{if } e = \rho_T, \\ \big(P^e(T_l),\ R^e(T_l) \vee T_r\big), & \text{if } e \in \mathrm{Int}(T_l), \\ \big(T_l \vee P^e(T_r),\ R^e(T_r)\big), & \text{if } e \in \mathrm{Int}(T_r). \end{cases} \tag{20}$$

Remark 9.

1. The elementary cut in Definition 8 is not an admissible cut in the sense of the Connes–Kreimer coproduct [12, 31] or the Loday–Ronco Hopf coproduct [33, 43].
2. The $\big(P^e(T), R^e(T)\big)$ is close to the lightning decompositions appearing in [3, 8], but the grading behavior is different. In those constructions the number of internal vertices is conserved, whereas in the present elementary cut the vertex e is removed.
3. With respect to the grading by the number of internal vertices, the product $\backslash$ is homogeneous of degree 0, whereas the coproduct $\Delta^{\epsilon}_{\mathrm{LR}}$ is homogeneous of degree -1. This is a major difference from the Loday–Ronco Hopf coproduct, which is homogeneous of degree 0.

The following formula shows that these elementary cuts give exactly the coproduct $\Delta^{\epsilon}_{\mathrm{LR}}$.

Theorem 10. *For every planar binary tree $T \in \mathcal{Y}$, one has*

$$\Delta^{\epsilon}_{\mathrm{LR}}(T) = \sum_{e \in \mathrm{Int}(T)} P^e(T) \otimes R^e(T). \tag{21}$$

Proof. We prove (21) by induction on $|\mathrm{Int}(T)| \geqslant 0$. For the initial step of $|\mathrm{Int}(T)| = 0$, one has $T = |$ and $\mathrm{Int}(T) = \emptyset$. Hence

$$\Delta^{\epsilon}_{\mathrm{LR}}(T) = \Delta^{\epsilon}_{\mathrm{LR}}(|) = 0 = \sum_{e \in \mathrm{Int}(T)} P^e(T) \otimes R^e(T).$$

For the induction step of $|\mathrm{Int}(T)| \geqslant 1$, write $T = T_l \vee T_r$. Then

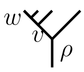
$$\Delta^{\epsilon}_{\mathrm{LR}}(T) \overset{(4)}{=} T_l \otimes T_r + (\mathrm{id} \otimes \vee)\big(\Delta^{\epsilon}_{\mathrm{LR}}(T_l) \otimes T_r\big)$$

$$
\begin{aligned}
&\quad + (\vee \otimes \mathrm{id})\big(T_l \otimes \Delta^{\epsilon}_{\mathrm{LR}}(T_r)\big)\\
&= T_l \otimes T_r + \sum_{u\in \mathrm{Int}(T_l)} P^u(T_l) \otimes \big(R^u(T_l) \vee T_r\big)\\
&\quad + \sum_{w\in \mathrm{Int}(T_r)} \big(T_l \vee P^w(T_r)\big) \otimes R^w(T_r) \qquad \text{(by the inductive hypothesis)}\\
&= P^{\rho_T}(T) \otimes R^{\rho_T}(T) + \sum_{u\in \mathrm{Int}(T_l)} P^u(T) \otimes R^u(T)\\
&\quad + \sum_{w\in \mathrm{Int}(T_r)} P^w(T) \otimes R^w(T) \qquad \text{(by Definition 8)}\\
&= \sum_{e\in \mathrm{Int}(T)} P^e(T) \otimes R^e(T) \qquad \text{(by the decomposition of } \mathrm{Int}(T)\text{)}.
\end{aligned}
$$

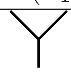
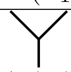
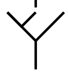
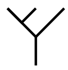
This proves the induction step and completes the proof. □

To make the combinatorial content of the coproduct formula transparent, we record the following examples.

Example 11. Let

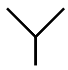
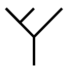
$T_1 =$ 

with $\mathrm{Int}(T_1) = \{\rho, v, w\}$. Then

e	$P^e(T_1)$	$R^e(T_1)$
v		
w	$\vert$	
ρ		$\vert$

Hence

$$\Delta^{\epsilon}_{\mathrm{LR}}(T_1) = P^v(T_1) \otimes R^v(T_1) + P^w(T_1) \otimes R^w(T_1) + P^{\rho}(T_1) \otimes R^{\rho}(T_1)$$

$=$  $\otimes$  $+ \, | \otimes$  $+$  $\otimes \, |.$

This agrees with Example 3.

We now state the resulting infinitesimal bialgebra structure.

