

A generalization of RSK to d -complete posets

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Abstract

The hook length formula for d -complete posets expresses the number of linear extensions of a d -complete poset P in terms of hooks of P . It generalizes the usual hook length formula for standard Young tableaux, as well as hook length formulas for shifted Young tableaux and trees. We give the first purely combinatorial proof of the hook length formula for d -complete posets. Our approach is to define a generalization of the Robinson-Schensted-Knuth bijection for d -complete posets, which may be of independent interest.

Mathematics Subject Classifications: 06A06, 06A07, 06A11

1 Introduction

Given a poset P , a natural question is to count the number of linear extensions of P . A poset P is said to have a *hook length formula* if the number of linear extensions has the form $\frac{|P|!}{\prod_{x \in P} h_P(x)}$, where h_P is some function from elements of P to the positive integers and $h_P(x)$ is called the *hook length* of x . A simple example is when P is a rooted tree, in which case the hook length of x is simply the number of elements nonstrictly dominated by x in the tree. Another example is when P is a (rotation of a) Young diagram of some partition λ . In this case, the number of linear extensions of P is the same as the number of standard Young tableaux (SYT) of shape λ . This is the celebrated hook length formula for SYTs. Several generalizations of this hook length formula (together with bijective proofs) have been given for shifted Young diagrams, skew Young diagrams, skew shifted Young diagrams, and other related posets; see [19, 9, 11, 7, 6].

One way to generalize the hook length formula is to find a more general family of posets to which it applies. There is a quite general family of posets, containing all Young diagrams, shifted Young diagrams, and rooted trees, called d -complete posets. They were introduced by Proctor [14, 15] and can be defined via a small set of local axioms [18]. We refer the readers to [12, Section 2.2] for a summary of the connection between

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d -complete posets and Weyl groups, which was their original motivation. By design, minuscule posets are d -complete posets, and Chaput–Perrin [2] used them to give a positive combinatorial formula for computing certain Λ -minuscule Schubert structure constants for general Kac-Moody flag varieties. See also [4] for a conjectural K -theoretic version of Chaput–Perrin’s formula. In addition, d -complete posets have the confluence property of jeu de taquin similar to SYTs [16].

With all these nice properties, it is not surprising that d -complete posets also have a hook length formula, described in Section 2 and first proved by Petersen and Proctor [17]. An additional proof was given by Kim–Yoo [8] and was the first completely detailed proof in the literature, previous proofs having only been sketched in conference proceedings. Their proof is very intricate, using the technique of q -integrals and verifying many cases by computer. Finally, Naruse–Okada [12] proved a general formula for skew d -complete posets via the machinery of equivariant K -theory. In addition to generalizing to skew d -complete posets, their method yields a *multivariate* generalization of the hook length formula which assigns each linear extension of a d -complete poset P a weight and expresses the sum of weights over all linear extensions in terms of certain hook polynomials.

The multivariate hook length formula for d -complete posets obtained by Naruse–Okada [12, Corollary 5.4] gives a combinatorial interpretation of the colored hook length formula for root systems of Nakada [10]. The special case of Young diagrams was independently obtained but not published by Postnikov; the proof is explained in Hopkins’s notes [3] and generalizes Pak’s [13] proof of the hook length formula for Young diagrams.

Our goal is to give a simple, purely combinatorial proof of the multivariate hook length formula for d -complete posets. To this end, we further generalize the proof of Pak [13], which considers the celebrated Robinson–Schensted–Knuth correspondence as a volume-preserving bijection between two polytopes. We extend this notion of RSK to d -complete posets, and this generalized RSK may be of independent interest.

Central to our argument is a generalization of the natural notion of *diagonals* from Young diagrams to d -complete posets; these correspond to the d -complete colorings of Proctor [15, Proposition 8.6]. In Section 2, we formally define diagonals and denote by $D(q)$ the diagonal of an element q of d -complete poset P . We also define a generalization of hooks to d -complete posets, where we view the hook $\mathbf{h}_P(p)$ of an element p as a vector with integer coordinates $h_P^{(D)}(p)$ indexed by diagonals of P summing to the hook length of p used by Proctor [17]. With this notation, we will use RSK to prove the following theorem, first obtained in the context of d -complete posets by Naruse–Okada [12].

Theorem 15. *Let P be a d -complete poset. Consider a family of indeterminates $\{x_D\}_D$ indexed by diagonals D of P . For each element p of P , define the hook polynomial $H_p(\mathbf{x}) = \sum_D h_P^{(D)}(p)x_D$. Also, for each linear extension T of P given by $p_1 > p_2 > \cdots > p_n$, define the weight of T to be the rational function with $\text{weight}(T)^{-1} = \prod_{i=1}^n \sum_{j=i}^n x_{D(p_i)}$. Then*

$$\sum_T \text{weight}(T) = \frac{1}{\prod_{p \in P} H_p(\mathbf{x})},$$

where the sum is over all linear extensions T of P .

Remark 1. Setting all indeterminates to 1 and multiplying both sides by $n!$ recovers Proctor’s [17] hook length formula. Moreover, Theorem 2 of [17] for colored P -partitions is a multivariate result for d -complete posets that is closely related to our Theorem 15. In fact, Naruse and Okada derived the identity in our Theorem 15 after they first re-obtained Theorem 2 of [17].

In the following subsection, we outline the argument of the proof and motivate our definition of RSK. We then formalize our definitions in Section 2 and prove preliminary lemmas about d -complete posets in Section 3. We define RSK for d -complete posets in Section 4, and give several of its properties in Section 5. Finally, in Section 6, we prove the multivariable hook length formula.

Proof sketch of Theorem 15

We now sketch our proof of the multivariate hook length formula. Our strategy follows Pak’s geometric proof of the hook length formula in [13].

To each d -complete poset P , we associate two polytopes in $\mathbb{R}^{|P|}$. The first polytope is P_{fillings} , given by $x_p \geq 0$ for all $p \in P$ and $\sum_{p \in P} H_p(\mathbf{x})x_p \leq 1$. The volume of P_{fillings} can be shown to be

$$\frac{1}{n!} \prod_{p \in P} \frac{1}{H_p(\mathbf{x})}.$$

The second polytope is P_{RPP} , given by $x_p \geq 0$ for all $p \in P$, $\sum_{p \in P} x_p \leq 1$, and $x_p \leq x_q$ if $p \geq_P q$. Here, RPP stands for *reverse plane partition*. The volume of P_{RPP} can be shown to be

$$\frac{1}{n!} \sum_T \text{weight}(T),$$

where the sum is over all linear extensions T of P .

Thus, to show Theorem 15, it suffices to show that P_{fillings} and P_{RPP} have the same volume. For this, we will define a piecewise-linear, volume-preserving maps that sends P_{fillings} to P_{RPP} . We call this map RSK, for reason explained in Section A. Our RSK map can be constructed via a series of *toggle operations*, each of which has determinant ± 1 when viewed as a linear map $\mathbb{R}^{|\lambda|} \rightarrow \mathbb{R}^{|\lambda|}$, whence RSK is volume-preserving (Theorem 32). It also follows that the toggle operations are invertible. Hence, it remains to show that RSK maps P_{fillings} into P_{RPP} . This (Theorem 35) is the main technical part of the proof.

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2 Definitions

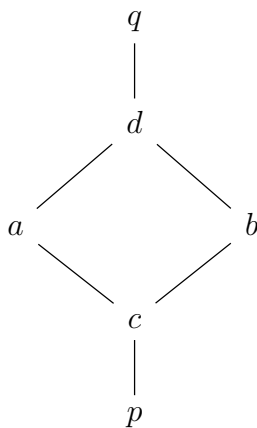
Proctor [14] defined d -complete posets in terms of simple posets denoted $d_k(1)$.

Definition 2 ([14]). Let $k \geq 3$. The poset $d_k(1)$ has $2k - 2$ elements, of which two are incomparable, $k - 2$ form a chain above the two incomparable elements, and $k - 2$ form a chain below the two incomparable elements (see Theorem 3).

We will use the following terminology of Proctor [14, 18]. Let P be a poset.

- The two incomparable elements of $d_k(1)$ are called the *side elements*.
- The $k - 2$ elements above the side elements are called *neck elements*. For $k \geq 4$, all but the lowest neck element are called *strict neck elements*.
- The $k - 2$ elements below the side elements are called *tail elements*. For $k \geq 4$, all but the highest tail element are called *strict tail elements*.
- An interval $[p, q]$ of P is called a d_k -interval if it is isomorphic to $d_k(1)$. The side, neck, and tail elements of $[p, q]$ are those that correspond respectively to side, neck, and tail elements of $d_k(1)$. We will often simplify terminology and refer simply to d -intervals when the value of k does not matter.
- A subposet of P is called d_k^- -convex if it is convex and isomorphic to $d_k(1) \setminus \{\sup d_k(1)\}$. Again, we refer simply to d^- -convex sets when the value of k does not matter.
- We call d_3 -intervals *diamonds*.

Example 3. A d_4 -interval is shown below. The neck elements are q and d , with q a strict neck element. The side elements are a and b , and the tail elements are c and p , with p a strict tail element. The interval $[p, d]$ is a d_4^- -convex set.



A d -complete poset is a poset which is built from d_k -intervals in a ‘coherent way’.

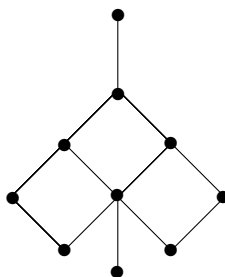
Definition 4 ([14]). A finite poset P is d -complete if it satisfies the following conditions:

- for each d_k^- -convex subset I of P , there exists an element z of P such that $I \cup \{z\}$ is a d_k -interval;

- for each d -interval $[w, z]$, the element z does not cover any elements outside $[w, z]$;
- there are no d^- -convex sets which differ only in the minimal elements.

