Counterexamples to Generalizations of the Erdős $B + B + t$ Problem

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Abstract

Following their resolution of the Erdős $B+B+t$ problem, Kra, Moreira, Richter, and Robertson posed a number of questions and conjectures related to infinite configurations in positive density subsets of the integers and other amenable groups. We give a negative answer to several of their questions and conjectures by producing families of counterexamples based on a construction of Ernst Straus.

Included among our counterexamples, we exhibit, for any $\varepsilon > 0$, a set $A \subseteq \mathbb{N}$ with multiplicative upper Banach density at least $1-\varepsilon$ such that A does not contain any dilated product set $\{b_1 b_2 t : b_1, b_2 \in B, b_1 \neq b_2\}$ for an infinite set $B \subseteq \mathbb{N}$ and $t \in \mathbb{Q}_{>0}$. We also prove the existence of a set $A \subseteq \mathbb{N}$ with additive upper Banach density at least $1 - \varepsilon$ such that A does not contain any polynomial configuration $\{b_1^2+b_2+t : b_1, b_2 \in B, b_1 < b_2\}$ for an infinite set $B \subseteq \mathbb{N}$ and $t \in \mathbb{Z}$. Counterexamples to some closely related problems are also discussed.

Mathematics Subject Classifications: 05D10, 11B13, 11B30

1 Introduction

In [11], Kra, Moreira, Richter, and Robertson provided a positive resolution to the Erdős $B + B + t$ problem (asked in [6, 7, 8]) by showing that any positive density subset of the integers contains a shift of an infinite sumset, i.e. a configuration of the form $B \oplus B + t = \{b_1 + b_2 + t : b_1, b_2 \in B, b_1 \neq b_2\}$ for some infinite set $B \subseteq \mathbb{N}$ and some $t \in \mathbb{Z}$ (and, moreover, B may be taken as a subset of A). One is then left with the natural question of which other infinite combinatorial configurations can be found in sets of positive density of the integers or of other amenable groups. The survey article [12] provides an extensive overview of questions along these lines, with some partial results in both the positive and negative direction. The goal of the present paper is to provide a negative answer to several of the questions and conjectures posed in [12].

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1.1 Sumsets in abelian groups

First, we address the generalization of the Erdős $B + B + t$ problem to abelian groups. Let Γ be a countable discrete abelian group. A *Følner sequence* in Γ is a sequence of finite subsets $(\Phi_N)_{N\in\mathbb{N}}$ of Γ such that for any $x \in \Gamma$,

$$\lim_{N\to\infty} \frac{|(\Phi_N + x)\triangle\Phi_N|}{|\Phi_N|} = 0.$$

The *upper and lower density of a subset* $A \subseteq \Gamma$ *along a Følner sequence* $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ are given by

$$\overline{d}_\Phi(A) = \limsup_{N\to\infty} \frac{|A\cap\Phi_N|}{|\Phi_N|} \qquad \text{and} \qquad \underline{d}_\Phi(A) = \liminf_{N\to\infty} \frac{|A\cap\Phi_N|}{|\Phi_N|}$$

The *upper and lower Banach density* of $A \subseteq \Gamma$ are defined by $d^*(A) = \sup_\Phi \overline{d}_\Phi(A)$ and $d_*(A) = \inf_\Phi \underline{d}_\Phi(A)$, where the supremum and infimum are taken over all Følner sequences Φ in Γ. In [5], Charamaras and Mountakis prove a generalization of the main result of [11] for abelian groups whose "even" elements form a finite index subgroup:

Theorem 1 ([5, Corollary 1.13][1])**.** *Let Γ be a countable discrete abelian group such that $2\Gamma = \{2x : x \in \Gamma\}$ is a finite index subgroup of Γ. If $A \subseteq \Gamma$ is a set of positive upper Banach density, then there exists an infinite set $B \subseteq \Gamma$ and a shift $t \in \Gamma$ such that*

$$B \oplus B + t = \{b_1 + b_2 + t : b_1, b_2 \in B, b_1 \neq b_2\} \subseteq A.$$

We prove that the Erdős $B + B + t$ problem for an abelian group Γ has a negative answer whenever 2Γ is of infinite index, disproving [12, Conjecture 5.15][2].

Theorem 2. *Suppose Γ is a countable discrete abelian group and 2Γ is a subgroup of infinite index. Then for any $\varepsilon > 0$, there exists $A \subseteq \Gamma$ such that $d^*(A) > 1 - \varepsilon$ and if $B \subseteq \Gamma$ and $t \in \Gamma$ with $B \oplus B + t \subseteq A$, then B is finite.*

In fact, we can prove a similar result for higher-order sumsets as well. For $k \in \mathbb{N}$ and $B \subseteq \Gamma$, let

$$B^{\oplus k} = \left\{ \sum_{x\in F} x : F \subseteq B, |F| = k \right\}.$$

Theorem 3. *Let $k \geqslant 2$. Suppose Γ is a countable discrete abelian group and $k\Gamma$ is a subgroup of infinite index. Then for any $\varepsilon > 0$, there exists $A \subseteq \Gamma$ such that $d^*(A) > 1-\varepsilon$ and if $B \subseteq \Gamma$ and $t \in \Gamma$ with $B^{\oplus k} + t \subseteq A$, then B is finite.*

[1] The main result of [5] applies to a class of amenable groups under a technical assumption on the set of squares. The result stated in Theorem 1 is the special case of their result in the setting of abelian groups.