Theorem 12. *The quadruple $(H^{\epsilon}_{\mathrm{LR}}, \backslash, |, \Delta^{\epsilon}_{\mathrm{LR}})$ is a unitary infinitesimal bialgebra.*

Proof. Notice that $(H^{\epsilon}_{\mathrm{LR}}, \backslash, |)$ is a unitary associative algebra [2]. Then the result follows from Proposition 4 and Lemma 7. □

Remark 13. The coproduct $\Delta^{\epsilon}_{\mathrm{LR}}$ is not counitary. Indeed, since $\Delta^{\epsilon}_{\mathrm{LR}}(|) = 0$, there cannot exist a linear map $\varepsilon : H^{\epsilon}_{\mathrm{LR}} \to \mathbf{k}$ satisfying

$$(\varepsilon \otimes \mathrm{id})\Delta^{\epsilon}_{\mathrm{LR}} = \mathrm{id}_{H^{\epsilon}_{\mathrm{LR}}} \quad \text{and} \quad (\mathrm{id} \otimes \varepsilon)\Delta^{\epsilon}_{\mathrm{LR}} = \mathrm{id}_{H^{\epsilon}_{\mathrm{LR}}}.$$

Applying the first identity to $|$ would give $0 = |$, a contradiction.

3 Free unitary infinitesimal $(\backslash, \vee)$-bialgebras

In this section we formulate a universal property for the unitary infinitesimal bialgebra obtained in Theorem 12.

3.1 Two one-sided grafting operators

The binary grafting operation $\vee$ gives two natural one-sided operators by fixing the trivial tree on the left or on the right.

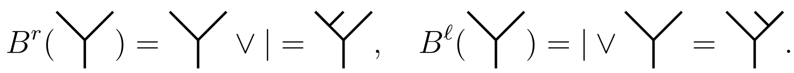
Definition 14. Define two linear maps

$$B^r, B^\ell : H^\epsilon_{\mathrm{LR}} \longrightarrow H^\epsilon_{\mathrm{LR}}$$

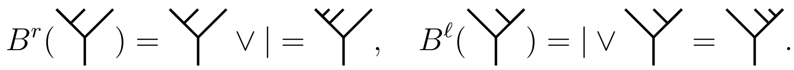
by setting, for every tree $T \in \mathcal{Y}$,

$$B^r(T) := T \vee |, \quad B^\ell(T) := | \vee T.$$

Example 15. For the one-vertex tree, one has



Iterating once more gives the two comb examples



The coproduct $\Delta^\epsilon_{\mathrm{LR}}$ behaves recursively with respect to these two one-sided grafting operators.

Proposition 16. *For every $T \in H^\epsilon_{\mathrm{LR}}$, one has*

$$\Delta^\epsilon_{\mathrm{LR}}(B^r(T)) = T \otimes | + (\mathrm{id} \otimes B^r)\Delta^\epsilon_{\mathrm{LR}}(T), \tag{22}$$

and

$$\Delta^\epsilon_{\mathrm{LR}}(B^\ell(T)) = | \otimes T + (B^\ell \otimes \mathrm{id})\Delta^\epsilon_{\mathrm{LR}}(T). \tag{23}$$

Proof. It is enough to verify the identities on $T \in \mathcal{Y}$. For (22), using Definition 14 and (4),

$$\begin{aligned}
\Delta^\epsilon_{\mathrm{LR}}(B^r(T)) &= \Delta^\epsilon_{\mathrm{LR}}(T \vee |) \\
&\overset{(4)}{=} T \otimes | + (\mathrm{id} \otimes \vee)\big(\Delta^\epsilon_{\mathrm{LR}}(T) \otimes |\big) + (\vee \otimes \mathrm{id})\big(T \otimes \Delta^\epsilon_{\mathrm{LR}}(|)\big) \\
&= T \otimes | + (\mathrm{id} \otimes \vee)\big(\Delta^\epsilon_{\mathrm{LR}}(T) \otimes |\big) \\
&= T \otimes | + (\mathrm{id} \otimes B^r)\Delta^\epsilon_{\mathrm{LR}}(T).
\end{aligned}$$

The proof of (23) is similar. This completes the proof. $\square$

For the comb trees introduced in (2) and (3), the one-sided grafting operators give

$$F_n^r = (B^r)^n(|), \quad F_n^\ell = (B^\ell)^n(|), \quad \forall\, n \geqslant 0.$$

As a consequence of Proposition 16, we obtain the coproduct formulas for these comb trees.