Remark 5. Any rooted tree is trivially a d -complete poset because it contains no d^- -convex sets. Young diagrams can be viewed as d -complete posets: there is an element corresponding to each box, with one element covering another if one is directly above or to the left of the other. Shifted Young diagrams can be viewed as d -complete posets in a similar way.

Example 6. The Hasse diagram of a d -complete poset which is not a tree, Young diagram, or shifted Young diagram is shown below.



Remark 7. An *upper set* of a poset P is a subposet P' of P such that if $x \in P'$ and $y \geq x$, then $y \in P'$. It follows from the axioms that upper sets of d -complete posets are themselves d -complete. The same is not true for lower sets, however. For example, a d_k^- -interval is a lower set of a d_k -interval, but it is not a d -complete poset.

The following generalization of the usual notion of diagonals in a Young diagram will be central in our generalized hook length formula.

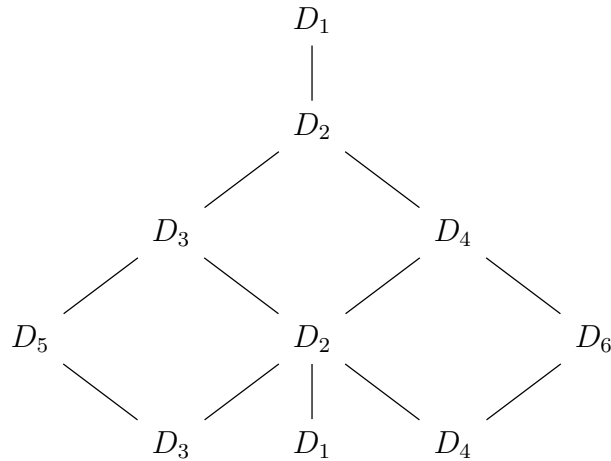
Definition 8. Let P be a d -complete poset. Define an equivalence relation on the elements of P as follows: for $p, q \in P$, let $p \sim q$ if $[p, q]$ is a d -interval and extend the equivalence relation transitively. We call the equivalence classes under $\sim$ *diagonals*, and if $p \sim q$, we say that p and q are in the same diagonal. For an element p of P , we let $D(p)$ denote the diagonal of P containing p . Also, we let $\mathcal{D}(P)$ denote the set of diagonals of P .

The partition of P into its diagonals corresponds to the coloring of P obtained in [15, Proposition 8.6].

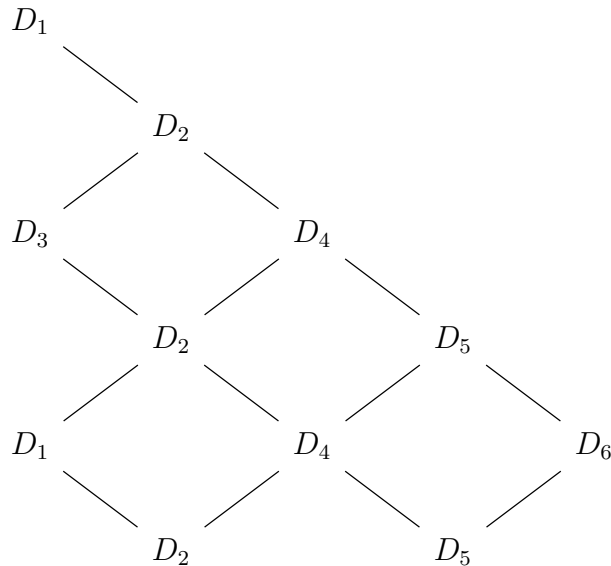
We also need a formal notion of diagonal adjacency.

Definition 9. We say two elements of P are *adjacent* if one element covers the other, and we say two diagonals D_1, D_2 of a d -complete poset P are adjacent if there exist $p_1 \in D_1, p_2 \in D_2$ which are adjacent.

Example 10. The following diagram labels the diagonals of the d -complete poset from Theorem 6. The pairs of adjacent diagonals are (D_1, D_2) , (D_2, D_3) , (D_2, D_4) , (D_3, D_5) , (D_4, D_6) .



The following diagram labels the diagonals of a d -complete poset arising from a shifted Young diagram.



We observe that for shifted Young diagrams, most diagonals are as expected from Young diagrams. However, the leftmost elements alternate between two diagonals, rather than forming a single diagonal.

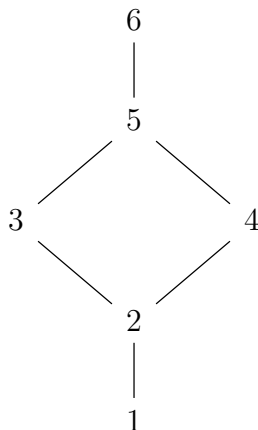
Finally, we need to formalize the notion of a hook vector in a d -complete poset.

Definition 11. For an element p of a d -complete poset P , we define the *hook vector* of p , denoted $\mathbf{h}_P(p) = [h_P^{(D)}(p)]_{D \in \mathcal{D}(P)}$, as an element of $\mathbb{Z}^{|\mathcal{D}(P)|}$ recursively as follows. If p is not a neck element of any d -interval, then $h_P^{(D)}(p)$ is the number of elements of D nonstrictly

dominated by p . If p is the maximal element of some d -interval $[p', p]$ with side elements w, z , then $\mathbf{h}_P(p) = \mathbf{h}_P(w) + \mathbf{h}_P(z) - \mathbf{h}_P(p')$. If the poset P is clear from context, we will simply write $\mathbf{h}(p)$. Also, for convenience, if $p \notin P$, we will set $\mathbf{h}_P(p) = \mathbf{0}$.

We will see in Theorem 23 that hook vectors are indeed well-defined.

Example 12. Consider the poset $d_4(1)$ with elements labeled as below. We will order the diagonals by $D(1), D(2), D(3), D(4)$.



The hook vectors of elements 1, 2, 3, 4 are, respectively, $[1, 0, 0, 0]$, $[1, 1, 0, 0]$, $[1, 1, 1, 0]$, $[1, 1, 0, 1]$, because these elements are not top elements of any d -interval. Now, 5 is the top element of the d_3 -interval $[2, 5]$ with side elements 3 and 4, so the hook vector of 5 is $[1, 1, 1, 1]$. Finally, 6 is the top element of $d_4(1)$ with side elements 3 and 4, so the hook vector of 6 is $[1, 2, 1, 1]$.

To assist with calculations involving hook vectors, we introduce some additional notation.

Definition 13. For an element $p \in P$, let $\mathbf{1}_p \in \mathbb{Z}^{|\mathcal{D}(P)|}$ be given by $\mathbf{1}_p = [\delta_{D(p), D}]_{D \in \mathcal{D}(P)}$. For a subset $S \subseteq P$, let $\mathbf{1}_S = \sum_{p \in S} \mathbf{1}_p$, which we call the *indicator vector* of S .

We observe that the sum of the entries of $\mathbf{h}_P(p)$ is the hook length defined by Proctor [17]. Thus with this definition, Proctor's hook length formula is as follows.

Theorem 14 ([17]). *The number of linear extensions of a d -complete poset P is given by*

$$\frac{|P|!}{\prod_{p \in P} \sum_{D \in \mathcal{D}(P)} h_P^{(D)}(p)}.$$

We prove the following generalization, proved in the context of root systems by Naruse-Okada [12, Corollary 5.4] and first stated for d -complete posets by Nakada [10].

Theorem 15. *Let P be a d -complete poset. Consider a family of indeterminates $\{x_D\}_{D \in \mathcal{D}(P)}$ indexed by diagonals D of P . For each element p of P , define the hook polynomial $H_p(\mathbf{x}) = \sum_{D \in \mathcal{D}(P)} h_P^{(D)}(p) x_D$. Also, for each linear extension T of P given by*

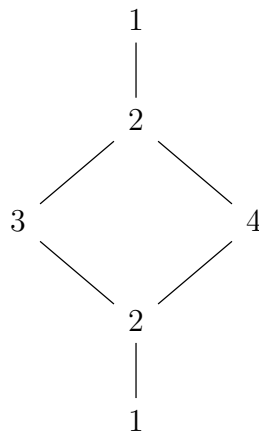
$p_1 > p_2 > \cdots > p_n$, define the weight of T to be the rational function with $\text{weight}(T)^{-1} = \prod_{i=1}^n \sum_{j=i}^n x_{D(p_i)}$. Then

$$\sum_T \text{weight}(T) = \frac{1}{\prod_{p \in P} H_p(\mathbf{x})},$$

where the sum is over all linear extensions T of P .

Remark 16. Setting all of the indeterminates x_D in the theorem to 1 recovers Proctor's original result as stated in Theorem 14.

Example 17. Consider the poset $d_4(1)$ from Theorems 3 and 12. In the diagram below we label elements by their diagonals.