[2] Terence Tao has also observed that the condition $[\Gamma : 2\Gamma] < \infty$ in Theorem 1 is necessary (see [14]).

1.2 Product sets in $(\mathbb{N}, \times)$

The semigroup $(\mathbb{N}, \times)$ of positive integers under multiplication has the property that the subsemigroup of kth powers, $\{n^k : n \in \mathbb{N}\}$, has zero (multiplicative) upper Banach density for every $k \geqslant 2$. With a small amount of additional work, we can use Theorem 3 to construct a counterexample in $(\mathbb{N}, \times)$ for all $k \geqslant 2$ simultaneously:

Theorem 4. *Let $\varepsilon > 0$. There exists $A \subseteq \mathbb{N}$ with $d^*_\times(A) > 1 - \varepsilon$ such that if $B \subseteq \mathbb{N}$, $t \in \mathbb{Q}_{>0}$, and $k \geqslant 2$ with*

$$tB^{\odot k} = \left\{ t \prod_{i=1}^{k} b_i : b_1, \dots, b_k \in B, b_1 < \cdots < b_k \right\} \subseteq A,$$

then B is finite.

The case $k = 2$ in Theorem 4 gives a disproof of [12, Conjecture 5.1].

1.3 Polynomial configurations

Our next counterexample deals with polynomial patterns in the integers. Seeking a sumset analogue of the Furstenberg–Sárközy theorem [10, 13], Kra, Moreira, Richter, and Robertson asked whether every set of positive density in the integers contains a configuration of the form

$$\{b_1^2 + b_2 : b_1, b_2 \in B, b_1 < b_2\} + t$$

for some infinite set $B \subseteq \mathbb{N}$ and some $t \in \mathbb{Z}$ (see [12, Question 3.13]). We show that the answer is negative and remains negative for any polynomial of degree at least two in place of the squares:

Theorem 5. *Let $P(x) \in \mathbb{Q}[x]$ be an integer-valued polynomial[3] with $\deg P \geqslant 2$. Then for any $\varepsilon > 0$, there exists a set $A \subseteq \mathbb{N}$ with $d^*(A) > 1 - \varepsilon$ such that if $B \subseteq \mathbb{N}$, $t \in \mathbb{Z}$, and*

$$P(B) \mathbin{\triangleleft\!\!+} B + t = \{P(b_1) + b_2 + t : b_1, b_2 \in B, b_1 < b_2\} \subseteq A,$$

then B is finite.

Remark 6. Let us comment briefly on the choice of ordering for the configuration appearing in Theorem 5. One could also consider the related configuration $P(B) \mathbin{+\!\!\triangleright} B = \{P(b_1) + b_2 : b_1, b_2 \in B, b_1 > b_2\}$. In [12, Example 3.14], a counterexample of an even stronger form is given for this configuration. Namely, there exists a set $A \subseteq \mathbb{N}$ with $d(A) = 1$ such that A does not contain any set of the form $B^2 \mathbin{+\!\!\triangleright} C = \{b^2 + c : b \in B, c \in C, b > c\}$ with $B, C \subseteq \mathbb{N}$ infinite, and the only property of the squares used for the proof is that the set of squares $\{n^2 : n \in \mathbb{N}\}$ is of zero density, so the argument applies equally well to any polynomial of degree at least 2.

[3] By an *integer-valued polynomial*, we mean a polynomial taking integer values on the integers. That is, $P(\mathbb{Z}) \subseteq \mathbb{Z}$.

The configuration $P(B) \lhd B$ in Theorem 5 is better behaved in several respects. First, if $A \subseteq \mathbb{N}$ and $d^*(A) = 1$, then A contains $P(B) \lhd B$ for some infinite set $B \subseteq \mathbb{N}$. Second, if $A \subseteq \mathbb{N}$ and $d^*(A) > 0$, then there are infinite sets $B, C \subseteq \mathbb{N}$ such that $P(B) \lhd C \subseteq A$, and, moreover, one may choose $C \subseteq A - P(0)$; see [12, Corollary 3.33].

Theorem 5 shows that, despite the improved behavior of ordered sumsets with the polynomial input being of smaller size, one may still avoid infinite configurations of the form $P(B) \lhd B$ in all shifts of a set of positive density. We should note that our proof makes use of the algebraic structure of polynomials in addition to the fact that $P(\mathbb{N})$ has zero density. A natural lingering question is whether one can construct counterexamples without this algebraic structure; see Question 20 and the accompanying discussion at the end of the paper.

1.4 Product-sum sets

Finally, we show that [12, Question 3.18], which is a variant of [12, Question 3.13], also has a negative answer:

Theorem 7. *For any $\varepsilon > 0$, there exists a set $A \subseteq \mathbb{N}$ with $d^*(A) > 1 - \varepsilon$ such that if $B \subseteq \mathbb{N}$, $t \in \mathbb{Z}$, and*

$$B \lhd B \lhd B + t = \{b_1 \cdot b_2 + b_3 + t : b_1, b_2, b_3 \in B, b_1 < b_2 < b_3\} \subseteq A,$$

then B is finite.