Corollary 17. *For every $n \geqslant 0$, one has*

$$\Delta_{\mathrm{LR}}^{\epsilon}(F_n^r) = \sum_{i=0}^{n-1} F_{n-1-i}^r \otimes F_i^r, \tag{24}$$

$$\Delta_{\mathrm{LR}}^{\epsilon}(F_n^\ell) = \sum_{i=0}^{n-1} F_i^\ell \otimes F_{n-1-i}^\ell, \tag{25}$$

where the sum is understood to be zero for $n = 0$.

Proof. We prove (24) by induction on $n \geqslant 0$. For the initial step of $n = 0$, we have $F_0^r = |$, and hence

$$\Delta_{\mathrm{LR}}^{\epsilon}(F_0^r) = \Delta_{\mathrm{LR}}^{\epsilon}(|) = 0.$$

For the induction step of $n \geqslant 1$, assume that

$$\Delta_{\mathrm{LR}}^{\epsilon}(F_{n-1}^r) = \sum_{i=0}^{n-2} F_{n-2-i}^r \otimes F_i^r.$$

Since $F_n^r = B^r(F_{n-1}^r)$, Proposition 16 gives

$$\begin{aligned}
\Delta_{\mathrm{LR}}^{\epsilon}(F_n^r) &\overset{(22)}{=} F_{n-1}^r \otimes | + (\mathrm{id} \otimes B^r)\Delta_{\mathrm{LR}}^{\epsilon}(F_{n-1}^r) \\
&= F_{n-1}^r \otimes F_0^r + \sum_{i=0}^{n-2} F_{n-2-i}^r \otimes B^r(F_i^r) \qquad \text{(by the inductive hypothesis)} \\
&= F_{n-1}^r \otimes F_0^r + \sum_{i=0}^{n-2} F_{n-2-i}^r \otimes F_{i+1}^r \\
&= \sum_{i=0}^{n-1} F_{n-1-i}^r \otimes F_i^r.
\end{aligned}$$

Thus (24) holds. The proof of (25) is similar, using (23). $\square$

3.2 Unitary $(\backslash, \vee)$-algebras of planar binary trees

We now isolate the algebraic structure carried by planar binary trees with the operations $\backslash$ and $\vee$. This allows us to formulate the universal property satisfied by $(H_{\mathrm{LR}}^{\epsilon}, \backslash, |, \vee)$.

The following definition is inspired by the operated-algebra and cocycle bialgebra viewpoints used for rooted forests [24, 42, 44] and by the compatibility between the under product $\backslash$ and the binary grafting operation $\vee$ on planar binary trees [2].

Definition 18. A **unitary $(\backslash, \vee)$-algebra** is a unitary associative algebra $(H, \backslash_H, 1_H)$ together with a bi-linear map
$$\vee_H : H \otimes H \to H$$
such that
$$(x \vee_H y)\backslash_H z = x \vee_H (y \backslash_H z), \quad \forall\, x, y, z \in H. \tag{26}$$
A morphism of unitary $(\backslash, \vee)$-algebras is defined as usual.

Lemma 19. *The quadruple $(H^{\epsilon}_{\mathrm{LR}}, \backslash, |, \vee)$ is a unitary $(\backslash, \vee)$-algebra.*

Proof. The triple $(H^{\epsilon}_{\mathrm{LR}}, \backslash, |)$ is a unitary associative algebra by Aguiar–Sottile [2], and the compatibility (26) follows from (10). ☐

Now we arrive at our main conclusion of this subsection.

Theorem 20. *The unitary $(\backslash, \vee)$-algebra $(H^{\epsilon}_{\mathrm{LR}}, \backslash, |, \vee)$ is the free unitary $(\backslash, \vee)$-algebra on the empty set.*