The poset $d_4(1)$ has two linear extensions, determined by a choice of ordering of the two side elements. This gives the sum of weights to be

$$\begin{aligned}
& \frac{1}{x_1(x_1+x_2)(x_1+x_2+x_3)(x_1+x_2+x_3+x_4)(x_1+x_2+x_3+x_4+x_2)(x_1+x_2+x_3+x_4+x_2+x_1)} \\
& + \frac{1}{x_1(x_1+x_2)(x_1+x_2+x_4)(x_1+x_2+x_3+x_4)(x_1+x_2+x_3+x_4+x_2)(x_1+x_2+x_3+x_4+x_2+x_1)} \\
& = \frac{1}{x_1(x_1+x_2)(x_1+x_2+x_3+x_4)(x_1+2x_2+x_3+x_4)(2x_1+2x_2+x_3+x_4)} \left(\frac{1}{x_1+x_2+x_3} + \frac{1}{x_1+x_2+x_4} \right) \\
& = \frac{2x_2+2x_2+x_3+x_4}{x_1(x_1+x_2)(x_1+x_2+x_3)(x_1+x_2+x_4)(x_1+x_2+x_3+x_4)(x_1+2x_2+x_3+x_4)(2x_1+2x_2+x_3+x_4)} \\
& = \frac{1}{x_1(x_1+x_2)(x_1+x_2+x_3)(x_1+x_2+x_4)(x_1+2x_2+x_3+x_4)},
\end{aligned}$$

which by Theorem 12 is the reciprocal of the product of the hook polynomials, demonstrating Theorem 15 for $d_k(4)$.

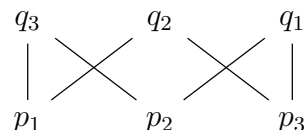
3 Preliminary results

Before defining RSK for d -complete posets and proving the multivariable hook length formula, we give some simple structural facts about d -complete posets. In particular, we show that the hook vectors from Theorem 11 are well-defined, and we show some

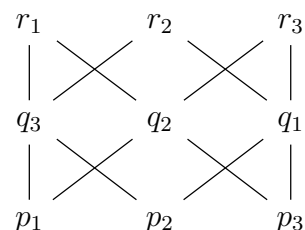
useful facts about the diagonals of d -complete posets. Throughout this section, let P be a d -complete poset.

Theorem 18, Theorem 19, and Theorem 20 appear in Proctor's works [14, 15]; for completeness, we give their proofs.

Lemma 18 ([14, Section 3]). *The poset P cannot contain a subset $\{p_1, p_2, p_3, q_1, q_2, q_3\}$ such that for $i \in [3]$, the sets $\{p_i, q_{i+1}, q_{i+2}\}$ with indices taken mod 3 are d_3^- -convex, as in the diagram below.*



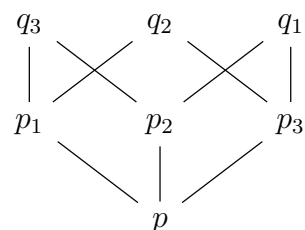
Proof. Suppose for the sake of contradiction that such a subset does exist, and let $\{p_1, p_2, p_3, q_1, q_2, q_3\}$ be one with maximal q_1 . Because P is d -complete, there exist r_1, r_2, r_3 in P such that for each $i \in [3]$, the set $\{p_i, r_i, q_{i+1}, q_{i+2}\}$ is a d_3 -interval, as shown below.



No r_i may cover an element outside its respective d_3 -interval, so in particular r_i cannot cover q_i . Thus, the elements r_1, r_2 , and r_3 are distinct. Also, because each set $\{p_i, r_i, q_{i+1}, q_{i+2}\}$ is an interval, there are no elements of P between q_i and r_j for $i \neq j$. Thus, for each i , the set $\{q_i, r_{i+1}, r_{i+2}\}$ is convex. It follows that the set $\{q_1, q_2, q_3, r_1, r_2, r_3\}$ is a set satisfying the conditions of the lemma, which since $r_3 \geq q_1$ contradicts maximality. $\square$

Proposition 19 ([14, Section 3]). *Each element of a d -complete poset is covered by at most two elements.*

Proof. Suppose element p is covered by more than two elements, and let p_1, p_2, p_3 be elements covering p . Since they cover p , the elements p_1, p_2 , and p_3 are incomparable. Thus for each $i \in [3]$ the set $\{p, p_{i+1}, p_{i+2}\}$ is d_3^- -convex, where we take indices mod 3. Thus there exist elements q_1, q_2 , and q_3 such that $\{p, p_{i+1}, p_{i+2}, q_i\}$ is a d_3 -interval for $i \in [3]$.



For $i, j \in [3]$ with $i \neq j$, the element q_i covers p_j . Thus for $i \in [3]$ the set $\{p_i, q_{i+1}, q_{i+2}\}$ is d_3^- -convex, contradicting Theorem 18. $\square$

Proposition 20 ([15, Lemma 8.3]). *A neck element of a d -interval does not cover any elements outside the d -interval, and a tail element of a d -interval is not covered by any elements outside the d -interval.*

Proof. By Theorem 4, a neck element of a d -interval does not cover any elements outside the d -interval. Now suppose for the sake of contradiction that there exists a tail element of some d -interval I covered by some $x \notin I$. Let p be a maximal such element in P , and let p be the tail element of d -interval I with side elements w and z , and let x be an element not in I covering p . By Theorem 19, p must be a strict tail element of I since otherwise p would be covered by w , z , and x . Thus, there is a unique element p' of I covering p , and the elements $\{p, p', x\}$ form a d_3^- -convex set. Thus there must exist an element x' covering p' and q , and x' cannot be in I because I is convex and $x \notin I$. This contradicts the maximality of p . $\square$

Proposition 21. *If an element p of P is the maximal element of some d -interval, then p is the maximal element of a unique d -interval I , and any d -interval having p as a neck element contains I . Similarly, if p is the minimal element of some d -interval, then p is the minimal element of a unique d -interval I , and any d -interval having p as a tail element contains I .*

Proof. Suppose, for the sake of contradiction, that P contains some element which is maximal in multiple d -intervals, and let q be minimal among such elements. Let $[p, q]$ and $[p', q]$ be two distinct d -intervals. By Theorem 4, the intervals $[p, q]$ and $[p', q]$ cannot differ in only their minimal elements, so we have $[p, q] \setminus \{p\} \neq [p', q] \setminus \{p'\}$. Also by Theorem 4, the elements of P covered by q must be in both $[p, q]$ and $[p', q]$. In particular, q is a strict neck element of $[p, q]$ if and only if it is a strict neck element of $[p', q]$; in this case, letting q' be the unique element covered by q , we have distinct d -intervals $[p, q'] \setminus \{p\}$ and $[p', q'] \setminus \{p'\}$, contradicting the minimality of q . Thus $[p, q]$ and $[p', q]$ are diamonds differing only in their minimal elements, contradicting Theorem 4. The statement about d -intervals containing q as a neck element follows; such intervals contain a d -interval with q as maximal neck element, and thus contain the unique such interval.

Now suppose, for the sake of contradiction, that P contains elements which are minimal in multiple d -intervals, and let p be maximal among such elements. Let $[p, q]$ and $[p, q']$ be two distinct d -intervals. By Theorem 20, the elements of P covering p are contained in both $[p, q]$ and $[p, q']$. In particular, p is a strict tail element of $[p, q]$ if and only if it is a strict tail element of $[p, q']$, in which case letting p' be the unique element covering p gives distinct d -intervals $[p', q] \setminus \{q\}$ and $[p', q'] \setminus \{q'\}$, contradicting the maximality of p . Thus $[p, q]$ and $[p, q']$ are diamonds sharing side elements w and z . Then $\{w, q, q'\}$ and $\{z, q, q'\}$ are d_3^- -convex sets differing only in their minimal elements, contradicting Theorem 4. As before, the statement about d -intervals containing p as a neck element follows. $\square$

In light of Theorem 21, we can make the following definition:

Definition 22. Let p be an element of d -complete poset P . We define $p^\uparrow$ to be the unique element such that $[p, p^\uparrow]$ is a d -interval, if such $p^\uparrow$ exists. Similarly, we define $p^\downarrow$ to be the unique element such that $[p^\downarrow, p]$ is a d -interval, if such $p^\downarrow$ exists. By definition, p is in the same diagonal as $p^\uparrow$ and $p^\downarrow$.

As a consequence of Theorem 21, we see that hook vectors are well-defined.

Corollary 23. *The notion of hook vectors given in Theorem 11 is well-defined.*

Proof. If p is not a neck element of a d -interval, then $\mathbf{h}_P(p)$ is clearly well-defined. If p is a neck element of some d -interval, then by Theorem 21 there is a unique $p' \in P$ such that $[p', p]$ is a d -interval, so the elements p' , w , and z in Theorem 11 are uniquely determined. $\square$

Finally, we collect several facts about the diagonals of a d -complete P .

Proposition 24. *The following statements hold for diagonals of P :*

- (1) *The elements of a diagonal D form a chain under the relation of P , with all covering relations given by $p < p^\uparrow$;*
- (2) *Elements of a single diagonal D cannot be adjacent;*
- (3) *If P' is an upper set of P , then two elements of P' are in the same diagonal of P' if and only if they are in the same diagonal of P . Thus we may identify diagonals of P with diagonals of P' ;*
- (4) *If C, D are adjacent diagonals such that the minimal element of C is minimal in P , then each element of D is adjacent to an element of C ;*
- (5) *Two diagonals having nonempty intersection with upper set P' are adjacent in P' if and only if they are adjacent in P ;*
- (6) *If C, D are adjacent diagonals of P with minimal elements c and d , respectively, then at most one of c and d is minimal in P .*

Proof. (1) This follows from Theorem 21.

- (2) Suppose for the sake of contradiction that p, q are elements of D with q covering p in P . Then q also covers p in the subposet D , so, by statement 24, $[p, q]$ is a d -interval of two elements, a contradiction.
- (3) It is immediate from the definition that if two elements of P' are in the same diagonal of P' , then they are in the same diagonal of P , so we only prove the converse. Let $p < q$ be two elements of P' in the same diagonal of P , so that by statement 24 there exists a sequence $p = p_0 < \cdots < p_n = q$ of elements of P such that each $[p_{i-1}, p_i]$ is a d -interval. Each element of each $[p_{i-1}, p_i]$ dominates p in P , and thus is in P' , so p and q are in the same diagonal of P' .

