Anti-Concentration with respect to Random Permutations

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Abstract

Classical anti-concentration results focus on the random sum $S := \sum_{i=1}^{n} \xi_i v_i$, where ξ_i are independent random variables and v_i are real numbers. In this paper, we prove new concentration results concerning the random sum $S := \sum_{i=1}^{n} w_{\pi_i} v_i$, where w_i, v_i are real numbers and π is a random permutation.

Mathematics Subject Classifications: 05D40, 60C05

1 Introduction

Anti-concentration is a well known and important phenomenon in probability. An anti-concentration inequality asserts that (under certain conditions) a random variable S does not have too much mass in a small region or on a single point. We follow Levy [7] and define the concentration function of a real random variable Y as

$$Q(S,t) = \sup_{|I|=t} \mathbb{P}(S \in I), \tag{1}$$

where I runs over the set of all closed intervals of length t.

Many prominent mathematicians, including Littlewood, Offord, Erdős, Levy, Kolmogorov, Rogozin, and Esseen made significant contribution to the early study of the concentration function and the anti-concentration phenomenon [5, 2, 12, 3, 4, 7, 1]. Since the early 2000s, the study of anti-concentration has been revitalized, with motivations and applications coming from various areas, including random matrix theory, combinatorics, random functions, and data science. New, stronger, anti-concentration inequalities have been discovered, and these play essential role in the solution of many long standing

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problems, such as the Circular Law conjecture in random matrix theory; see [13] for a survey.

In most of the above mentioned works, the random variable in question is a linear combination of many independent atom random variables. The Erdős-Littlewood-Offord inequality, one of the earliest and most well-known results in this area, has the following form.

Theorem 1 (Erdős-Littlewood-Offord). *Let $v \in \mathbb{R}^n$ satisfy $|v_i| \geqslant 1$ for all $i \in [n]$ and let ξ_i be iid Rademacher variables. Then for any interval I*

$$\mathbb{P}(\sum_{i=1}^{n} \xi_i v_i \in I) = O\left(\frac{|I|+1}{\sqrt{n}}\right).$$

In combinatorial applications, one often uses the following corollary

Corollary 2. *Let $v_1, \ldots, v_n$ be non-zero numbers and ξ_i be iid Rademacher variables. Then for any x*

$$\mathbb{P}(\sum_{i=1}^{n} \xi_i v_i = x) = O\left(\frac{1}{\sqrt{n}}\right).$$

This bound is sharp, as shown by choosing $v_1 = \cdots = v_n = 1$, and using Sterling's formula. For many refinements of result, we refer to the survey [13].

Notice that the random variable $\sum_{i=1}^{n} \xi_i v_i$ can be written as the inner product of the random vector $w := (\xi_1, \ldots, \xi_n)$ with the deterministic vector $v = (v_1, \ldots, v_n)$.

In this, and many other theorems (see [13]), w is a random vector distributed according to the product measure generated by independent atom random variables. In applications, another frequently used probability space is the space generated by random permutations. The goal of this paper is to initiate the study of anti-concentration results in the random permutation space.

Let $w = (w_1, \ldots, w_n)$ be a fixed vector and π a permutation in S_n. We denote by w_π the vector $(w_{\pi(1)}, \ldots, w_{\pi(n)})$ obtained by permuting the coordinates of w by π. Fix $v \in \mathbb{R}^n$. Our object of study is the random variable $w_\pi \cdot v = \sum_i w_i v_{\pi(i)}$, where π is chosen uniformly at random from S_n.

In a recent paper [10], Soze, motivated by an application concerning random polynomials, considered a special case wheren $v = (1, 2, \ldots, n)$, and proved

Theorem 3. *Let $v = (1, 2, \ldots, n)$ and $w \in \mathbb{R}^n$ be a unit vector such that $w \cdot \mathbf{1} = 0$. Then for any $L \in \mathbb{R}$*

$$\mathbb{P}\left(|w_\pi \cdot v - Ln| \leqslant 1\right) = O\left(\frac{1}{n} e^{-c|L|}\right),$$

for some absolute constant $c > 0$.