1.5 Remarks about partition regularity versus density regularity

Consider the following families of sets discussed in the above results:

- For an abelian group Γ and $k \geqslant 2$:

$$\mathscr{C}_{k\text{-sumset}} = \{A \subseteq \Gamma : \exists B \subseteq \Gamma \text{ infinite}, B^{\oplus k} \subseteq A\}.$$

- For $k \geqslant 2$:

$$\mathscr{C}_{k\text{-product set}} = \left\{A \subseteq \mathbb{N} : \exists B \subseteq \mathbb{N} \text{ infinite}, B^{\odot k} \subseteq A\right\},$$

  and

$$\mathscr{C}_{\text{product}} = \bigcup_{k \geqslant 2} \mathscr{C}_{k\text{-product set}}.$$

- For an integer-valued polynomial $P(x) \in \mathbb{Q}[x]$:

$$\mathscr{C}_P = \{A \subseteq \mathbb{N} : \exists B \subseteq \mathbb{N} \text{ infinite}, P(B) \lhd B \subseteq A\}.$$

- $\mathscr{C}_{\times,+} = \{A \subseteq \mathbb{N} : \exists B \subseteq \mathbb{N} \text{ infinite}, B \lhd B \lhd B \subseteq A\}$.

Theorems 3, 4, 5, and 7 provide examples of sets with density arbitrarily close to 1 not containing any shift of an element of $\mathscr{C}_{k\text{-sumset}}$, $\mathscr{C}_{\text{product}}$, $\mathscr{C}_P$, or $\mathscr{C}_{\times,+}$, respectively. We can therefore rephrase the main theorems as showing that these families of configuration together with their shifts are not *density regular*. This is made more meaningful by the following observation, which is a simple consequence of Ramsey's theorem.

Proposition 8. *The families $\mathscr{C}_{k\text{-sumset}}$, $\mathscr{C}_{\text{product}}$, $\mathscr{C}_P$, and $\mathscr{C}_{\times,+}$ are partition regular. That is, for each $\mathscr{C} \in \{\mathscr{C}_{k\text{-sumset}}, \mathscr{C}_{\text{product}}, \mathscr{C}_P, \mathscr{C}_{\times,+}\}$, if $C \in \mathscr{C}$ and $C = \bigcup_{i=1}^{r} C_i$, then $C_i \in \mathscr{C}$ for some $i \in \{1, \dots, r\}$.*

Remark 9. One can check that each of the families $\mathscr{C}_{k\text{-sumset}}$, $\mathscr{C}_{\text{product}}$, $\mathscr{C}_P$, and $\mathscr{C}_{\times,+}$ contains all thick sets[4] and deduce that every piecewise syndetic set[5] contains a shift of each of the configurations we have considered. The counterexamples we produce below will therefore be examples of non-piecewise syndetic sets with positive density. Such sets have been explored previously in [4, 3].

2 Revisiting the Straus example

All of the constructions in this paper are based on Straus's counterexample to a density version of Hindman's theorem:

Theorem 10 (Ernst Straus, cf. [2, Theorem 2.20])**.** *Let $\varepsilon > 0$. There is a set $A \subseteq \mathbb{N}$ with $d^*(A) > 1 - \varepsilon$ such that A does not contain any configuration of the form*

$$FS\left((x_n)_{n\in\mathbb{N}}\right) + t = \left\{x_{i_1} + \cdots + x_{i_k} + t : k \in \mathbb{N}, i_1 < \cdots < i_k\right\}.$$

for an infinite sequence $x_1 < x_2 < \cdots \in \mathbb{N}$ and $t \in \mathbb{Z}$.

Our goal is prove a similar result for the combinatorial configurations mentioned in the introduction, and the basic approach will be very similar. In order to motivate the new constructions in this paper, let us briefly review the main ideas in the proof of Theorem 10.

The first observation to make is that IP-sets (sets of the form $FS((x_n)_{n\in\mathbb{N}})$ for an infinite sequence $x_1 < x_2 < \cdots \in \mathbb{N}$) are "divisible" in the following sense: if A contains an IP-set, then $A \cap k\mathbb{N}$ contains an IP-set for every $k \in \mathbb{N}$. The idea to prove Theorem 10 is then to remove from $\mathbb{N}$ an infinite arithmetic progression $k_t\mathbb{N} + t$ for each $t \in \mathbb{Z}$ with k_t growing sufficiently quickly so that $\bigcup_{t\in\mathbb{Z}}(k_t\mathbb{N}+t)$ has small density. Isolating the essential properties utilized by Straus, we can prove the following general result:

Theorem 11. *Let Γ be an abelian group. Let $\mathscr{C}$ be a family of infinite subsets of Γ, and suppose there is a family $\mathscr{D}$ of subsets of Γ and a Følner sequence Φ with the following properties:*

[4] A set is *thick* if it contains a shift of every finite set, or, equivalently, if it has upper Banach density equal to 1.

[5] A set is *syndetic* if it has non-empty intersection with every thick set, or, equivalently, if it has positive lower Banach density. A *piecewise syndetic set* is an intersection of a syndetic set with a thick set. The family of piecewise syndetic sets is partition regular.

(1) For any $C \in \mathscr{C}$ and $D \in \mathscr{D}$, one has $C \cap D \in \mathscr{C}$, and

(2) $\inf_{D\in\mathscr{D}} \overline{d}_\Phi(D) = 0$.

Then for any $\varepsilon > 0$, there exists $A \subseteq \Gamma$ with $\underline{d}_\Phi(A) > 1 - \varepsilon$ such that for any $t \in \Gamma$, $A - t \notin \mathscr{C}$.

Taking $\mathscr{C} = \{A \subseteq \mathbb{N} : A \text{ contains an IP-set}\}$ and $\mathscr{D} = \{k\mathbb{N} : k \in \mathbb{N}\}$ reproduces Theorem 10.