Proof. (**Existence**). Let $(H, \backslash_H, 1_H, \vee_H)$ be a unitary $(\backslash, \vee)$-algebra. Define the linear map
$$\Phi : H^{\epsilon}_{\mathrm{LR}} \longrightarrow H$$
on basis trees recursively by
$$\Phi(|) := 1_H, \quad \Phi(T_l \vee T_r) := \Phi(T_l) \vee_H \Phi(T_r), \quad \forall\, T_l, T_r \in \mathcal{Y}. \tag{27}$$
We now verify that Φ preserves $\backslash$
$$\Phi(S\backslash T) = \Phi(S)\backslash_H \Phi(T), \quad \forall\, S, T \in \mathcal{Y}, \tag{28}$$
by induction on $|\mathrm{Int}(S)| \geqslant 0$. For the initial step of $|\mathrm{Int}(S)| = 0$, one has $S = |$, and hence
$$\Phi(|\backslash T) = \Phi(T) = 1_H \backslash_H \Phi(T) = \Phi(|)\backslash_H \Phi(T).$$
For the induction step of $|\mathrm{Int}(S)| \geqslant 1$, write $S = S_l \vee S_r$. Then
$$\begin{aligned}
\Phi(S\backslash T) &\overset{(10)}{=} \Phi\big(S_l \vee (S_r \backslash T)\big) \\
&\overset{(27)}{=} \Phi(S_l) \vee_H \Phi(S_r \backslash T) \\
&= \Phi(S_l) \vee_H \big(\Phi(S_r)\backslash_H \Phi(T)\big) \qquad \text{(by the inductive hypothesis)} \\
&\overset{(26)}{=} \big(\Phi(S_l) \vee_H \Phi(S_r)\big)\backslash_H \Phi(T) \\
&\overset{(27)}{=} \Phi(S_l \vee S_r)\backslash_H \Phi(T) \\
&= \Phi(S)\backslash_H \Phi(T).
\end{aligned}$$
Thus Φ is compatible with the products $\backslash$ and $\backslash_H$. By (27), it also preserves the unit and the grafting operation. Hence Φ is a morphism of unitary $(\backslash, \vee)$-algebras.

(**Uniqueness**). It follows from (27). ☐

3.3 Unitary infinitesimal $(\backslash, \vee)$-bialgebras

We now add the infinitesimal coproduct to the preceding $(\backslash, \vee)$-algebraic structure and require it to be compatible with the grafting operation.

Definition 21. A **unitary infinitesimal $(\backslash, \vee)$-bialgebra** is a quintuple

$$(H, \backslash_H, 1_H, \Delta_H, \vee_H)$$

such that:

(a) $(H, \backslash_H, 1_H, \vee_H)$ is a unitary $(\backslash, \vee)$-algebra;

(b) $(H, \backslash_H, 1_H, \Delta_H)$ is a unitary infinitesimal bialgebra;

(c) The operation $\vee_H$ satisfies the compatibility

$$\Delta_H(x \vee_H y) = x \otimes y + (\mathrm{id} \otimes \vee_H)\big(\Delta_H(x) \otimes y\big) + (\vee_H \otimes \mathrm{id})\big(x \otimes \Delta_H(y)\big), \quad \forall\, x, y \in H. \tag{29}$$

A morphism of unitary infinitesimal $(\backslash, \vee)$-bialgebras is defined as usual.

Proposition 22. *The quintuple $(H^{\epsilon}_{\mathrm{LR}}, \backslash, |, \Delta^{\epsilon}_{\mathrm{LR}}, \vee)$ is a unitary infinitesimal $(\backslash, \vee)$-bialgebra.*

Proof. By Lemma 19, $(H^{\epsilon}_{\mathrm{LR}}, \backslash, |, \vee)$ is a unitary $(\backslash, \vee)$-algebra. By Theorem 12, $(H^{\epsilon}_{\mathrm{LR}}, \backslash, |, \Delta^{\epsilon}_{\mathrm{LR}})$ is a unitary infinitesimal bialgebra. The compatibility (29) follows directly from (4). □

Having identified the planar binary-tree object inside this category, we now record its corresponding universal property.

Theorem 23. *The unitary infinitesimal $(\backslash, \vee)$-bialgebra $(H^{\epsilon}_{\mathrm{LR}}, \backslash, |, \Delta^{\epsilon}_{\mathrm{LR}}, \vee)$ is the free unitary infinitesimal $(\backslash, \vee)$-bialgebra on the empty set.*

Proof. Let $(H, \backslash_H, 1_H, \Delta_H, \vee_H)$ be a unitary infinitesimal $(\backslash, \vee)$-bialgebra. By Theorem 20, there exists a unique morphism $\Phi : H^{\epsilon}_{\mathrm{LR}} \longrightarrow H$ of unitary $(\backslash, \vee)$-algebras satisfying

$$\Phi(|) = 1_H, \quad \Phi(T_l \vee T_r) = \Phi(T_l) \vee_H \Phi(T_r), \quad \forall\, T_l, T_r \in \mathcal{Y}.$$

It remains to prove that Φ is compatible with the coproducts, namely,

$$(\Phi \otimes \Phi)\Delta^{\epsilon}_{\mathrm{LR}}(T) = \Delta_H(\Phi(T)), \quad \forall\, T \in \mathcal{Y}. \tag{30}$$