To give some intuition for the assumption that $w \cdot \mathbf{1} = 0$, let us point out that if all the coordinates of w are the same, then $w_\pi \cdot v$ is constant and no nontrivial anti-concentration statement holds. Thus, in statements involving random permutations, the anti-concentration of $w_\pi \cdot v$ will depend on how close w and v are to being parallel to $\mathbf{1}$. Furthermore, as any vector can be decomposed as $c\mathbf{1} + u$, where u is orthogonal to $\mathbf{1}$, we can reduce the general case to the case where $w \cdot \mathbf{1} = 0$.

Corollary 4. *Let $v = (1, 2, \ldots, n)$ and $w \in \mathbb{R}^n$ be a unit vector such that $w \cdot \mathbf{1} = 0$. Then for any point $x \in \mathbb{R}$*

$$\mathbb{P}(w_\pi \cdot v = x) = O\left(\frac{1}{n}\right).$$

The bound $O(1/n)$ here is tight, as shown by $w = (1, -1, 0, \ldots, 0)$. The problem of bounding this point mass was recently considered by Huang, McKinnon, and Satriano [14], who obtained the correct asymptotics and optimal bounds in the case $n = p$ or $n = p + 1$, where p is a prime. Subsequently, Pawlowski [15] obtained optimal bounds for all n, showing that under the assumptions that the coordinates of v are all distinct then for all $x \in \mathbb{R}$ we have

$$\mathbb{P}[w_\pi \cdot v = x] \leqslant \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd,} \\ \frac{1}{n-1} & \text{if } n \text{ is even.} \end{cases}$$

In both cases there exist choices of v, w for which equality holds.

We are going to develop a bound that applies for arbitrary vectors v and w. Our approach will be entirely different from that of Soze, which is tailored to the special case $v = (1, 2, \ldots, n)$. We say that w is *increasing* if $w_1 \leqslant w_2 \leqslant \ldots \leqslant w_n$. Without loss of generality, we can assume that both w and v are increasing and define

$$\Delta(v) = \min_{i \neq j} |v_i - v_j|.$$

Our main result is the following

Theorem 5 (Anti-concentration for random permutations)**.** *Let $w, v \in \mathbb{R}^n$ be increasing with $\Delta(v) > 0$. For any interval I and indices i_1, i_2 such that $i_1 + i_2 \leqslant n$ and $w_{n-i_2} - w_{i_1} > 0$, we have*

$$\mathbb{P}(w_\pi \cdot v \in I) = O\left(1 + \frac{|I|}{\Delta(v)(w_{n-i_2} - w_{i_1})}\right) \frac{1}{(i_1 + i_2)\sqrt{\min(i_1, i_2)}}.$$

This gives a strong bound if by some careful choice we can make i_1, i_2, and $w_{n-i_2} - w_{i_1}$ large simultaneously. In the case when there are repetition among the v_i, or $\Delta(v)$ is small, one can still get a bound by conditioning on a subset which has large Δ. For instance, (by choosing $i_i = i_2 = \varepsilon n/3$ in what follows) we obtain the following improvement of Corollary 4. This result can also be seen as an analogue of Corollary 2 in the random permutation setting.

Corollary 6. *Assume v has all distinct coordinates and that no w_i is repeated more than $(1-\epsilon)n$ times, for some $\epsilon > 0$. Then for any $x \in \mathbb{R}$,*

$$\mathbb{P}(w_\pi \cdot v = x) = O\left(\frac{1}{\epsilon\ n^{3/2}}\right).$$

Through a slight additional work (see Section 3), we also obtain the following

Corollary 7. *Let $w, v \in \mathbb{R}^n$ with $\Delta(v) > 0$. Let $\overline{w} = n^{-1}\sum w_i$ and $\sigma^2 = \sum(w_i - \overline{w})^2$. Then for any interval I,*

$$\mathbb{P}(w_\pi \cdot v \in I) = O\left(\frac{|I|}{n\sigma\Delta(v)}\sqrt{\log n} + \frac{1}{n}\right).$$

The main tools in the proof of Theorem 5 are the following, which are of independent interest