- (4) Suppose for the sake of contradiction that not all elements of D are adjacent to an element of C . Let d be the minimal element of D that is not.

We first claim that d is minimal in D . Suppose for the sake of contradiction that $d^\downarrow$ exists. Then $d^\downarrow$ is adjacent to some element c of C by assumption of minimality. We consider the possible positions of c with respect to the d -interval $[d^\downarrow, d]$.

If c is a side element of $[d^\downarrow, d]$, then d is adjacent to c , a contradiction. If c is contained in $[d^\downarrow, d]$ but is not a side element, then c is a tail element dominating $d^\downarrow$, and so $c^\uparrow$ exists and is dominated by d . In this case d is adjacent to $c^\uparrow$, again a contradiction.

Finally, if c is not in $[d^\downarrow, d]$, then $d^\downarrow$ must dominate c because by Theorem 20 elements outside $[d^\downarrow, d]$ cannot dominate $d^\downarrow$. If $S = [d^\downarrow, d] \sqcup \{c\}$ is d^- -convex, then $c^\uparrow$ exists and dominates d , whence again d is adjacent to $c^\uparrow$, a contradiction. On the other hand, if S is not d^- -convex, then there is some element x outside of S covering c and dominated by an element of S . Now $\{c, d^\downarrow, x\}$ is d_3^- -convex, so there exists y covering $d^\downarrow$ and x such that $[c, y]$ is a d_3 -interval. Now since y covers $d^\downarrow$, by Theorem 20 it must be in $[d^\downarrow, d]$ and thus either y or $y^\uparrow$ is covered by d , depending on whether y is a side element or a tail element. Also, y is in C , and thus $y^\uparrow$ is as well, and so again d is adjacent to an element of C , a contradiction. This proves our claim that d is minimal in D .

Now let d', c' be adjacent elements of D and C , respectively, such that (d', c') is minimal in the product poset $P \times P$. Such a pair exists because D and C are adjacent diagonals. We have $d' \neq d$ and d is minimal in D , so the element $d'^\downarrow$ exists. By assumption of minimality of (d', c') , the element $d'^\downarrow$ cannot be adjacent to c' . Since c' is adjacent to d' but not $d'^\downarrow$, the element c' is either a neck element of $[d'^\downarrow, d']$ or covers d' . Thus if $c'^\downarrow$ exists, then by Theorems 20 and 21 $[c'^\downarrow, c']$ either is contained in or contains $[d'^\downarrow, d']$, whence $c'^\downarrow$ is adjacent to $d'^\downarrow$, contradicting the minimality of (d', c') . So c' is minimal in C , and thus in P by the assumption on C .

It follows that d' covers a minimal element in P , and thus is minimal in D , a contradiction.

- (5) Let the two diagonals be C and D . If C and D are adjacent in P' , then by definition they are adjacent in P , so we have only to prove the other direction. Suppose C and D are adjacent in P .

Consider the upper set P'' consisting of all elements of P which dominate some element of $C \cap P'$ or $D \cap P'$. The set $P' \setminus P''$ cannot contain any elements of C or D . Also, any minimal element of P'' is in C or D . Without loss of generality, let an element of C be minimal in P'' . Then by 24, any element d of $D \cap P'$ is adjacent to an element c of $C \cap P'$. Suppose $c \notin P'$, in which case we must have $d \succ c$. Since by assumption P' contains elements of C , there must exist elements of C dominating c . Thus $c^\uparrow$ exists, and must dominate d , whence $c^\uparrow \in P'$. Also, by Theorem 20, we have $d \in [c, c^\uparrow]$. If $[c, c^\uparrow]$ is a diamond, then $c^\uparrow$ is adjacent to d . Otherwise, $c^\uparrow \succ d^\uparrow$. Either way, C and D are adjacent in P' .

- (6) This follows from statement 24; if c and d were both minimal, each would have to be adjacent to an element of the other diagonal. Since an element covering a minimal element of P cannot be in the neck of a d -interval and thus is minimal in its diagonal, this implies that one of c and d is not minimal in P , a contradiction. $\square$

4 Defining RSK for d -complete posets

As described in Section A, Pak [13] conceives of RSK as a bijection from fillings of Young diagrams by nonnegative real numbers to fillings of Young diagrams with nonnegative real numbers that increase from left to right and top to bottom. We will define RSK for d -complete posets similarly.

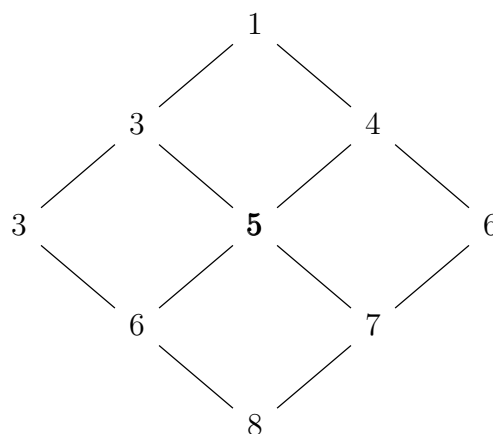
Definition 25. A *filling* of P is a vector $[t_p]_{p \in P}$ of nonnegative reals indexed by the elements of P . We say that a filling $[s_p]_{p \in P}$ is *order-reversing* if $p \leq q$ implies that $s_p \geq s_q$. We will denote the set of fillings of P by $\mathcal{F}_P$ and the set of order-reversing fillings of P by $\mathcal{F}'_P$.

RSK can be described as a composition of certain *toggle operations*, first defined by Berenstein and Kirillov [5]. We define toggle operations in the d -complete setting.

Definition 26 (Toggle operations). Let $[s_p]_{p \in P}$ be a filling of a d -complete poset P . *Toggling* at an element $p \in P$ is defined as follows. Of the elements covering p , let x be such that s_x is maximal; of the elements covered by p , let y be such that s_y is minimal. If p is not covered by any elements, we set $s_x = 0$, and similarly for s_y . We then change the label of p to $s'_p = s_x + s_y - s_p$.

When P corresponds to a Young diagram, toggle operations are exactly the toggle operations defined by Berenstein and Kirillov on Young tableaux.

Example 27. Consider a toggle operation at the bolded central element of the filling below.



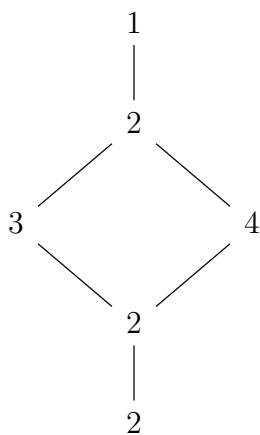
We have $s_x = 4$, $s_y = 6$, and $s'_p = 4 + 7 - 5 = 6$.

We now define RSK_P as a bijection from fillings $\mathcal{F}_P$ to order-reversing fillings $\mathcal{F}'_P$.

Definition 28 (RSK). Let $[t_p]_{p \in P}$ be a filling of a d -complete P . We define $\text{RSK}_P([t_p]_{p \in P})$ recursively. If P has only one element, RSK_P is the identity map. Otherwise, pick a linear extension $p_1 > p_2 \cdots > p_n$ of P , and let $P^{(j)} = \{p_i : i \leq j\}$. Suppose inductively that we have defined $\text{RSK}_{P^{(j)}}([t_p]_{p \in P^{(j)}}) = [s_p]_{p \in P^{(j)}}$; we wish to define $\text{RSK}_{P^{(j+1)}}([t_p]_{p \in P^{(j+1)}})$. To do so, we first label p_i for $i \leq j$ with s_{p_i} and p_{j+1} with $-t_{p_{j+1}}$. Then, we toggle every element in the diagonal of p_{j+1} (because by Theorem 2424 elements of $D(p_{j+1})$ are not adjacent, this is well-defined without specifying an order for the toggles). The result is $\text{RSK}_{P^{(j+1)}}([t_p]_{p \in P^{(j+1)}})$, and we can continue until we have defined $\text{RSK}_P([t_p]_{p \in P})$.

This definition relies on a choice of linear extension of P , but we will show shortly (Theorem 31) that this choice does not matter.

Example 29. Let us consider how RSK acts on the following filling with the left neck element preceding the right neck element in the linear ordering:



Step 1

1

Step 2

1

3

Step 3

1

3

6

Step 4

1

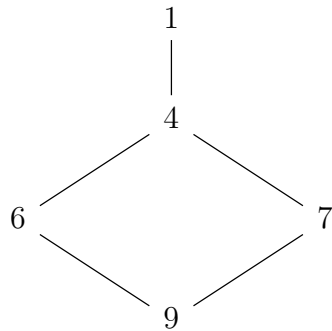
3

6

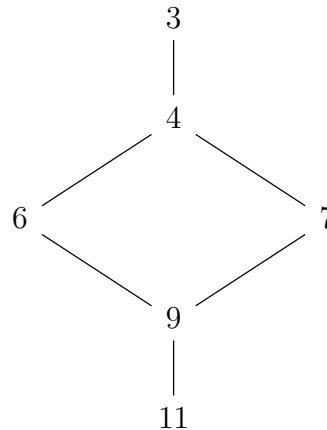
7

Considering $P^{(i)}$ for $i \leq 4$ is simple, because each element is in its own diagonal. We start by labeling p_1 with 1. Then, we add p_2 , label it with -2 , and toggle its value to $3 = 1 - (-2)$. We add p_3 , label it with -3 , and toggle its value to $3 - (-3)$. We add p_4 , label it with -4 , and toggle its value to $3 - (-4)$.