3 Key lemma and proof of Theorem 11

The key lemma for proving Theorem 11 can be interpreted as an approximate version of countable subadditivity for the upper density along a Følner sequence:

Lemma 12. *Let Γ be a countable discrete abelian group with a Følner sequence Φ. Let $(A_n)_{n\in\mathbb{N}}$ be a countable family of subsets of Γ. There exists $A \subseteq \Gamma$ such that*

1. *for any $n \in \mathbb{N}$, $A_n \setminus A$ is finite, and*
2. $\overline{d}_\Phi(A) \leqslant \sum_{n\in\mathbb{N}} \overline{d}_\Phi(A_n)$.

Proof. Let $\delta_n = \overline{d}_\Phi(A_n)$, and let $\delta = \sum_{n\in\mathbb{N}} \delta_n$. If $\delta \geqslant 1$, there is nothing to prove (one may take $A = \Gamma$), so assume $\delta < 1$. By the definition of upper density along Φ, we have

$$\frac{|A_n \cap \Phi_k|}{|\Phi_k|} \leqslant \delta_n + \varepsilon(n,k)$$

for some $\varepsilon(n,k) \in [0,1]$ with $\varepsilon(n,k) \to 0$ as $k \to \infty$.

We will construct a sequence $(k_n)_{n\in\mathbb{N}}$ by induction, put $A'_n = A_n \setminus \bigcup_{k<k_n} \Phi_k$, and then let $A = \bigcup_{n\in\mathbb{N}} A'_n$. Note that property (1) is automatically satisfied (regardless of the choice of $(k_n)_{n\in\mathbb{N}}$) since $A_n \setminus A \subseteq \bigcup_{k<k_n} \Phi_k$.

Let $k_1 = 1$. For $n \in \mathbb{N}$, given $k_1, \ldots, k_{n-1}$, choose $k_n > k_{n-1}$ such that

$$\sum_{j=1}^{n} \varepsilon(j,k) \leqslant \frac{1}{n}$$

for all $k \geqslant k_n$. We check that property (2) is satisfied for this choice of $(k_n)_{n\in\mathbb{N}}$. For $k \in \mathbb{N}$, let $n_k = \max\{n \in \mathbb{N} : k_n \leqslant k\}$. Note that $A \cap \Phi_k = \bigcup_{j=1}^{n_k} A'_j \cap \Phi_k \subseteq \bigcup_{j=1}^{n_k} A_j \cap \Phi_k$. Therefore

$$\frac{|A \cap \Phi_k|}{|\Phi_k|} \leqslant \sum_{j=1}^{n_k} \frac{|A_j \cap \Phi_k|}{|\Phi_k|} \leqslant \sum_{j=1}^{n_k} \delta_j + \sum_{j=1}^{n_k} \varepsilon(j,k) \leqslant \delta + \frac{1}{n_k},$$

so

$$\overline{d}_\Phi(A) \leqslant \delta + \limsup_{k\to\infty} \frac{1}{n_k} = \delta. \qquad \square$$

Remark 13. The proof of Lemma 12 does not use that Φ is a Følner sequence nor does it utilize any algebraic structure of Γ. The same result holds for any notion of upper density on a set obtained by taking a limit of finitely-supported probability measures.

We can now give a very short proof of Theorem 11.

Proof of Theorem 11. Let $\varepsilon > 0$. By property (2), we may choose a family of sets $(D_t)_{t\in\Gamma}$ in $\mathscr{D}$ such that $\sum_{t\in\Gamma} \overline{d}_\Phi(D_t) < \varepsilon$. Then by Lemma 12, let $S \subseteq \Gamma$ such that $\overline{d}_\Phi(S) < \varepsilon$ and $(D_t + t) \setminus S$ is finite for each $t \in \Gamma$. Let $A = \Gamma \setminus S$. Then $\underline{d}_\Phi(A) = 1 - \overline{d}_\Phi(S) > 1 - \varepsilon$. Moreover, for each $t \in \Gamma$, the intersection $(A-t)\cap D_t = D_t \setminus (S-t)$ is finite. In particular, $(A - t) \cap D_t \notin \mathscr{C}$, so by property (1), $A - t \notin \mathscr{C}$. □

4 Constructing the counterexamples

Using Theorem 11, all that remains in order to prove Theorems 3, 4, 5, and 7 is to find the appropriate family of sets $\mathscr{D}$ for each corresponding collection of combinatorial configurations $\mathscr{C}$.

4.1 Sumsets in abelian groups

Let Γ be a countable discrete abelian group. Recall $\mathscr{C}_{k\text{-sumset}} = \{A \subseteq \Gamma : \exists B \subseteq \Gamma \text{ infinite}, B^{\oplus k} \subseteq A\}$.