We prove (30) by induction on $|\mathrm{Int}(T)| \geqslant 0$. For the initial step of $|\mathrm{Int}(T)| = 0$, we have $T = |$. Since $\Delta^{\epsilon}_{\mathrm{LR}}(|) = 0$, we get

$$(\Phi \otimes \Phi)\Delta^{\epsilon}_{\mathrm{LR}}(|) = 0.$$

Also, the infinitesimal compatibility in H gives

$$\Delta_H(1_H) = \Delta_H(1_H \backslash_H 1_H) \overset{(1)}{=} 1_H \backslash_H \Delta_H(1_H) + \Delta_H(1_H) \backslash_H 1_H = 2\Delta_H(1_H),$$

hence $\Delta_H(1_H) = 0$. Since $\Phi(|) = 1_H$, we get

$$\Delta_H(\Phi(|)) = 0.$$

Thus (30) holds for $|\mathrm{Int}(T)| = 0$.

For the induction step of $|\mathrm{Int}(T)| \geqslant 1$, write $T = T_l \vee T_r$. We compute

$$\begin{aligned}
(\Phi\otimes\Phi)\Delta_{\mathrm{LR}}^{\epsilon}(T) &\overset{(4)}{=} (\Phi\otimes\Phi)\Big(T_l\otimes T_r + (\mathrm{id}\otimes\vee)\big(\Delta_{\mathrm{LR}}^{\epsilon}(T_l)\otimes T_r\big)\\
&\qquad + (\vee\otimes\mathrm{id})\big(T_l\otimes\Delta_{\mathrm{LR}}^{\epsilon}(T_r)\big)\Big)\\
&\overset{(27)}{=} \Phi(T_l)\otimes\Phi(T_r) + (\mathrm{id}\otimes\vee_H)\big((\Phi\otimes\Phi)\Delta_{\mathrm{LR}}^{\epsilon}(T_l)\otimes\Phi(T_r)\big)\\
&\qquad + (\vee_H\otimes\mathrm{id})\big(\Phi(T_l)\otimes(\Phi\otimes\Phi)\Delta_{\mathrm{LR}}^{\epsilon}(T_r)\big)\\
&= \Phi(T_l)\otimes\Phi(T_r) + (\mathrm{id}\otimes\vee_H)\big(\Delta_H(\Phi(T_l))\otimes\Phi(T_r)\big)\\
&\qquad + (\vee_H\otimes\mathrm{id})\big(\Phi(T_l)\otimes\Delta_H(\Phi(T_r))\big) \qquad \text{(by the inductive hypothesis)}\\
&\overset{(29)}{=} \Delta_H\big(\Phi(T_l)\vee_H\Phi(T_r)\big)\\
&\overset{(27)}{=} \Delta_H\big(\Phi(T_l\vee T_r)\big)\\
&= \Delta_H(\Phi(T)).
\end{aligned}$$

This proves (30) for $|\mathrm{Int}(T)| \geqslant 1$ and completes the induction.

Therefore Φ preserves the coproduct. Since Φ is already a morphism of unitary $(\backslash, \vee)$-algebras, it is a morphism of unitary infinitesimal $(\backslash, \vee)$-bialgebras. Finally, uniqueness follows from Theorem 20. This completes the proof. $\square$

4 An infinitesimal bialgebra isomorphism from planar binary trees to planar rooted forests

In this section, we establish an infinitesimal bialgebra isomorphism from the infinitesimal bialgebra $(H_{\mathrm{LR}}^{\epsilon}, \backslash, |, \Delta_{\mathrm{LR}}^{\epsilon})$ of planar binary trees constructed in last section to the infinitesimal bialgebra $(\mathbf{k}\mathcal{F}, \cdot, \mathbf{1}, \Delta_{\mathrm{RT}}^{\epsilon})$ of planar rooted forests [44].

Let $\mathcal{T}$ and $\mathcal{F}$ denote the sets of planar rooted trees and planar rooted forests, respectively. The grafting operation

$$B^+ : \mathcal{F} \longrightarrow \mathcal{T} \subseteq \mathcal{F}$$

sends a forest to the tree obtained by grafting its roots to a new common root. We recall from [44] that the $\mathbf{k}$-module $\mathbf{k}\mathcal{F}$, equipped with forest concatenation product $\cdot$, unit $\mathbf{1}$, and the infinitesimal coproduct

$$\Delta_{\mathrm{RT}}^{\epsilon} : \mathbf{k}\mathcal{F} \longrightarrow \mathbf{k}\mathcal{F}\otimes\mathbf{k}\mathcal{F},$$

is a unitary infinitesimal bialgebra (of weight zero). Moreover,

$$\Delta_{\mathrm{RT}}^{\epsilon}(FG) = F\cdot\Delta_{\mathrm{RT}}^{\epsilon}(G) + \Delta_{\mathrm{RT}}^{\epsilon}(F)\cdot G, \quad \forall\, F, G \in \mathbf{k}\mathcal{F}, \tag{31}$$

and

$$\Delta_{\mathrm{RT}}^{\epsilon}(B^+(F)) = F \otimes \mathbf{1} + (\mathrm{id} \otimes B^+)\Delta_{\mathrm{RT}}^{\epsilon}(F), \quad \forall\, F \in \mathbf{k}\mathcal{F}. \tag{32}$$