Step 5



Step 6



When we add p_5 , we set its value to -2 . We toggle its value to $7 - (-2) = 9$, but we also toggle the other element in its diagonal from 3 to $1 + 6 - 3 = 4$. Finally, when we add p_6 , we set its value to -2 . Its value gets toggled to $9 - (-2)$, and the value of its other diagonal element gets toggled from 1 to $4 - 1 = 3$.

Remark 30. When the poset element y exists in the toggle operation, we have $s'_p = s_x + s_y - s_p$, which if $s_x \leq s_p \leq s_y$ satisfies $s_x \leq s'_p \leq s_y$. The only step of RSK when s_y does not exist is when we toggle c , in which case $s'_c = s_x + t_c \geq s_x$. Thus RSK_P maps $\mathcal{F}_P$ into $\mathcal{F}'_P$. Also, toggle operations are reversible, and since RSK_P consists of repeated applications of toggle operations, RSK_P is indeed a bijection $\mathcal{F}_P \rightarrow \mathcal{F}'_P$.

We now verify that RSK_P is well-defined.

Proposition 31. *The output of RSK_P does not depend on the choice of linear extension of P .*

Proof. Let $P' = P \sqcup \{a, b\}$ be a d -complete poset, with P an upper set and a, b distinct, non-adjacent elements. Then for an ordering $\{p_i\}_{i=1}^n$ of the elements of P , appending a and b to the end in either order gives a valid insertion order for P' . We wish to show that the two orders give the same result, which will prove the claim because any linear extension can be obtained from any other by a sequence of such transpositions (see, for example, [1, Lemma 2.3]). In general, for a labelled d -complete poset $Q' = Q \sqcup \{c\}$ with Q an upper set, the computation of $\text{RSK}_{Q'}([t_p]_{p \in Q'})$ from $\text{RSK}_Q([t_p]_{p \in Q})$ depends only on the labels of $\text{RSK}_Q([t_p]_{p \in Q})$ at elements in $D(c)$ and adjacent diagonals, and only changes the values at elements in $D(c)$. Since $D(a)$ and $D(b)$ are disjoint by Theorem 2424 and non-adjacent by Theorem 2424, the order of addition of a and b does not matter. $\square$

5 Properties of RSK

In this section, we give several properties of RSK for d -complete posets, focusing particularly on the relationship between RSK and hook vectors. The results proved in this

section, namely Theorem 32 and Theorem 35, are the main technical tools in our proof of the multivariate hook length formula.

Viewing $\mathcal{F}_P$ and $\mathcal{F}'_P$ as subsets (cones, in fact) of $\mathbb{R}^n$, we observe that RSK_P is a piecewise-linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Proposition 32. *Let P be a d -complete poset with n elements. Then the piecewise linear map $\text{RSK}_P: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is volume-preserving.*

Proof. It suffices to prove that RSK_P is volume-preserving on regions on $\mathbb{R}^n$ where it is linear. Since RSK_P is a composition of toggle operations, it suffices to prove that a toggle at $c \in P$ effects a volume-preserving map $\mathbb{R}^n \rightarrow \mathbb{R}^n$. The matrix describing the linear map induced by a toggle at c has all 1s on the diagonal except for a -1 at position (c, c) , some number of 1s in the c th row, and 0s everywhere else. This has determinant -1 , and thus is volume-preserving. $\square$

The crux of the geometric proof of the usual hook length formula was the following lemma (see [3, Proposition 13], generalizing a computation of [13, Section 4]), which shows that RSK actually sends one polytope into the other.

Lemma 33. *If $[a_{ij}]_{(i,j) \in \lambda}$ is a point of P_{fillings} and $[r_{ij}]_{(i,j) \in \lambda}$ is its image under RSK , then, denoting the hook length of cell (i, j) by h_{ij} , we have*

$$\sum_{(i,j) \in \lambda} r_{ij} = \sum_{(i,j) \in \lambda} h_{ij} a_{ij}.$$

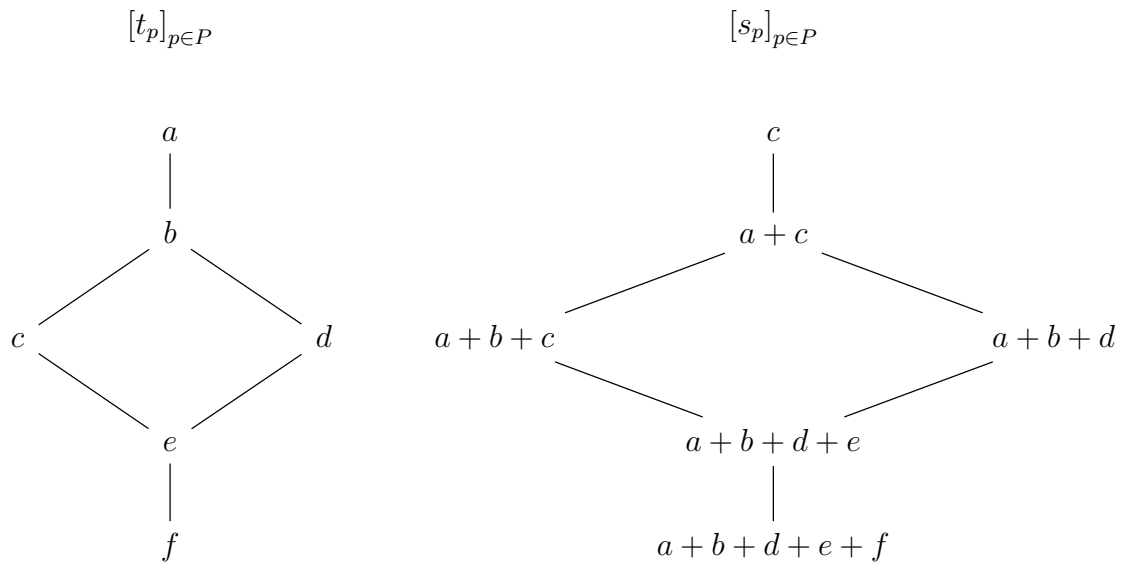
To prove the multivariate generalization of the hook length formula, we will need a stronger fact involving the diagonals of P . First, we introduce some notation.

Definition 34. Fix $[t_p]_{p \in P} \in \mathcal{F}_P$, and let $[s_p]_{p \in P} = \text{RSK}_P([t_p]_{p \in P})$. For each diagonal D of P , define the *diagonal sum* $S_P(D) = \sum_{p \in D} s_p$.

With this notation, our strengthening of Theorem 33 is as follows.

Proposition 35. *Let P be a d -complete poset. Fix $[t_p]_{p \in P} \in \mathcal{F}_P$, and let $[s_p]_{p \in P} = \text{RSK}_P([t_p]_{p \in P})$. Then for every diagonal D of P , we have $S_P(D) = \sum_{p \in P} h_P^{(D)}(p) t_p$.*

Example 36. The diagram below shows $[t_p]_{p \in P}$ and $[s_p]_{p \in P}$ for a filling of $d_4(1)$, where $c < d$. The hook vectors for this poset, computed in Theorem 12, for the poset elements initially containing a, b, c, d, e , and f are, respectively, $[1, 2, 1, 1]$, $[1, 1, 1, 1]$, $[1, 1, 1, 0]$, $[1, 1, 0, 1]$, $[1, 1, 0, 0]$, and $[1, 0, 0, 0]$.



We have

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a+b+c+d+e+f \\ 2a+b+c+d+e \\ a+b+c \\ a+b+d \end{bmatrix},$$

equal to the vector of diagonal sums.

Remark 37. For Young diagrams, summing over all diagonals gives Theorem 33.

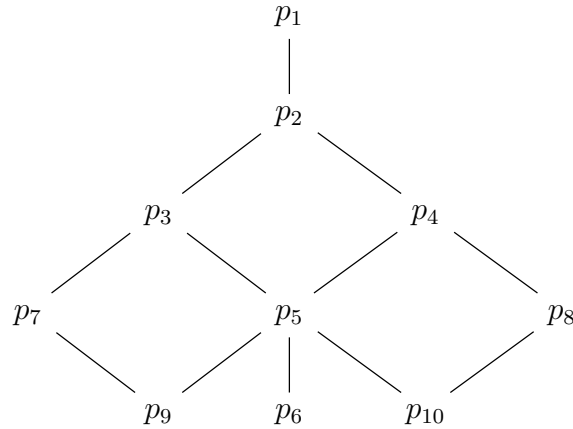
Proving Theorem 35 is the most technical step of our proof, and will require a couple intermediary results.

Since we may choose any linear extension of P when computing RSK_P will be useful for us to choose a particularly nice linear extension.

Definition 38. A linear extension $p_1 > p_2 > \dots > p_n$ of P is called a *stable insertion order* if whenever p_i is a bottom element of a d -interval, the side elements of that d -interval are not neck elements of any other d -interval in $P^{(i)} = \{p_1, \dots, p_i\}$.

Informally, this condition says that we should finish inserting the tail of a d -interval that we have already started before moving on to d -intervals that are lower down in P .

Example 39. A stable insertion order for the d -complete poset from Theorem 6 is shown below.



Although p_3 and p_4 are neck elements of d -intervals in P , they are not in $P^{(5)}$ and $P^{(6)}$. Also, p_5 is not a neck element of P , and thus is not a neck element of $P^{(9)}$ or $P^{(10)}$.