Lemma 14. *Let Γ be a countable discrete abelian group and $k \in \mathbb{N}$. If $C \in \mathscr{C}_{k\text{-sumset}}$ and $\Lambda \leqslant \Gamma$ is a finite index subgroup with $k\Gamma \subseteq \Lambda$, then $C \cap \Lambda \in \mathscr{C}_{k\text{-sumset}}$.*

Proof. Let $B \subseteq \Gamma$ be an infinite set such that $B^{\oplus k} \subseteq C$. By the pigeonhole principle, there exists $x \in \Gamma$ such that $B' = B \cap (\Lambda + x)$ is infinite. Since $kx \in \Lambda$, we conclude $B'^{\oplus k} \subseteq C \cap \Lambda$. □

Proof of Theorem 3. Let $\mathscr{D} = \{\Lambda \leqslant \Gamma : k\Gamma \subseteq \Lambda \text{ and } [\Gamma : \Lambda] < \infty\}$. A finite index subgroup $\Lambda \leqslant \Gamma$ has uniform density $\frac{1}{[\Gamma:\Lambda]}$, so by Theorem 11 and Lemma 14, it suffices to show $\sup_{\Lambda\in\mathscr{D}}[\Gamma : \Lambda] = \infty$. By assumption, $[\Gamma : k\Gamma] = \infty$, so $\Gamma/k\Gamma$ is an infinite group. Since each element of $\Gamma/k\Gamma$ has order dividing k, we may write $\Gamma/k\Gamma$ as an infinite direct sum of cyclic groups $\Gamma/k\Gamma = \bigoplus_{i\in\mathbb{N}} \mathbb{Z}/n_i\mathbb{Z}$ with $n_i \mid k$ for each $i \in \mathbb{N}$ (see, e.g., [9, Theorem 17.2]). The subgroups $\widetilde{\Lambda}_j = \{(x_i)_{i\in\mathbb{N}} : x_i = 0 \text{ for } i < j\} \leqslant \Gamma/k\Gamma$ satisfy $[\Gamma/k\Gamma : \widetilde{\Lambda}_j] = \prod_{i<j} n_i$, so $\sup_{j\in\mathbb{N}}[\Gamma/k\Gamma : \widetilde{\Lambda}_j] = \infty$. Lifting $\widetilde{\Lambda}_j$ to a subgroup $\Lambda_j \leqslant \Gamma$, we have $\Lambda_j \in \mathscr{D}$ and $\sup_{j\in\mathbb{N}}[\Gamma : \Lambda_j] = \infty$. □

4.2 Product sets in $(\mathbb{N}, \times)$

Proof of Theorem 4. Consider the group isomorphism $\varphi : (\mathbb{Q}_{>0}, \times) \to (\bigoplus_{n\in\mathbb{N}} \mathbb{Z}, +)$ that maps an element to its prime factorization. That is, if $q = \prod_{i=1}^{s} p_i^{e_i}$, where $p_1 < p_2 < \dots$ is the increasing enumeration of the primes, then $\varphi(q) = (e_1, e_2, \dots, e_s, 0, \dots)$.

Let Φ be a Følner sequence in $(\mathbb{N}, \times)$, and let $\widetilde{\Phi}$ be the Følner sequence in $(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}, +)$ given by $\widetilde{\Phi}_N = \varphi(\Phi_N)$. By Theorem 3, we may find, for each $k \geqslant 2$, a set $A_k \subseteq \bigoplus_{n\in\mathbb{N}} \mathbb{Z}$

such that $\underline{d}_{\widetilde{\Phi}}(A_k) > 1 - 2^{-k}\varepsilon$ and no shift of A_k contains an infinite k-fold sumset $B^{\oplus k}$. (As stated in the introduction, Theorem 3 only guarantees $d^*(A_k) > 1 - 2^{-k}\varepsilon$. However, from the conclusion of Theorem 11, we see that one may replace the upper Banach density with the lower density along the Følner sequence $\widetilde{\Phi}$.)

Put $N_k = (\bigoplus_{n\in\mathbb{N}} \mathbb{Z}) \setminus A_k$. Then $\overline{d}_{\widetilde{\Phi}}(N_k) < 2^{-k}\varepsilon$, so by Lemma 12, there exists a set $N \subseteq \bigoplus_{n\in\mathbb{N}} \mathbb{Z}$ such that $N_k \setminus N$ is finite for each $k \geqslant 2$ and $\overline{d}_{\widetilde{\Phi}}(N) < \varepsilon$.

Let $A = \varphi^{-1}\left((\bigoplus_{n\in\mathbb{N}} \mathbb{Z}) \setminus N\right) \cap \mathbb{N}$. Then $d^*_\times(A) \geqslant \underline{d}_\Phi(A) = 1 - \overline{d}_{\widetilde{\Phi}}(N) > 1 - \varepsilon$. Suppose $B \subseteq \mathbb{N}$, $t \in \mathbb{Q}_{>0}$, $k \geqslant 2$, and $tB^{\odot k} \subseteq A$. Then $\varphi\left(tB^{\odot k}\right) \cap N = \emptyset$. But φ is a group isomorphism, so we have $\varphi\left(tB^{\odot k}\right) = \varphi(t) + \varphi(B)^{\oplus k}$. Therefore, from the construction of N, the set $\left(\varphi(t) + \varphi(B)^{\oplus k}\right) \cap N_k$ is finite. Hence, removing finitely elements from B, we obtain a cofinite subset $B' \subseteq B$ such that $\left(\varphi(t) + \varphi(B')^{\oplus k}\right) \subseteq A_k$. By construction, the set $A_k - \varphi(t)$ does not contain any infinite k-fold subset, so we conclude that B' (and hence B) is a finite set. □

4.3 Polynomial configurations

Lemma 15. *Let $P(x) \in \mathbb{Q}[x]$ be an integer-valued polynomial. Let $k \in \mathbb{N}$. If $C \in \mathscr{C}_P$, then there exists $n \in \mathbb{N}$ such that $C \cap (k\mathbb{N} + P(n) + n) \in \mathscr{C}_P$.*