Lemma 8 (Random subset sum with replacement)**.** *Let $A := \{a_1, \ldots, a_n\}$ be a set of distinct real numbers. Set S be the sum of k elements from A, chosen uniformly randomly with replacement. Then for any $x \in \mathbb{R}$,*

$$\mathbb{P}(S = x) = O\left(\frac{1}{n\sqrt{k}}\right).$$

Lemma 9 (Random subset sum without replacement)**.** *Let $\{a_1, \ldots, a_n\}$ be a set of distinct real numbers. Set S be the sum of k elements from A, chosen uniformly randomly without replacement. Then for any $x \in \mathbb{R}$,*

$$\mathbb{P}(X = x) = O\left(\frac{1}{n\sqrt{\min(k, n-k)}}\right).$$

Remark 10. First commenting on the tightness of the bounds, we note that Lemmas 8 and 9 are both sight (up to the implied constant) as witnessed when the set A consists of an arithmetic progression (e.g., $A = \{1, 2, 3, \ldots, n\}$). In that case, if σ denotes the standard deviation of the sum in question, then $1/\sigma$ is the same order as the general upper bound (and for integer-valued random variables, at least one atom probability is at least C/σ by—say—Chebyshev's inequality).

The tightness in Theorem 5 is somewhat less clear. For instance, if we take $v = (1, 2, 3, \ldots, n)$ and $w \in \{0, 1\}^n$ is chosen so that w consists of $k = \min(i_1, i_2)$ zeroes and $n - k$ ones, then this matches the setting of Lemma 9 with $A = \{1, 2, \ldots, n\}$, so as discussed, this has $1/\sigma = C/n\sqrt{k}$. Thus when $|I|$ is small and centered at μ (e.g., $|I| \ll n\sqrt{k}$), this gives a probability of $\mathbb{P}(w_\pi \cdot v \in I)$ being roughly $(i_1 + i_2)/n$ times the bound given in Theorem 5. For our applications, this is not an issue as the ratio $(i_1 + i_2)/n$ will be bounded below, but for small values of $i_1 + i_2$, Theorem 5 may have room for improvement.

Remark 11. One referee pointed out to us that results of the flavour of Lemma 1.8 (anti-concentration for sums of random fixed-size subsets of a set) were considered in Section 4.2.2 of [6].

Remark 12. We originally proved the main result of this note around 2019–2020, as a tool to study random polynomials. The application we had in mind (see Section 4), however, has turned out to be more technical than anticipated, so we did not publish our findings at the time. Since then, we found out about recent developments [14, 15] and have decided to publish this first as it is of independent interest. A very recent paper [9] provides inverse Littlewood-Offord theorems (in spirit of [16]), which strengthens Corollary 6, and characterizes all pairs of vectors v, w where the mass probability $\mathbb{P}(w_\pi \cdot v = x)$ is polynomially large for some x.

Remark 13. When considering random variables defined in terms of random permutations, it is very common (e.g. Hoeffding's combinatorial CLT) to define random variables of the form $\sum_i A_{i,\pi(i)}$, for a general matrix A with entries A_{ij}. The setting in this paper corresponds to the case when A has rank 1. Results for general A were recently obtained in [9].

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2 Proofs of the main lemmas

We make use of the following result of Rogozin [12]. Let Y be a real random variable; recall the notion of concentration function

$$Q(Y, t) := \max_y \mathbb{P}(y \leqslant Y \leqslant y + t).$$

Theorem 14 (Rogozin)**.** *There is a constant $C_0 > 0$ such that the following holds. Let $X_1, \ldots, X_k$ be independent random variables and set $X = \sum_I^k X_i$. Then*

$$Q(X, t) \leqslant C_0 (\sum_{i=1}^{k} (1 - Q(X_i, t)))^{-1/2}.$$

With this in place, we are ready to prove the relevant lemmas.

Proof of Lemma 8. Without loss of generality, we can assume that $n \geqslant 10$. We use an *induction* on k. Set $C = 6(C_0 + 1)$. Since $\mathbb{P}(X = x) \leqslant n^{-1}$ for any x, the claim is trivial for $k \leqslant 4$. From now we can assume $k > 4$.