Lemma 40. *Every d -complete poset has at least one stable insertion order.*

Proof. We proceed by strong induction under poset inclusion; the unique ordering on a singleton P establishes the base case. Now suppose P has more than one element. If $P = P' \sqcup \{c\}$ for minimal element c of P that is not in any d -interval, then appending c to the end of a stable ordering for P' gives a stable ordering for P .

Now suppose that all minimal elements of P are in d -intervals. We may identify each maximal d -interval of P with the top element of its diamond; this gives an injective map into P because by Theorem 20 distinct maximal d -intervals must have distinct diamonds. Thus, this injection induces a partial ordering on the maximal d -intervals of P . Let $[c, p]$ be minimal under this ordering. First, we note that the side elements of $[c, p]$ cannot be neck elements of other d -intervals, since then those d -intervals would contradict the minimality of $[c, p]$.

Now we claim that c is minimal in P . If not, c covers some element c' of P . Since $[c, p]$ is a maximal d -interval (under inclusion), $\{c'\} \cup [c, p]$ cannot be d^- -convex. Thus, c' must be covered by an element y of P not in $[c, p]$, and since $\{c', c, y\}$ is d_3^- -convex, there must exist x covering c and y . Since c is in the tail of $[c, p]$, by Theorem 20, x must be in $[c, p]$, and in particular is a tail element or a side element. Either way, the top element of the d_3 -interval $[c', x]$ is dominated by the top element of the diamond of $[c, p]$, contradicting the minimality assumption on $[c, p]$.

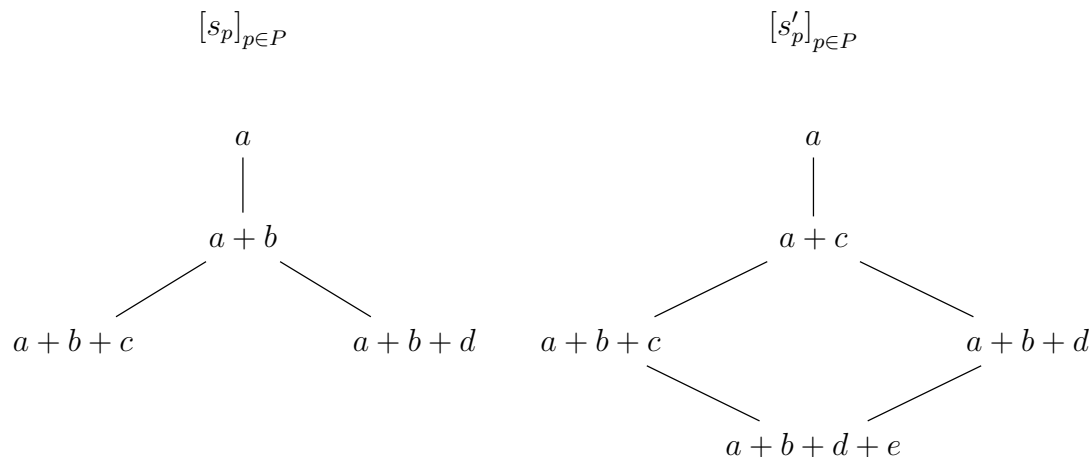
Now by the induction hypothesis, appending c to the end of a stable insertion order for $P \setminus \{c\}$ gives a stable insertion order. $\square$

To prove Theorem 35, we need to understand how diagonal sums are changed by insertion in RSK. If P is an upper set of d -complete poset P' , then by Theorem 2424 diagonals of P' correspond to (possibly empty) diagonals of P and by Theorem 2424 a pair of diagonals is adjacent in P if and only if it is adjacent in P' . Thus for a filling

$[t_p]_{p \in P'}$ of P' and corresponding $[s_p]_{p \in P} = \text{RSK}_P([t_p]_{p \in P})$ and $[s'_p]_{p \in P'} = \text{RSK}_{P'}([t_p]_{p \in P'})$ we may consider both $S_P(D)$ and $S_{P'}(D)$, where the sum in the former is only over $p \in P$. The following lemma relates the two quantities.

Lemma 41. *Let $P' = P \sqcup \{c\}$ be d -complete with minimal element c . Let $[t_p]_{p \in P'}$ be a filling of P' . Let $[s_p]_{p \in P} = \text{RSK}_P([t_p]_{p \in P})$ and $[s'_p]_{p \in P'} = \text{RSK}_{P'}([t_p]_{p \in P'})$. Let $\mathcal{D}$ denote the set of diagonals of P' which are adjacent to $D(c)$. Then $S_P(D(c)) + S_{P'}(D(c)) = t_c + \sum_{D \in \mathcal{D}} S_P(D)$.*

Example 42. The diagram below shows an RSK insertion step for a copy of $d_4(1)$ with the bottom element removed, filled with a, b, c, d, e , where $c < d$ (Theorem 36 shows the result of RSK for $d_4(1)$, which uses this as a step).



The diagonal sum relevant to Theorem 41 initially is $a + b$, and after insertion is $2a + b + c + d + e$. These sum to $3a + 2b + c + d + e = e + a + (a + b + c) + (a + b + d)$, which since all other diagonals are in $\mathcal{D}$ is what is predicted by Theorem 41.

Proof of Theorem 41. For convenience, we set $s_c = -t_c$. Redefining S_P accordingly for the rest of this proof, we wish to show that $S_P(D(c)) + S_{P'}(D(c)) = \sum_{D \in \mathcal{D}} S_P(D)$. Also, we assume that all s_p are distinct; the case when some s_p coincide will then follow from continuity of the piecewise-linear map $\text{RSK}_{P'}$. By Theorem 31 we may assume that c was inserted last in RSK, so that for each $d \in D(c)$, we have $s'_d + s_d = s'_{m_d} + s'_{n_d}$, where m_d and n_d are the elements covering and covered by d , respectively, with maximal s'_{m_d} and minimal s'_{n_d} (if m_d or n_d does not exist, we set the corresponding $s'_\bullet$ to be 0). By definition, we have $D(m_d), D(n_d) \in \mathcal{D}$ for $d \in D(c)$. We wish to show that for each element p of P with $D(p) \in \mathcal{D}$, the term s_p occurs exactly once in the formal sum $S_P(D(c)) + S_{P'}(D(c)) = \sum_{d \in D(c)} (s_d + s'_d) = \sum_{d \in D(c)} (s_{m_d} + s_{n_d})$.

By Theorem 2424, the element p is adjacent to some element of $D(c)$. We claim that p cannot be covered by two distinct elements of $D(c)$, and cannot cover two distinct elements of $D(c)$. Suppose first that p covers two elements c' and c'' of $D(c)$. Since $D(c)$

is a chain by Theorem 2424, we may assume $c' < c''$. Because c' is not the largest element in its diagonal, it is the tail element of a d -interval; then by Theorem 20, p is in the d -interval $[c', c'^\uparrow]$, so $c'^\uparrow > p$. But $c'' \geq c'^\uparrow$, which contradicts p covering c'' . Suppose now that p is covered by two elements c' and c'' of $D(c)$, and assume $c' > c''$. Because c' is not the smallest element in its diagonal, it is the neck element of a d -interval; then by Theorem 20, p is in the d -interval $[c'^\downarrow, c']$, and so $p > c'^\downarrow$. Since $c'' \leq c'^\downarrow$, this contradicts p being covered by c'' .

We are now ready to prove that s_p occurs exactly once in the formal sum $\sum_{d \in D(c)} (s_{m_d} + s_{n_d})$. We consider a few cases.

- The element p is adjacent to only one element d of $D(c)$, and p covers d . We claim that p is the only element covering d ; this will complete the proof since then $p = m_d$ and p is adjacent to no other element of $D(c)$. If there were another element p' covering d , then $\{p, p', d\}$ would be d_3^- -convex, so there would be $d' \in D(c)$ such that $\{d, p, p', d'\}$ is a diamond. This contradicts the assumption that p is adjacent to only one element of $D(c)$.
- The element p is adjacent to only one element d of $D(c)$, and p is covered by d . We claim that p is the only element covered by d ; this will complete the proof since then $p = n_d$ and p is adjacent to no other element of $D(c)$. Since d covers an element of P , we have $d \neq c$, so $d^\downarrow$ exists. Then d cannot cover any elements outside of the d -interval $[d^\downarrow, d]$, so if d covers an element $p' \neq p$ of P , the interval $[d^\downarrow, d]$ must be a diamond with side elements p and p' . But then p is adjacent to $d^\downarrow$, a contradiction.
- The element p covers d and is covered by d' in $D(c)$. The smallest possible distance between the maximal and minimal elements of a d -interval is 2, so there cannot be elements of $D(c)$ between d and d' . Thus $[d, d']$ is a diamond, one of whose side element is p ; let the other be p' . If $s_p < s_{p'}$, then $m_d = p$ and $n_{d'} = p'$; if $s_p > s_{p'}$, then $m_d = p'$ and $n_{d'} = p$. Either way, s_p occurs exactly once in the formal sum $\sum_{d \in D(c)} (s_{m_d} + s_{n_d})$.

Since these cases are exhaustive, this completes the proof. $\square$

We may now prove Theorem 35.

Proof of Theorem 35. We proceed by strong induction under poset inclusion; the statement for singleton P is immediate because P has only one element and one diagonal. Let P' be a d -complete poset with filling $[t_p]_{p \in P'}$, and let c be the last element of some stable insertion order on P' , which exists by Theorem 40. Let P be such that $P' = P \sqcup \{c\}$, and assume that the statement of the proposition is true for P , that is, that for each diagonal D , we have

$$S_P(D) = \sum_{p \in P} h_P^{(D)}(p) t_p.$$

We wish to obtain the analogous equality for P' , whence the result will follow by induction. We consider the cases when $D = D(c)$ and when $D \neq D(c)$ separately. Within each case, we study hook vectors of individual elements p and consider three subcases:

- the element p is a neck element of both P' and P ;
- the element p is a neck element of P' , but not of P ;
- the element p is not a neck element of P' .