Proof. Let $B \subseteq \mathbb{N}$ be an infinite set such that $P(B) \trianglelefteq B \subseteq A$. Let $D \in \mathbb{N}$ such that $D \cdot P(x) \in \mathbb{Z}[x]$ has integer coefficients. Then for any $k \in \mathbb{N}$, if $n \equiv m \pmod{Dk}$, then $P(n) \equiv P(m) \pmod{k}$. By the pigeonhole principle, there exists $n \in \{0, 1, \ldots, Dk-1\}$ such that $B' = B \cap (Dk\mathbb{N} + n)$ is infinite. Then $P(B') \trianglelefteq B' \subseteq A \cap (k\mathbb{N} + P(n) + n)$. □

As suggested by Lemma 15, the relevant family $\mathscr{D}$ for applying Theorem 11 will be the family of sets of the form $D_k = \bigcup_{n\in\mathbb{N}} (k\mathbb{N} + P(n) + n)$ for $k \in \mathbb{N}$. In order to show that such sets have arbitrarliy small density (condition (2) in Theorem 11), we use the following property of polynomials:

Lemma 16. *Let $P(x) \in \mathbb{Q}[x]$ be an integer-valued polynomial with $\deg P \geqslant 2$. Let $\mathbb{P}_P$ be the set of prime numbers $p \in \mathbb{P}$ such that $P(\mathbb{Z})$ represents every residue mod p. That is,*

$$\mathbb{P}_P = \{p \in \mathbb{P} : \forall r \in \mathbb{Z}, \exists n \in \mathbb{Z}, P(n) \equiv r \pmod{p}\}.$$

Then

$$\sum_{p\in\mathbb{P}\setminus\mathbb{P}_P} \frac{1}{p} = \infty.$$

Proof. Let us first reduce to the case that P has integer coefficients. Let $D \in \mathbb{N}$ such that $Q(x) = D \cdot P(x) \in \mathbb{Z}[x]$. For any prime $p \in \mathbb{P}$ with $p \nmid D$, we have $p \in \mathbb{P}_P$ if and only if $p \in \mathbb{P}_Q$, since D is invertible mod p. Therefore, $\mathbb{P}_P \subseteq \mathbb{P}_Q \cup \{p \in \mathbb{P} : p \mid D\}$. The set $\{p \in \mathbb{P} : p \mid D\}$ is finite, so

$$\sum_{p\in\mathbb{P}\setminus\mathbb{P}_P} \frac{1}{p} = \infty \iff \sum_{p\in\mathbb{P}\setminus\mathbb{P}_Q} \frac{1}{p} = \infty.$$

We may therefore assume without loss of generality that P has integer coefficients. In this case, P is well-defined mod p for every $p \in \mathbb{P}$, so $p \in \mathbb{P}_P$ if and only if P acts as a permutation on $\mathbb{Z}/p\mathbb{Z}$; that is, P is a permutation polynomial mod p. If $\mathbb{P}_P$ is finite, there is nothing to prove, so suppose $\mathbb{P}_P$ is infinite. Then P is a composition of Dickson polynomials[6] and linear polynomials by [15, Theorem 2]. Hence, we may assume $P = P_1 \circ P_2 \circ \cdots \circ P_r$, where each $P_i(x)$ is either linear or a Dickson polynomial. Since $\deg P \geqslant 2$, at least one of the polynomials P_i has $\deg P_i \geqslant 2$ and is therefore a Dickson polynomial. Let $i \in \{1, \ldots, r\}$ such that P_i is a Dickson polynomial, say $P_i = D_{n_i}(a_i, x)$ for some $a_i \in \mathbb{Z}$ and $n_i \geqslant 2$. If $p \in \mathbb{P}_P$, then P_i must be a permutation polynomial mod p, so either $p \mid a_i$ and $\gcd(n_i, p-1) = 1$ or $p \nmid a_i$ and $\gcd(n_i, p^2-1) = 1$. In particular, if q is a prime factor of n_i and $p \in \mathbb{P}_P$, then $q \nmid p-1$. Therefore, for any prime factor $q \in \mathbb{P}$ of n_i,

$$\{p \in \mathbb{P} : p \equiv 1 \pmod{q}\} \subseteq \mathbb{P} \setminus \mathbb{P}_P.$$

By Dirichlet's theorem on primes in arithmetic progressions, it follows that

$$\sum_{p \in \mathbb{P} \setminus \mathbb{P}_P} \frac{1}{p} = \infty. \qquad \square$$

Proof of Theorem 5. For each $k \in \mathbb{N}$, let $D_k = \bigcup_{n \in \mathbb{N}} (k\mathbb{N} + P(n) + n)$, and let $\mathscr{D} = \{D_k : k \in \mathbb{N}\}$. Note that D_k has uniform density equal to $\frac{|R_k|}{k}$, where

$$R_k = \{P(n) + n \bmod k : n \in \mathbb{Z}\} \subseteq \mathbb{Z}/k\mathbb{Z}.$$

Therefore, by Theorem 11 and Lemma 15, it suffices to show $\inf_{k \in \mathbb{N}} \frac{|R_k|}{k} = 0$.