Now let I be the shortest closed interval such that $|I \cap v| = \lceil n/2 \rceil$ and set $t = |I|$ (if there are many such intervals, take any). By the definition of I and t, $Q(X_j, t) \leqslant \lceil n/2 \rceil / n < 2/3$, for all $1 \leqslant j \leqslant k$. Notice that

$$\mathbb{P}(X = x) = \sum_{i=1}^{n} \mathbb{P}(X' = x - a_i \wedge X_k = a_i) = n^{-1} \sum_{i=1}^{n} \mathbb{P}(X' = x - a_i)$$

Let $J := x - I := \{x - y, y \in I\}$. Then

$$\sum_{i=1}^{k} \mathbb{P}(X' = x - a_i) = \mathbb{P}(X' \in J) + \sum_{j, a_j \notin I} \mathbb{P}(X' = x - a_j).$$

By Theorem 14, $\mathbb{P}(X' \in J) \leqslant C_0\sqrt{3/(k-1)}$. Furthermore, by the induction hypothesis

$$\mathbb{P}(X' = x - a_j) \leqslant C(n\sqrt{k-1})^{-1}.$$

Thus,

$$n^{-1} \sum_{j, a_j \notin I} \mathbb{P}(X' = x - a_j) \leqslant \frac{C}{2}(n\sqrt{k-1})^{-1},$$

which—thanks to the fact that $k \geqslant 5$ and the definition of C—implies that

$$\mathbb{P}(X = x) \leqslant \frac{1}{n\sqrt{k-1}}(C_0\sqrt{3} + C/2) \leqslant C\frac{1}{n\sqrt{k}},$$

as desired. □

Proof of Lemma 9. We consider two cases.

Case 1 ($k \leqslant n^{0.49}$): Suppose $k \leqslant n^{.49}$. Let $X_1, \ldots, X_k$ be chosen uniformly from A (with replacement). Let $\mathcal{E}$ be the event that the values of $X_1, \ldots, X_k$ are different. As $k = o(\sqrt{n})$, by the birthday paradox, $\mathbb{P}(\mathcal{E}) = 1 - o(1)$. On the other hand, if $\mathcal{E}$ occurs then the set formed by $X_1, \ldots, X_k$ is uniformly random among the set of all subsets of size k. Therefore

$$\mathbb{P}(S = x) = \mathbb{P}(\sum_{i=1}^{k} X_i = x|\mathcal{E}) = \mathbb{P}(\sum_{i=1}^{k} X_i = x \wedge \mathcal{E})\mathbb{P}(\mathcal{E})^{-1} \leqslant (1 + o(1))\mathbb{P}(\sum_{i=1}^{k} = x),$$

and the claim follows from Lemma 8.

Case 2 ($k > n^{0.49}$): We now turn our attention to the case that $k > n^{.49}$. Without less of generality, we can assume $k \leqslant n/2$. We use the following result from Inverse Littlewood-Offord theory (see [13] for a general discussion and clarifying details).

Lemma 15. *(Sparse Inverse Erdős-Littlewood-Offord) [13] Let $c < 1$ and C be positive constants. There are constants R, A depending on c, C such that the following holds.*

Let x_i be iid $(0, 1)$ random variables with $n^{-c} \leqslant \mathbb{P}(x_i = 1) := \alpha \leqslant 1/2$. Let $b_1, \ldots, b_n$ be integers such that

$$\rho := \sup_a \mathbb{P}(b_1x_1 + \cdots + b_nx_n = a) \geqslant n^{-C}.$$

Then there exists a proper symmetric generalized arithmetic progression Q of rank $r \leqslant R$ which contains at least $0.9n$ of the v_i (counting multiplicities) and where $|Q| \leqslant A\rho^{-1}(\alpha n)^{-r/2} + 1$.

In order to use the above result, we set $b_i = a_i + N$ where N is a large integer to be chosen later. Towards a contradiction, assume that for some x

$$\mathbb{P}(S = x) \geqslant B\frac{1}{n\sqrt{k}}, \tag{2}$$

where $B > 0$ is a large constant to be chosen. Define x_i as in Lemma 15 with $\mathbb{P}(x_i = 1) = \alpha k/n$. The probability that exactly k of the x_i are non-zero is at least $c_1 k^{-1/2}$ for some constant $c_1 > 0$. Thus, (2) implies that

$$\rho \geqslant \mathbb{P}(b_1 x_1 + \cdots + b_n x_n = x + Nk) \geqslant B\frac{1}{n\sqrt{k}} c_1 k^{-1/2} = Bc_1(nk)^{-1}.$$

Now we apply Lemma 15. By choosing N sufficiently large (say $N \geqslant 2^n$), it is easy to see that there is no symmetric arithmetic progression of length $O(n^2)$ containing at least half of the b_i. Thus, the rank r in this lemma is at least 2. By the conclusion of the lemma

$$|Q| \leqslant AB^{-1}c_1^{-1}nk(\alpha n)^{-1} + 1.$$

By setting $B \geqslant 2Ac_1^{-1}$, and recalling that $\alpha = k/n$, we have

$$|Q| \leqslant \frac{Ac_1}{B}n + 1 \leqslant \frac{n}{2} + 1 < 0.9n,$$

a contradiction. □

3 Proof of Theorem 5 and Corollary 7

Proof of Theorem 5. Set $\beta(n,k) := \frac{1}{n\sqrt{\min(k,n-k)}}$. Recall that $w = (w_1, \ldots, w_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are increasing and $\Delta = \min_{i \neq j} |v_i - v_j| \neq 0$.