Since d -intervals of P are d -intervals in P' , these subcases are exhaustive.

Case 1: $D \neq D(c)$. Since only labels in $D(c)$ are toggled when c is inserted in RSK, we have $S_P(D) = S_{P'}(D)$. We will prove that for all $p \in P'$, we have $h_P^{(D)}(p) = h_{P'}^{(D)}(p)$, from which it will follow that $S_{P'}(D) = \sum_{p \in P'} h_{P'}^{(D)}(p)t_p$ as needed.

We proceed by strong induction on p , with the base case being minimal elements of P' ; for such $p \neq c$ we have $h_P^{(D)}(p) = \delta_{D(p), D} = h_{P'}^{(D)}(p)$, and for c we have $H_P^{(D)}(c) = 0 = \delta_{D(c), D} = h_{P'}^{(D)}(c)$. Now suppose the claim is true for all elements strictly dominated by a given p . We consider several cases.

- (a) The element p is a neck element of both P' and P . Then there is an element $q \in P$ such that $[q, p]$ is a d -interval with side elements w and z . Then $\mathbf{h}_P(p) = \mathbf{h}_P(w) + \mathbf{h}_P(z) - \mathbf{h}_P(q)$ and $\mathbf{h}_{P'}(p) = \mathbf{h}_{P'}(w) + \mathbf{h}_{P'}(z) - \mathbf{h}_{P'}(q)$, so

$$\begin{aligned} h_{P'}^{(D)}(p) &= h_{P'}^{(D)}(w) + h_{P'}^{(D)}(z) - h_{P'}^{(D)}(q) \\ &= h_P^{(D)}(w) + h_P^{(D)}(z) - h_P^{(D)}(q) \\ &= h_P^{(D)}(p) \end{aligned}$$

by the induction hypothesis.

- (b) The element p is a neck element of P' but not of P , so $[c, p]$ is a d -interval of P' . Let w, z be the side elements of $[c, p]$. Since the insertion order is stable, w and z are not neck elements in P' . Thus, $\mathbf{h}_P(w)$ and $\mathbf{h}_P(z)$ are the indicator vectors of the sets of elements dominated by w and z , respectively, in P and $\mathbf{h}_{P'}(w) = \mathbf{h}_P(w) + \mathbf{1}_c$ and $\mathbf{h}_{P'}(z) = \mathbf{h}_P(z) + \mathbf{1}_c$. Then $\mathbf{h}_{P'}(p) = \mathbf{h}_{P'}(w) + \mathbf{h}_{P'}(z) - \mathbf{1}_c = \mathbf{h}_P(w) + \mathbf{h}_P(z) + \mathbf{1}_c$.

Letting T be the set of elements of P dominated by both w and z and N be the set of elements of P dominated by p but not w or z , we have $\mathbf{h}_P(p) = \mathbf{h}_P(w) + \mathbf{h}_P(z) + \mathbf{1}_N - \mathbf{1}_T$. Now N is the neck of $[c, p]$ and $T \sqcup \{c\}$ is the tail, and each tail element of $[c, p]$ corresponds bijectively to a neck element in the same diagonal. Thus, $\mathbf{1}_N - \mathbf{1}_T = \mathbf{1}_c$, giving $\mathbf{h}_P(p) = \mathbf{h}_{P'}(p)$ as needed.

- (c) The element p is not a neck element of P' . Then p is not a neck element of P either. By definition, $\mathbf{h}_P(p)$ and $\mathbf{h}_{P'}(p)$ are indicator vectors of the sets of elements nonstrictly dominated by p in P and P' , respectively. Thus, $\mathbf{h}_{P'}(p) = \mathbf{h}_P(p) + \mathbf{1}_c$ or $\mathbf{h}_{P'}(p) = \mathbf{h}_P(p)$ depending on whether p dominates c , and since $c \notin D$, we have $h_{P'}^{(D)}(p) = h_P^{(D)}(p)$ either way.

Case 2: $D = D(c)$. Let $\mathcal{D}$ be the set of diagonals of P' which are adjacent to $D(c)$. Then $S_P(D(c)) + S_{P'}(D(c)) = t_c + \sum_{D' \in \mathcal{D}} S_P(D')$ by Theorem 41. We have $h_{P'}^{(D)}(c) = 1$ and we

claim that for $p \in P$, we have

$$h_P^{(D)}(p) + h_{P'}^{(D)}(p) = \sum_{D' \in \mathcal{D}} h_P^{(D')}(p).$$

The induction step will then follow because

$$\begin{aligned} S_{P'}(D) &= t_c - S_P(D) + \sum_{D' \in \mathcal{D}} S_P(D') \\ &= t_c + \sum_{p \in P} \left(-h_P^{(D)}(p) + \sum_{D' \in \mathcal{D}} h_P^{(D')}(p) \right) t_p \\ &= t_c + \sum_{p \in P} h_{P'}^{(D)}(p) t_p \\ &= \sum_{p \in P'} h_{P'}^{(D)}(p) t_p. \end{aligned}$$

We again proceed by strong induction on p . When p is minimal in P , we have $h_P^{(D)}(p) = \delta_{D(p), D} = 0$, and $\mathbf{h}_{P'}(p)$ is $\mathbf{1}_p$ or $\mathbf{1}_p + \mathbf{1}_c$ depending on whether p dominates c or not; since p dominates c if and only if it is in a diagonal adjacent to D , this proves the claim for minimal p . Now suppose the claim is true for all elements strictly dominated by a given $p \neq c$. We consider the same cases as before.

- (a) The element p is a neck element of both P' and P . Then there is an element $q \in P$ such that $[q, p]$ is a d -interval with incomparable elements w and z . Then $\mathbf{h}_P(p) = \mathbf{h}_P(w) + \mathbf{h}_P(z) - \mathbf{h}_P(q)$ and $\mathbf{h}_{P'}(p) = \mathbf{h}_{P'}(w) + \mathbf{h}_{P'}(z) - \mathbf{h}_{P'}(q)$, so

$$\begin{aligned} h_{P'}^{(D)}(p) + h_P^{(D)}(p) &= h_{P'}^{(D)}(w) + h_{P'}^{(D)}(z) - h_{P'}^{(D)}(q) \\ &\quad + h_P^{(D)}(w) + h_P^{(D)}(z) - h_P^{(D)}(q) \\ &= \sum_{D' \in \mathcal{D}} \left(h_P^{(D')}(w) + h_P^{(D')}(z) - h_P^{(D')}(q) \right) \\ &= \sum_{D' \in \mathcal{D}} h_P^{(D')}(p) \end{aligned}$$

by the induction hypothesis.

- (b) The element p is a neck element of P' but not of P , so $[c, p]$ is a d -interval of P' . Let w, z be the side elements of $[c, p]$. Since the insertion order is stable, w and z are not neck elements in P' . Thus, $\mathbf{h}_P(w)$ and $\mathbf{h}_P(z)$ are indicator vectors of the sets of elements dominated by w and z , respectively, in P and $\mathbf{h}_{P'}(w) = \mathbf{h}_P(w) + \mathbf{1}_c$ and $\mathbf{h}_{P'}(z) = \mathbf{h}_P(z) + \mathbf{1}_c$. Then $\mathbf{h}_{P'}(p) = \mathbf{h}_{P'}(w) + \mathbf{h}_{P'}(z) - \mathbf{1}_c = \mathbf{h}_P(w) + \mathbf{h}_P(z) + \mathbf{1}_c$. Since w and z do not dominate $c^\uparrow = p$, we have $h_P^{(D)}(w) = h_P^{(D)}(z) = 0$, whence $h_{P'}^{(D)}(p) = 1$.

Letting T be the set of elements of P dominated by both w and z and N be the set of elements of P dominated by p but not w or z , we have $\mathbf{h}_P(p) = \mathbf{h}_P(w) + \mathbf{h}_P(z) + \mathbf{1}_N - \mathbf{1}_T$. Since $c^\uparrow = p$ we have $h_P^{(D)}(w) = h_P^{(D)}(z) = |T \cap D| = 0$ and $|N \cap D| = 1$, whence $h_P^{(D)}(p) = 1$.

It follows that $h_P^{(D)}(p) + h_{P'}^{(D)}(p) = 2$. Also, $\mathbf{h}_P(p)$ is the indicator vector of $[c, p] \setminus \{c\}$ and by Theorem 2424, the only elements of $[c, p] \setminus \{c\}$ in a diagonal adjacent to D are those which are adjacent to c or p . There are two such elements, giving $\sum_{D' \in \mathcal{D}} h_P^{(D')}(p) = 2$, as needed.

- (c) The element p is not a neck element of P' . By definition, $\mathbf{h}_P(p)$ and $\mathbf{h}_{P'}(p)$ are indicator vectors of the sets of elements nonstrictly dominated by p in P and P' , respectively. We now consider two cases based on whether p dominates c .

Subcase (i): The element p does not dominate c . Then p cannot dominate any element of D , so $h_P^{(D)}(p) + h_{P'}^{(D)}(p) = 0$. Now suppose for the sake of contradiction that there exists some $d \in D' \in \mathcal{D}$ dominated by p . By Theorem 2424, the element d is adjacent to some element c' of $D(c)$, and by assumption cannot dominate c' . Thus c' covers d . It follows by Theorem 20 that d is in the d -interval $[c'^\downarrow, c']$, whence p dominates $c'^\downarrow$ and hence c , a contradiction. So $\sum_{D' \in \mathcal{D}} h_P^{(D')}(p) = 0$, as desired.