Let $\mathbb{P}_{P+x}$ be the set as in Lemma 16 for the polynomial $P(x) + x$. We will consider k of the form $k = \prod_{i=1}^{s} p_i$ with $p_1, \ldots, p_s \in \mathbb{P} \setminus \mathbb{P}_{P+x}$ distinct. By the Chinese remainder theorem, $P(n) + n \equiv r \pmod{k}$ if and only if $P(n) + n \equiv r \pmod{p_i}$ for every $i \in \{1, \ldots, s\}$. Therefore, $|R_k| \leqslant \prod_{i=1}^{s} |R_{p_i}|$. By the definition of $\mathbb{P}_{P+x}$, we have $|R_{p_i}| \leqslant p_i - 1$, so

$$\frac{|R_k|}{k} \leqslant \prod_{i=1}^{s} \frac{|R_{p_i}|}{p_i} \leqslant \prod_{i=1}^{s} \left(1 - \frac{1}{p_i}\right).$$

We have thus shown

$$\inf_{k \in \mathbb{N}} \frac{|R_k|}{k} \leqslant \prod_{p \in \mathbb{P} \setminus \mathbb{P}_{P+x}} \left(1 - \frac{1}{p}\right).$$

But by Lemma 16, the infinite product $\prod_{p \in \mathbb{P} \setminus \mathbb{P}_{P+x}} \left(1 - \frac{1}{p}\right)$ is equal to 0, so we are done. $\square$

Remark 17. The family $\{P(x) \in \mathbb{Q}[x] : P \text{ is integer-valued and } \deg P \geqslant 2\}$ is countable, so the set A in Theorem 5 can be chosen independent of the polynomial P by another application of Lemma 12.

[6]The Dickson polynomials are defined recursively by $D_0(a, x) = 2$, $D_1(a, x) = x$, and $D_n(a, x) = xD_{n-1}(a, x) - aD_{n-2}(a, x)$, and a Dickson polynomial $D_n(a, x)$ (considered as a polynomial in x with $a \in \mathbb{Z}$ fixed) is a permutation polynomial mod p if and only if either $p \mid a$ and $\gcd(n, p-1) = 1$ or $p \nmid a$ and $\gcd(n, p^2-1) = 1$; see [15, Lemma 1.4].

4.4 Product-sum sets

Lemma 18. *Let $C \in \mathscr{C}_{\times,+}$. Then for any $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $A \cap (k\mathbb{N} + n^2 + n) \in \mathscr{C}_{\times,+}$.*

Proof. Let $B \subseteq \mathbb{N}$ be an infinite set such that $B \lhd B \lhd B \subseteq A$. By the pigeonhole principle, let $n \in \{0, 1, \ldots, k-1\}$ such that $B' = B \cap (k\mathbb{N} + n)$ is infinite. Then $B' \lhd B' \lhd B' \subseteq A \cap (k\mathbb{N} + n^2 + n)$. □

The family $\mathscr{D} = \left\{\bigcup_{n\in\mathbb{N}} (k\mathbb{N} + n^2 + n) : k \in \mathbb{N}\right\}$ is the same family of sets that appeared in the proof of Theorem 5 for $P(x) = x^2$, so the same argument proves Theorem 7.

5 Concluding remarks

Each of Theorems 3, 4, 5, and 7 can be strengthened as follows. Instead of simply concluding that B is a finite set, we may make the stronger conclusion that $|B| \leqslant M_t$, where M_t is some finite bound depending on t. That is, each shift of the set A not only does not contain any infinite configuration of the desired type, but it in fact only contains configurations of bounded size. This can be deduced using the following variant of Theorem 11:

Theorem 19. *Let Γ be an abelian group. For each $m \in \mathbb{N}$, let $\mathscr{C}_m$ be a family of subsets of Γ and of cardinality at least m such that $\mathscr{C}_1 \supseteq \mathscr{C}_2 \supseteq \ldots$. Suppose there are families $\mathscr{D}_1 \subseteq \mathscr{D}_2 \subseteq \ldots$ of subsets of Γ and a Følner sequence Φ with the following properties:*

(1) For any $m, r \in \mathbb{N}$, there exists $M = M(m, r) \in \mathbb{N}$ such that if $C \in \mathscr{C}_M$ and $D \in \mathscr{D}_r$, then $C \cap D \in \mathscr{C}_m$, and

(2) $\inf_{r\in\mathbb{N}} \inf_{D\in\mathscr{D}_r} \overline{d}_\Phi(D) = 0$.

Then for any $\varepsilon > 0$, there exists $A \subseteq \Gamma$ with $\underline{d}_\Phi(A) > 1 - \varepsilon$ and $(M_t)_{t\in\Gamma}$ in $\mathbb{N}$ such that for any $t \in \Gamma$, $A - t \notin \mathscr{C}_{M_t}$.

Theorem 19 can be proved along exactly the same lines as Theorem 11, and the claimed strengthenings of Theorems 3, 4, 5, and 7 hold by similar arguments to those presented in Section 4. To avoid significant repetition, we omit the proofs and leave the details to the interested reader.

In the proof of Theorem 5, the reason we work with polynomials $P(x) \in \mathbb{Q}[x]$ with $\deg P \geqslant 2$ is that the set of values $P(\mathbb{Z}) \subseteq \mathbb{Z}$ is sparse (of zero upper Banach density). This is a key aspect of what allows for property (2) in Theorem 11 to hold. A natural question, then, is whether the polynomial structure is really needed, or if the same result holds for arbitrary sparse sequences.