In particular, $\Delta_{\mathrm{RT}}^{\epsilon}(\mathbf{1}) = 0$ [44]. Under the correspondence used below, the grafting operator B^+ corresponds to the operator B^r on planar binary trees, and forest concatenation corresponds to the associative product $\backslash$ on $H_{\mathrm{LR}}^{\epsilon}$. Hence the infinitesimal coproduct $\Delta_{\mathrm{RT}}^{\epsilon}$ should correspond, via a bridge map, to $\Delta_{\mathrm{LR}}^{\epsilon}$.

The key point is that on the rooted-forest side the coproduct is governed by the infinitesimal 1-cocycle condition for B^+ together with the derivation law for concatenation, whereas on the planar binary tree side $\Delta_{\mathrm{LR}}^{\epsilon}$ is governed by the corresponding recursive identity for B^r together with the derivation law for $\backslash$.

Remark 24. There are two reasons for using B^r, rather than B^ℓ, in the forthcoming proposition.

1. The B^r-recursion (33) below has the same one-sided form as the infinitesimal 1-cocycle formula for planar rooted forests [44]:

   $$\Delta_{\mathrm{RT}}^{\epsilon}(B^+(F)) = F \otimes \mathbf{1} + (\mathrm{id} \otimes B^+)\Delta_{\mathrm{RT}}^{\epsilon}(F).$$

   By contrast, the B^ℓ-identity (23) has the opposite form

   $$\Delta_{\mathrm{LR}}^{\epsilon}\big(B^\ell(T)\big) = | \otimes T + (B^\ell \otimes \mathrm{id})\Delta_{\mathrm{LR}}^{\epsilon}(T),$$

   and therefore does not match the rooted-forest 1-cocycle pattern.

2. The choice of B^r is made to be compatible with the progressive tree:

   $$\mathrm{P}(\mathcal{Y}) = \{B^r(U) \mid U \in \mathcal{Y}\}.$$

   Here B^r parametrizes the generators on which the uniqueness argument starts, whereas B^ℓ does not.

This leads to the following uniqueness characterization of $\Delta_{\mathrm{LR}}^{\epsilon}$.

Proposition 25. *The coproduct $\Delta_{\mathrm{LR}}^{\epsilon}$ is uniquely determined by the following three conditions:*

$$\begin{aligned}
\Delta_{\mathrm{LR}}^{\epsilon}(|) &= 0,\\
\Delta_{\mathrm{LR}}^{\epsilon}\big(B^r(T)\big) &= T \otimes | + (\mathrm{id} \otimes B^r)\Delta_{\mathrm{LR}}^{\epsilon}(T), \quad \forall\, T \in H_{\mathrm{LR}}^{\epsilon}, &(33)\\
\Delta_{\mathrm{LR}}^{\epsilon}(S\backslash T) &= S\backslash\Delta_{\mathrm{LR}}^{\epsilon}(T) + \Delta_{\mathrm{LR}}^{\epsilon}(S)\backslash T, \quad \forall\, S \in H_{\mathrm{LR}}^{\epsilon},\ T \in \{|\} \sqcup \mathrm{P}(\mathcal{Y}). &(34)
\end{aligned}$$

Proof. Let

$$\widetilde{\Delta} : H_{\mathrm{LR}}^{\epsilon} \to H_{\mathrm{LR}}^{\epsilon} \otimes H_{\mathrm{LR}}^{\epsilon}$$

be a coproduct satisfying the same three conditions. We want to prove that $\Delta_{\mathrm{LR}}^{\epsilon} = \widetilde{\Delta}$. Notice that $(H_{\mathrm{LR}}^{\epsilon}, \backslash, |)$ is freely generated by progressive trees as a unitary associative algebra [2]. So we are left to verify that

$$\widetilde{\Delta}(T) = \Delta_{\mathrm{LR}}^{\epsilon}(T), \quad \forall\, T \in \{|\,\} \sqcup \mathrm{P}(\mathcal{Y}),$$

which is done by induction on $|\mathrm{Int}(T)| \geqslant 0$.