Subcase (ii): The element p dominates c . Then $h_P^{(D)}(p) + h_{P'}^{(D)}(p) = 2h_P^{(D)}(p) + 1$. Let $c = c_0 < \dots < c_k$ be the elements of D dominated by p in P' , so that $k = h_P^{(D)}(p)$. Each d -interval $[c_i, c_{i+1}]$ contains two elements in some $D' \in \mathcal{D}$. Since p is not a neck element in P' , we have $p > c_k$ with strict inequality. Also, c_k is covered by exactly one element which is in the principal ideal generated by p , since otherwise $c_k^\uparrow$ would exist and thus need to be in the principal ideal generated by p . This gives $2k + 1$ elements counting towards $\sum_{D' \in \mathcal{D}} h_P^{(D')}(p)$. Also, every element counting towards this sum has been accounted for; if $d \in D'$ for $D' \in \mathcal{D}$ is dominated by p then d is adjacent to $c' \in C$ by Theorem 2424, and if c' is not dominated by p then d is either a side element of $[c'^\downarrow, c']$, in which case d covers $c'^\downarrow = c_k$ and thus has been counted, or d is a neck element of $[c'^\downarrow, c']$ which is dominated by p and hence covered by some element other than c' , contradicting Theorem 20. It follows that $\sum_{D' \in \mathcal{D}} h_P^{(D')}(p) = 2k + 1 = 2h_P^{(D)}(p) + 1$, as desired.

As remarked earlier, these cases are exhaustive, so the proof is complete. $\square$

Remark 43. It follows from the analysis of the evolution of $\mathbf{h}_P(p)$ over the RSK insertion process in the proof of Theorem 35 that $\mathbf{h}_P(p)$ has nonnegative entries. Thus we may interpret hook vectors as multisets of diagonals.

6 Proof of multivariate hook length formula

From this point, the proof of Theorem 15 follows that of Pak [13].

Proof of Theorem 15. For a d -complete poset P with n elements, we define two polytopes $P_{\text{fillings}} = P_{\text{fillings}}(\mathbf{x})$ and $P_{\text{RPP}} = P_{\text{RPP}}(\mathbf{x})$ depending on positive real parameters x_D corresponding to diagonals of P in the n -dimensional real space of vectors with entries indexed by elements of P . The polytope P_{fillings} will be a subset of $\mathcal{F}_P$, and the polytope P_{RPP} will be a subset of $\mathcal{F}'_P$. We will find that RSK_P gives a volume-preserving map between P_{fillings} and P_{RPP} which yields our result.

The polytope P_{fillings} consists of all vectors $[t_p]_{p \in P}$ satisfying:

- $t_p \geq 0$ for all $p \in P$;
- $\sum_{p \in P} H_p(\mathbf{x}) t_p \leq 1$.

Since P_{fillings} is a rescaling of the standard n -simplex, the volume of P_{fillings} is

$$\frac{1}{n!} \prod_{p \in P} \frac{1}{H_p(\mathbf{x})}.$$

The polytope P_{RPP} consists of all vectors $[s_p]_{p \in P}$ satisfying:

- $s_p \geq 0$ for all $p \in P$;
- $s_p \geq s_q$ whenever $p \leq_P q$;
- $\sum_{p \in P} x_{D(p)} s_p \leq 1$.

This is the union of polytopes defined for each linear extensions T of P given by $p_1 > \dots > p_n$ as the set of all vectors satisfying:

- $0 \leq s_{p_1} \leq \dots \leq s_{p_n}$;
- $\sum_{i=1}^n x_{D(p_i)} s_{p_i} \leq 1$

We prove by induction on the dimension of space that such a polytope has volume $\frac{1}{n!} \text{weight}(T)$. The only vertex with nonzero s_{p_1} coordinate is given by $s_{p_1} = \dots = s_{p_n}$ and $\sum_{i=1}^n x_{D(p_i)} s_{p_i} = 1$, and thus is located at $(\sum_{i=1}^n x_{D(p_i)})^{-1} \mathbf{1}$, where $\mathbf{1}$ is the vector of all 1s. Inductively this gives a volume of

$$\frac{1}{n!} \prod_{i=1}^n \left(\sum_{j=i}^n x_{D(p_j)} \right)^{-1} = \frac{1}{n!} \text{weight}(T),$$

as claimed. Thus P_{RPP} has volume

$$\frac{1}{n!} \sum_T \text{weight}(T),$$

where the sum is over linear extensions T of P .

The map RSK_P is volume-preserving and bijective between $\mathcal{F}_P$ and $\mathcal{F}'_P$. We claim that RSK_P gives a bijection between P_{fillings} and P_{RPP} . Let $[t_p]_{p \in P} \in P_{\text{fillings}}$, and let $[s_p]_{p \in P} = \text{RSK}_P([t_p]_{p \in P})$. Let $\mathcal{D}$ be the set of diagonals of P . By Theorem 35, we have

$$\begin{aligned} \sum_{p \in P} x_{D(p)} s_p &= \sum_{D \in \mathcal{D}} x_D S_P(D) \\ &= \sum_{D \in \mathcal{D}} \sum_{p \in P} x_D h_P^{(D)}(p) t_p \\ &= \sum_{p \in P} t_p \sum_{D \in \mathcal{D}} x_D h_P^{(D)}(p) \\ &= \sum_{p \in P} H_p(\mathbf{x}) t_p, \end{aligned}$$

so $[t_p]_{p \in P} \in P_{\text{fillings}}$ if and only if $[s_p]_{p \in P} \in P_{\text{RPP}}$, as desired.

It follows that

$$\sum_T \text{weight}(T) = \prod_{p \in P} \frac{1}{H_p(\mathbf{x})},$$

completing the proof. □

A Toggle operations and RSK

We refer the readers to [20] for the background on RSK. We recall that RSK is a bijective map between $n \times n$ matrices of nonnegative integers, which can be viewed as fillings of a square Young diagram, and pairs of semistandard Young tableaux of the same shape.

Example 44. Consider the following filling of $\lambda = (3, 3, 3)$:

1	0	2
0	2	0
1	1	0

Its corresponding two-line array is

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 3 & 3 & 2 & 2 & 1 & 2 \end{pmatrix}.$$

After insertion, we obtain two SSYT's

$$P = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 & & \\ \hline 3 & & & \\ \hline \end{array}, \quad Q = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & & \\ \hline 3 & & & \\ \hline \end{array}.$$

On the other hand, pick any ordering of the squares of λ $(i_1, j_1), (i_2, j_2), \dots$ such that (i, j) appears before $(i+1, j)$ and $(i, j+1)$ for all i, j . Let $(t_{i,j})$ be any filling of λ with nonnegative integers. One can construct a RPP of shape λ with entries $(p_{i,j})$ recursively as follows.

1. If $\lambda = (1)$, set $p_{1,1} = t_{1,1}$.
2. Suppose we have built a RPP with the squares $(i_1, j_1), \dots, (i_{\ell-1}, j_{\ell-1})$,
 - (a) set $p_{i_\ell, j_\ell} = \max(p_{i-1, j}, p_{i, j-1}) + t_{i_\ell, j_\ell}$,
 - (b) for any other square (i, j) in the same diagonal as (i_ℓ, j_ℓ) , toggle this square by replacing $p_{i,j}$ with

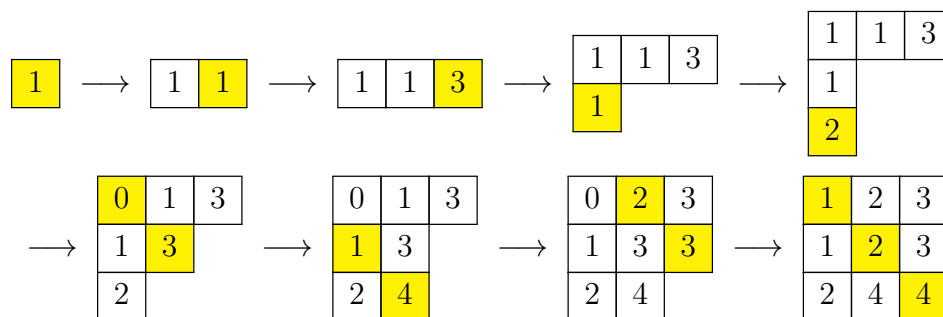
$$\max(p_{i-1, j}, p_{i, j-1}) + \min(p_{i+1, j}, p_{i, j+1}) - p_{i,j}.$$

If any of $p_{i-1, j}, p_{i, j-1}, p_{i+1, j}, p_{i, j+1}$ is not in the RPP, we take the value to be 0.

Example 45. Consider the same filling as in Theorem 44, and order the squares as follows:

1	2	3
4	6	8
5	7	9

Then, the above procedure is carried out as follows.



In fact, the resulting RPP corresponds to the two SSYT in Theorem 44. The lower triangular part of this RPP is the Gelfand–Tsetlin pattern

$$\begin{array}{ccc} 4 & 2 & 1 \\ & 4 & 1 \\ & & 2 \end{array}.$$

There is a unique SSYT such that the squares containing $1, \dots, i$ form the shape given by the i th row (from bottom to top) of the Gelfand–Tsetlin pattern. This SSYT is the SSYT P in Theorem 44. Similarly, the upper triangular part of this RPP is the Gelfand–Tsetlin pattern

$$\begin{array}{ccc} 4 & 2 & 1 \\ & 3 & 2 \\ & & 3 \end{array}.$$

The corresponding SSYT is the SSYT Q in Theorem 44.

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