For any t, let $A(t) = \{i : w_i < t\}$, $B(t) = \{i : w_i \geqslant t\}$. For a permutation σ, let $S(\sigma) = \sum_{i=1}^n w_{\sigma(i)} v_i$. Fix two numbers $t_1 < t_2$ and let $A := A(t_1)$ and $B := B(t_2)$. Then we define a permutaiton σ via the following process:

(i) pick $\sigma(i)$ for each $i \notin A \cup B$;

(ii) determine the relative ordering of $\sigma(i)$ and $\sigma(j)$ for each $\{i,j\} \subseteq A$ and each $\{i,j\} \subseteq B$;

(iii) determine the set $\{\sigma(i) \ : \ i \in A\}$.

Notice that if σ is chosen uniformly at random from S_n, then given any outcomes for (i) and (ii), the set in (iii) is distributed uniformly from among all $|A|$-element subsets of $\sigma(A) \cup \sigma(B)$.

Suppose the outcomes of (i) and (ii) are given. Let $X = \sigma(A) \cup \sigma(B)$, and for each $U \subseteq X$ with $|U| = |A|$, let $S(U)$ denote the sum $\sum_{i=1}^n w_i v_{\sigma(i)}$ where σ is the permutation that would result from chosing $\sigma(A) = U$ in step (iii).

Define the following partial order P on $|A|$-element subsets of X: $U \preceq V$ iff for all x we have $|U \cap (-\infty, x]| \leqslant |V \cap (-\infty, x]|$. Stanley [11] proved that this poset has Sperner property, which implies that its width is given by

$$width(P) = \max_t \left| \left\{ U \subseteq \{1, 2, \ldots, n\} \ : \ |U| = k, \text{ and } \sum_{u \in U} u = t \right\} \right|.$$

Furthermore, Lemma 9 implies that the width of P is $O(\binom{n}{k}\beta(k,n))$.

Next, notice that $U \preceq V$ iff for all i, the i^{th} largest element of U is at least that of V. Thus, if $U \prec V$ and there are no sets W strictly between U and V, then U must be of the form $U = \{y\} \cup V \setminus \{x\}$ for some $y > x$. In this case, we have

$$S(V) - S(U) = (v_x - v_y)(w_i - w_j),$$

for some elements $i \in A$ and $j \in B$. It follows that if $U \prec V$, then we have

$$S(V) - S(U) \geqslant \Delta(t_2 - t_1) > 0. \tag{3}$$

Let E be the collection of sets U for which $S(U) \in I$. By Dilworth's theorem, we can decompose E into chains $C_1, C_2, \ldots, C_l$, where l is the width of E. By (3), we know that each chain in E has size at most $1 + \frac{|I|}{\Delta(t_2 - t_1)}$. On the other hand, as argued above, the width is at most $O\binom{|X|}{|A|}\beta(|A|, |A| + |B|)$.

Therefore,

$$|E| = O(\left(1 + \frac{|I|}{\Delta(t_2 - t_1)}\right) \binom{|X|}{|A|} \beta(|A|, |A| + |B|))$$

which implies

$$\mathbb{P}(S(U) \in I) = O(\left(1 + \frac{|I|}{\Delta(t_2 - t_1)}\right) \beta(|A|, |A| + |B|)).$$

This proves the theorem, via a routine conditioning argument. $\square$

Proof of Corollary 7. Without loss of generality we may assume that $\overline{w} = 0$, so $\sigma^2 = \sum w_i^2$. Let j be the largest index among negative entries of w, and assume without loss of generality that $j \leqslant n/2$. Our first guess might be to choose $i_1 = k < j + 1 = i_2$. If there is some index $k \leqslant j$ such that $w_k^2 > \frac{\sigma^2}{100k \log n}$ then by 5 this choice yields the claimed bound. Otherwise, we have

$$\sum_{k \leqslant j} w_k^2 \leqslant \sum_{k \leqslant j} \frac{\sigma^2}{100k \log n} \leqslant \frac{\sigma^2}{100}. \tag{4}$$