For the initial step of $|\mathrm{Int}(T)| = 0$, one has $T = |$. Hence

$$\widetilde{\Delta}(T) = \widetilde{\Delta}(|) = 0 = \Delta^{\epsilon}_{\mathrm{LR}}(|) = \Delta^{\epsilon}_{\mathrm{LR}}(T).$$

For the induction step of $|\mathrm{Int}(T)| \geqslant 1$, the tree T is non-trivial and progressive. Hence there exists a unique $U \in \mathcal{Y}$ such that

$$T = B^r(U) = U \vee |.$$

By the defining condition for $\widetilde{\Delta}$ and by (33),

$$\widetilde{\Delta}(T) = U \otimes | + (\mathrm{id} \otimes B^r)\widetilde{\Delta}(U), \tag{35}$$
$$\Delta^{\epsilon}_{\mathrm{LR}}(T) = U \otimes | + (\mathrm{id} \otimes B^r)\Delta^{\epsilon}_{\mathrm{LR}}(U). \tag{36}$$

Write the unique \-factorization of U as

$$U = U_1 \backslash \cdots \backslash U_m, \text{ where } U_i \in \mathrm{P}(\mathcal{Y}) \text{ and } m \geqslant 0.$$

Here we employ the convention that $U = |$ if $m = 0$. For $m \geqslant 1$, one has $|\mathrm{Int}(U_i)| < |\mathrm{Int}(T)|$, and hence the induction hypothesis gives

$$\widetilde{\Delta}(U_i) = \Delta^{\epsilon}_{\mathrm{LR}}(U_i), \quad \forall\, i = 1, \ldots, m.$$

Since both coproducts satisfy the same derivation rule, the factorization $U = U_1 \backslash \cdots \backslash U_m$ gives

$$\begin{aligned}
\widetilde{\Delta}(U) &= \sum_{j=1}^{m} (U_1 \backslash \cdots \backslash U_{j-1}) \backslash \widetilde{\Delta}(U_j) \backslash (U_{j+1} \backslash \cdots \backslash U_m) \\
&= \sum_{j=1}^{m} (U_1 \backslash \cdots \backslash U_{j-1}) \backslash \Delta^{\epsilon}_{\mathrm{LR}}(U_j) \backslash (U_{j+1} \backslash \cdots \backslash U_m) \qquad \text{(by the inductive hypothesis)} \\
&= \Delta^{\epsilon}_{\mathrm{LR}}(U),
\end{aligned}$$

where the sum is understood to be zero for $m = 0$. Hence (35) and (36) imply $\widetilde{\Delta}(T) = \Delta^{\epsilon}_{\mathrm{LR}}(T)$. This completes the induction and hence the proof. ☐

The following recalls Knuth's rotation correspondence, under the name left-child/right-sibling representation in [30].

Lemma 26. *[30] There is a set bijection $\Theta : \mathcal{Y} \longrightarrow \mathcal{F}$ determined recursively by*

$$\Theta(|) := \mathbf{1}, \quad \Theta(T_1 \vee T_2) := B^+\big(\Theta(T_1)\big)\Theta(T_2), \quad \forall\, T_1, T_2 \in \mathcal{Y}. \tag{37}$$

The planar rooted-tree version of Knuth's rotation correspondence used in [15] is given by the composition map $\mathcal{Y} \xrightarrow{\Theta} \mathcal{F} \xrightarrow{B^+} \mathcal{T}$. Now we arrive at the main result of this section.

Theorem 27. *There is an isomorphism of unitary infinitesimal* $(\backslash, \vee)$*-bialgebras:*

$$\overline{\Theta} : \big(H_{\mathrm{LR}}^{\epsilon}, \backslash, |, \Delta_{\mathrm{LR}}^{\epsilon}, \vee\big) \longrightarrow \big(\mathbf{k}\mathcal{F}, \backslash_{\mathcal{F}}, \mathbf{1}, \Delta_{\mathrm{RT}}^{\epsilon}, \vee_{\mathcal{F}}\big). \tag{38}$$

In particular, $\overline{\Theta}$ *is an isomorphism of unitary infinitesimal bialgebras*

$$\overline{\Theta} : \big(H_{\mathrm{LR}}^{\epsilon}, \backslash, |, \Delta_{\mathrm{LR}}^{\epsilon}\big) \longrightarrow \big(\mathbf{k}\mathcal{F}, \cdot, \mathbf{1}, \Delta_{\mathrm{RT}}^{\epsilon}\big). \tag{39}$$