Question 20. Does there exist an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} \frac{f(n)}{n} = \infty$ with the following property: for any $A \subseteq \mathbb{N}$ with $d(A) > 0$, there exists an infinite set $B \subseteq \mathbb{N}$ and $t \in \mathbb{Z}$ such that

$$f(B) \lhd B + t = \{f(b_1) + b_2 + t : b_1, b_2 \in B, b_1 < b_2\} \subseteq A?$$

Under the stronger assumption that the set $\{f(n) + n : n \in \mathbb{N}\}$ is sparse in the Bohr topology, we can apply Theorem 11 to obtain a counterexample. Let us make this precise. Let Γ be a countable discrete abelian group, and let $\mathbb{G}$ denote the dual group $\mathbb{G} = \widehat{\Gamma}$. We will write $\mathbb{G}_d$ for the group $\mathbb{G}$ considered with the discrete topology. The *Bohr compactification* of Γ is the group $b\Gamma = \widehat{\mathbb{G}_d}$. Note that $b\Gamma$ is a compact Hausdorff topological group, and Γ embeds as a dense subset of $b\Gamma$ by $\iota(g) = e_g$, where $e_g \in \widehat{\mathbb{G}_d}$ is the evaluation map $e_g(\chi) = \chi(g)$. Say that a set $S \subseteq \Gamma$ is *Bohr-sparse* if $\overline{\iota(S)} \subseteq b\Gamma$ is a set of measure zero with respect to the Haar probability measure $m_{b\Gamma}$ on $b\Gamma$. In general, $d^*(A) \leqslant m_{b\Gamma}\left(\overline{\iota(A)}\right)$ for a subset $A \subseteq \Gamma$, so Bohr-sparse sets are sets of density zero. However, not all density zero sets are Bohr-sparse (for instance, [1, Section 5] gives several interesting examples of quickly growing sequences that are dense in $b\mathbb{Z}$), so Proposition 21 below does not fully resolve Question 20. Nevertheless, Proposition 21 includes Theorems 2 and 5 as special cases: subgroups of infinite index are Bohr-sparse (one can check that distinct cosets of a subgroup have disjoint closures in the Bohr compactification), and polynomials are Bohr-sparse in the integers (this follows from the argument in Section 4.3).

Proposition 21. *Let $f : \Gamma \to \Gamma$ such that $\{f(g) + g : g \in \Gamma\}$ is Bohr-sparse. Then for any $\varepsilon > 0$, there exists $A \subseteq \Gamma$ with $d^*(A) > 1 - \varepsilon$ such that if $B = \{b_1, b_2, \dots\} \subseteq \Gamma$, $t \in \Gamma$, and*

$$f(B) \lhd B + t = \{f(b_i) + b_j + t : i < j\} \subseteq A,$$

then B is finite.

Proof. Let $\mathscr{C} = \{A \subseteq \Gamma : \exists B = \{b_1, b_2, \dots\} \text{ infinite}, f(B) \lhd B \subseteq A\}$. Consider the set $K = \overline{\{\iota(f(g) + g) : g \in \Gamma\}} \subseteq b\Gamma$, and let $\mathscr{D} = \{\iota^{-1}(U) : U \subseteq b\Gamma \text{ open}, K \subseteq U\}$. We will check that the families $\mathscr{C}$ and $\mathscr{D}$ satisfy the hypotheses of Theorem 11.

(1) Let $C \in \mathscr{C}$ and $D \in \mathscr{D}$. Let $B = \{b_1, b_2, \dots\} \subseteq \Gamma$ be an infinite set such that $f(B) \lhd B \subseteq C$. We will assume that the elements $b_1, b_2, \dots$ are distinct. Let $U \subseteq b\Gamma$ be an open set, $K \subseteq U$, such that $D = \iota^{-1}(U)$. Let (x, y) be an accumulation point for the sequence $(\iota(f(b_n)), \iota(b_n))_{n\in\mathbb{N}}$. Then $x + y$ is an accumulation point for the sequence $\iota(f(b_n) + b_n)$, so $x + y \in K$. Since $U \supseteq K$ is an open set and $b\Gamma$ is a topological group, there exist open neighborhoods $V_1 \ni x$ and $V_2 \ni y$ such that $V_1 + V_2 \subseteq U$. Let $S = \{n \in \mathbb{N} : (\iota(f(b_n)), \iota(b_n)) \in V_1 \times V_2\}$, and note that S is infinite. Enumerate $S = \{s_1 < s_2 < \dots\}$, and let $B' = \{b'_1, b'_2, \dots\}$ with $b'_n = b_{s_n}$. Then for $i < j$, we have $f(b'_i) + b'_j = f(b_{s_i}) + b_{s_j} \in C$, and $\iota(f(b'_i)) + \iota(b'_j) \in V_1 + V_2 \subseteq U$, so $f(b'_i) + b'_j \in \iota^{-1}(U) = D$. Thus, $f(B') \lhd B' \subseteq C \cap D$, so $C \cap D \in \mathscr{C}$.

(2) By assumption, $m_{b\Gamma}(K) = 0$. Therefore, by outer regularity of the Haar measure, for any $\varepsilon > 0$, there exists an open set $U \supseteq K$ such that $m_{b\Gamma}(U) < \varepsilon$. Since $b\Gamma$ is a

compact Hausdorff space (hence a normal topological space), there exists an open set $V \subseteq b\Gamma$ such that $K \subseteq V \subseteq \overline{V} \subseteq U$. Noting that $d^*(\iota^{-1}(V)) \leqslant m_{b\Gamma}(\overline{V}) \leqslant m_{b\Gamma}(U) < \varepsilon$, we conclude that $\inf_{D\in\mathscr{D}} d^*(D) = 0$.

Applying Theorem 11 completes the proof. □

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