By Cauchy-Schwarz we conclude $\left|\sum_{k \leqslant j} w_k\right| \leqslant \sigma\sqrt{j}/10 \leqslant \sigma\sqrt{n}/10$. On the other hand, since $\overline{w} = 0$ we have

$$\frac{\sigma\sqrt{n}}{10} \geqslant \left|\sum_{k\leqslant j} w_k\right| = \left|\sum_{k>j} w_k\right| \geqslant \frac{n}{2} w_{n/2}.$$

So we see $w_{n/2} \leqslant \sigma/(5\sqrt{n})$. Since $w_{n/2}$ is so small, our next guess will be to take $i_1 = w_{n/2}$, $i_2 = w_{n-k}$ for some $k < n/2$. If $(w_k - w_{n/2})^2 > \frac{\sigma^2}{10k\log n}$ then by 5 this choice yields the corollary. Otherwise, $w_{n-k} \leqslant \frac{\sigma}{5\sqrt{n}} + \frac{\sigma}{\sqrt{10k\log n}}$ and

$$\sum_{k\leqslant n/2} w_{n-k}^2 \leqslant \sum_{k\leqslant n/2} \frac{\sigma^2}{10n} + \frac{\sigma^2}{5k\log n} \leqslant \frac{3\sigma^2}{10}. \tag{5}$$

Finally, observe

$$\sum_{k\in[j,n/2]} w_k^2 \leqslant \frac{1}{2}\sum_{k\geqslant j} w_k^2 \leqslant \frac{\sigma^2}{2}. \tag{6}$$

Combining (4), (5), and (6) we see

$$\sigma^2 = \sum_k w_k^2 \leqslant \frac{\sigma^2}{100} + \frac{\sigma^2}{2} + \frac{3\sigma^2}{10} < \sigma^2,$$

which is a contradiction, completing the proof. □

4 Applications

A (Kac) random poplynomial is a function of the form $P_n(x) = \sum_{i=0}^n \xi_i x^i$, where ξ_i are iid random real variables. The study of random polynomials started in the 1940s, with famous works from Littewood-Offord and Kac, who investigated the number of real roots of P_n. Progresses made by Kac, Erdos-Offord, and Ibragimov and Mashlova, through the 1950s and 1960s, led to the solution in the case when ξ has bounded variance. In this case, it has been shown that the expectation of the number of real roots is of order $O(\log n)$. The case when the ξ_i do not have bounded variance was opened until 2017, when Soze, using Theorem 3, proved that the same bound $O(\log n)$ holds in this case as well. We refer to [10] for a discussion of these results.

It is interesting to point out that while both Theorems 1 and 5 were invented to serve the study of the number of real roots of random polynomials, they were used in very different ways.

Another important object in the theorem of random functions is the number of extremal points (i.e., the number of real roots of the derivative of P_n). This is a harder problem, as the coefficients of $P_n'(x)$ are no longer iid variables. All of the above mentioned works made significant use of the iid assumption. By the mid-point theorem, we know that the number real roots of P_n' is at least that of P_n minus one. However, classical results form analysis do not yield any non-trivial upper bound. In [8], Mashlova proved that in the bounded variance case, the expectation of the number of extremal points is also of order $O(\log n)$. However, her results also showed that the hidden constants in the big "oh"'s are different. It had turned out that the number of extremal points is significantly larger than the number of real roots.

Our application mentioned in Remark 12 concerns the number of extremal points in Soze's setting, when the ξ_i can have unbounded variance. We conjectured that the bound $O(\log n)$ still holds, but this was harder to prove than anticipated, leading to the delay of the publication of this paper. Soze's approach no loner applies, again due to the fact that the coefficients of P_n' are no longer iid variables. However, we are able to prove a weaker bound $O(\log^c n)$, for some constant $c > 1$. The detailed proof of this is already technical, and will appear in a later paper.

As the space of random permutation is considered frequently in combinatorics and probability, we believe that these new anti-concentration results have high potential for applications in many other areas.

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