Proof. Define bi-linear maps

$$\backslash_{\mathcal{F}}, \vee_{\mathcal{F}} : \mathbf{k}\mathcal{F} \otimes \mathbf{k}\mathcal{F} \longrightarrow \mathbf{k}\mathcal{F}$$

by setting

$$F \backslash_{\mathcal{F}} G := FG, \quad F \vee_{\mathcal{F}} G := B^{+}(F)G, \quad \forall\, F, G \in \mathcal{F}.$$

Since

$$(F \vee_{\mathcal{F}} G) \backslash_{\mathcal{F}} H = B^{+}(F)GH = F \vee_{\mathcal{F}} \big(G \backslash_{\mathcal{F}} H\big),$$

the quadruple $(\mathbf{k}\mathcal{F}, \backslash_{\mathcal{F}}, \mathbf{1}, \vee_{\mathcal{F}})$ is a unitary $(\backslash, \vee)$-algebra. By (31), $(\mathbf{k}\mathcal{F}, \backslash_{\mathcal{F}}, \mathbf{1}, \Delta_{\mathrm{RT}}^{\epsilon})$ is a unitary infinitesimal bialgebra. Moreover, for $F, G \in \mathcal{F}$,

$$\begin{aligned}
\Delta_{\mathrm{RT}}^{\epsilon}\big(F \vee_{\mathcal{F}} G\big) &= \Delta_{\mathrm{RT}}^{\epsilon}\big(B^{+}(F)G\big) \\
&\overset{(31)}{=} B^{+}(F) \cdot \Delta_{\mathrm{RT}}^{\epsilon}(G) + \Delta_{\mathrm{RT}}^{\epsilon}\big(B^{+}(F)\big) \cdot G \\
&\overset{(32)}{=} F \otimes G + (\mathrm{id} \otimes \vee_{\mathcal{F}})\big(\Delta_{\mathrm{RT}}^{\epsilon}(F) \otimes G\big) + (\vee_{\mathcal{F}} \otimes \mathrm{id})\big(F \otimes \Delta_{\mathrm{RT}}^{\epsilon}(G)\big).
\end{aligned}$$

Hence $(\mathbf{k}\mathcal{F}, \backslash_{\mathcal{F}}, \mathbf{1}, \Delta_{\mathrm{RT}}^{\epsilon}, \vee_{\mathcal{F}})$ is a unitary infinitesimal $(\backslash, \vee)$-bialgebra.

Since $\big(H_{\mathrm{LR}}^{\epsilon}, \backslash, |, \Delta_{\mathrm{LR}}^{\epsilon}, \vee\big)$ is the free unitary infinitesimal $(\backslash, \vee)$-bialgebra on the empty set by Theorem 23, there exists a unique morphism of unitary infinitesimal $(\backslash, \vee)$-bialgebras

$$\overline{\Theta} : \big(H_{\mathrm{LR}}^{\epsilon}, \backslash, |, \Delta_{\mathrm{LR}}^{\epsilon}, \vee\big) \longrightarrow \big(\mathbf{k}\mathcal{F}, \backslash_{\mathcal{F}}, \mathbf{1}, \Delta_{\mathrm{RT}}^{\epsilon}, \vee_{\mathcal{F}}\big).$$

Notice that $\overline{\Theta}(|) = \mathbf{1}$ and

$$\overline{\Theta}(T_1 \vee T_2) = \overline{\Theta}(T_1) \vee_{\mathcal{F}} \overline{\Theta}(T_2) = B^{+}\big(\overline{\Theta}(T_1)\big)\overline{\Theta}(T_2), \quad \forall\, T_1, T_2 \in \mathcal{Y}.$$

Together with Lemma 26, we have

$$\overline{\Theta}|_{\mathcal{Y}} = \Theta : \mathcal{Y} \longrightarrow \mathcal{F}$$

is a set bijection, and so $\overline{\Theta}$ is a linear isomorphism. $\square$

We conclude this section with a final remark.

Remark 28.

1. Recall the Aguiar–Sottile rooted-forest/binary-tree bijection $\varphi : \mathcal{F} \longrightarrow \mathcal{Y}$ characterized by [2]

$$\varphi(\mathbf{1}) = |, \quad \varphi(FG) = \varphi(F) \backslash \varphi(G), \quad \varphi\big(B^{+}(F)\big) = \varphi(F) \vee |, \quad \forall\, F, G \in \mathcal{F}.$$

   We have $\overline{\Theta}|_{\mathcal{Y}} = \varphi^{-1}$. The underlying bijection is classical, but the algebraic structures identified here are different.

2. Since Aguiar–Sottile's framework also connects planar binary trees with permutations, it is natural to ask whether one can study an infinitesimal bialgebra structure on permutations compatible with these maps.

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